Model Universes with Spherical Symmetry.

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In honour of Professor Beniamino SEGRE.

Summary. – *Spherically symmetric universes are defined, and spherically symmetric solutions of Einstein's field equations in vacuo are explored in terms of suitable coordinates. The Kruskal metric is thus obtained in a systematic way, with possibilities of generalisation.*

1. – Introduction.

The formula

$$\Phi = (1-2m/r)^{-1}dr^2 + r^2 d\sigma^2 - (1-2m/r)\,dt^2\,, \tag{1.1}$$

where

$$d\sigma^2 = d\theta^2 + \sin^2\theta\, d\varphi^2\,, \tag{1.2}$$

is one of the most famous in the general theory of relativity. It was obtained by SCHWARZSCHILD in 1916 as the metric form for the gravitational field outside a sphere of mass m (in appropriate units). The metric tensor g_{ij} contained in (1.1) is an exact solution of Einstein's equation $R_{ij}=0$. The locus $r=2m$ is called the « Schwarzschild singularity ». Not long after the form (1.1) appeared, the question was raised as to whether this is a real singularity. The discussion proceeded sporadically for some forty years until finally KRUSKAL (1) showed how to remove the apparent singularity by a simple transformation $(r, t)\to(u, v)$. Kruskal's method is open to a methodological criticism: he uses the « bad » coordinates (r, t) to obtain the « good » coordinates (u, v). In the present paper I start with « good » coordinates and investigate the vacuum field equations systematically.

The first step is to define what we mean by spherically symmetric space-time V_4. I take it to be the product of a unit sphere S_2 and a 2-space U_2 in the sense that an event (or point) of V_4 corresponds to an ordered pair of points, one on S_2 and one on U_2. On S_2 I take the usual polar coordinates (θ, φ) and the metric $d\sigma^2$ as

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(1) M. D. KRUSKAL, Phys. Rev., **119** (1960), p. 1743, where references will be found to earlier work by others. See also S. MAVRIDÈS, *L'univers relativiste*, Masson (Paris, 1973), p. 338.

in (1.2). For the metric of V_4 I take the sum of the metric of U_2 and the metric of S_2, multiplied by a *positive* factor which is a function of position in U_2. To get the correct signature $(+2)$ for the metric of V_4, the metric of U_2 must be indefinite. Thus null lines exist in U_2: let us take their equations to be $u=\text{const}$, $v=\text{const}$. All this sums up to the following statement:

The most general metric form for spherically symmetric space-time is

$$\Phi = -2f\,du\,dv + r^2\,d\sigma^2, \tag{1.3}$$

where f and r are functions of (u, v).

Note that, in this approach to spherical symmetry, no mention is made of a *centre*. Except where f or r vanishes, the signature is correct. The minus sign in front of the first term is of no particular significance, but merely a notational convenience. Note that r is not a coordinate, but some function of the coordinates (u, v); it occurs only in the form r^2, and so there is no metrical distinction between positive and negative values of r.

The coordinates (u, θ, φ, v) may be called *null coordinates*. It is convenient to have an indicial notation for them:

$$x^1 = u\,, \qquad x^2 = \theta\,, \qquad x^3 = \varphi\,, \qquad x^4 = v\,. \tag{1.4}$$

For brevity I put $\sin\theta = s$, $\cos\theta = c$.

It is doubtful whether anyone really understands what a singularity in space-time means. One tends to proceed formally, examining for zeros and infinities the coefficients in the metric form, and, in particular, the determinant formed from them. Thus we are to treat with respect events at which the functions $f(u, v)$ and $r(u, v)$ vanish or become infinite. But we must not jump to conclusions about such events, as the following elementary case shows.

Consider Minkowskian space-time with metric form

$$\Phi = dx^2 + dy^2 + dz^2 - dt^2. \tag{1.5}$$

Transforming the spatial part to spherical polars relative to $x=y=z=0$, we get

$$\Phi = dr^2 + r^2\,d\sigma^2 - dt^2. \tag{1.6}$$

Making the transformation

$$u = \tfrac{1}{2}(r+t)\,, \qquad v = \tfrac{1}{2}(r-t)\,, \tag{1.7}$$

we arrive at

$$\Phi = 4\,du\,dv + r^2\,d\sigma^2, \qquad r = u + v\,. \tag{1.8}$$

This is a particular case of (1.3). It would be absurd to attach importance to the locus $r=0$. There are no singularities in Minkowskian space-time, and $r=0$ is merely a timelike geodesic.

2. – Some formulae.

It is convenient to write out some formulae for the metric

$$\Phi = -2f\,du\,dv + r^2\,d\sigma \tag{2.1}$$

as in (1.3), without restriction on the functions $f(u, v)$, $r(u, v)$ except that neither vanishes and the indicated derivatives exist. Indices are as in (1.4), and the subscripts 1 and 4 attached to f and r indicate partial derivatives with respect to x^1 and x^4 (equivalently, u and v). Note the symmetry of the formulae with respect to interchange of x^1 and x^4.

