Modeling, Analysis and Control of Networked Evolutionary Games

Daizhan Cheng, Fenghua He, Hongsheng Qi, and Tingting Xu

Abstract

This paper considers a networked evolutionary game. According to strategy updating rules, a method is proposed to calculate the dynamics of the profile, which is expressed as a k-valued logical dynamic network. This dynamic model is then used as a framework to analyze the dynamic behaviors, such as emergence *etc*., of the networked evolutionary game. To apply the method to arbitrary size networks, the homogeneous networked games are then investigated. Certain interesting general results are obtained. Finally, the control of networked evolutionary games is considered, network consensus is explored. The basic tool for this approach is the semi-tensor product of matrices, which is a generalization of conventional matrix product.

Index Terms

Networked evolutionary game, strategy updating rule, k-valued logical dynamic network, Probabilistic network, Emergence, Consensus.

I. Introduction

EVOLUTIONARY game was firstly introduced by biologists for describing the evolution of lives in nature [3, 32]. In recent years, the investigation of complex networks have attracted much attention from physical, social, and system and control communities [16, 23]. Early studies on evolutionary games were based on uniformly mixed form, *i.e.*, assume each player plays with all other players or randomly with some others, and various game-types for pairwise game were used. For instance, Prisoner's Dilemma [2, 25, 34], Public Goods Game [12, 14], Snowdrift Game (or Hawk-Dove Game) [15, 33], *etc*.

In the last few years, the investigation on evolutionary game on graphs, or networked evolutionary game (NEG), becomes a very appealing research because the evolution of biological systems is naturally over a networked environment [6]. Practically, it has very wide background in biological system, economical system, social system *etc*. [15, 24, 30, 36]. The theoretical interest comes from the observation that it merges two important ideas together: (i) the interactions over a network [6]; (ii) the dynamics forms an evolutionary approach [26].

In an NEG, a key issue is the strategy updating rule (learning rule). That is, how a player to choose his next strategy based on his information about his neighborhood players. There are several commonly used rules such as Unconditional Imitation (UI) [24], Femi Rule [34, 37], Moran Rule [19, 29].

In last several decades, another kind of networks, called the finitely valued logical network, have also been studied widely. Particular interest has been put on Boolean networks [17], and k-valued logical dynamic networks [1, 38], and their corresponding probabilistic networks [31]. The strong interest on these networks is mainly also caused by biological systems, because they can be used to describe gene regulatory network, metabolic network *etc*. [18].

A recent development in the study of logical networks is the application of semi-tensor product (STP) approach [8, 20]. STP is a new matrix product, which generalizes the conventional matrix product to arbitrary two matrices. We refer to Appendix for a brief review. STP approach uses STP to express a logical equation into a matrix form, which makes it possible to convert a logical dynamic system into a conventional discrete time system. Then the conventional analysis tools can be used to analyze

This work was supported in part by the National Natural Science Foundation (NNSF) of China under Grants 61074114, 61273013 and 61104065.

D. Cheng, H. Qi and T. Xu are with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, P.R.China (E-mail: dcheng@iss.ac.cn, qihongsh@amss.ac.cn, xutingtinghello@126.com). F. He is with the Institute of Astronautics, Harbin Institute of Technology, Harbin 150080, P.R.China (E-mail: hefenghua@gmail.com).

and design control for logical dynamic networks. This approach has been proved very successful. [9] gives a comprehensive introduction to this.[1]

Consider a networked evolutionary game (NEG), where each player plays same game pairwise with each of his neighborhood players, and assume the strategy updating rule is the same for all players. The most challenging problem is the global behaviors of the evolution, say, emergence. Currently, there are some efficient methods to model this. One is the statistic-based approximation. Replicator dynamics is one of the most studied dynamics in evolutionary game theory [35, 36]. Another one is the simulation-based analysis [24, 34]. Some experiments have also been performed to verify the emergence [27].

As mentioned in [40], since the short of proper mathematical tools, analyzing the dynamics of NEGs directly is difficult. This paper proposes to use STP approach to the modeling, analysis, and control of NEG. Using the strategy updating rule, we first convert the evolutionary dynamics of a networked game into a k-valued logical network, where k is the number of actions (strategies) for each player. This converting is extremely useful because in this way we are able to provide a rigorous mathematical model for the profile evolution of NEGs. Then the tools developed and results obtained for logical networks are available for NEGs. Moreover, using the rigorous dynamic equation of NEGs, we are also able to consider the control of NEGs theoretically.

The rest of this paper is organized as follows: Section 2 gives a formulation for NEGs. Three main factors of an NEG: (i) network graph, (ii) fundamental network game, (iii) strategy updating rule are precisely introduced. Section 3 considers the modeling of NEG. In the light of semi-tensor product approach, it is revealed that the evolutionary dynamics of a NEG can be expressed as a k-valued logical dynamic network. This discovery provides a precise mathematical model for NEGs. Based on their mathematical models, the dynamical properties of NEGs are analyzed in Section 4. Section 5 devotes to homogeneous NEGs. We intend to find some general properties for arbitrary size NEGs. The control problems of NEGs are explored in Section 6. Particularly, the consensus of NEGs is considered. Section 7 contains some concluding remarks. A brief review of STP is given in Appendix.

