Notre Dame Journal of Formal Logic
Volume XIX, Number 2, April 1978
NDJFAM

MODEL THEORY OF ALTERNATIVE RINGS

BRUCE I. ROSE

Introduction Recently much work has been done in applying various techniques developed in logic to the study of associative rings [4, 6, 9, 18, 42]. As a result of this inquiry we have a better understanding of what certain general model theoretic properties mean in well-known mathematical contexts. In this paper*, although we continue this program of examining logic in the context of ring theory, we are concerned with a larger class of rings- alternative rings. The class of alternative rings is axiomatizable by the standard axioms of ring theory with the associative axiom replaced by the sentence:

$$\forall x \forall y((xx)y = x(xy) \wedge (yx)x = y(xx)).$$

Note that an alternative ring may be associative. A very useful characterization of alternative rings which shows their relationship to associative rings is Artin's Theorem [29]:

A ring is alternative if and only if all of its subrings generated by two elements are associative.

The canonical examples of alternative rings are the Cayley-Dickson algebras. Section 1 contains a brief introduction to Cayley-Dickson algebras.

We begin the mathematics of this paper in section 2 with a model theoretic exploration of split Cayley-Dickson algebras. We first show that

*This paper was the author's Ph.D. thesis in Mathematics, University of Chicago, 1976. The dissertation committee consisted of Professors Israel Herstein, William Tait and Robert Soare. The author appreciates their support. He also wishes to acknowledge his debt to Professor John Baldwin for many substantive discussions about this paper and more generally for his help and encouragement. The many helpful suggestions by Professor Michael Slater were appreciated and have been incorporated in this paper.

The author is grateful for the financial support received by way of a National Science Foundation graduate fellowship (1972-1975). He thanks Professor Angus Macintyre and Maureen Fitz-Gerald for their help in his development as a mathematician.

Received April 4, 1977

this class of structures is finitely axiomatizable. We then axiomatize split Cayley-Dickson algebras over algebraically closed fields and show that this theory is $\aleph_1$-categorical, model complete, and the model completion of the theory of Cayley-Dickson algebras. We then use these results to prove a Hilbert Nullstellenstatz-like theorem for Cayley-Dickson algebras. No algebraic proof of this result is known.

Our primary concern in section **3** is with stability in alternative rings. Stability is a model theoretic concept which was developed by Morley [23] and Shelah [30, 31] in attempting to determine the number of models of a first order theory. Understanding the properties of stable theories is a major occupation of model theorists. To this end we restrict our attention to alternative rings. In particular we show that if R is a stable ring and J(R) is the Jacobson radical of R, then R/J(R) has the descending chain condition on right ideals. This and the information obtained in section 2 enable us to deduce theorems on the structure of stable, ω-stable and $\aleph_1$-categorical rings.

Section **4** is concerned with $\aleph_0$-categorical alternative rings. In it we generalize many of the theorems of [4] on $\aleph_0$-categoricity in associative rings to $\aleph_0$-categoricity in alternative rings. For example, we show that an $\aleph_0$-categorical alternative ring of arbitrary infinite cardinality can have neither the ascending nor the descending chain condition on right ideals.

This paper was written to be accessible to anyone familiar with the Wedderburn-Artin structure theory for associative rings and model theory as contained in the first half of [28]. For ring theoretic concepts or notation see [11]. For further information on alternative rings see [21] or [29]. For further model theoretic information see [5] or [28]. In this paper we will take the term nonassociative to mean not associative. All rings, unless stated otherwise, are assumed to be alternative. At times this will be emphasized by our using the phrase "alternative ring" rather than just "ring". Recall that an alternative ring may be associative. Fields are assumed to be both associative and commutative. Many ring theoretic concepts do not involve associativity and have the same definition for alternative rings as they do for associative rings (e.g., right ideals). One ring theoretic property central to this paper is that of a ring satisfying the ascending (descending) chain condition on right ideals. A ring has the *ascending (descending) chain condition on right ideals* if it contains no infinite properly ascending (descending) chain of right ideals.

A first order formula of ring theory is a formula built up in the natural way using only logical connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\forall$, $\exists$ (quantifiers over elements of the ring), = (equality), the ring theoretic function symbols +, $\cdot$, 0, and variables $x, y, z, x_1, \ldots, x_n$. We let R denote ambiguously the set of members of a ring and the structure $\langle R, +, \cdot, 0\rangle$. For a finite sequence $a_1, \ldots, a_n (x_1, \ldots, x_n)$ of elements (variables) we write $\bar{a}(\bar{x})$. If φ is a formula of ring theory (possibly with names for ring elements $a_1, \ldots, a_n$) we write $R \models \varphi(\bar{a})$ to mean φ is true in R of the elements $a_1, \ldots, a_n$. A subset B of R is *first order definable* or definable in first order logic if there is a first order formula of ring theory with one variable, $\varphi(x)$, such that $a \in B$ if and only if $R \models \varphi(a)$. Frequently we denote the subset B of elements of the ring R which satisfy a formula φ, by $\varphi(R)$. A *theory* is any set of sentences (formulas without free variables). The theory of a ring R, Th(R), is the set of all first order sentences of ring theory which are true in R. Two rings are *elementarily equivalent* if they satisfy exactly the same sentences of ring theory. If $A \subset B$ are rings then B is an *elementary extension* of A if any sentence of ring theory with parameters from A that is true in A is also true in B. A ring R is $\aleph_\alpha$-*categorical* if, up to isomorphism, Th(R) has at most one model of cardinality $\aleph_\alpha$. Note that any finite ring is $\aleph_\alpha$-categorical.

Let T be a complete consistent theory and let $F_1(T)$ be the set of formulas in the language of T that contain at most x as a free variable. Let **p** be a maximal subset of $F_1(T)$ such that any finite subset of **p** is consistent with T (i.e., satisfiable in some model of T). The set **p** is called a *type* of T. Let $S(T)$ denote the set of types of T. If A is a model of T, let Th(A, A) denote the set of sentences, in the language of T together with names for all elements of A, that are true in A. A theory T is said to be κ-stable if for any model A of T of cardinality κ, $S(\text{Th}(A, A))$ has cardinality κ. A theory is *stable* if it is κ-stable in some cardinality κ. A ring R is stable (κ-stable) if Th(R) is stable (κ-stable).

1 *Cayley-Dickson algebras* In this section we will present a development of the construction of Cayley-Dickson algebras over an arbitrary field. As a result of the work of Kleinfeld [15] and others, one could simply define Cayley-Dickson algebras as the class of simple (having no proper two-sided ideals) alternative, nonassociative rings. It is hoped that the following exposition will promote a more concrete understanding of what Cayley-Dickson algebras are than does the preceeding statement. A complete and detailed development of Cayley-Dickson algebras may be found in [21] or [29]. The proofs of any assertions that we make in what follows may also be found there. The adjective "Cayley-Dickson" in the phrase "Cayley-Dickson algebra" refers to a procedure for constructing algebras. By Cayley-Dickson algebras (hence-forth **CD** *algebras*) we will mean the alternative, not associative algebras constructed over any given field by this process.

Some **CD** algebras are division algebras; that is, their multiplicative structure is a group (not necessarily associative). For alternative algebras, being a division algebra is equivalent to having an identity element and each non-zero element having an inverse. It is also equivalent to being a simple algebra with no zero divisors. **CD** algebras with zero divisors are called *split Cayley-Dickson algebras*. Split **CD** algebras may be constructed over any field. The construction of a **CD** division algebra over a given field depends, as we shall see, on the elementary properties of the given field.

Fix a field F. We show how to construct **CD** algebras over F. Let $Z_1 = F \oplus F$ and $Z_2 = F(s)$ be a separable quadratic field over F. We may assume that s satisfies an equation of the form $x^2 - x - a$ for some a in F such that $-4a \neq 1$ and $x^2 - x - a$ is irreducible over F. From Z_1 we will eventually construct a split **CD** algebra over F. Constructions on Z_2 will sometimes give us **CD** division algebras.

An involution on an algebra A over F is a linear operator $x \to \bar{x}$ on A satisfying $\overline{(xy)} = \bar{y}\bar{x}$ and $\bar{\bar{x}} = x$ for all x and y in A. Let an involution on Z_1 be given by $\overline{\langle x, y\rangle} = \langle y, x\rangle$ and an involution on Z_2 be given by $\overline{f_1 s + f_2} = f_1(1 - s) + f_2$ for f_1 and f_2 in F. Let Q_i be the set of ordered pairs $\langle x_1, x_2\rangle$ where x_1 and x_2 are in Z_i. We put an algebra structure on Q_i by defining addition and multiplication by scalars in F componentwise and multiplication by

$$\langle x_1, x_2\rangle\langle x_3, x_4\rangle = \langle x_1x_3 + b_1x_4\bar{x}_2, \bar{x}_1x_4 + x_3x_2\rangle \tag{1.1}$$

for all x_1, x_2, x_3, and x_4 in Z_i and some $b_1 \neq 0$ in F. The element $\langle 1, 0\rangle$ is an identity element for Q_i and the set $\{\langle x, 0\rangle : x \in Z_i\}$ is a subalgebra of Q_i isomorphic to Z_i and hence we may identify Z_i with its image. Let $\mathbf{v} = \langle 0, 1\rangle$, then $\mathbf{v}^2 = b_1$ and Q_i may be written as the vector space direct sum $Z_i + \mathbf{v}Z_i$ and given the multiplication:

$$\langle x_1 + \mathbf{v}x_2\rangle\langle x_3 + \mathbf{v}x_4\rangle = \langle x_1x_3 + b_1x_4\bar{x}_2\rangle + \mathbf{v}\langle \bar{x}_1x_4 + x_3x_2\rangle. \tag{1.2}$$

Q_i has an involution given by $\overline{x + \mathbf{v}y} = \bar{x} - \mathbf{v}y$. Each Q_i is a simple associative algebra which is 4-dimensional over its center F. By the classical theorem of Wedderburn [11], Q_i is a division ring or a complete 2×2 matrix ring over F. Q_1 is never a division ring since Z_1 has zero divisors. Necessary and sufficient conditions may be found for Q_2 to be a division ring. These are that there exist no elements c and d in F such that $b_1 = d^2 + cd - ac^2$.

Our goal is near. To construct **CD** algebras C_i, repeat this entire process again. That is, let C_i be the set of ordered pairs $\langle x_1, x_2\rangle$ where x_1 and x_2 are elements of Q_i. Choose a nonzero element $b_2 \in F$. Extend addition and multiplication by scalars of F componentwise. Define multiplication in C_i by (1.1) using b_2 instead of b_1. C_1 and C_2 are **CD** algebras. Each C_i is a simple algebra which is 8-dimensional over F. The noncommutativity of Q_i entails the nonassociativity of C_i. However, C_i is alternative. Were we to repeat this process again, the resulting algebra would not be alternative.

C_2 is a division ring if and only if Q_2 is a division ring and there do not exist elements c, d, e, and f in F, such that

$$b_2 = c^2 + cd - ad^2 - b_1e^2 - b_1ef + ab_1f^2.$$

We again emphasize that whether or not a **CD** division algebra exists over a particular field depends on whether or not a certain first order sentence does or does not hold in that field. Further, two **CD** division algebras over elementarily equivalent fields are elementarily equivalent if the set of formulas of ring theory satisfied in a given field by the three elements chosen to define the other **CD** algebra is the same as the set of formulas satisfied by the three elements used to define the other **CD** algebra (i.e., the defining coefficients realize the same 3-type).