Metric tensor and its inverse.

$$\begin{aligned} &g_{14} = -f\,, && g_{22} = r^2, && g_{33} = r^2 s^2, \\ &g^{14} = -f^{-1}, && g^{22} = r^{-2}, && g^{33} = r^{-2}s^{-2}. \end{aligned} \tag{2.2}$$

Connection.

$$\begin{aligned} &\Gamma^1_{11} = f_1/f\,, && \Gamma^1_{22} = rr_4/f\,, && \Gamma^1_{33} = rr_4 s^2/f\,, \\ &\Gamma^2_{33} = -sc\,, && \Gamma^2_{12} = r_1/r\,, && \Gamma^2_{24} = r_4/r\,, \\ &\Gamma^3_{23} = c/s\,, && \Gamma^3_{31} = r_1/r\,, && \Gamma^3_{34} = r_4/r\,, \\ &\Gamma^4_{22} = rr_1/f\,, && \Gamma^4_{33} = rr_1 s^2/f\,, && \Gamma^4_{44} = f_4/f\,. \end{aligned} \tag{2.3}$$

Riemann tensor (1).

$$\begin{aligned} &R_{2323} = r^2 s^2(1 + 2r_1 r_4/f)\,, && R_{3131} = s^2 R_{1212}\,, && R_{1212} = r(-r_{11} + r_1 f_1/f)\,, \\ &R_{3134} = -s^2 R_{1224}\,, && R_{1224} = rr_{14}\,, && \\ &R_{1414} = -f_{14} + f_1 f_4/f\,, && R_{2424} = r(-r_{44} + r_4 f_4/f)\,, && R_{3434} = s^2 R_{2424}\,. \end{aligned} \tag{2.4}$$

(1) In respect to signs, the Riemann tensor and the Ricci tensor are defined as in J. L. SYNGE, *Relativity: the General Theory*, North-Holland, Amsterdam 1964, pp. 15-17.

Ricci tensor.

$$
(2.5)\qquad
\begin{aligned}
&R_{11} = 2r^{-1}(r_{11} - r_1 f_1/f) ,\\
&R_{22} = -1 - 2f^{-1}(rr_{14} + r_1 r_4) , \qquad R_{33} = s^2 R_{22} ,\\
&R_{44} = 2r^{-1}(r_{44} - r_4 f_4/f) ,\\
&R_{14} = f^{-1}(f_{14} - f_1 f_4/f) + 2r_{14}/r ,
\end{aligned}
$$

In terms of the Riemann tensor, the components of the Ricci tensor are as follows:

$$
(2.6)\qquad
\begin{aligned}
&R_{11} = -2r^{-2}R_{1212}\\
&R_{22} = -r^{-2}s^{-2}R_{2323} - 2f^{-1}R_{1224} , \qquad R_{33} = s^2 R_2 ,\\
&R_{44} = -2r^{-2}R_{2424} ,\\
&R_{14} = -f^{-1}R_{1414} + 2r^{-2}R_{1224} .
\end{aligned}
$$

3. – The vacuum field with spherical symmetry.

We are now to solve Einstein's field equations in vacuo, $R_{ij} = 0$. By (2.5) there are four equations:

$$
(3.1)\qquad
\begin{aligned}
&fr_{11} - r_1 f_1 = 0 ,\\
&fr_{44} - r_4 f_4 = 0 ,\\
&f + 2(rr_{14} + r_1 r_4) = 0 ,\\
&rff_{14} + 2f^2 r_{14} - rf_1 f_4 = 0 .
\end{aligned}
$$

The first two of these tell us that

$$
(3.2)\qquad f = 2B(x^4)\, r_1 , \qquad f = 2A(x^1)\, r_4 ,
$$

the functions A and B being arbitrary. We can now write the third of (3.1) in the alternative forms

$$
(3.3)\qquad (rB + rr_4)_1 = 0 , \qquad (rA + rr_1)_4 = 0 ,
$$

and so

$$
(3.4)\qquad r_4 = -B + G/r , \qquad r_1 = -A + F/r ,
$$

the functions $F(x^1)$ and $G(x^4)$ being arbitrary. But the consistency of these two equations requires $Fr_4 = Gr_1$. Comparing with (3.2), we have $F/A = G/B$. Here the left hand side is a function of x^1 only and the right hand side a function of x^4 only. Thus a constant k exists so that $F = kA$, $G = kB$, and we have for r the consistent partial differential equations

$$r_1 = -A(x^1)(1-k/r)\,, \qquad r_4 = -B(x^4)(1-k/r)\,; \tag{3.5}$$

if r has been found to satisfy these, f is given by the equivalent expressions

$$f = 2A(x^1)\,r_4 = 2B(x^4)\,r_1 = -2AB(1-k/r)\,. \tag{3.6}$$

It is easy to verify by direct calculation that (3.5) and (3.6) together imply the last of (3.1): thus the problem of the spherically symmetric vacuum field is reduced to the solution of (3.5) and (3.6), the functions A and B being arbitrary.