II. Formulation of Networked Evolutionary Games

For statement ease, we first give some notations.

1) $\mathcal{M}_{m\times n}$ is the set of $m\times n$ real matrices;
2) $\mathrm{Col}_i(M)$ is the i-th column of matrix M; $\mathrm{Col}(M)$ is the set of columns of M;
3) $\mathcal{D}_k := \{1,2,\cdots,k\}$;
4) $\delta_n^i := \mathrm{Col}_i(I_n)$, *i.e.*, it is the i-th column of the identity matrix;
5) $\Delta_n := \mathrm{Col}(I_n)$;
6) $M\in\mathcal{M}_{m\times n}$ is called a logical matrix if $\mathrm{Col}(M)\subset\Delta_m$, the set of $m\times n$ logical functions is denoted by $\mathcal{L}_{m\times n}$;
7) Assume $L\in\mathcal{L}_{m\times n}$, then
$$L=\left[\delta_m^{i_1}\;\delta_m^{i_2}\;\cdots\;\delta_m^{i_n}\right];$$
and its shorthand form is
$$L=\delta_m\left[i_1\;i_2\;\cdots\;\delta_m\right].$$
8) $r=(r_1,\cdots,r_k)^T\in\mathbb{R}^k$ is called a probabilistic vector, if $r_i\geq 0$, $i=1,\cdots,k$, and
$$\sum_{i=1}^k r_i=1.$$
The set of k dimensional probabilistic vectors is denoted by Υ_k.
9) If $M\in\mathcal{M}_{m\times n}$ and $\mathrm{Col}(M)\subset\Upsilon_m$, M is called a probabilistic matrix. The set of $m\times n$ probabilistic matrices is denoted by $\Upsilon_{m\times n}$.

[1] A Mablab® Toolbox has been created for the STP computation at http://lsc.amss.ac.cn/~dcheng/stp/STP.zip. Numerical computation of the examples in this paper is based on this toolbox.

10) A k dimensional vector with all entries equal to 1 is denoted by

$$\mathbf{1}_k := (\underbrace{1\ 1\ \cdots\ 1}_{k})^T.$$

11) $A \ltimes B$ is the semi-tensor product (STP) of two matrices A and B. (Please refer to the Appendix for STP. Throughout this paper the matrix product is assumed to be STP. Moreover, the symbol "$\ltimes$" is mostly omitted. That is, express

$$AB := A \ltimes B.$$

12) $W_{[m,n]}$ is a swap matrix. (Please refer to the Appendix for swap matrix.)

Since there are many articles on NEG and we are interested in a general framework, to avoid ambiguity, we give a general definition for NEG. We start by specifying (i) the network graph, (ii) the fundamental network game, and (iii) the strategy updating rule.

A. Network Graph

Given a set N and $E \subset N \times N$, (N, E) is called a graph, where N is the set of nodes and E the set of edges. If $(i, j) \in E$ implies $(j, i) \in E$ the graph is undirected, otherwise, it is directed. Let $N' \subset N$, and $E' = (N' \times N') \cap E$. Then (N', E') is called a sub-graph of (N, E). Briefly, N' is a subgraph of N. A network graph can be briefly called a network.

Definition II.1. *A network is called a homogeneous network, if either its directed network and all its nodes have same number of in-degree and same number of out-degree, or it is undirected and all its nodes have same number of degree. A network, which is not homogeneous, is called a heterogeneous network.*

We give some examples of networks.

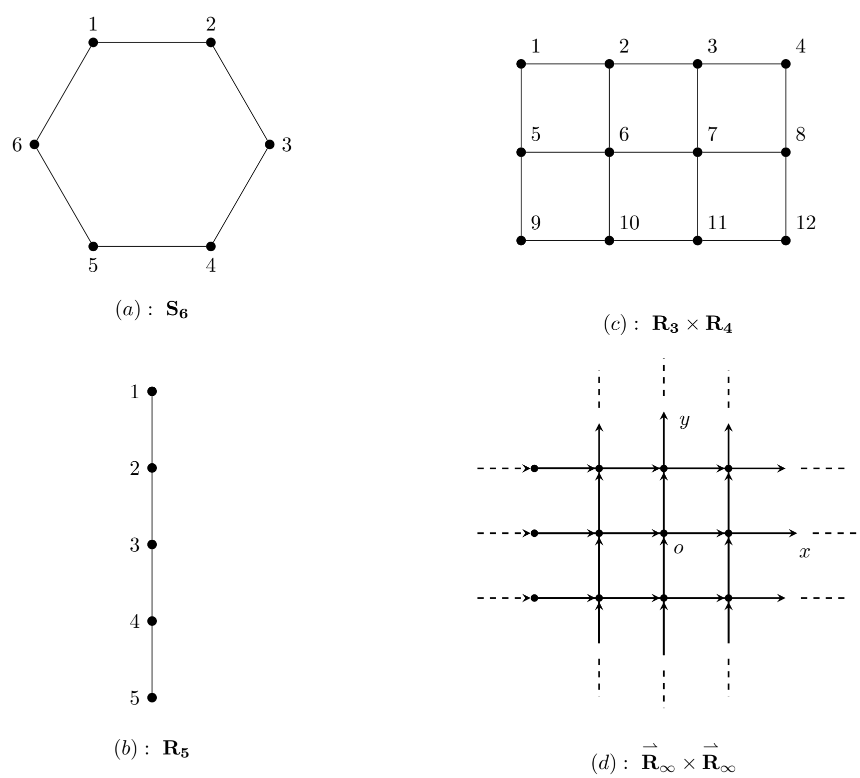

Fig. 1: Some Standard Networks

Example II.2. (i) *S_n is a cycle with n nodes. S_6 is shown in Fig. 1 (a);*

(ii) *R_n is a line with n nodes. R_5 is shown in Fig. 1 (b), note that we use R_∞ for a line with infinite nodes (on both directions).*