C_1, for any choice of multiplication constants, is always a split **CD** algebra and can be constructed from any field. An amazing and useful fact about split **CD** algebras is that the split **CD** algebra over any field is unique [27; 52]. Hence we may think of a split **CD** algebra as having a standard construction. Let F_2 be the complete 2×2 matrix ring over F with involution given by $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Any split **CD** algebra over F is *isomorphic to* $F_2 + \mathbf{v}F_2$ *where* $\mathbf{v}$ *is a new formal element and multiplication* is given by (1.2). Another important property of **CD** algebras is that the only elements of a **CD** algebra, R over F, which associate with all pairs of elements of R, are the elements of F. F is also the set of elements which commutes with all elements of R.

2 *Model theory of split Cayley-Dickson algebras* We begin this section by showing that the theory of Cayley-Dickson algebras is finitely axiomatizable in the language of ring theory. Using this axiomatization we may axiomatize the theory of split **CD** algebras over fields modeling a given theory. This axiomatizable construction preserves many model theoretic properties. As an example, we show that it preserves completeness of theories. In the later part of this section we restrict our attention to the theory of split **CD** algebras over an algebraically closed field. We show that this theory is $\aleph_1$-categorical (fixed characteristic), model complete, the model completion of the theory of **CD** algebras, but does not admit elimination of quantifiers. From the model completeness of this theory we deduce a Nullstellensatz-like theorem for Cayley-Dickson algebras. We have been unable to find in the literature an algebraic proof of this same result.

Presently we will present an axiom system for Cayley-Dickson algebras. We are aided in our attempt by the very strong theorems that have been proved about alternative rings. In particular, we know that the only simple alternative nonassociative rings are **CD** algebras. Unfortunately, in general the property of being a simple ring is not axiomatizable [3].

In the axiomatization to follow, we are guided by the above characterization of **CD** algebras and we replace the property of being simple by other axiomatizable properties that imply simplicity in this case. More specifcally, we require *our ring to be semisimple (that is, its Jacobson radical* is zero. The Jacobson radical of an alternative ring is that ring's maximal quasi-regular ideal. A detailed discussion of the Jacobson radical may be found in section **3**), and an algebra no more than 8-dimensional over its center. We also require its center to be a field. In [32] Slater proved that any semisimple ring satisfying the descending chain condition on right ideals is a finite direct sum of simple associative rings and Cayley-Dickson algebras. Since any finite dimensional algebra satisfies the descending chain condition on right ideals, our ring has this property. By requiring our ring to be nonassociative, it forces at least one summand to be a **CD** algebra. By dimension counting, since a **CD** algebra is exactly

8-dimensional over its center and its center is contained in the center of our ring, our ring is a **CD** algebra.

Another difficulty that we have is that the radical of an alternative ring is not *a priori* definable. (We discuss this problem in section **3**.) However, in a finite dimensional algebra, the radical is nilpotent [29]. (An ideal is *nilpotent* if there exists an integer n such that the product of any n elements, associated in any manner, is zero.) In a non-zero nilpotent ideal there exists a non-zero element, say x, such that for any element y of the ring $(xy)^2 = 0$ (Artin's Theorem is used here). Since **CD** algebras contain no non-zero elements with this property, we may require that such bad elements do not exist and thus force our ring to be semisimple. These ideas can be put together to axiomatize the class of **CD** algebras. To obtain axioms for split **CD** algebras, we require the existence of non-zero elements whose product is zero. More concretely, consider the following sentences.

Let φ_1 be the conjunction of axioms for the theory of rings (except for the associative law). (We note that this may be written as a universal-existential sentence.) Let φ_2 be the sentence

$$\forall x_1 \forall x_2 \exists x_3 \exists x_4 \exists x_5 ((x_1x_1)x_2 = x_1(x_1x_2) \wedge (x_1x_2)x_2 = x_1(x_2x_2) \wedge (x_3x_4)x_5 \neq x_3(x_4x_5)).$$

φ_2 forces a ring to be alternative but not associative. Let $\psi(x)$ be the formula $\forall y(xy = yx)$. The center of an arbitrary alternative ring is defined to be the set of elements which both commutes and associates (left, middle, and right) with every element of the ring. However, the properties postulated for the set $\psi(x)$ are sufficient to force R to be a **CD** algebra. This in turn justifies our referring to the set $\psi(\mathrm{R})$ as the center of R [27:48]. Let φ_3 be the sentence

$$\forall x \forall z \exists y (\neg\psi(x) \vee \neg\psi(z) \vee x = 0 \vee xy = z).$$

φ_3 enables us to deduce that the center of our ring is a field.

Let φ_4 be the sentence: $\exists x(\psi(x) \wedge x \neq 0)$. The sentence φ_4 states that the center of our ring is not zero. Let φ_5 be the sentence

$$\forall x_1 \forall x_2 \ldots \forall x_9 \exists y_1 \ldots \exists y_9 \left[\left(\bigwedge_{i=1}^{9} \psi(y_i)\right] \wedge \left[\bigvee_{i=1}^{9} y_i \neq 0\right] \wedge \left[\sum_{i=1}^{9} y_i x_i = 0\right]\right).$$

φ_5 states that the ring has dimension at most eight over its center. Let φ_6 be the sentence

$$\forall x \exists y (x = 0 \vee (xy)^2 \neq 0).$$

As explained above, the role of φ_6 is to ensure semisimplicity. Let φ' be the conjunction of the φ_i, $i = 1, \ldots, 6$. Let φ_7 be the sentence

$$\exists x \exists y (x \neq 0 \wedge y \neq 0 \wedge xy = 0).$$

φ_7 keeps a ring from being a division algebra.

Finally, let φ be $\varphi' \wedge \varphi_7$. We have already argued,

Theorem 2.1 *If* R *is a structure in the language of rings, then* $\mathrm{R} \models \varphi$ *if and only if* R *is a split Cayley-Dickson algebra over its center. Further* $\mathrm{R} \models \varphi'$ *if and only if* R *is a Cayley-Dickson algebra over its center.*

The notion of the *relativization* of one formula by another is well-known [5]. Relativization by a formula $\psi(x)$ with one free variable is defined on all formulas in the following recursive way. If ψ is atomic then the relativization of φ by ψ, φ^{ψ}, is φ; $(\varphi \vee \theta)^{\psi} = \varphi^{\psi} \vee \theta^{\psi}$; $(\varphi \wedge \theta)^{\psi} = \varphi^{\psi} \wedge \theta^{\psi}$; $(\neg\varphi)^{\psi} = \neg\varphi^{\psi}$; $(\forall x\varphi)^{\psi} = \forall x(\psi(x) \rightarrow \varphi^{\psi})$; and $(\exists x\varphi)^{\psi} = \exists x(\psi(x) \wedge \varphi^{\psi})$.

Let $\psi(x)$ be the formula $\forall y(xy = yx)$. If T is a theory of fields, let T^{ψ} denote the collection of sentences in T relativized by ψ. $\varphi' \cup T^{\psi}$ ($\varphi \cup T^{\psi}$) is a theory whose models are (split) Cayley-Dickson algebras over fields that satisfy the axiom set T.

As the split Cayley-Dickson algebra construction preserves various model theoretic properties, this approach to axiomatizing theories of split **CD** algebras enables us to find new "natural" theories that are axiomatizable and have desirable model theoretic properties. As an example, we will show that if T is a complete theory of fields, then $\varphi \cup T^{\psi}$ is a complete theory of split **CD** algebras. (A theory T is *complete* if for any A and B, models of T, A is elementarily equivalent to B.) We require the following results: Let SCD(F) denote the split **CD** algebra over the field F. Recall that the split **CD** algebra over any field is unique to within isomorphism [29; 52]. See [5] for the definition of an ultrapower.

Lemma 2.1 *The functor* SCD *commutes with taking ultrapowers, i.e., if* D *is an ultrafilter on* I, *then* $\mathrm{SCD}(F^I/D) \simeq \mathrm{SCD}(F)^I/D$.

Proof: Let $R = \mathrm{SCD}(F^I/D)$ and $S = \mathrm{SCD}(F)^I/D$. By Łos's Theorem [5] and Theorem 2.1, S is a split **CD** algebra. Łos's Theorem may also be used to show that the center of both R and S is the field F^I/D. By the uniqueness of a split **CD** algebra over a given field we have the desired isomorphism.

Theorem 2.A (Keisler and Shelah [5]) *Two structures are elementarily equivalent if and only if they have isomorphic ultrapowers.*

Theorem 2.2 *Let T be a theory of fields. T is complete if and only if $\varphi \cup T^{\psi}$ is complete.*

Proof: Since any model F of T may be extended to a model R of $\varphi \cup T^{\psi}$ such that F is the center of R, if $\varphi \cup T^{\psi}$ is a complete theory, then so is T.

Conversely, if T is complete, let R and S be models of $\varphi \cup T^{\psi}$. The centers of R and S, say F and G respectively, as models of T are elementarily equivalent. By Theorem 2.A there is some ultrafilter D such that $F^I/D \simeq G^I/D$. This isomorphism may be extended so that $\mathrm{SCD}(F^I/D) \simeq \mathrm{SCD}(G^I/D)$. An application of Lemma 2.1 allows us to conclude that $\mathrm{SCD}(F)^I/D \simeq \mathrm{SCD}(G)^I/D$. Theorem 2.A implies that SCD(F) is elementarily equivalent to SCD(G). Since split **CD** algebras over a given field are unique we have $R \simeq \mathrm{SCD}(F)$ and $S \simeq \mathrm{SCD}(G)$. In particular, R is elementarily equivalent to S, entailing the completeness of the theory $\varphi \cup T^{\psi}$.

In the remainder of this section we will be concerned with split **CD** algebras over algebraically closed fields. Let ACF denote the theory of algebraically closed fields and let ACF_n denote ACF of characteristic n. Since ACF_n is complete and recursively axiomatizable [28], by Theorem 2.2 we know that the theory of split **CD** algebras over models of ACF_n is complete and we have a recursive axiomatization of it. It is also well-known that ACF_n is $\aleph_1$-categorical. The following Theorem is central to our interest in split **CD** algebras and will play a role in the results of section **3**.

Theorem 2.3 *The theory of split Cayley-Dickson algebras over an algebraically closed field of characteristic n is $\aleph_1$-categorical.*

Proof: Let R and S be models of $\varphi \cup ACF_n^{\psi}$ of cardinality $\aleph_1$. The centers of R and S are algebraically closed fields of cardinality $\aleph_1$ and hence isomorphic. As split **CD** algebras over a given field are unique, this isomorphism may be extended to one between R and S.

A theory T is *model complete* if whenever R and S are models of T and $R \subset S$, then S is an elementary extension of R. We next prove that the theory of split **CD** algebras over a model of ACF is model complete. A formula is universal-existential if it is of the form—$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m \psi$, where ψ is quantifier-free. A theory T is universal-existential if each of its sentences is equivalent in T to a universal-existential sentence. In [16] Lindström proved that an $\aleph_1$-categorical universal-existential theory is model complete. Thus to show that $T' = \varphi \cup ACF^{\psi}$ is model complete, it suffices to find a universal-existential axiomatization for T'.