If $k = 0$, substitution from (3.5) and (3.6) in (2.4) reduces the Riemann tensor to zero, so that space-time is flat. Let us then assume $k \neq 0$, and define

$$Z = r/k\,, \qquad C(x^1) = A/k\,, \qquad D(x^4) = B/k\,. \tag{3.7}$$

The basic equations become

$$Z_1 = -C(1-Z^{-1})\,, \qquad Z_4 = -D(1-Z^{-1})\,, \tag{3.8}$$

and

$$f = 2k^2 C Z_4 = 2k^2 D Z_1\,. \tag{3.9}$$

The indicial notation is somewhat clumsy here, so let us change back from (x^1, x^4) to (u, v). At the same time we may combine (3.8) into the single equation

$$Z\,dZ/(Z-1) = -C(u)\,du - D(v)\,dv\,, \tag{3.10}$$

while (3.9) reads

$$f = 2k^2 C(u) Z_v = 2k^2 D(v) Z_u\,, \tag{3.11}$$

the subscripts u, v indicating partial derivatives.

4. – The essential functional relationship.

Let a real variable H be related to Z by the differential equation

$$dH/H = Z\,dZ/(Z-1)\,. \tag{4.1}$$

All solutions of this equation are comprised in the functional relationship

$$H(Z, b) = b(Z-1)\,e^{Z}, \tag{4.2}$$

where b is a real constant. Some of thse curves are sketched in Fig. 1. By (4.2) Z is a function of H and b, $Z = Z(H, b)$, but not always single-valued, as is evident from Fig. 1.

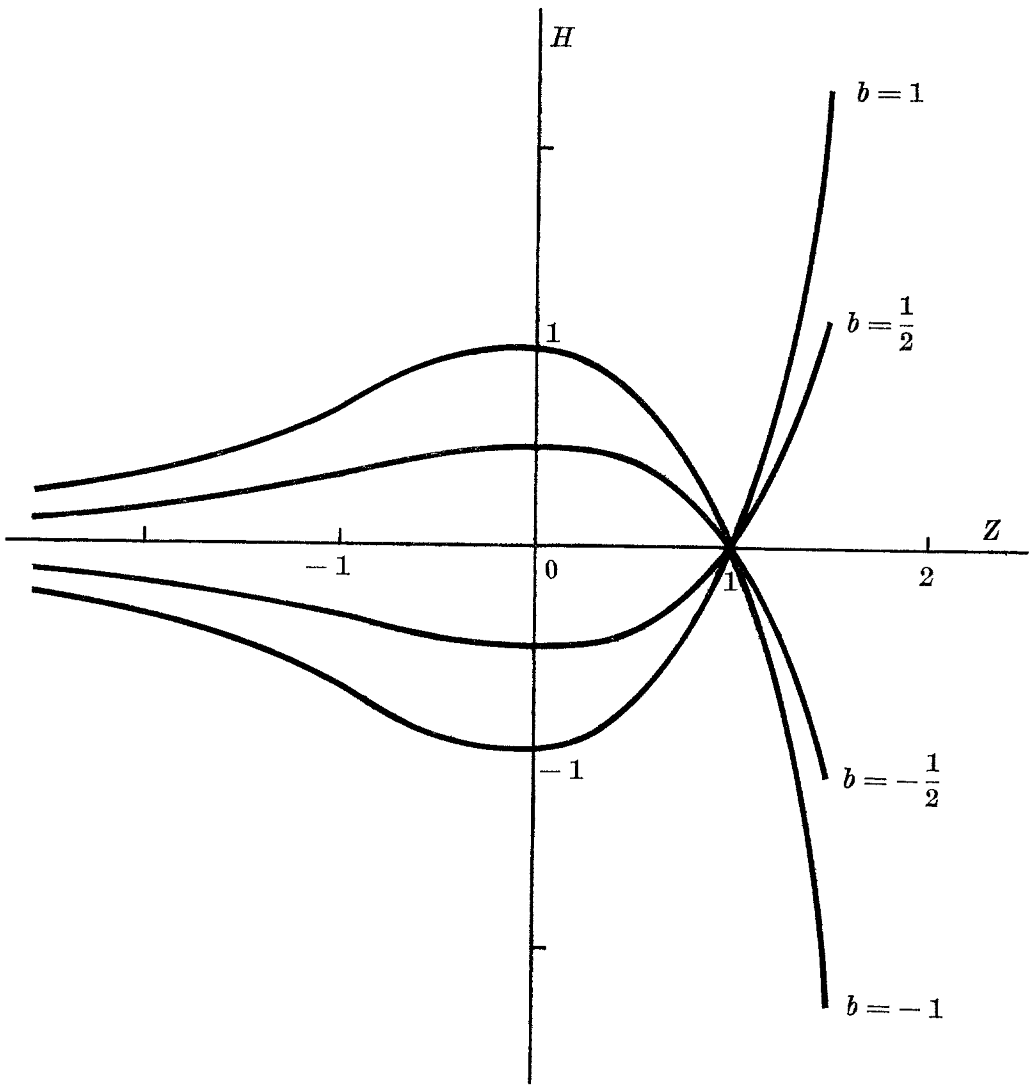

Fig. 1. – Graphs of the (H, Z) functional relationship. In the later argument, only the curve for $b = 1$ is used.