(iii) *$R_m \times R_n$ is a rectangle with height m and width n. $R_3 \times R_4$ is shown in Fig. 1 (c);*

(iv) *In Fig. 1 (c) if we identify two nodes in pairs $\{1,9\}$, $\{2,10\}$, $\{3,11\}$, $\{4,12\}$, we have $S_2 \times R_4$. If we further identify two nodes in pairs $\{1,4\}$, $\{3,8\}$, $\{9,12\}$, then we have $S_2 \times S_3$;*

(v) *A directed line with unified direction is denoted by $\vec{R}_n$. Similarly, we can define $\vec{R}_m \times \vec{R}_n$. $\vec{R}_\infty \times \vec{R}_\infty$ is shown in Fig. 1 (d), where all the horizontal edges are directed to the right and all vertical edges are directed up.*

Note that S_n, R_∞, $S_m \times S_n$, and $\vec{R}_\infty \times \vec{R}_\infty$ are homogeneous networks, while R_5, $R_m \times R_n$, $S_2 \times R_4$ are not.

Next, we define the neighborhood of a node. Note that when the networked games are considered the direction of an edge is used to determine (in a non-symmetric game) who is player one and who is player two. Hence when the neighborhoods are considered the direction of an edge is ignored.

Definition II.3. *Let N be the set of nodes in a network, $E \subset N \times N$ the set of edges.*

(i) *$j \in N$ is called a neighborhood node of i, if either $(i,j) \in E$ or $(j,i) \in E$. The set of neighborhood nodes of i is called the neighborhood of i, denoted by $U(i)$. Throughout this paper it is assumed that $i \in U(i)$.*

(ii) *Ignoring the directions of edges, if there exists a path from i to j with length less than or equal to r, then j is said to be an r-neighborhood node of i, the set of r-neighborhood nodes of i is denoted by $U_r(i)$.*

Consider R_∞, and assume the nodes are labeled by $\cdots,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \cdots$. Then

$$U(0) = \{-1\ ,0,\ 1\}; \quad U_2(0) = \{-2,\ -1,\ 0,\ 1,\ 2\}; \quad U_3(0) = \{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}; \quad \cdots$$

Consider $\vec{R}_\infty \times \vec{R}_\infty$. Then (ignoring the directions of edges)

$$\begin{array}{rcl} U(O) & = & \{(-1,0),(0,0),(0,1),(0,-1),(1,0)\}; \\ U_i(O) & = & \{(\alpha,\beta) | \alpha,\beta \in \mathbb{Z},\ \text{and}\ |\alpha|+|\beta| \le i\}, \quad i = 2,3,\cdots. \end{array}$$

B. Fundamental Network Game (FNG)

A normal finite game we considered consists of three factors [13]:

(i) n players $N = \{1,2,\cdots,n\}$;

(ii) Player i has $S_i = \{1,\cdots,k_i\}$ strategies, $i = 1,\cdots,n$, $S := \prod_{i=1}^n S_i$ is the set of profiles;

(iii) Player i has its payoff function $c_i : S \to \mathbb{R}$, $i = 1,\cdots,n$, $c := (c_1,c_2,\cdots,c_n)$.

Definition II.4. (i) *A normal game with two players is called a fundamental network game (FNG), if*

$$S_1 = S_2 := S_0 = \{1,2,\cdots,k\}.$$

(ii) *An FNG is symmetric, if*

$$c_1(x,y) = c_2(y,x), \quad \forall x,y \in S_0.$$

Example II.5. *Consider the Prisoner's Dilemma [28]. Two players can choose strategies from*

$$S_0 = \{1,2\},$$

where 1 means "cooperate" and 2 means "defect". The payoff bi-matrix is as in Table I.

TABLE I: Payoffs for Prisoner's Dilemma

$P_1 \backslash P_2$	1	2
1	$(R,\ R)$	$(S,\ T)$
2	$(T,\ S)$	$(P,\ P)$

This game can be chosen as an FNG, which is symmetric. In fact, it is a popularly used FNG in NEG.

C. Strategy Updating Rule

Definition II.6. *A strategy updating rule for an NEG, denoted by Π, is a set of mappings:*

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \big| j \in U(i)\}\right), \quad t \geq 0, \quad i \in N. \tag{1}$$

That is, the strategy of each player at time $t+1$ depends on its neighborhood players' information at t, including their strategies and payoffs.

Note that (i) f_i could be a probabilistic mapping, which means a mixed strategy is used by player i; (ii) when the network is homogeneous, f_i, $i \in N$, are the same.

We have mentioned several different strategy updating rules in Introduction. We refer to [12] for a detailed description. Throughout this paper we only use the following three rules, denoted by $\Pi - I$, $\Pi - II$, and $\Pi - III$ respectively.