We will use the following notion in obtaining our universal-existential axiomatization. Let R be a split **CD** algebra. Consider the subset of R, $\mathbf{C} = \{e_{ij}, e_{ij}\mathbf{v}: i, j = 1, 2\}$. We will call C an **SCD** set if the elements of **C** are distinct and have the following multiplication in R, where δ_{jk} is the Kronecker delta:

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \ .$$

$$(\mathbf{v}e_{ij})e_{kl} = \delta_{li}\mathbf{v}e_{kj} \ .$$

$$e_{ij}(\mathbf{v}e_{kl}) = \begin{cases} -\delta_{jk}\mathbf{v}e_{il}, & \text{if } i \neq j \\ \delta_{2k}\mathbf{v}e_{2l}, & \text{if } i = j = 1 \\ \delta_{1k}\mathbf{v}e_{1l}, & \text{if } i = j = 2. \end{cases}$$

$$(\mathbf{v}e_{ij})(\mathbf{v}e_{kl}) = \begin{cases} -\delta_{lj}e_{ki}, & \text{if } i \neq j \\ \delta_{l2}e_{k2}, & \text{if } i = j = 1 \\ \delta_{l1}e_{k1}, & \text{if } i = j = 2. \end{cases}$$

We will see that if **C** is any **SCD** set contained in R, then **C** generates R as a vector space over the center of R, F. By computation that is built into the multiplication table that **C** satisfies, the subalgebra generated by **C** over F is isomorphic in the obvious way to $F_2 + \mathbf{v}F_2$ (as in section **1**) where F_2 is the 2×2 matrix ring over F and multiplication in $F_2 + \mathbf{v}F_2$ is given by (1.2). In particular, **C** is linearly independent over F. Thus since **C** contains 8 elements and R is 8-dimensional over F, C generates R as a vector space over F. As this is true for any **SCD** set, an element is in the center of R if and only if it commutes with all of the elements in some **SCD** set.

We use this fact to get a universal-existential axiomatization of split **CD** algebras. Let $\psi'(y; \overline{y}_{ij}, w)$ be a quantifier-free formula which says that $\{\overline{y}_{ij}, w\overline{y}_{ij}\}$ is a Cayley-Dickson set and that y commutes with each element in it. Let φ_4' be the sentence

$$\exists x \exists \overline{e}_{ij}\, \exists v \left(\bigwedge_{k=1}^{9} \psi'(x; \overline{e}_{ij}, v) \wedge x \neq 0 \right).$$

φ_4' is equivalent to φ_4 in a split **CD** algebra but is universal-existential. Similarly let φ_5' be the sentence

$$\forall \overline{x}_k \exists \overline{y}_k \exists \overline{e}_{ij}\, \exists v \left(\left[\bigwedge_{k=1}^{9} \psi'(y_k; \overline{e}_{ij}, v) \right] \wedge \left[\bigvee_{k=1}^{9} y_k \neq 0 \right] \wedge \left[\sum_{k=1}^{9} y_k x_k = 0 \right] \right).$$

The sentences φ_i, $i = 1, \ldots, 7$ with φ_4' and φ_5' replacing φ_4 and φ_5 are a universal-existential axiomatization of split **CD** algebras. These sentences together with the sentences:

$$\forall y_1 \ldots \forall y_n \exists x \exists \overline{e}_{ij}\, \exists v \left(\bigwedge_{l=1}^{n} \psi'(y_l; \overline{e}_{ij}, v) \rightarrow \psi'(x; \overline{e}_{ij}, v) \wedge (x^n + y_1 x^{n-1} + \ldots + y_n = 0) \right)$$

for each $n > 0$, are a universal-existential axiomatization for the theory of split **CD** algebras over an algebraically closed field. We have shown

Theorem 2.4 *The theory of split Cayley-Dickson algebras over an algebraically closed field is model complete.*

We could have initially developed the idea of a **SCD** set to axiomatize split **CD** algebras. The major disadvantage to this approach is that our proof of the model completeness of $\varphi \cup ACF^{\psi}$ would not work.

A notable application of model theory to algebra has been to use the model completeness of algebraically closed fields to prove Hilbert's Nullstellensatz. We prove a similar theorem here. In this situation, however, we know of no algebraic proof of the same result. We first require the following lemma.

Lemma 2.2 *If* R *is a* **CD** *algebra over an algebraically closed field then* R *is a split* **CD** *algebra.*

Proof: Let F be the algebraically closed field which is the center of R. If R is not a split **CD** algebra then R is a **CD** division algebra. Consider the subalgebra G of R generated by any non-central element over F. By Artin's Theorem, G is associative. Since R is finite dimensional over F and since R is a division ring, G is a division ring. As G was generated over R by one element, G is commutative. In summary, G is a finite dimensional field extension of F. Since F is algebraically closed, this implies that R = F - a contradiction.

Lemma 2.3 *If* R *and* S *are Cayley-Dickson algebras and* R *is contained in* S, *then the center of* R *is contained in the center of* S.

Proof: If R is a split **CD** algebra then R contains a **SCD** set C. The center of a split **CD** algebra is the subset which commutes with any **SCD** set contained in it. The set C is also a **SCD** set for S. Thus the center of R is contained in the center of S.

Suppose now that R is a **CD** division algebra. Let F be the center of R and G be the center of S. Let x and y be elements of R which do not commute. Let D be the subalgebra of S generated by F, G, x, and y. By Artin's Theorem, D is an associative noncommutative subring of S. Since x and y are elements of R and R is a finite-dimensional division algebra, D contains no zero-divisors and all non-zero elements of D have inverses in D. Necessarily, D is an associative division ring. The center of D is the subfield of S generated by F and G, FG. The dimension of a finite-dimensional division algebra over its center is a square [11; 96]. Since the dimension of S over G is 8, the dimension of D over FG must be 4. By similar dimension counting, FG = G. That is, the center of R is contained in the center of S.

Corollary 1 *Let* R *be a Cayley-Dickson algebra over* F. *Let* P *be a finite set of polynomial equations and inequations over* R *in* d *variables.* P *has a solution in some Cayley-Dickson algebra containing* R *if and only if* P *has a solution in the split Cayley-Dickson algebra over the algebraic closure of* F.

Proof: The property of P having a solution is expressible by some sentence θ of the form $\exists x_1 \ldots \exists x_d \psi$, where ψ is a quantifier-free formula in the language of ring theory. Suppose that θ holds in some **CD** algebra S containing R. Let G be the center of S and let $\tilde{G}$ be the algebraic closure of the field G. Let S′ be the tensor product of S and $\tilde{G}$ over G. S′ is an 8-dimensional algebra over its center $\tilde{G}$. Since S′ is an alternative ring but is not associative, S′ must be a **CD** algebra [32]. By Lemma 2.2 we conclude that S′ is a split **CD** algebra since $\tilde{G}$ is an algebraically closed field. Further, by the construction of S′, S is naturally contained in S′.

Let R′ be the split **CD** algebra over $\tilde{F}$, the algebraic closure of F. ACF_n has the joint embedding property since it is a complete theory; that is, if A and B are algebraically closed fields of characteristic n, then there is an algebraically closed field of characteristic n containing both of them. Since R ⊂ S are **CD** algebras and hence simple rings, R and S have the same characteristic. In particular, $\tilde{F}$ and $\tilde{G}$ are algebraically closed fields of the same characteristic. Let H be an algebraically closed field containing both $\tilde{F}$ and $\tilde{G}$. Let T be the split **CD** algebra over H. Construct maps from R′ and S; into T′ by mapping a basis for R′ and a basis for S′ onto a basis for T′.

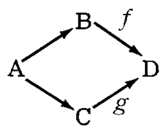
Since θ is an existential sentence holding in S, θ also holds in T′ since T′ ⊃ S′ ⊃ S. Since the theory of split **CD** algebras over an algebraically closed field is model complete (by Theorem 2.4), R′, under the mapping above, is an elementary substructure of T′. In particular, θ must hold in R′. Thus P has a solution in the split **CD** algebra over the algebraic closure of F.

By a result of Slater's [34] we may weaken the hypothesis of Corollary 1 to consider polynomials over prime alternative nonassociative rings (characteristic $\neq 3$). (Also see [46].) Our conclusion would then be that P has a solution in some prime alternative not associative ring containing R if and only if it has a solution in the split **CD** algebra over the algebraic closure of the field of quotients of the center of R.

Let T and T' be theories in the same language. T' is called the *model completion* of T if the following conditions are satisfied:

(1) Any model of T' is a model of T.
(2) Any model of T may be extended to a model of T'.
(3) If A is a model of T, and B and C are models of T' containing A, then there exists a model D of T', elementary monomorphisms f and g such that the following diagram commutes:

(2.1)



Corollary 2 *The theory of split Cayley-Dickson algebras over an algebraically closed field is the model completion of the theory of Cayley-Dickson algebras.*

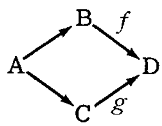
Proof: Condition (1) follows since any split **CD** algebra over an algebraically closed field is, *a fortiori*, a **CD** algebra. Condition (2) follows since, as shown in the proof of Corollary 1, the tensor product of any **CD** algebra with the algebraic closure of its center is a split **CD** algebra over an algebraically closed field.

To see that condition (3) is satisfied, let T be the theory of **CD** algebras and let T' be the theory of split **CD** algebras over an algebraically closed field. By Lemma 2.3, if A is a model of T and B and C are models of T' containing A, then the center of A is contained in the centers of B and C. Let F, G, and H be the centers of A, B, and C respectively. As models of T', B, and C have centers that are algebraically closed fields. Since the theory of algebraically closed fields is the model completion of the theory of fields and F is a field contained in the algebraically closed fields G and H, there exists an algebraically closed field and elementary monomorphisms f and g such that the diagram as in (2.1) commutes.

Let $R = K_2 + \mathbf{v}K_2$ be the standard split **CD** algebras over K. The matrix units of K_2 and $\mathbf{v}$ give us a **SCD** set for R. Pick any **SCD** set in A. The image of this **SCD** set in B and C is an **SCD** set and generates B and C over their centers. Extend the maps f and g from G and H into K to maps f', g' from B and C into R by mapping a **SCD** set of A contained in B and C to the given **SCD** set in R. The maps f' and g' are then monomorphisms from B and C into R. Since, B, C, and R models of the theory of split **CD** algebras over an algebraically closed field and this theory is model complete by Theorem 2.4, f' and g' are elementary monomorphisms.

A theory T admits *elimination of quantifiers* if in T every formula in the language of T is equivalent to a formula without quantifiers. At times, being a model completion can be used to show that a theory admits elimination of quantifiers [28]. We will show that the theory T of split **CD** algebras over an algebraically closed field does *not* admit elimination of quantifiers. We will do this by showing that T is not substructure complete, a property equivalent to admitting elimination of quantifiers [28]. A theory T of rings is *substructure complete* if whenever B and C are models of T and A is a subring contained in both B and C, then there exists a model D of T, a monomorphism f, and an elementary monomorphism g such that the following diagram commutes.