If we substitute from (4.1) in (3.10), we get

$$dH/H = -C(u)\,du - D(v)\,dv\,, \tag{4.3}$$

and this tells us that H is the product of a function of u by a function of v, say

$$H = U(u)V(v)\,. \tag{4.4}$$

Then

$$-C(u) = H_u/H = U'/U\,, \qquad -D(v) = H_v/H = V'/V\,. \tag{4.5}$$

By (4.1) we have

$$\begin{aligned} H_u/H &= Z(Z-1)^{-1}Z_u\,, \qquad & H_v/H &= Z(Z-1)^{-1}Z_v\,,\\ Z_u &= Z^{-1}(Z-1)H_u/H\,, \qquad & Z_v &= Z^{-1}(Z-1)H_v/H\,. \end{aligned} \tag{4.6}$$

As for the metric form (2.1), that is,

$$\Phi = -2f\,du\,dv + r^2\,d\sigma^2, \tag{4.7}$$

we have by (3.7)

$$r^2 = k^2Z^2, \tag{4.8}$$

where Z is determined as a function of H (and b) by (4.2) and H is as in (4.4), so that r^2 comes out as a function of (u, v). For f we have (3.11), and we can write it in a number of equivalent forms as a function of (u, v). If we substitute for $C(u)$ from (4.5) and for Z_v from (4.6), we get

$$f = -2k^2Z^{-1}(Z-1)H_uH_v/H^2. \tag{4.9}$$

Then the metric (4.7) reads

$$\Phi = k^2[4Z^{-1}(Z-1)H_uH_vH^{-2}du\,dv + Z^2d\sigma^2]\,. \tag{4.10}$$

There are two points of interest to note here. First, the constant k which is responsible for the curvature of our space-time, occurs only in the form k^2, so that the sign of k does not affect the metric. Second, k^2 appears as a factor multiplying a quadratic form, and we may put $k^2 = 1$ without loss of generality; it merely amounts to a change of units, or, equivalently, to multiplying all proper times by the same constant factor.

Let us then put $k^2 = 1$, so that henceforth the metric reads

$$\Phi = -2f\,du\,dv + Z^2\,d\sigma^2, \tag{4.11}$$

where

$$f = -2Z^{-1}(Z-1)H_u H_v H^{-2}. \tag{4.12}$$

To sum up: *The most general vacuum field with spherical symmetry is constructed as follows*:

i) *Choose two functions* $U(u)$, $V(v)$ *and define* H *as in* (4.4).

ii) *Choose a constant* b *and define* Z *as a function of* H (*and hence of* u *and* v) *by inverting the functional equation* (4.2).

iii) *Express* f *by* (4.12) *as a function of* u *and* v.

Although the constant b came in naturally as a constant of integration for the differential equation (4.1), it appears redundant in the sense that it can be absorbed into H. Accordingly I shall in future take $b=1$, so that the (H, Z) relationship (4.2) is now simply

$$H = (Z-1)\,e^Z, \qquad dH/dZ = Z\,e^Z. \tag{4.13}$$

The graph is now the curve labelled $b=1$, and it is important to note that it may be thought of as having two branches. In the *positive* branch Z runs from 0 to $+\infty$, and H from -1 to $+\infty$. In the *negative* branch Z runs from $-\infty$ to 0, and H from 0 to -1. Note the important inequality

$$H \geqslant -1\,. \tag{4.14}$$

5. – The Kruskal form.

Following the above scheme, let us choose

$$U(u) = u\,, \qquad V(v) = v\,. \tag{5.1}$$

Then by (4.4)

$$H = uv\,, \qquad H_u H_v = H\,. \tag{5.2}$$

By (4.13) Z is to be determined as a function of uv by the equation

$$(Z-1)\,e^Z = uv\,. \tag{5.3}$$

But now an important question arises: *which branch of the* (H, Z) *curve are we to use?* Let us examine each in turn. But first note that, for either, (4.12) gives

$$f = -2Z^{-1}(Z-1)/H = -2Z^{-1}e^{-Z}. \tag{5.4}$$

Note also that, since the field equations $R_{ij}=0$ have been satisfied, it follows from (2.6) that there is essentially only one surviving components of the Riemann tensor, namely R_{1224}. In terms of it we have (since $k=1$, $r=Z$)

$$R_{1414}=2fZ^{-2}R_{1224}\,,\qquad R_{2323}=-2f^{-1}Z^2s^2R_{1224}\,. \tag{5.5}$$