(i) $\Pi - I$: Unconditional Imitation [24] with fixed priority: The strategy of player i at time $t+1$, $x_i(t+1)$, is selected as the best strategy from strategies of neighborhood players $j \in U_i$ at time t. Precisely, if

$$j^* = \text{argmax}_{j \in U(i)}\, c_j(x(t)), \tag{2}$$

then

$$x_i(t+1) = x_{j^*}(t). \tag{3}$$

When the players with best payoff are not unique, say

$$\text{argmax}_{j \in U(i)}\, c_j(x(t)) := \{j_1^*, \cdots, j_r^*\},$$

then we may choose one corresponding to a priority as

$$j^* = \min\{\mu | \mu \in \text{argmax}_{j \in U(i)}\, c_j(x(t))\}. \tag{4}$$

This method leads to a deterministic k-valued dynamics.

(ii) $\Pi - II$: Unconditional Imitation with equal probability for best strategies. When the best payoff player is unique, it is the same as $\Pi - I$. When the players with best payoff are not unique, we randomly choose one with equal probability. That is,

$$x_i(t+1) = x_{j_\mu^*}(t), \quad \text{with probability} \;\; p_\mu^i = \frac{1}{r}, \quad \mu = 1, \cdots, r. \tag{5}$$

This method leads to a probabilistic k-valued dynamics.

(iii) $\Pi - III$: Use a simplified Femi Rule [34, 37]. That is, randomly choose a neighborhood $j \in U(i)$. Comparing $c_j(x(t))$ with $c_i(x(t))$ to determine $x_i(t+1)$ as

$$x_i(t+1) = \begin{cases} x_j(t), & c_j(x(t)) > c_i(x(t)) \\ x_i(t), & \text{otherwise.} \end{cases} \tag{6}$$

This method leads to a probabilistic k-valued dynamics.

In fact, the method developed in this paper is applicable to any strategy updating rules.

D. Networked Evolutionary Game

Definition II.7. *A networked evolutionary game, denoted by $((N,E), G, \Pi)$, consists of*

(i) *a network (graph) (N,E);*

(ii) *an FNG, G, such that if $(i,j) \in E$, then i and j play FNG repetitively with strategies $x_i(t)$ and $x_j(t)$ respectively. Particularly, if the FNG is not symmetric, then the corresponding network must be directed to show i is player one and j is player two;*

(iii) *a local information based strategy updating rule, which can be expressed as (1).*

Finally, we have to specify the overall payoff for each player.

Definition II.8. *Let $c_{i,j}$ be the payoff of the FNG between i and j. Then the overall payoff of player i is*

$$c_i(t) = \sum_{j \in U(i)\backslash i} c_{ij}(t), \quad i \in N. \tag{7}$$

Remark II.9. *If the network is heterogeneous, the number of times played by i needs to be considered. In this case, (7) is replaced by*

$$c_i(t) = \frac{1}{|U(i)| - 1} \sum_{j \in U(i)\backslash i} c_{ij}(t), \quad i \in N. \tag{8}$$

For notational ease, throughout this paper, we use (7) for homogeneous case, and use (8) for heterogeneous case.

Definition II.10. *Consider a networked evolutionary game $((N, E), G, \Pi)$.*

(i) *If the network (graph) is homogeneous, the game is called a homogeneous NEG.*
(ii) *If N' is a sub-graph of N, then the evolutionary game over N' with same FNG and same strategy updating rule, that is, $((N', E'), G, \Pi)$, is called a sub-game of the original game.*

III. Model of Networked Evolutionary Games

Theorem III.1. *The evolutionary dynamics can be expressed as*

$$x_i(t+1) = f_i(\{x_j(t) \big| j \in U_2(i)\}), \quad i \in N. \tag{9}$$

Proof: Observing (2), $c_j(t)$ depends on $x_k(t)$, $k \in U(j)$, and this is independent of strategy updating rule. According to (1), $x_i(t+1)$ depends on $x_j(t)$ and $c_j(t)$, $j \in U(i)$. But $c_j(t)$ depends on $x_k(t)$, $k \in U(j)$. We conclude that $x_i(t+1)$ depends on $x_j(t)$, $j \in U_2(i)$, which leads to (9). ■

Remark III.2. (i) *As long as the strategy updating rule is assigned, the f_i, $i \in N$ can be determined. Then (9) becomes a k-valued logical dynamic network. It will be called the fundamental evolutionary equation.*
(ii) *For a homogeneous network all f_i are the same.*

We use some examples to show how to use strategy updating rule to determine the fundamental evolutionary equation. Note that since (9) is a k-valued logical dynamic network, we can use the matrix expression for k-valued logical equations. Please refer to the Appendix.