(2.2)



We proceed by showing that the theory of 2×2 matrix rings over an algebraically closed field is not substructure complete. Since any split **CD** algebra over F is isomorphic to $F_2 + \mathbf{v}F_2$ as in section **1**, our proof may be easily adapted to show that the theory of split **CD** algebras over an algebraically closed field is not substructure complete. Let A be the algebraically closed field F. Let B and C be 2×2 matrix rings over F. Consider A contained in B in standard way as the set of scalar matrices over F. Consider A contained in C as the subring generated over F by the matrix unit e_{11}. For no ring D can we obtain a commuting diagram as in (2.2). We have:

Theorem 2.5 *The theory of split Cayley-Dickson algebras over an algebraically closed field does not admit elimination of quantifiers.*

A theory T is called *universal* if every sentence in T is equivalent in T to a sentence of the form $\forall x_1 \ldots \forall x_n \varphi$ where φ is quantifier-free. A result due to Łoś and Tarski is that a theory is universal if and only if all substructures of models of T are models of T [28]. From this it is immediate that the theory of split **CD** algebras is not universal. We may also obtain this result as a corollary of Theorems 2.4 and 2.5 without using the Łoś-Tarski result. Instead, we apply a theorem due to Robinson [28; 67]:

The model completion of a universal theory admits elimination of quantifiers.

To conclude this section we remark that many of the results proved here are also true for the theory of matrix rings over fields. In particular, the theory of an $n \times n$ matrix ring over an algebraically closed field is complete (characteristic m), $\aleph_1$-categorical (characteristic m), model complete, the model completion of the theory of $n \times n$ matrices over a field, and does not admit elimination of quantifiers ($n > 1$). The proof of these facts is somewhat easier than for split Cayley-Dickson algebras since one can prove the model completeness of this theory using algebraic facts without resorting to Lindström's Theorem.

3 *Structure theory for stable rings* The core of this section is our proof of the following two results.

(1) If R is stable and J(R) is the Jacobson radical of R, then J(R) is definable in first order logic.
(2) If R is stable, then R/J(R) has the descending chain condition on right and left ideals.

The nonassociativity of R adds considerable difficulties to the proof of (2) not present in the associative case [4, 6, 9, and 41]. Unlike the associative case in which principal right ideals are definable by a first order formula, in an alternative ring it is difficult to construct right ideals. (For example, in the split Cayley-Dickson algebra $R = F_2 + \mathbf{v}F_2$, the principal right ideal generated by any non-zero element of R equals R. However, if e_{12} is a matrix unit in F_2, then $\mathbf{v}e_{12} \notin e_{12}R$.) In order to construct the right ideals that we will need, we require much algebra. Also, our approach in examining the Jacobson radical involves showing that a stable alternative ring is a Zorn ring. We use this algebraic consequence of stability in defining the Jacobson radical of a stable ring (the defining formula is dependent on the ring). Our proof of the fact that a stable ring is a Zorn ring (and hence of statement [1]) is intertwined with our proof of statement (2). Together statements (1) and (2) yield a structure theorem for stable alternative rings. We then proceed to explore $\aleph_1$-categorical and ω-stable rings. We deduce that any infinite $\aleph_1$-categorical alternative division ring is an algebraically closed field. This extends the result of Macintyre and Shelah that an infinite $\aleph_1$-categorical associative division ring is an algebraically closed field.

Continuing in this vein we then show that an infinite nonassociative simple ring is $\aleph_1$-categorical if and only if it is ω-stable if and only if it is a split Cayley-Dickson algebra over an algebraically closed field. This fact and statement (2) enable us to completely describe the class of semisimple $\aleph_1$-categorical rings. In this section we will use the following characterization of stability, due to Shelah [31].

Theorem 3.A *A first order theory T is unstable (i.e., T is not stable) if and only if there is a formula* $\varphi(x; y_1, \ldots, y_k)$, *a model* A *of T, elements* $b_0, b_1, \ldots$ *of* A, *and k-ary sequences* $\bar{a}_0, \bar{a}_1, \ldots$ *of* A^k *such that*

$$A \models \varphi(b_n; \bar{a}_j) \text{ if and only if } j > n.$$

We will begin the work of this section by deducing some information about the Jacobson radical of a stable ring. Before doing this we present some relevant definitions and show how the Jacobson radical of an alternative ring differs from the Jacobson radical of an associative ring.

Let $\mathsf{r}(a)$ denote the principal right ideal generated by an element a in R. We note that $\mathsf{r}(a)$ consists of the totality of elements of R of the form

$ia + \sum aU_j$, where i is a non-negative integer and each U_j is a product of a product of a finite number of right multiplications, R_x: $a \mapsto ax$. Because of the nonassociativity of R, $r(a)$ is generally not definable by a formula in first order logic. An element a in R is called *right quasi-regular*, if there is an element b in R (its *right quasi-inverse*) such that $a + b + ab = 0$. We define the *Jacobson* (Jacobson-Kleinfeld-Smiley) *radical* of an alternative ring R to be the maximal ideal, J(R), such that $a \in$ J(R) implies that each element of $r(a)$ is right quasi-regular. J(R) exists for any ring R and contains all of the elements of R that generate right quasi-regular right ideals [36]. A ring is semisimple if J(R) = (0). By the work of Zhevlakov [38, 40, and 41], it is equivalent to define J(R) as the intersection of the regular maximal right ideals of R. (A right ideal I of R is *regular* if there exists an element e in R such that $x - ex \in$ I for all x in R. In a ring with identity every right ideal is regular.)

Thus, the definitions of the Jacobson radical of associative and alternative rings are identical—in English (or higher order logic). In contrast, although the Jacobson radical of an associative ring is easily seen to be definable by the simple first order formula: $\forall y \exists z (xy + z + xyz = 0)$, the radical of an alternative ring is not *a priori* first order definable. From the definition of the radical of an alternative ring we can however, describe a countable set of formulas that is satisfied only by elements in the radical. This set consists of all formulas $\psi_{i,j,n_1,\ldots,n_k}(x)$:

$$\forall y \forall x_{11} \ldots \forall x_{1n_1} \forall x_{21} \ldots \forall x_{kn_k} \exists z \left(y = x + \ldots + x + \sum_{i=1}^{k} ((x x_{ii_1}) \ldots) x_{in_i} \rightarrow y + z + yz = 0 \right).$$

As we will refer to this result in a later section, we label it as a theorem.

Theorem 3.1 *There is a countable set of formulas of first order logic* $\{\psi_i\}_{i\in\omega}$ *such that given any alternative ring* R *and* $a \in$ R: $a \in$ J(R) *if and only if* R $\models \psi_i(a)$ *for all* $i \in \omega$.

We require additional definitions. A right ideal I is *nil* provided each element $a \in$ I is *nilpotent*; that is, there is an integer $n = n(a)$ such that $a^n = 0$. A right ideal I is *nilpotent* provided there is an integer n such that any product of n elements of I *in any association* is equal to zero. A nilpotent right ideal is nil, but the converse is not true. J(R) contains all nil and hence nilpotent right ideals of R [36]. A ring is *locally nilpotent* provided any finite subset of the ring generates a nilpotent subring. An element $a \in$ R is (right) *properly nilpotent* provided for all $r \in$ R, ar is nilpotent. It is an open question due to Zorn (equivalent to the Köthe conjecture) as to whether the set of properly nilpotent elements of a ring is an ideal of that ring. (Later in this section we settle this question for the class of stable rings.) We may now prove:

Theorem 3.2 *If* R *is stable, then* J(R) *is nil of bounded index, that is, there is an integer* n *such that for all* x *in* J(R), $x^n = 0$. J(R) *is also locally nilpotent.*

Proof: Let $x \in J(R)$. Let $a_i = x^i$, $i = 1, 2, \ldots$, and let $\varphi(z; y)$ be the formula $\exists w(y = zw \wedge y \neq z)$. Clearly $R \models \varphi(a_n; a_j)$ if $j < n$. Since R is stable, by Theorem 3.A, the converse to this statement is false. Hence there exist $j < n$ such that $R \models \varphi(a_n; a_j)$. That is for some w in R, we have

$$x^j = x^n w = (x^j x^{n-j})w \,. \tag{3.1}$$

By Artin's Theorem any two elements of an alternative ring generate an associative subring. This and the fact that J(R) is a right ideal enable us to rewrite (3.1) as $x^j = x^j y$, where $y \in J(R)$. In an alternative ring this situation is impossible unless $x^j = 0$. To see this, assume that R has a unity element. (If R has no unity, we may adjoin one in the standard manner since this construction preserves Jacobson radicals [20; 8].) Let v be the right quasi-inverse of $-y$, i.e., $-y + v - yv = 0$. Observe that in an alternative ring with unity, the identity $(xy)y^{-1} = x(yy^{-1})$ holds [29, 38]. Then $0 = x^j(1 - y) = [x^j(1 - y)](1 + v) = x^j[(1 - y)(1 + v)] = x^j$. We have shown that any element in the radical of a stable ring must be nilpotent.

To see that elements of J(R) are nilpotent of bounded index, assume that this is not the case. Our argument here goes back to [6]. For each integer n there exists an element $x \in J(R)$ such that $x^n \neq 0$. Let a be a new constant symbol. Let B be the infinite set of sentences that say that every element in the principal right ideal generated by a, $\mathrm{r}(a)$, is right quasi-regular. Let T be set of sentences $\mathrm{Th}(R) \cup B \cup \{a^n \neq 0 \mid n = 1, 2, \ldots\}$. T is a consistent set of sentences since it is finitely satisfiable in R. Thus by the Compactness Theorem, T has a model, S. S, restricted to the language of ring theory, is elementarily equivalent to R, and thus is a stable ring. By construction, J(S) contains an element which is not nilpotent, contradicting the first part of this theorem. Thus J(R) is nil of bounded index. J(R) is locally nilpotent by a theorem [22] which says that an alternative nil ring of bounded index is locally nilpotent. Using compactness arguments like the above we could further conclude that there is a function, f, on the natural numbers such that any n elements of J(R) generate a nilpotent subring with index of nilpotence bounded by $f(n)$.

Eventually we will demonstrate that in a stable ring the radical is definable. More specifically, we will show that given a stable ring R, there exists an integer n such that J(R) is the set of elements picked out by the formula $\forall y((xy)^n = 0)$. We will do this by showing that a stable ring is a Zorn ring and then applying a theorem about the radical of a Zorn ring. A *Zorn ring* is a ring in which every element is either nilpotent or divides an idempotent. (An element e is *idempotent* if $e^2 = e$.) As alluded to in the introduction to this section, we obtain this information through (or with) our proof of the fact that a semisimple stable ring has the descending chain condition.

We now move closer to the proofs of these theorems as we prove the following lemma which enables us to make algebraic applications of the model theoretic concept—stability. We need a definition to make the lemma understandable. We will say that a theory T has the *ascending* [*descending*] *chain condition on uniformly definable subsets*: if given any formula $\psi(x; y_1, \ldots, y_n)$ and any model R of T, there does not exist an infinite sequence of n-tuples of R, $\bar{a}_i$, such that for all i:

(3.2) $\quad R \models \forall x(\psi(x; \bar{a}_i) \rightarrow \psi(x; \bar{a}_{i+1})) \wedge \exists y(\neg\psi(y; \bar{a}_i) \wedge \psi(y; \bar{a}_{i+1}))$

(3.3) $\quad [R \models \forall x(\psi(x; \bar{a}_i) \leftarrow \psi(x; \bar{a}_{i+1}) \wedge \exists y(\psi(y; \bar{a}_i) \wedge \neg\psi(y; \bar{a}_{i+1}))]$.