Substituting f from (5.4) in (2.4), we get

$$R_{1224}=Z^{-2}\,e^{-Z}. \tag{5.6}$$

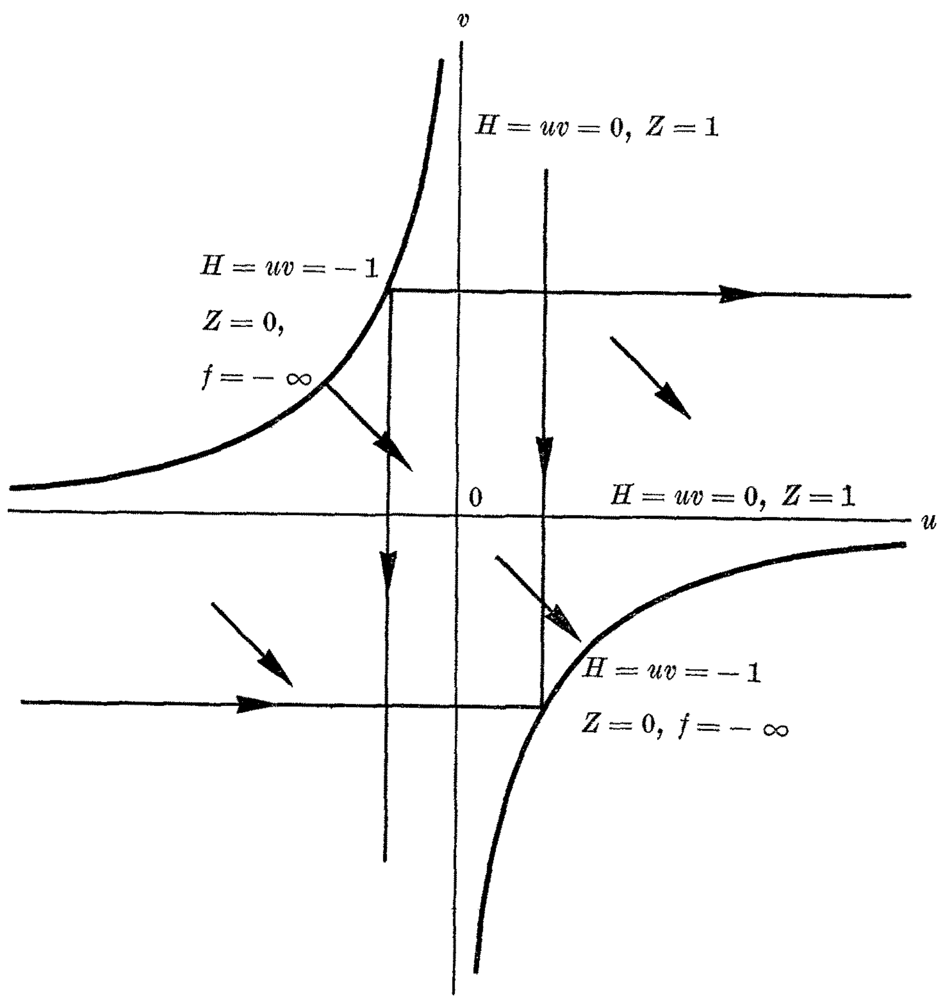

Fig. 2. – The relevant domain for Kruskal I. The arrows indicate consistent past-future directions; they might all be reversed.

It is convenient to display the coordinates (u, v) as rectangular Cartesian coordinates in a plane. The null lines in U_2 then appear as the straight lines $u = \text{const}$, $v = \text{const}$. These null lines are null geodesics in U_2 and also in V_4. But of course they do not represent all null geodesics in V_4—we may call them *radial* null geodesics,

If we use the *positive* branch of the (H, Z) curve, we have

$$H = uv \geqslant -1 . \tag{5.7}$$

Thus the domain D in which we are to operate is the part of the (u, v) plane which contains the origin and is bounded by the two branches of the hyperbola $uv = -1$ (Fig. 2). This is the *Kruskal universe.* It looks a little different, being turned through 45°, because he preferred to use $(u+v, u-v)$ where I have used (u, v). The arrows in Fig. 2 indicate consistent past-future directions; these are not determined by the metric and might be reversed. The « Schwarzschild singularity » is represented by the coordinate axes of u and v, and on them $H = 0$, $Z = 1$. How smooth are Z and f on those axes? By differentiating (5.3) and (5.4) with respect to u and v, we find that derivatives of all orders exist. If the problem of spherical

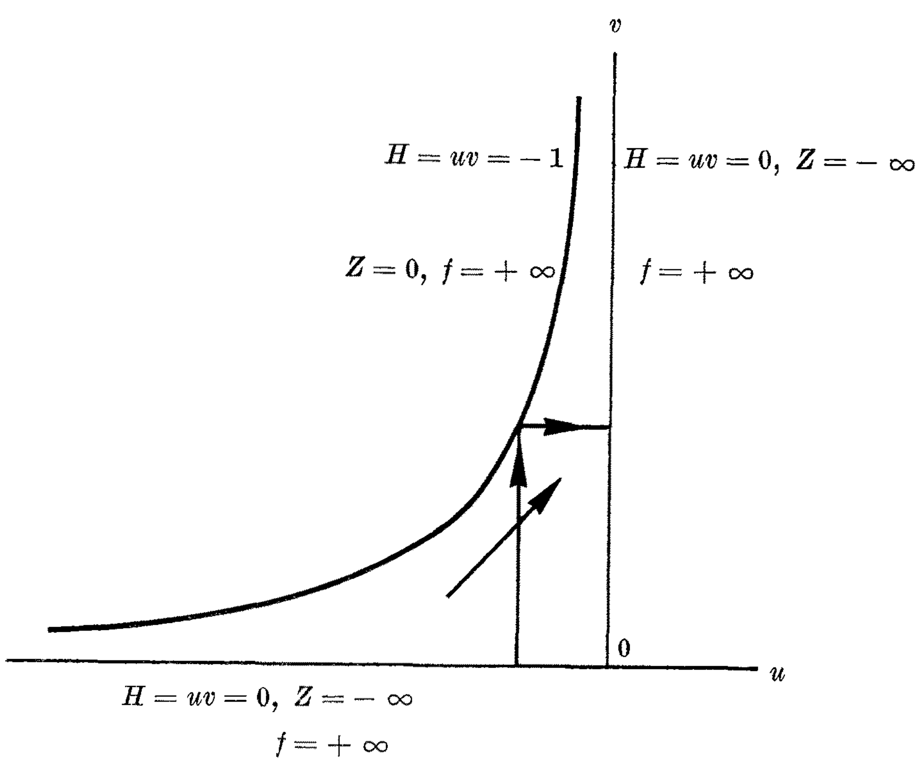

Fig. 3. – The relevant domain for Kruskal II, which is the field of a particle of negative mass. The arrows indicate consistent past-future directions; they might all be reversed.