Example III.3. *Assume the network is R_3 and the FNG is the game of Rock-Scissors-Cloth. The payoff bi-matrix is shown in Table II.*

TABLE II: Payoff Bi-matrix (Rock-Scissors-Cloth)

$P_1\backslash P_2$	$R=1$	$S=2$	$C=3$
$R=1$	(0, 0)	(1, −1)	(−1, 1)
$S=2$	(−1, 1)	(0, 0)	(1, −1)
$C=3$	(1, −1)	(−1, 1)	(0, 0)

(i) *Assume the strategy updating rule is $\Pi - I$:*
If $x_1(t)$, $x_2(t)$, $x_3(t)$ are known, then $x_i(t+1) = f_i(x_1(t), x_2(t), x_3(t))$ can be calculated. For instance, assume $x_1(t) = 1$, $x_2(t) = 2$, $x_3(t) = 3$, then

$$\begin{aligned} &c_1(t) = 1, \\ &c_{21}(t) = -1, \quad c_{23}(t) = 1, \ \Rightarrow \ c_2(t) = 0, \\ &c_3(t) = -1. \end{aligned}$$

Hence,

$$\begin{aligned} x_1(t+1) &= f_1(x_1(t), x_2(t), x_3(t)) \\ &= x_{\text{argmax}_j\{c_1(t), c_2(t)\}}(t) = x_1(t) = 1, \\ x_2(t+1) &= f_2(x_1(t), x_2(t), x_3(t)) \\ &= x_{\text{argmax}_j\{c_1(t), c_2(t), c_3(t)\}}(t) = x_1(t) = 1, \\ x_3(t+1) &= f_3(x_1(t), x_2(t), x_3(t)) \\ &= x_{\text{argmax}_j\{c_2(t), c_3(t)\}}(t) = x_2(t) = 2, \end{aligned}$$

Using the same argument for each profile (x_1, x_2, x_3)*,* f_i*,* $i = 1, 2, 3$*, can be figured out as in Table III.*

TABLE III: Payoffs $\to$ Dynamics

Profile	111	112	113	121	122	123	131	132	133
C_1	0	0	0	1	1	1	-1	-1	-1
C_2	0	1/2	-1/2	-1	-1/2	0	1	0	1/2
C_3	0	-1	1	1	0	-1	-1	1	0
f_1	1	1	1	1	1	1	3	3	3
f_2	1	1	3	1	1	1	3	2	3
f_3	1	1	3	1	2	2	3	2	3
Profile	211	212	213	221	222	223	231	232	233
C_1	-1	-1	-1	0	0	0	1	1	1
C_2	1/2	1	0	-1/2	0	1/2	0	-1	-1/2
C_3	0	-1	1	1	0	-1	-1	1	0
f_1	1	1	1	2	2	2	2	2	2
f_2	1	1	3	1	2	2	2	2	2
f_3	1	1	3	1	2	2	3	2	3
Profile	311	312	313	321	322	323	331	332	333
C_1	1	1	1	-1	-1	-1	0	0	0
C_2	-1/2	0	-1	0	1/2	1	1/2	-1/2	0
C_3	0	-1	1	1	0	-1	-1	1	0
f_1	3	3	3	2	2	2	3	3	3
f_2	3	3	3	1	2	2	3	2	3
f_3	1	1	3	1	2	2	3	2	3

Identifying $1 \sim \delta_3^1$*,* $2 \sim \delta_3^2$*,* $3 \sim \delta_3^3$*, we have the vector form of each* f_i *as*

$$x_i(t+1) = f_i(x_1(t), x_2(t), x_3(t)) = M_i x_1(t) x_2(t) x_3(t), \quad i = 1, 2, 3, \tag{10}$$

where

$$\begin{aligned} M_1 &= \delta_3[1\ 1\ 1\ 1\ 1\ 1\ 3\ 3\ 3\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 2\ 2\ 2\ 3\ 3\ 3]; \\ M_2 &= \delta_3[1\ 1\ 3\ 1\ 1\ 1\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 2\ 2\ 2\ 3\ 3\ 3\ 1\ 2\ 2\ 3\ 2\ 3]; \\ M_3 &= \delta_3[1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 2\ 3\ 2\ 3]. \end{aligned}$$

(ii) *Assume the strategy updating rule is* $\Pi - II$*: Since player 1 and player 3 have no choice,* f_1 *and* f_3 *are the same as in* Π *is BNS. That is,*

$$M_1' = M_1, \quad M_3' = M_3.$$

Consider player 2, who has two choices: either choose 1 *or choose* 3*, and each choice has probability* 0.5*. Using similar procedure, we can finally figure out* f_2 *as:*

$$M_2' = \\ \begin{bmatrix} 1 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & 1 \end{bmatrix}$$

Next, we give another example, where the payoff bi-matrix is not symmetric, and hence the network graph is directed.

Example III.4. *Consider a networked game: Assume the fundamental network game is the Boxed Pigs Game [28]. The strategy set is $\{P = 1,\ W = 2\}$, where P means press the panel and W means wait. It is not a symmetric game and P_1 and P_2 represent smaller pig and bigger pig respectively. Then the payoffs are shown in Table IV.*

TABLE IV: Payoff Bi-matrix for the Boxed Pigs Game

$P_1\backslash P_2$	P	W
P	(2, 4)	(0, 6)
W	(5, 1)	(0, 0)

Next, assume there are 4 pigs, labeled by 1, 2, 3 and 4, in which Pig 1 is the smallest pig, Pig 3 is the biggest one, and Pig 2 and Pig 4 are mid-size pigs. The network is shown in Fig. 2.