This merely says that in a model of such a theory there are no properly infinite chains of uniformly definable subsets. For example, in an associative ring whose theory has this property there are no infinitely descending chains of principal right ideals since a principal right ideal, aR, is definable by the formula $\exists y(x = ay)$.

Lemma 3.1 *If* R *is a stable ring, then* Th(R) *has both the ascending and descending chain condition on uniformly definable subsets.*

Proof: Assume to the contrary that there is some formula $\psi(x; \bar{y})$, some ring $S \models$ Th(R) and tuples $\bar{a}_i$, $i = 1, 2, \ldots$ in S such that for all i, (3.2) with S replacing R, holds. Let $\psi(S; \bar{a}_i)$ denote the set of $b \in S$ such that $S \models \psi(b; \bar{a}_i)$. By assumption we have the infinite properly ascending chain $\psi(S; \bar{a}_1) \subset \psi(S; \bar{a}_2) \subset \ldots$. Let $b_i \in \psi(S; \bar{a}_{i+1}) \setminus \psi(S; \bar{a}_i)$. Combining this information, $S \models \psi(b_i, a_j)$ if and only if $i > j$. By Theorem 3.A this contradicts the fact that R is a stable ring.

From this argument we may also conclude that Th(R) satisfies the descending chain condition on uniformly definable subsets. This is because one can show by a standard compactness argument that if a structure R does not have the descending chain condition on uniformly definable subsets, then there is a structure elementarily equivalent to it that does not have the ascending chain condition on uniformly definable subsets. Since this lemma will be applied frequently, some further comments on it will be useful. In the course of this paper we will be concerned with right ideals defined by the formulas,

$$\psi_1(x; y_1, y_2): \exists z(x = (y_1y_2 - y_2y_1)^4 z),$$
$$\psi_2(x; y): \exists z(x = yz),$$
$$\psi_3(x; y): (yx = 0).$$

Usually we will be concerned with the right ideals picked out by $\psi_i(x; \bar{y})$ when the parameters $\bar{y}$ are restricted to some subset of our ring (for example, we might only be concerned with parameters chosen from some given right ideal). We will want to know that in this situation, for example, minimal subsets of this form exist. Since chains of uniformly definable subsets with parameters chosen freely from the ring in question can only be of finite length, *a fortiori* chains of subsets obtained by that formula with parameters restricted to some subset of the ring can only be finite in length. In this way we conclude that minimal elements of our set of uniformly defined subsets with restricted parameters, exist. We also need the following algebraic lemmas. The proofs involve primitive rings. As in the associative case, a ring is *primitive* if it contains a maximal regular right ideal containing no non-zero two-sided ideal [14].

Lemma 3.2 *Let* R *be a semisimple ring. If* $(xy - xy)^4 = 0$ *for all* x *and* y *in* R, *then* R *is associative.*

Proof: We first show that this condition forces R to be commutative and then show that any commutative semisimple ring is associative. By Theorem 4.1 (a result of Herstein, Kaplansky, and Kleinfeld) of [15], in an alternative ring R in which $(xy - yx)^4 = 0$ for all x and y, the nilpotent elements form an ideal. Since any nil ideal is contained in the radical of R which is (0) and since $xy - yx$ is nilpotent for all x and y in R, $xy = yx$ for all x and y in R. Thus R is commutative. Since R is semisimple, R is a subdirect sum of primitive rings each of which is a homomorphic image of R [14]. As an homomorphic image of R, each subdirect summand is commutative. By [14], a primitive nonassociative alternative ring is a Cayley-Dickson algebra. Since **CD** algebras are not commutative, each subdirect summand of R must be associative. As a subdirect sum of associative rings, R is associative.

We require a few new definitions. The *nucleus* of a ring R, N(R), is the set of elements of R which associate with all pairs of elements of R: N(R) = $\{x\colon \forall y \forall z ((xy)z = x(yz))\}$. A *nuclear element* of R is an element in the nucleus of R. Because of identities true in alternative rings [29], if $a \in$ N(R), then $(ya)z = y(ax)$ and $(yz)a = y(za)$ for all y and z in R. The following lemma is essentially Lemma 6.7 of [34].

Lemma 3.3 *If* R *is an alternative semisimple ring and* A *is a non-zero right ideal of* R, *then either* A *contains a non-zero right ideal* B *of* R *such that* B $\subset$ N(R) *or* A *contains a non-zero two-sided ideal of* R.

Theorem 3.3 *Let* R *be a stable alternative ring. Let* $\overline{R}$ = R/J(R). *If* A *is a right ideal of* R *and is not nil, then* $\overline{A}$, *the image of* A *in* $\overline{R}$, *contains a nuclear idempotent.*

Proof: Let (x, y) denote the commutator of the elements x and y: $(x, y) = xy - yz$. In an alternative ring, the fourth power of any commutator lies in the nucleus of that ring [15]. Thus the subsets of R defined by the formula $\psi_1(x; a_1, a_2)\colon (a_1a_2 - a_2a_1)^4x$, for a_1 and a_2 parameters from R, are right ideals of R. Let $\mathbf{V} = \{(a_1, a_2)^4\mathrm{R}\colon a_1, a_2 \in \mathrm{A} \text{ and } (\bar{a}_1, \bar{a}_2)^4\overline{\mathrm{R}} \neq (\bar{0})\}$. If **V** is non-empty, then **V** is a collection of right ideals contained in A and not contained in J(R). By Lemma 3.1 and the discussion following it, **V** has minimal elements. Let B be a minimal element of **V** if **V** is non-empty and let B = A otherwise. By construction, $\overline{B} \neq (\bar{0})$. If $\overline{B}$ is a minimal right ideal of $\overline{R}$, then $\overline{B}$ contains a nuclear idempotent [35; 449]. Since B $\subset$ A, in this case we are done.

Consider the remaining case in which $\overline{B}$ is not a minimal right ideal of $\overline{R}$. Then there exists a right ideal $\overline{B}' \neq (\bar{0})$, properly contained in $\overline{B}$. Our objective will be to find a non-zero right ideal $\overline{B}''$ which is contained both in $\overline{B}'$ and in N($\overline{R}$). By Lemma 3.3, either $\overline{B}'$ contains a non-zero right ideal of $\overline{R}$ which is contained in N($\overline{R}$); this is a suitable $\overline{B}''$, or $\overline{B}'$ contains a non-zero two-sided ideal of $\overline{R}$. Let $\overline{B}''$ denote this two-sided ideal. We proceed to show that $\overline{B}''$ is nuclear. By our choice of B, for any $\bar{a}_1, \bar{a}_2$ in

$\bar{B}''$, $(\bar{a}_1, \bar{a}_2)^4\bar{R} = (\bar{0})$. This implies that $(\bar{a}_1, \bar{a}_2)^4$ generates a nil right ideal in $\bar{R}$. The semisimplicity of $\bar{R}$ forces $(\bar{a}_1, \bar{a}_2)^4 = \bar{0}$ for all $\bar{a}_1$ and $\bar{a}_2$ in $\bar{B}''$. As a two-sided ideal in a semisimple ring, $\bar{B}''$ is semisimple [40]. These conditions on $\bar{B}''$ imply that $\bar{B}''$ is an associative ring by Lemma 3.2. Recall that N(S) is the nucleus of the ring S. Slater [33] proved that if S is a two-sided ideal of a semisimple ring R, then $N(S) = S \cap N(R)$. Since $\bar{B}''$ is associative, $N(\bar{B}'') = \bar{B}''$. Slater's result then implies that $\bar{B}'' \subset N(\bar{R})$. We have shown that if $\bar{B}$ is not a minimal right ideal of $\bar{R}$, then there exists a right ideal $\bar{B}''$ contained in the nucleus of $\bar{R}$. These nuclear elements of $\bar{B}''$ will provide a plentiful source of definable right ideals contained in $\bar{A}$.

Let $V' = \{bR: \bar{b} \in \bar{B}''$ and $\overline{bR} \neq (\bar{0})\}$. The elements of V' are uniformly defined by the formula $\psi_2(x; b)$: $\exists z(x = bz)$. By Lemma 3.1 since R is stable, V' has minimal elements. Let b_0R be such a minimal element. Since $\bar{b}_0 \in N(\bar{R})$, $\bar{b}_0\bar{R}$ is a non-zero right ideal of $\bar{R}$. To see that $\bar{b}_0\bar{R}$ is a minimal right ideal of $\bar{R}$, let $\bar{C}$ be a right ideal contained in $\bar{b}_0\bar{R}$ and let $\bar{c} \neq \bar{0}$ be an element of $\bar{C}$. By our choice of b_0R, since $\bar{c}$ is an element of $\bar{B}''$ and since $cR \subset C \subset b_0R$, either $\overline{cR} = (0)$ or $cR = b_0R$. By arguments similar to ones given earlier, since $\bar{R}$ contains no non-zero nil right ideals, $\overline{cR} \neq (\bar{0})$. Thus $cR = b_0R$ which implies that $\bar{C} = \bar{b}_0\bar{R}$. Hence $\bar{b}_0\bar{R}$ is a minimal right ideal of $\bar{R}$. As before, we resort to [35; 449] in order to find a nuclear idempotent in $\bar{b}_0\bar{R} \subset \bar{A}$.

Theorem 3.4 *Let* R *be a stable ring and* A *be a right ideal of* R. *Let* $\bar{A}$ *denote the image of* A *in* $\bar{R} = R/J(R)$. $\bar{A}$ *is generated as a principal right ideal in* $\bar{R}$ *by a nuclear idempotent.*

Proof: The proof of this theorem mirrors that of 1.4.2 in [11]. If A is nil then $\bar{A} = (\bar{0})$ and there is nothing to prove. If A is not nil, then by Theorem 3.3, $\bar{A}$ contains a non-zero nuclear idempotent, $\bar{e}$. Let Ann(e) be the set $\{x \in R: ex = 0\}$. Let V be the collection of sets-{Ann(e): $\bar{e}$ is a non-zero nuclear idempotent contained in $\bar{A}$}.

Elements of V are definable by the formula $\psi_3(x; e)$: $ex = 0$. Since R is stable, by Lemma 3.1, V has minimal elements. Let Ann(e_0) be a minimal element of V. If $\overline{\text{Ann}(e_0) \cap A} = \text{Ann}(\bar{e}_0) \cap \bar{A} = (\bar{0})$, then since for any $\bar{x} \in \bar{A}$; $\bar{e}_0(\bar{x} - \overline{e_0x}) = \bar{0}$, $\bar{x} - \bar{e}_0\bar{x} \in \text{Ann}(\bar{e}_0) \cap \bar{A} = (\bar{0})$. We then have $\bar{x} = \bar{e}_0\bar{x}$ for all $\bar{x}$ in $\bar{A}$. This implies that $\bar{A} = \bar{e}_0\bar{A} \subset \bar{e}_0\bar{R} \subset \bar{A}$ or $\bar{A} = \bar{e}_0\bar{R}$ as desired. If Ann$(\bar{e}_0) \cap \bar{A} \neq (\bar{0})$, then $\overline{\text{Ann}(e_0) \cap A}$ is not a nil right ideal and by Theorem 3.3 contains a non-zero nuclear idempotent, $\bar{e}_1$. As in the proof of Theorem 1.4.2 in [11] one can show that $\bar{e}^* = \bar{e}_0 + \bar{e}_1 - \bar{e}_1\bar{e}_0$ is a non-zero nuclear idempotent in $\bar{A}$ such that Ann(e^*) is properly contained in Ann(e_0), contradicting the minimality of Ann(e_0).