symmetry had been attacked originally in this way, it would never have occurred to anyone to think of a singularity here. This does not mean that the Kruskal, universe is without its mysteries, but these are associated with the hyperbola, $uv = -1$, not with the coordinate axes $uv = 0$.

What shall we find if we use the *negative* branch of the (H, Z) curve? For it we have

$$(5.8) \qquad -1 \leqslant H = uv < 0 \, .$$

Thus the domain D in which we can operate is bounded by one branch of the hyperbola $uv = -1$ and by a pair of coordinate axes as shown in Fig. 3. There are of course two such domains, but it suffices to consider the one shown, namely the one for which $u < 0$, $v > 0$. As in Fig. 2, the arrows indicate consistent past-future directions, and might be reversed. Note that, whereas in the Kruskal universe, the singularities $uv = -1$ were *spacelike*, now $uv = -1$ is *timelike*, while the axes are null.

For reference purposes, I shall call the Kruskal universe K_1 and this other universe K_2. I shall now show that K_2 is something familiar but usually rejected in physics. The simplest thing to do is to resort to those coordinates which have caused so much confusion in regard to the Schwarzschild singularity. We have

$$(5.9) \qquad \begin{aligned} v\,du + u\,dv &= dH = HZ(Z-1)^{-1}dZ \, , \\ dv/v + du/u &= Z(Z-1)^{-1}dZ \, . \end{aligned}$$

Define τ by

$$(5.10) \qquad \tau = \ln(-v/u) \, ;$$

then

$$(5.10) \qquad dv/v - du/u = d\tau \, .$$

From (5.9) and (5.11)

$$(5.12) \qquad 4\,du\,dv/H = Z^2(Z-1)^{-2}dZ^2 - d\tau^2 .$$

By (5.4) the first part of the metric form is

$$(5.13) \qquad \begin{aligned} -2f\,du\,dv &= 4Z^{-1}e^{-Z}du\,dv = Z^{-1}e^{-Z}H[Z^2(Z-1)^{-2}dZ^2 - d\tau^2] \\ &= Z^{-1}(Z-1)[Z^2(Z-1)^{-2}dZ^2 - d\tau^2] \\ &= (1-Z^{-1})^{-1}dZ^2 - (1-Z^{-1})\,d\tau^2 . \end{aligned}$$

The complete metric form is then

$$(5.14) \qquad \Phi = (1-Z^{-1})^{-1}dZ^2 + Z^2 d\sigma^2 - (1-Z^{-1})\,d\tau^2 .$$

Comparing this with the familiar Schwarzschild metric (1.1), we recognise that *this K_2 universe is simple the field of a point-particle of negative mass, since in* (5.14) *Z is negative.*

To sum up: having chosen the functions U and V as in (5.1), we get the Kruskal universe K_1 by using the positive branch of the (H, Z) curve, and the field K_2 of a negative mass by using the negative branch. It is interesting to note that, although the constant k, introduced after (3.4), might have seemed to be a mass-factor (positive or negative), this constant did not in fact distinguish between the two universes. In (4.10) it appeared only in the form k^2 and was eliminated by change of scale.

6. – Further developments.

In the metric (4.10) the two functions $U(u)$ and $V(v)$ were arbitrary. To get the universes K_1 and K_2 those functions were chosen as in (5.1). By using other functions, can we create other spherically symmetric universes worthy of study?

Let us collect the essential formulae from Sect. 4:

$$\begin{aligned} H &= U(u)\,V(v)\,, \qquad H = (Z-1)\,e^Z, \\ \Phi &= -2f\,du\,dv + Z^2\,d\sigma^2, \\ f &= -2Z^{-1}(Z-1)H_u H_v H^{-2}. \end{aligned} \tag{6.1}$$

The exploration of any such universe call for the following steps:

i) Find the relevant (u, v) domain under the condition $H \geqslant -1$.

ii) Decide which branch of the (H, Z) curve to use, taking $b = 1$ in Fig. 1.

iii) Examine the behaviour of the affine parameter on the lines $u = \text{const}$, $v = \text{const}$, these being radial null geodesics.

iv) Examine the behaviour of proper time on timelike radial geodesics.

v) Extend iii) to all null geodesics in V_4.

vi) Extend iv) to all timelike geodesics in V_4.