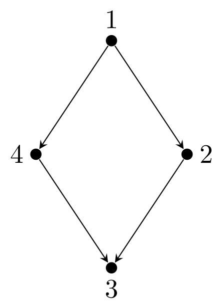

Fig. 2: The Boxed Pigs Game over a Uniformed Network

By comparing the payoffs and using $\Pi - III$, we can obtain that

$$\begin{aligned} x_1(t+1) &= f_1(x_1(t), x_2(t), x_3(t), x_4(t)) \\ &= \delta_2[1\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] x_1(t)x_2(t)x_3(t)x_4(t) \\ &:= M_1 x(t), \end{aligned} \tag{11}$$

where $x(t) = \ltimes_{i=1}^4 x_i(t)$, and

$$M_1 = \delta_2[1\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2].$$

$$\begin{aligned} x_2(t+1) &= f_2(x_1(t), x_2(t), x_3(t), x_4(t)) \\ &= \begin{cases} f_2^1 = \delta_2[1,1,2,2,2,2,2,2,2,1,2,2,1,2,2,2]x_1(t)x_2(t)x_3(t)x_4(t), & p_2^1 = 0.25 \\ f_2^2 = \delta_2[1,1,2,2,2,2,2,2,2,1,2,2,2,2,2,2]x_1(t)x_2(t)x_3(t)x_4(t), & p_2^2 = 0.25 \\ f_2^3 = \delta_2[1,1,2,2,2,2,2,2,2,2,2,2,1,2,2,2]x_1(t)x_2(t)x_3(t)x_4(t), & p_2^3 = 0.25 \\ f_2^4 = \delta_2[1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2]x_1(t)x_2(t)x_3(t)x_4(t), & p_2^4 = 0.25 \end{cases} \\ &:= M_2 x(t), \end{aligned} \tag{12}$$

where

$$M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.5 & 1 & 1 & 0.5 & 1 & 1 & 1 \end{bmatrix}.$$

Similarly, we have

$$x_3(t+1) = f_3(x_1(t), x_2(t), x_3(t), x_4(t)) := M_3 x(t), \tag{13}$$

where

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0.5 & 1 & 1 & 0.5 & 1 & 1 & 1 \end{bmatrix}.$$

$$x_4(t+1) = f_4(x_1(t), x_2(t), x_3(t), x_4(t)) := M_4 x(t), \tag{14}$$

where

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0.5 & 1 & 1 & 0.5 & 1 & 1 & 1 \end{bmatrix}.$$

IV. Analysis of Networked Evolutionary Games

A. Algebraic Form of the Evolutionary Dynamics

As we mentioned in previous section that (9) is a k-valued logical network. Hence, as long as (9) is obtained, the techniques developed for k-valued logical networks are applicable to the analysis of evolutionary games. We briefly review this.

In an evolutionary dynamic game assume $S_0 = \{1, \cdots, k\}$. Identifying $i \sim \delta_k^i$, $i = 1, \cdots, k$, then Theorem A.8 says that for each equation in (9), we can find for each i a matrix $M_i \in \mathcal{L}_{k \times k^{\ell_i}}$ (where $\ell_i = |U_2(i)|$), such that (9) can be expressed into its vector form as

$$x_i(t+1) = M_i \ltimes_{j \in U_2(i)} x_j(t), \quad i \in N, \tag{15}$$

where M_i is the structure matrix of f_i, $i \in N$. (In probabilistic case, $M_i \in \Upsilon_{k \times k^{\ell_i}}$.)

Next, assume $|N| = n < \infty$. To add some dummy variables into each equation of (15), we need the following lemma, which can be proved by a straightforward computation.

Lemma IV.1. *Assume $X \in \Upsilon_p$ and $Y \in \Upsilon_q$. We define two dummy matrices, named by "front-maintaining operator" (FMO) and "rear-maintaining operator"(RMO) respectively, as:*

$$D_f^{p,q} = \delta_p[\underbrace{1 \cdots 1}_{q} \; \underbrace{2 \cdots 2}_{q} \; \cdots \; \underbrace{p \cdots p}_{q}],$$

$$D_r^{p,q} = \delta_q[\underbrace{\underbrace{1\,2 \cdots q}\; \underbrace{1\,2 \cdots q} \; \cdots \; \underbrace{1\,2 \cdots q}}_{p}].$$

Then we have

$$D_f^{p,q} XY = X. \tag{16}$$

$$D_r^{p,q} XY = Y. \tag{17}$$

Using Lemma IV.1, we can express each equation in (15) as a function with all x_i as its arguments. We use a simple example to depict this.

Example IV.2. *Assume*

$$x_i(t+1) = M_i x_{i-2}(t) x_{i-1}(t) x_i(t) x_{i+1}(t) x_{i+2}(t). \tag{18}$$

Using Lemma IV.1, we have

$$\begin{aligned} x_i(t+1) &= M_i D_r^{k^{i-3},k^5} x_1(t) x_2(t) \cdots x_{i+2}(t) \\ &= M_i D_r^{k^{i-3},k^5} D_f^{k^{i+2},k^{n-i-2}} \ltimes_{j=1}^n x_j(t) \\ &:= \tilde{M}_i x(t), \end{aligned} \tag{19}$$

where $x(t) = \ltimes_{j=1}^n x_j(t)$.