Corollary *Let* R *be a stable ring and let* $\bar{R} = R/J(R)$.

(a) *If* A *is a two-sided ideal of* R, *then* $\bar{A} = \overline{eR} = \overline{Re}$ *where* e *is a nuclear idempotent in the center of* R.
(b) *If* $\bar{R} \neq (\bar{0})$, *then* $\bar{R}$ *has a unity element.*

Proof: The proof of these statements is identical to the standard proof for associative semisimple Artinian rings [11; 30].

We now arrive at the main theorem of this section.

Theorem 3.5 *Let* R *be a stable ring.* $\overline{R} = R/J(R)$ *satisfies the ascending and descending chain condition on left and right ideals.*

Proof: By Theorem 3.4, each right ideal of $\overline{R}$ is of the form $\overline{eR}$, where $\overline{e}$ is a nuclear idempotent of $\overline{R}$. By Lemma 3.1, R satisfies the ascending and descending chain condition on uniformly defined subsets of R. In particular R satisfies the ascending and descending chain condition on the set $V = \{eR \mid \overline{e}$ is a non-zero nuclear idempotent$\}$. Thus any ascending or descending chain of right ideals of $\overline{R}$ must be finite. The corresponding chain conditions on left ideals follow *mutatis mutandis*.

Before stating the various corollaries to this theorem, we will now show that the radical of a given stable ring is definable in first order logic. We do this here as it will permit us to derive significantly stronger corollaries than otherwise possible. We need the following lemma.

Lemma 3.4 *Let* R *be an alternative ring and suppose for some* $a \in R$, $a^2 - a$ *is nilpotent. If* a *is not nilpotent, then there is some polynomial* $q(x)$ *with integer coefficients such that* $e = aq(a)$ *is a non-zero idempotent.*

Proof: For associative rings this is Lemma 1.3.2 of [11]. The proof in [11] uses only the power associativity of R and thus by Artin's Theorem holds for alternative rings.

Theorem 3.6 *If* R *is a stable ring then* J(R) *is the set of properly nilpotent elements of* R.

Proof: We will show that any stable ring is a Zorn ring. In a Zorn ring the radical consists of the ring's properly nilpotent elements [13; 46]. Recall that a ring is Zorn if given any $x \in R$ either x is nilpotent or there exists $y \in R$ such that xy is an idempotent. Since $\overline{R}$ satisfies the descending chain condition on right ideals and is semisimple, $\overline{R} = \overline{R}_1 \oplus \ldots \oplus \overline{R}_n$ is a finite direct sum of its minimal ideals, each of which is a **CD** algebra over its center or is a matrix ring over a division ring [32]. By inspection each $\overline{R}_i$ is a Zorn ring. By the corollary to Theorem 3.4, each $\overline{R}_i$ is generated over $\overline{R}$ by a nuclear idempotent in the center of $\overline{R}$.

To see that R is Zorn, let $x \in R$. If x is nilpotent, we have nothing to show. Assume that x is not nilpotent. Let $\overline{x}$ be the image of x in $\overline{R}$, $\overline{x} = \overline{x}_1 + \ldots + \overline{x}_n$ where $\overline{x}_i \in \overline{R}_i$. Since $\overline{x}$ is not nilpotent and $\overline{x}_i\overline{x}_j = \overline{0}$ for $i \neq j$, there exists an i such that $\overline{x}_i$ is not nilpotent. Let $\overline{e}_i$ be a central nuclear idempotent generating $\overline{R}_i$. $\overline{xe_i} = \overline{x}_i$. Since $\overline{R}_i$ is a Zorn ring and $\overline{x}_i$ is not nilpotent, there exists an element $\overline{y} \in \overline{R}_i$ such that $\overline{x}_i\overline{y}$ is an idempotent. Let $\overline{z} = \overline{e}_i\overline{y}$, we have $\overline{xz} = \overline{x}(\overline{e}_i\overline{y}) = (\overline{xe}_i)\overline{y} = \overline{x}_i\overline{y}$, since $\overline{e}_i$ is in the nucleus of $\overline{R}$. The hypothesis of Lemma 3.4 is satisfied; $(xz)^2 - (xz)$ is nilpotent since the radical of R is nil and xz is not nilpotent since $\overline{xz}$ is idempotent. Thus there is a polynomial with integer coefficients $q(x)$ such that $e = (xz)q(xz)$ is an idempotent. Since any two elements of R generate an associative subring by Artin's Theorem, we may rewrite $e = x(zq(xz))$. Thus R is a Zorn ring.

It is probably worth noting that being Zorn does not imply stability. Any field is Zorn, but it is well known that the field of real numbers is unstable [28].

We remark that Theorem 3.6 settles an open ring theoretic question due to Zorn for the class of stable alternative rings. Zorn asked whether or not the set of properly nilpotent elements of a ring was an ideal of that ring. This question is equivalent to another famous open question in ring theory—the Köthe conjecture (see page 481 of [44]). The Köthe conjecture asks whether a ring which has a non-zero nil one-sided ideal must have a non-zero nil two-sided ideal [11; 21]. We have:

Corollary 1 *The Köthe conjecture holds for the class of stable alternative rings.*

Corollary 2 *Let* R *be a stable ring, then there exists a positive integer* n *such that the formula* $\forall y((xy)^n = 0)$ *defines the Jacobson radical of* R.

Proof: By Theorem 3.2, J(R) is nil of bounded index. Let n be an integer such that $x \in$ J(R) implies that $x^n = 0$. If $x \in$ J(R) then clearly $\forall x((xy)^n = 0)$. Conversely, if for all y in $R(xy)^n = 0$, then x is properly nilpotent. By Theorem 3.6, x is an element of J(R).

Theorem 3.6 together with Theorem 3.5 enable us to deduce structure theorems for κ-stable rings. We require the following lemma whose proof is a special case of Theorem 1.1 of [6].

Lemma 3.5 *Let* R *be a* κ*-stable ring and let* $\psi(x)$ *be a formula with one free variable (possibly containing parameters from* R) *such that* $\psi(R)$ *is an ideal of* R. *Then* $\psi(R)$ *and* $R/\psi(R)$ *are* κ*-stable rings.*

Theorem 3.7 *If* R *is a* κ*-stable alternative ring then* J(R) *is nil and* $\overline{R} = R/J(R)$ *is a finite direct sum* $\overline{R} = \overline{R}_1 \oplus \ldots \oplus \overline{R}_n$ *of minimal ideals* $\overline{R}_i$. *Each* $\overline{R}_i$ *is generated over* $\overline{R}$ *by a central nuclear idempotent of* $\overline{R}$ *and is isomorphic to either a Cayley-Dickson algebra over a* κ*-stable field or a complete* $n_i \times n_i$ *matrix ring over a* κ*-stable associative division ring.*

Proof: We again employ Slater's generalization of the classical Wedderburn Theorem for Artinian rings [32]. As a semisimple ring satisfying the descending chain condition on right ideals, $\overline{R}$ is a finite direct sum of minimal ideals $\overline{R}_i$, each of which is a **CD** algebra over its center or an associative simple Artinian ring. By the corollary to Theorem 3.4, each $\overline{R}_i$ is generated by a central nuclear idempotent over $\overline{R}$. In particular each R_i is definable by a formula with a parameter from R $(\exists y(x = ey))$. As a definable subring of R, each R_i is a κ-stable ring (Lemma 3.5). The radical of R_i is definable, by the preceeding corollary, and thus $\overline{R}_i$ is κ-stable (again by Lemma 3.5). If $\overline{R}_i$ is a **CD** algebra, its center is definable and hence is also κ-stable. Finally, if $\overline{R}_i$ is a matrix ring over an associative division ring, the division ring is again definable and κ-stable.

We will soon specialize this theorem to ω-stable and $\aleph_1$-categorical rings. Before we do this, we will obtain some additional information about ω-stable and $\aleph_1$-categorical alternative rings. In [18] Macintyre showed that the only ω-stable fields were the finite and algebraically closed fields. Using this and a result of Baldwin's [2], Shelah (unpublished) was able to show that an $\aleph_1$-categorical associative division ring had to be a field. (It is unknown as to whether or not this is also true of ω-stable associative division rings.) The following theorem and its corollary generalize the above result of Shelah's.

Theorem 3.8 *If* R *is an* ω*-stable alternative division ring, then* R *is associative.*

Proof: In [15] Kleinfeld proved that any alternative division ring is either associative or a Cayley-Dickson algebra over its center. Assume that R is not associative, then R is a **CD** algebra over its center F. Since the center of R is definable, it is ω-stable by Lemma 3.5. F is a field and thus, by the result of Macintyre [18] mentioned above, is either finite or algebraically closed. F cannot be algebraically closed by Lemma 2.2.

Suppose F is finite. Let x and y be two noncommuting elements of R (these exist by the definition of a **CD** algebra). By Artin's Theorem, since F is the nucleus of R (see section **1**), x and y generate an associative algebra Q over F. Since R is finite dimensional over F and since R is a division ring, Q is a division ring. Since F is finite, Q is finite. However, a finite associative division ring, by the celebrated theorem of Wedderburn on this subject, is commutative [11]. This is a contradiction.

Corollary *If* R *is an* $\aleph_1$*-categorical alternative division ring then* R *is a finite or algebraically closed field.*

It is easy to see that for alternative nonassociative primitive rings the notions of ω-stability and $\aleph_1$-categorically coincide. We have:

Theorem 3.9 *Let* R *be an alternative not associative primitive ring. The following are equivalent:*

(1) R *is* ω*-stable.*
(2) R *is* $\aleph_1$*-categorical.*
(3) R *is a split Cayley-Dickson algebra over a finite or algebraically closed field.*

Proof: By Theorem 2.3 a split **CD** algebra over an algebraically closed field is $\aleph_1$-categorical. Since any $\aleph_1$-categorical structure is ω-stable [28] to prove this proposition it suffices to show that (1) implies (3). In [14] Kleinfeld showed that an alternative nonassociative primitive ring was a **CD** algebra over its center. Since R is ω-stable, so is the center of R (Lemma 3.5). The center of R, being a field, is finite or algebraically closed. By Theorem 3.8 since R is not associative, R cannot be a division algebra. Hence R is a split **CD** algebra over a finite or algegraically closed field.

Theorem 3.10 *Let* R *be an* ω*-stable ring, then* $R/J(R) = \overline{R} = \overline{R}_1 \oplus \ldots \oplus \overline{R}_n$ *where* $\overline{R}_i$ *is a complete* $n_i \times n_j$ *matrix ring over a* ω*-stable division ring or a split Cayley-Dickson algebra over a finite or algebraically closed field.*

Proof: Immediate from Theorems 3.7, 3.8, and 3.9.

Theorem 3.11 *Let* R *be an infinite semisimple ring.* R *is* $\aleph_1$*-categorical if and only if* $R = R_1 \oplus \ldots \oplus R_n \oplus S$ *where each* R_i *is a complete matrix ring or split* **CD** *algebra over a finite field and* S *is a complete matrix ring or a split* **CD** *algebra over an algebraically closed field.*

Proof: Since any $\aleph_1$-categorical structure is ω-stable, by Theorem 3.10 $R = R_1 \oplus \ldots \oplus R_n$ where each R_i is a split **CD** algebra or matrix ring over a field. To prove this theorem, it suffices to show that at most one R_i is infinite.