In dealing with geodesics, it is best not use the Γ's of (2.3). We have the Lagrangian

$$L = -2fu'v' + Z^2(\theta'^2 + \sin^2\theta\varphi'^2)\,, \tag{6.2}$$

where the primes indicate derivatives with respect to an affine parameter for a null geodesic and with respect to proper time for a timelike geodesic. Confining attention to radial geodesics, we have the simple Lagrangian

$$L = -2fu'v', \tag{6.3}$$

and the equations of motion

$$(fv')' - f_u u'v' = 0 , \qquad (fu')' - f_v u'v' = 0 , \tag{6.4}$$

with the integral

$$2fu'v' = \varepsilon , \tag{6.5}$$

where $\varepsilon = 0$ for a null geodesic, $\varepsilon = 1$ for a timelike geodesic.

The line $v = \text{const}$ is a radial null geodesic. The first of (6.4) is satisfied and the second gives

$$fu' = \text{function of } v . \tag{6.6}$$

Hence, if dw is an element of an affine parameter,

$$dw = C(v)\, f(u, v)\, du , \tag{6.7}$$

and w is obtained as an integral which may converge or diverge as the limits of integration tend to infinity or to a boundary of the domain D in the (u, v) plane. In (6.7) the function $C(v)$ is arbitrary.

As an illustration of at least the first steps, consider the universe for which

$$U(u) = e^u, \qquad V(v) = e^v, \qquad H = e^{u+v}, \tag{6.8}$$

so that

$$H_u = H_v = H . \tag{6.9}$$

Since H is positive for all finite values of (u, v), the limitation $H \geqslant -1$ does not operate; the domain D is the whole of the (u, v) plane. We have to choose the positive branch of the (H, Z) curve in Fig. 1, and moreover only the upper part, for which $H > 0$, $Z > 1$. Now Z is given as a function of $(u + v)$ by the equation

$$e^{u+v} = (Z - 1)\, e^Z, \tag{6.10}$$

and, by (4.12),

$$f = -2Z^{-1}(Z - 1) . \tag{6.11}$$

Let us now move along the radial null geodesic $v = \text{const}$, and apply (6.7) to see how the affine parameter changes. From (6.10) we have

$$e^{u+v}\, du = Ze^Z\, dZ , \qquad du = Z(Z - 1)^{-1}\, dZ , \tag{6.12}$$

and so, by (6.7), (6.11) and (6.12),

$$dw = -2C(v)\,dZ\,. \tag{6.13}$$

Since an affine parameter is always undetermined to within a linear transformation, we may write

$$w = Z\,. \tag{6.14}$$

Proceeding then from $u = -\infty$ to $u = +\infty$, we see that w increases monotonically from unity to $+\infty$. The universe is incomplete in the sense that, in coming from $u = -\infty$ to any finite value of u, w increases by a finite amount. The same is true for $u = \text{const}$.

7. – Canonical null coordinates.

The results of the preceding section may appear somewhat vacuous, since all that was done was to push the singularity to infinity by the monotonic transformation (6.8). However, before that was done, a useful formula (6.7) was obtained for the affine parameter on a null line $v = \text{const}$, with of course a similar formula for a null line $u = \text{const}$.

Let us look at the choice of coordinates in V_4. As for (θ, φ), they are the usual polar angles on a sphere, and we know how to transform them; all such systems of polar coordinates are equally good.

But in the case of (u, v), all that was demanded was that the null lines in U_2 should have the equations $u = \text{const}$, $v = \text{const}$. Given the null lines as geometrical objects, that leaves considerable freedom in the choice of (u, v). I shall now suggest a way in which canonical coordinates (u, v) may be assigned.

Assume in U_2 the existence of two families of null lines, M and N. Assume further that no two members of M intersect, and no two members of N intersect. But each member of M intersects all members of N, and each member of N intersects all members of M. These properties may only hold locally, but for expository purposes they will be assumed in general.

Let M_0 be a member of M and N_0 a member of N. Let M_0 and N_0 intersect at P. Let u be an affine parameter on M_0 and v an affine parameter on N_0, chosen so that $u = v = 0$ at P. To any point Q on U_2 assign coordinates (u, v) as follows. Through Q draw the null lines, M_Q, N_Q, of the two families. Then N_Q intersects M_0 at a point to which a coordinate u has already been assigned and M_Q intersects N_0 at a point to which a coordinate v has already been assigned (Fig. 4). Let us assign to Q those coordinates (u, v). It is clear that the null lines in M and N have the equations $v = \text{const}$, $u = \text{const}$. I shall call these *canonical null coordinates*. Their assignment is of course not unique: first, we have chosen M_0 and N_0 from the respective families, and, second, an affine parameter is always undetermined to within a linear transformation. But this seems to be as far as we can go to define canonical coordinates.