Now, instead of (15), we can assume that the strategy dynamics has the following form

$$x_i(t+1) = M_i x(t), \quad i = 1, \cdots, n. \tag{20}$$

Multiplying all the n equations together, we have the algebraic form of the dynamics as [10]

$$x(t+1) = M_G x(t), \tag{21}$$

where $M_G \in \mathcal{L}_{k^n \times k^n}$ (in probabilistic case, $M_G \in \Upsilon_{k^n \times k^n}$) is called the transition matrix of the game, and is determined by

$$\mathrm{Col}_j(M_G) = \ltimes_{k=1}^n \mathrm{Col}_j(M_k), \quad j = 1, \cdots, k^n.$$

Let $M \in \mathcal{M}_{p\times s}$ and $N \in \mathcal{M}_{q\times s}$. Define the Khatri-Rao product of M and N, denoted by $M * N$, as [10, 22]

$$M * N = [\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N) \;\; \mathrm{Col}_2(M) \ltimes \mathrm{Col}_2(N) \;\; \cdots \;\; \mathrm{Col}_s(M) \ltimes \mathrm{Col}_s(N)\;] \in \mathcal{M}_{pq\times s}. \tag{22}$$

Using this notation, we have

$$M_G = M_1 * M_2 * \cdots * M_n. \tag{23}$$

B. Emergence of Networked Games

Now all the properties of the evolutionary dynamics can be revealed from (21), equivalently, from the game transition matrix. For instance, we have the following result.

Theorem IV.3. *[21] Consider a k-valued logical dynamic network*

$$x(t+1) = Lx(t), \tag{24}$$

where $x(t) = \prod_{i=1}^n x_i(t)$, $L \in \mathcal{L}_{k^n\times k^n}$. Then

(i) *δ_k^i is its fixed point, if and only if the diagonal element ℓ_{ii} of L equals to 1. It follows that the number of equilibriums of (24), denoted by N_e, is*

$$N_e = \mathrm{Trace}(L). \tag{25}$$

(ii) *The number of length s cycles, N_s, is inductively determined by*

$$\begin{cases} N_1 = N_e \\ N_s = \frac{\mathrm{Trace}(L^s) - \sum\limits_{t\in\mathcal{P}(s)} tN_t}{s}, \quad 2 \le s \le k^n. \end{cases} \tag{26}$$

Note that in (26) $\mathcal{P}(s)$ is the set of proper factors of s. For instance, $\mathcal{P}(6) = \{1, 2, 3\}$, $\mathcal{P}(125) = \{1, 5, 25\}$.

We refer to [21] for many other dynamic properties of k-valued logical dynamic networks. Then we use some examples to demonstrate the dynamics of NEGs.

Example IV.4. *Recall Example III.3.*

(i) *Consider the case when $\Pi - I$ is used: Using (10), we can get the game transition matrix immediately as*

$$\begin{aligned} M_G &= M_1 * M_2 * M_3 \\ &= \delta_{27}[1\ 1\ 9\ 1\ 2\ 2\ 27\ 23\ 27\ 1\ 1\ 9\ 10\ 14\ 14\ 15\ 14\ 15\ 25\ 25\ 29\ 10\ 14\ 14\ 27\ 23\ 27]. \end{aligned} \tag{27}$$

Then we have the evolutionary dynamics as

$$x(t+1) = M_G x(t). \tag{28}$$

(Because of the space limitation, we skip M_G.) Since

$$M_G^k = \delta_{27}[1, 1, 27, 1, 1, 1, 27, 14, 27, 1, 1, 27, 1, 14, 14, 14, 14, 14, 27, 27, 27, 1, 14, 14, 27, 14, 27], \quad k \ge 2,$$

We can figure out that:

- *if $x(0) \in \{\delta_{27}^1, \delta_{27}^2, \delta_{27}^4, \delta_{27}^5, \delta_{27}^6, \delta_{27}^{10}, \delta_{27}^{11}, \delta_{27}^{13}, \delta_{27}^{22}\}$, then $x(\infty) = x(2) = \delta_{27}^1 \sim (1, 1, 1)$:*
- *if $x(0) \in \{\delta_{27}^8, \delta_{27}^{14}, \delta_{27}^{15}, \delta_{27}^{16}, \delta_{27}^{17}, \delta_{27}^{18}, \delta_{27}^{23}, \delta_{27}^{24}, \delta_{27}^{26}\}$, then $x(\infty) = x(2) = \delta_{27}^{14} \sim (2, 2, 2)$;*
- *if $x(0) \in \{\delta_{27}^3, \delta_{27}^7, \delta_{27}^9, \delta_{27}^{12}, \delta_{27}^{19}, \delta_{27}^{20}, \delta_{27}^{21}, \delta_{27}^{25}, \delta_{27}^{27}\}$, then $x(\infty) = x(2) = \delta_{27}^{27} \sim (3, 3, 3)$.*

So the network converges to one of three uniformed strategy cases with equal probability (as the initial strategies are randomly chosen with equal probability).

(ii) *Consider the other case when $\Pi - II$ is used: we have the transition matrix as*

$$M_G = M_1 * M_2' * M_3. \tag{29}$$

Then the dynamics of NEG is

$$x(t+1) = M_G x(t). \tag{30}$$

(Here M_G is also skipped.) We can show that

$$M_G^k = \delta_{27}[1,1,27,1,1,1,27,14,27,1,1,27,1,14,14,14,14,14,27,27,27,1,14,14,27,14,27], \quad k \geq 16.$$

This means the emergence is exactly the same as in case 1, where $\Pi - I$ is used. The only difference is: using $\Pi - II$, the profile converges much slower than that of $\Pi - I$. Obviously, it is reasonable.