Assume to the contrary that both R_1 and R_2 are infinite. Also assume that R, R_1, and R_2 all have cardinality $\aleph_1$. Let R_1' be a countable ring elementarily equivalent to R_1. If $R' = R_1' \oplus R_2 \oplus \ldots \oplus R_n$, then R' is elementarily equivalent to R by a well-known theorem of Feferman and Vaught [8]. Since R and R' have cardinality $\aleph_1$, they must be isomorphic. Since the rings R_1 and R_1' are matrix rings and split **CD** algebras over fields, their unit elements have unique finite decompositions into sums of primitive (i.e., not the sum of two non-zero orthogonal idempotents) central orthogonal idempotents. The sum of the primitive central orthogonal idempotents in the decomposition of each R_i is a unique decomposition of the unity element of R into primitive central orthogonal idempotents. A similar statement is true for R'. R cannot be isomorphic to R' since if e is an idempotent in the decomposition of the unity element of R into primitive central orthogonal idempotents, it must map to an idempotent, say f, in the decomposition of the unity element of R' such that the cardinality of eR is the same as the cardinality of fR.

Note: Recently Dr. Michael Slater alerted me to an article by Kevin McCrimmon—"A characterization of the Jacobson-Smiley radical" [43]. In this article Dr. McCrimmon shows that the Jacobson-Smiley radical is definable by the formula $\forall y \exists z (xy + z + (xy)z = 0)$. (The proof involves showing that the proper quasi-invertibility of an element is equivalent to being quasi-invertible in all homotopes. Passage from an alternative algebra R to the special quadratic Jordan algebra R^+ is an important technique used in the proof.) Corollary 2 to Theorem 3.6 on the definability of the Jacobson radical of a stable ring (my reason for proving Theorem 3.6) duplicates (and is weaker than) McCrimmon's work.

4 $\aleph_0$*-categorical alternative rings* In this section we obtain results on the structure of $\aleph_0$-categorical alternative rings. We begin by using the Ryll-Nardzewski Theorem to quickly define and find the structure of the radical of an $\aleph_0$-categorical ring. We then proceed to examine the "nicer" classes of $\aleph_0$-categorical alternative rings. We show that an $\aleph_0$-categorical nonassociative alternative primitive ring is a split Cayley-Dickson algebra over a finite field. Macintyre and Rosenstein categorized the class of associative $\aleph_0$-categorical rings (with unity) without non-zero nilpotent elements in [19]. We show that the alternative analogue of this class of rings is associative. We then extend a theorem of Baldwin and Rose [4] to alternative rings. We prove that if R is an $\aleph_0$-categorical alternative ring of infinite cardinality, then R has neither the ascending or descending chain condition on right or left ideals. We finish this section by generalizing another theorem in Baldwin and Rose [4]:

If R *is stable and* $\aleph_0$*-categorical (in particular if* R *is categorical in all powers), then* R/J(R) *is finite and* J(R) *is nil of bounded index.*

We begin by presenting our main logical tool in the investigation of $\aleph_0$-categorical rings: The Ryll-Nardzewski Theorem. Let F_n denote all first order formulas in the language of ring theory with free variables among $x_1, x_2, \ldots, x_n$.

Theorem 4.A (Ryll-Nardzewski) [28] R *is* $\aleph_0$*-categorical if and only if for each integer n there exists finitely many formulas* $\psi_1, \ldots \psi_{k_n}$ *in* F_n *such that every formula of* F_n *is equivalent to one of them, i.e.,*

$$R \models \forall x_1 \ldots \forall x_n \left(\bigvee_{i=1}^{k_n} \psi_i(\bar{x}) \leftrightarrow \theta(\bar{x}) \right) \textit{ for any } \theta \textit{ in } \mathsf{F}_n.$$

The following well-known (*cf.* [25]) application of the Ryll-Nardzewski Theorem to associative rings is also true for alternative rings. A proof of it may be found in [4].

Lemma 4.1 *Let* R *be an* $\aleph_0$*-categorical alternative ring. There is a function f mapping* ω *to* ω *such that every n-generated subring of* R *has less than* $f(n)$ *elements.*

A reformulation of Lemma 4.1 will sometimes be quite useful to us.

Lemma 4.2 *Let* R *be an* $\aleph_0$*-categorical alternative ring. There exist integers m, n, and k greater than* 1 *such that for all x in* R, $mx = 0$ *and* $x^n = 0$ *or there is some integer* $1 < j \leq k$ *such that* $x^j = x$. *In particular* R *satisfies the equation* $p(x) = x^n(x^{k-1} - x) \ldots (x^2 - x) = 0$.

Proof: The first contention follows by applying Lemma 4.1 to 1-generated subrings. Since any 1-generated subring of an alternative ring is associative by Artin's Theorem, R satisfies $p(x) = 0$.

A succinct way of summarizing this information (as pointed out to me by I. Herstein) is:

If R *is an* $\aleph_0$*-categorical ring, then there exists an integer q such that for all* $x \in$ R; $qx = 0$ *and* $x^q = 0$ *or* $x^q = x$.

For suppose $x^q = x$ for some $j \leq k$. Let $r = (k - 1)(k - 2) \ldots (1) + 1$. We then have $x^r = x$. Choose q to be a sufficiently large iterated (r-th) power of r such that $qx = 0$ and if x is nilpotent such that $x^q = 0$. We also note that although Lemma 4.1 is true for an arbitrary nonassociative ring, Lemma 4.2 requires that the ring in question be power associative.

Before proceeding to the main Theorem of this section, we will use Lemma 4.2 to characterize $\aleph_0$-categorical rings as Zorn rings. From this we may define and characterize the Jacobson radical of an $\aleph_0$-categorical ring.

Theorem 4.1 *Let* R *be an* $\aleph_0$*-categorical alternative ring. Then* R *is a Zorn ring.* J(R) *consists of the set of properly nilpotent elements of* R. *In particular, there exists an integer* n *such that* J(R) *is defined by the formula* $\forall y((xy)^n = 0)$.

Proof: Recall that a ring is a Zorn ring if every element is either nilpotent or divides an idempotent. By Lemma 4.2, if an element a in R is not nilpotent then there is an integer j such that $a^j = a$. By computation, a^{j-1} is an idempotent. Thus R is a Zorn ring. By Theorem 8 of [13], J(R) is the set of properly nilpotent elements of R. In particular, J(R) is nil. From Lemma 4.2, obtain a bound, say n, for the maximum index of nilpotence of any element of R. We conclude that the formula $\forall y((xy)^n = 0)$ defines the radical of R.

Observe that J(R) is locally nilpotent. This may be seen in several ways. Probably the most direct way to make this observation is to note that any finite set of elements of J(R) generates a finite subring of J(R) (Lemma 4.1). This subring is nil of bounded index. A combinatorial calculation yields that this subring must be nilpotent. Alternatively as we did in section **3**, one may cite the Theorem of McCrimmon (and Shirshov) [22] which states that a nil ring of bounded index is locally nilpotent.

An alternate way of obtaining the definability of the Jacobson radical of an alternative ring is by using Theorem 3.1. Theorem 3.1 gives us a countable set of formulas $\{\psi_i\}$ such that an element in R is in the radical of R if and only if it satisfies all the ψ_i: If R is $\aleph_0$-categorical, then in R only finitely many ψ_i are inequivalent. Thus there exists an $n = n(\mathrm{R})$ such that $a \in \mathrm{J(R)}$ if and only if $\mathrm{R} \models \bigwedge_{i=1}^{n} \psi_i(a)$.

We take this opportunity to mention an open question, first raised for the associative case [4]: Does there exist an $\aleph_0$-categorical nil ring which is not nilpotent?

We now study the structure of $\aleph_0$-categorical rings under the constraints of various ring theoretic conditions.

Theorem 4.2 *If* R *is an* $\aleph_0$*-categorical primitive ring, then* R *is either isomorphic to a complete matrix ring over a finite field or a split Cayley-Dickson algebra over a finite field. In particular,* R *is finite.*

Proof: If R is associative, then since R satisfies the equation $p(x) = x^n(x^k - x) \dots (x^2 - x) = 0$, R is isomorphic to a matrix ring over a finite field (Theorem 1.2 of [4]). If R is not associative, then as noted earlier, R is a **CD** algebra over it center [14]. By Lemma 4.2 R satisfies the equation $p(x) = 0$, hence *a fortiori* does the center of R. However, the center of R is a field and in a field a polynomial can have no more roots than its degree. Thus the center of R is a finite field. In the course of proving Theorem 3.8, we showed that **CD** division algebras over finite fields do not exist. Thus R is a split **CD** algebra.

For our purposes, a more useful form of Theorem 4.2 is the following corollary of its proof.

Corollary 1 *If* R *is a primitive ring satisfying an equation of the form* $p(x) = x^n(x^k - x) \dots (x^2 - x) = 0$, *then* R *is a matrix ring over a finite field or a split* **CD** *algebra over a finite field.*

We recall the algebraic fact [14] that a semisimple ring, R, is a subdirect sum of primitive rings each of which is a homomorphic image of R. By the Compactness Theorem and Corollary 1, an $\aleph_0$-categorical ring cannot have primitive homomorphic images which are either matrix rings of arbitrarily high dimension or which have centers of arbitrarily large cardinality. We have:

Corollary 2 *If* R *is a semisimple* $\aleph_0$*-categorical ring, then* R *is a subdirect product of complete matrix rings over finite fields and split* **CD** *algebras over finite fields. Moreover, only finitely many different matrix rings and split* **CD** *algebras occur as subdirect factors.*

Macintyre and Rosenstein [19] have classified the class of associative $\aleph_0$-categorical rings with identity which have no nilpotent non-zero elements. Their proof invokes the Arens-Kaplansky representation of such rings as rings of continuous functions. Analogously, one might consider the class of alternative $\aleph_0$-categorical rings with identity and with no non-zero nilpotent elements. The added generality of this class of rings is vacuous. We have:

Theorem 4.3 *If* R *is an* $\aleph_0$*-categorical alternative ring without non-zero nilpotent elements, then* R *is associative.*

Proof: An alternative ring without non-zero nilpotent elements is a subdirect sum of rings without zero-divisors [10]. It suffices to show that each of these subdirect summands must be associative. If some summand, R_α, were not associative, it could be embedded in some **CD** division algebra, S_α, such that the center of S_α is the quotient field of the center of R_α [34]. As each subdirect summand is a homomorphic image of R, it satisfies the same polynomial equation as does R, $p(x) = 0$. This forces the center of R_α and hence of S_α to be finite. Since **CD** division algebras with finite centers do not exist, R is associative.

The next major result is that an $\aleph_0$-categorical alternative ring of arbitrary infinite cardinality has neither the ascending chain condition (a.c.c.) nor the descending chain condition (d.c.c.) on right ideals. This is a generalization to alternative rings of a theorem of Baldwin and Rose [4]. The proof for alternative rings follows closely the account given in [4] for associative rings. Hence in what follows, the proofs will mainly emphasize differences from the proofs for the associative case. The reader may refer to [4] for additional details.