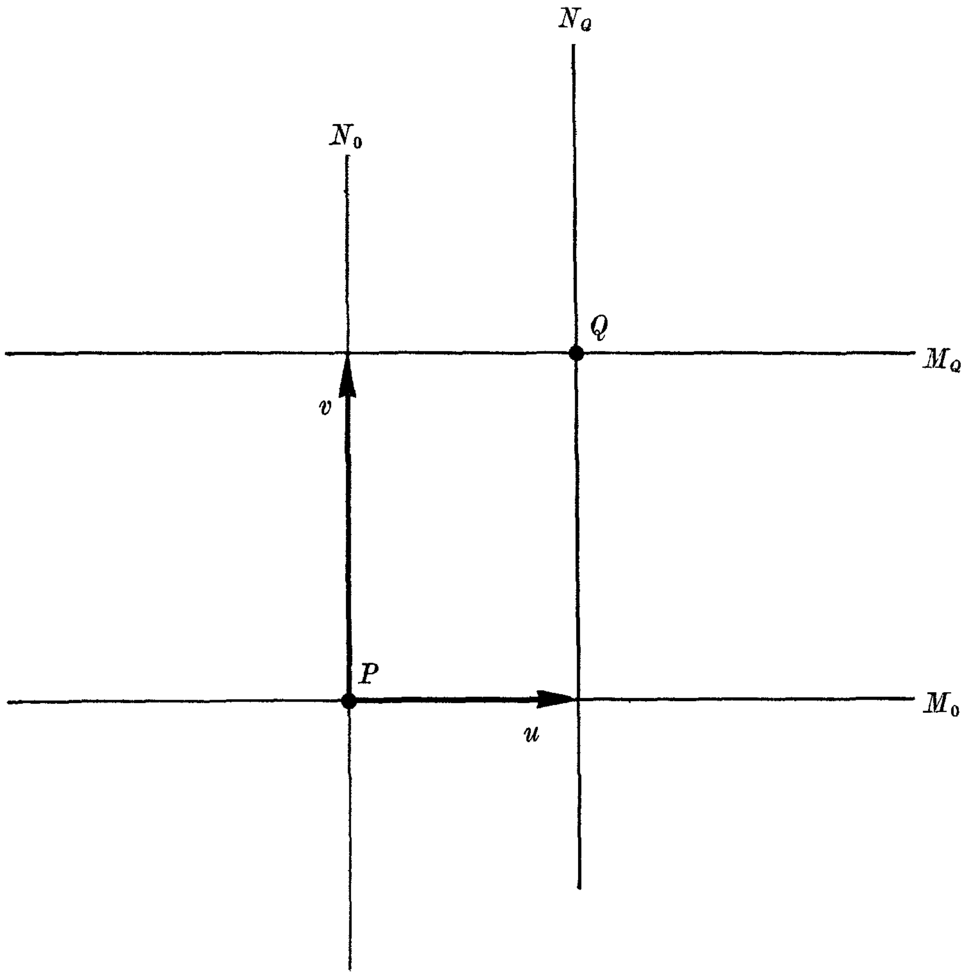

Fig. 4. – Canonical null coordinates in U_2.

Let us now turn to the formulae (6.1), noting in particular that

$$H = U(u)V(v) . \tag{7.1}$$

I shall now show that *if* (u, v) *are canonical coordinates as above, the functions* $U(u)$, $V(v)$ *cannot be arbitrarily chosen, but are determined to within a few arbitrary constants.*

Consider the null line M_0; u is an affine parameter on it, and $v=0$ on it. Thus in (6.7) $dw=du$, and so $f(u, 0)$ is independent of u. Using the same argument for N_0, we see that the function $f(u, v)$ must be such that

$$f(u,0)=C\,,\qquad f(0,v)=D\,, \tag{7.2}$$

where C and D are constants.

Now by (6.1), with H as in (7.1),

$$\begin{aligned} f(u,v) &= -2Z^{-1}(Z-1)H_uH_v/H^2 \\ &= -2Z^{-1}(Z-1)\,U'V'/H \\ &= -2Z^{-1}e^{-Z}\,U'V', \end{aligned} \tag{7.3}$$

where $U'=dU/du$, $V'=dV/dv$. Here we have used the (H, Z) relationship

$$H=(Z-1)\,e^{Z}. \tag{7.4}$$

Let us decide to use the positive or negative branch. Then (7.4) has a unique inverse

$$Z=Z(H)\,. \tag{7.5}$$

Define

$$P(u)=U(u)\,V(0)\,,\qquad Q(v)=U(0)\,V(v)\,. \tag{7.6}$$

Then

$$Z=Z(P)\quad \text{on } v=0\,,\qquad Z=Z(Q)\quad \text{on } u=0\,, \tag{7.7}$$

and hence by (7.2) and (7.3)

$$-2[Z(P)\exp(Z(P))]^{-1}[V'(0)/V(0)]\,dP/du=C\,, \tag{7.8}$$

and another equation with P, C replaced by Q, D. On carrying out the integrations, we have the following result:

If we use canonical null coordinates (u, v) in the 2-space U_2, so that u and v are affine parameters on the null geodesics $v=0$ and $u=0$ respectively, with $u=v=0$ at the origin, and if we assign values to the constants

$$U(0),\quad V(0),\quad U'(0),\quad V'(0),\quad C,\ D, \tag{7.9}$$

then the functions $U(u)$, $V(v)$ *in* (7.1) *are determined by the formulae*

$$-2[V'(0)/V(0)]\int_{U(0)V(0)}^{U(u)V(0)}[Z(x)\exp(Z(x))]^{-1}\,dx = Cu,$$

(7.10)

$$-2[U'(0)/U(0)]\int_{U(0)V(0)}^{U(0)V(v)}[Z(x)\exp(Z(x))]^{-1}\,dx = Dv\,,$$

the function Z *being as in* (7.5).