Theorem 4.4 *If* R *is a semisimple alternative ring with* a.c.c. *on right ideals with every element periodic or nilpotent, then* R *is finite. In fact* R *is a subring of a finite direct sum of matrix rings and split* **CD** *algebras over finite fields.*

Proof: The proof mirrors that of Proposition 2.1 of [4]. M. Slater in [45] showed that a semisimple alternative ring with a.c.c. is a subring of a finite direct sum of prime rings A_i, each of which is a homomorphic image of R. Each element of A_i is either periodic or nilpotent and thus if A_i is not semisimple (by an argument given previously) $J(A_i)$ is nil of bounded index. As a homomorphic image of R, A_i has the a.c.c. on right ideals. Thus by a result of Zhevlakov $J(A_i)$ is nilpotent. Since A_i is prime, it must therefore be semisimple.

As a semisimple ring, R is a subdirect product of primitive rings R which are homomorphic images of it [14; 730]. Let $\varphi\colon R \to \prod_{\alpha\in A} R_\alpha$ be the given subdirect product monomorphism. We will use the primeness of R and this representation as a subdirect product to embed R into a primitive ring which satisfies $p(x) = 0$. Let φ_α be the composition of φ with the projection map onto the α-th component. Let $T_r = \{\alpha\colon \varphi_\alpha(r) \neq 0\}$ T_r is the support of $\varphi(r)$. Since R is prime, the set $\{T_r\colon r \neq 0\}$ may be extended to a proper ultrafilter on A. By the choice of D, we can embed R in $S = \prod_{\alpha\in A} R_\alpha/D$. As an ultraproduct of primitive rings, S is a primitive ring [5; 175]. Further by Łoś's Theorem [5; 170], S satisfies $p(x) = 0$ since each R does. From these facts conclude by Corollary 1 to Theorem 4.2 that S is a split **CD** algebra or a matrix ring over a finite field.

Theorem 4.5 *If* R *is an* $\aleph_0$*-categorical alternative ring with* a.c.c., *then* R *is finite.*

Proof: By Theorem 4.1 J(R) is locally nilpotent. Since R has a.c.c., by a result of Zhevlakov [42], J(R) is nilpotent. Since R is a ring with a.c.c.; by the standard argument, J(R) is finitely generated as a right R-module. Since R is $\aleph_0$-categorical, the hypothesis of Theorem 4.4 is satisfied for R/J(R), and hence R/J(R) is finite. A combinatorial argument, as given in Theorem 2.2 of [4] using the facts that J(R) is nilpotent, J(R) is finitely generated over R, and R/J(R) is finite, shows that R is finite.

Theorem 4.6 *If* R *is an* $\aleph_0$*-categorical alternative ring with* d.c.c., *then* R *is finite.*

Proof: As observed in Lemma 4.2, R has finite characteristic. The idea of this proof is that the standard construction for adjoining a unity element to R preserves $\aleph_0$-categoricity. The new ring R′ has both d.c.c. and a unity element; thus by Hopkin's Theorem has a.c.c. (Given a structure theorem similar to that existing for Jacobson semisimple alternative rings with d.c.c. and the nilpotence of the Jacobson radical, Hopkin's Theorem is true for arbitrary nonassociative rings.) By Theorem 4.5, R′ is finite.

We may combine the information contained in this section with that of section 3 to obtain the following theorem.

Theorem 4.7 *Let* R *be an alternative ring that is both stable and* $\aleph_0$-*categorical. Then* R/J(R) *is finite.*

Proof: By Theorem 3.5, since R is stable, R/J(R) has the d.c.c. on right ideals and thus is a finite direct sum of **CD** algebras and simple Artinian associative rings [32]. Since R is $\aleph_0$-categorical, R satisfies a polynomial of the form $p(x) = x^n(x^k - x) \ldots (x^2 - x) = 0$ and hence so does R/J(R). By arguments given previously, this forces R/J(R) to be finite.

In particular, since $\aleph_1$-categorical rings are stable, and since the radical of a stable ring is nil, the class of rings categorical in all powers is nil by finite. Such a class of rings is a very natural class to examine. We raise the problem of finding a structure theorem for this class of rings. We also ask a simpler related question—if R is categorical in all powers, is J(R) nilpotent?

Examples: Probably the easiest method of constructing $\aleph_0$-categorical rings with particular properties is by taking reduced powers of finite rings. A full account of this approach to constructing examples may be found in [4]. Its mathematical basis is a theorem of Waszkiewicz and Węglorz [37].

Let D *be the filter of all confinite sets in* ω. *Let* A *be a finite ring, then* $R = A^\omega/D$, *the reduced power of* A *by* D, *is an* $\aleph_0$-*categorical ring. Further any Horn sentence that is true in* A *is true in* R. *Since the identity defining alternative rings is a Horn sentence (as are the other ring axioms)* R *is an alternative ring if* A *is.*

REFERENCES

[1] Amitsur, S. A., "A general theory of radicals, I," *American Journal of Mathematics*, vol. 74 (1952), pp. 474-486.

[2] Baldwin, J. T., "α_T is finite for ω_1-categorical T," *Transactions of the American Mathematical Society*, vol. 181 (1973), pp. 37-51.

[3] Baldwin, J. T., "Some EC_Σ classes of rings" (to appear).

[4] Baldwin, J. T., and B. Rose, "$\aleph_0$-categoricity and stability of rings," *Journal of Algebra*, vol. 45 (1977), pp. 1-16.

[5] Chang, C. C., and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam (1972).

[6] Cherlin, G. L., and J. Reineke, "Categoricity and stability of commutative rings," *Annals of Mathematical Logic*, vol. 10 (1976), pp. 367-399.

[7] Eklof, P., and E. Fisher, "The elementary theory of Abelian groups," *Annals of Mathematical Logic*, vol. 4 (1972), pp. 115-171.

[8] Feferman, S., and R. L. Vaught, "The first order properties of algebraic systems," *Fundamenta Mathematica*, vol. 47 (1959), pp. 57-103.

[9] Felgner, U., "Ringe, derer Theorien $\aleph_1$-categorich sind," *Fundamenta Mathematica*, vol. 72 (1975), pp. 331-346.

[10] Hentzel, I. R., "Alternative rings without nilpotent elements," *Proceedings of the American Mathematical Society*, vol. 42 (1974), pp. 373-376.

[11] Herstein, I. N., *Noncommutative Rings*, Mathematical Association of America, Carus Monograph Series No. 15, Rhode Island (1968).

[12] Herstein, I. N., *Topics in Ring Theory*, University of Chicago Press, Chicago (1969).

[13] Kaplansky, I., "Semi-simple alternative rings," *Portugaliae Mathematica*, vol. 10 (1951), pp. 37-50.

[14] Kleinfeld, E., "Primitive alternative rings and semi-simplicity," *American Journal of Mathematics*, vol. 77 (1955), pp. 725-730.

[15] Kleinfeld, E., "Simple alternative rings," *Annals of Mathematics*, vol. 58 (1953), pp. 544-547.

[16] Lindström, P., "On model completeness," *Theoria*, vol. 30 (1968), pp. 183-196.

[17] Macintyre, A., "On ω_1-categorical theories of Abelian groups," *Fundamenta Mathematica*, vol. 70 (1970), pp. 253-270.

[18] Macintyre, A., "On ω_1-categorical theories of fields," *Fundamenta Mathematica*, vol. 71 (1971), pp. 1-25.

[19] Macintyre, A., and J. G. Rosenstein, "$\aleph_0$-categoricity for rings without nilpotent elements and for Boolean structures," (to appear).

[20] McCoy, N. H., *The Theory of Rings*, Macmillan, New York (1968).

[21] McCrimmen, K., *Alternative Rings*, Marcel Dekker, New York (to appear).

[22] McCrimmon, K., "Alternative rings satisfying a polynomial identity," *Journal of Algebra*, vol. 24 (1973), pp. 283-292.

[23] Morley, M., "Categoricity in power," *Transactions of the American Mathematical Society*, vol. 114 (1965), pp. 514-538.

[24] Reineke, J., "Reduced powers of rings," (to appear).

[25] Rosenstein, J. G., "$\aleph_0$-categoricity of groups," *Journal of Algebra*, vol. 25 (1973), pp. 435-467.

[26] Sabbagh, G., "Categoricite et stabilite: quelques exemples parmi les groupes et anneaux," *Comptes Rendus des Sêances de l'Acadêmie des Sciences*, Series A, vol. 280 (1975), pp. 603-606.

[27] Sabbagh, G., and P. Eklof, "Definability problems for modules and rings," *The Journal of Symbolic Logic*, vol. 36 (1971), pp. 623-649.

[28] Sacks, G. E., *Saturated Model Theory*, W. A. Benjamin, Reading, Massachusetts (1972).

[29] Schafer, R. D., *An Introduction to Nonassociative Algebras*, Academic Press, New York (1966).

[30] Shelah, S., "The number of non-isomorphic models of an unstable first-order theory," *Israel Journal of Mathematics*, vol. 9 (1971), pp. 473-487.

[31] Shelah, S., "Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory," *Annals of Mathematical Logic*, vol. 3 (1971), pp. 271-362.

[32] Slater, M., "Alternative rings with d.c.c., I," *Journal of Algebra*, vol. 11 (1969), pp. 102-110.

[33] Slater, M., "Ideals in semiprime alternative rings," *Journal of Algebra*, vol. 8 (1968), pp. 60-76.

[34] Slater, M., "Prime alternative rings, I," *Journal of Algebra*, vol. 15 (1970), pp. 229-243.

[35] Slater, M., "The socle of an alternative ring," *Journal of Algebra*, vol. 14 (1970), pp. 443-463.

[36] Smiley, M. F., "The radical of an alternative ring," *Annals of Mathematics*, vol. 49 (1948), pp. 702-709.

[37] Waszkiewicz, J., and B. Węglorz, "On ω_0-categoricity of powers," *Bulletin de l'Academie Polonaise des Sciences*, vol. 17 (1969), pp. 195-199.

[38] Zhevlakov, K. A., "Coincidence of Smiley and Kleinfeld radicals in alternative rings," *Algebra and Logic*, vol. 8 (1969), pp. 175-181.

[39] Zhevlakov, K. A., "A note on locally nilpotent rings with chain conditions," *Mathematical Notes*, vol. 12 (1972), pp. 507-509.

[40] Zhevlakov, K. A., "Quasiregular ideals in finitely generated alternative rings," *Algebra and Logic*, vol. 11 (1972), pp. 81-93.

[41] Zhevlakov, K. A., "On radicals and Von Neumann ideals," *Algebra and Logic*, vol. 8 (1969), pp. 244-252.

[42] Zil'ber, B. I., "Rings with $\aleph_1$-categorical theories," *Algebra and Logic*, vol. 13 (1974), pp. 95-104.

[43] McCrimmon, K., "A characterization of the Jacobson-Smiley radical," *Journal of Algebra*, vol. 18 (1971), pp. 565-573.

[44] Slater, M., "Alternative rings with d.c.c., II," *Journal of Algebra*, vol. 14 (1970), pp. 464-484.

[45] Slater, M., "Goldie theorems for alternative rings," (to appear).

[46] Slater, M., "Strongly prime alternative rings," (to appear).

University of Notre Dame
Notre Dame, Indiana