Example IV.5. *Recall Example III.4. We have*

$$x(t+1) = M_G x(t),$$

where

$$\begin{aligned} M_G &= M_1 * M_2 * M_3 * M_4 \\ &= \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&\frac{1}{8}&0&0&\frac{1}{8}&0&0&0 \\ 0&0&0&0&0&0&0&0&0&\frac{1}{8}&0&0&\frac{1}{8}&0&0&0 \\ 0&0&0&0&0&0&0&0&0&\frac{1}{8}&0&0&\frac{1}{8}&0&0&0 \\ 0&1&0&0&0&0&0&0&0&\frac{1}{8}&0&0&\frac{1}{8}&0&0&0 \\ 0&0&0&0&0&0&0&0&0&\frac{1}{8}&0&0&\frac{1}{8}&0&0&0 \\ 0&0&0&0&0&0&0&0&1&\frac{1}{8}&0&0&\frac{1}{8}&0&0&0 \\ 0&0&0&0&1&0&0&0&0&\frac{1}{8}&0&0&\frac{1}{8}&0&0&0 \\ 0&0&0&1&0&1&1&1&0&\frac{1}{8}&1&1&\frac{1}{8}&1&1&1 \end{bmatrix}. \end{aligned}$$

One sees easily that there are two fixed points: $(1\ 1\ 1)$ and $(2\ 2\ 2)$.

To see how the evolutionary dynamics converges, it is easy to check that

$$M_G^k = \delta_{16}[1,16,16,16,16,16,16,16,16,16,16,16,16,16,16,16], \quad k \geq 8.$$

From this fact we conclude that

(i) *if $x(0) = \delta_{16}^1$, then $x(t) = \delta_{16}^1$, $\forall t > 0$. That is, if all $x_i(0) = \delta_2^1$, then $x_i(t) = \delta_2^1$ forever. In other words, if the initial strategies of all players are the first strategy, then they will stick on the first strategy forever;*

(ii) *if at least one player took the second strategy as his initial strategy, then with probabilistic 1 all the players will eventually take the second strategy (precisely, it happens after 8 iterations).*

V. Homogeneous NEGs

From Examples IV.4, IV.5 it is clear that in principle, for a networked evolutionary game $((N,E),G,\Pi)$, if $|N|<\infty$, and the network (N,E), the fundamental network game G, and the strategy updating rule Π are known, the evolutionary dynamics is computable; and then the behaviors of this NEG are determined. But because of the computational complexity, only in small scale case the dynamics of NEGs can be computed and used to analyze the dynamic behaviors of the networked game. This section considers the homogeneous NEGs, because in homogeneous case the dynamics of every nodes are the same. Then investigating local dynamic behaviors of the network may reveal the overall behaviors of the whole network.

We start from a simple model.

Example V.1. *Consider a networked evolutionary game $((N,E),G,\Pi)$, where $(N,E)=S_n$; G is the Prisoner's Dilemma defined in Example II.5 with parameters $R=-1$, $S=-10$, $T=0$, $P=-5$; and the strategy updated rule $\Pi-I$ is chosen. (In fact, in this case $\Pi-I$ and $\Pi-II$ lead to same dynamics.)*

We first calculate the fundamental evolutionary equation (9) for an arbitrary node i. Note that on S_n the neighborhoods of i are $U(i)=\{i-1,i,i+1\}$, $U_2(i)=\{i-2,i-1,i,i+1,i+2\}$, hence (9) becomes

$$x_i(t+1)=f(x_{i-2}(t),x_{i-1}(t),x_i(t),x_{i+1}(t),x_{i+2}(t)),\quad i=1,2,\cdots,n. \tag{31}$$

Note that on S_n we identify $x_{-k}=x_{n-k}$, and $x_{n+k}(t)=x_k$, $1\le k\le n-1$.

Using the same argument as in Example III.3, f can be figured out as in Table V

TABLE V: Payoffs $\to$ Dynamics

Profile	11111	11112	11121	11122	11211	11212	11221	11222
C_{i-1}	-2	-2	-2	-2	-11	-11	-11	-11
C_i	-2	-2	-11	-11	0	0	-5	-5
C_{i+1}	-2	-11	0	-5	-11	-20	-5	-10
f	1	1	2	1	2	2	2	2
Profile	12111	12112	12121	12122	12211	12212	12221	12222
C_{i-1}	0	0	0	0	-5	-5	-5	-5
C_i	-11	-11	-20	-20	-5	-5	-10	-10
C_{i+1}	-2	-11	0	-5	-11	-20	-5	-10
f	2	2	2	2	2	2	2	2
Profile	21111	21112	21121	21122	21211	21212	21221	21222
C_{i-1}	-11	-11	-11	-11	-20	-20	-20	-20
C_i	-2	-2	-11	-11	0	0	-5	-5
C_{i+1}	-2	-11	0	-5	-11	-20	-5	-10
f	1	1	1	2	2	2	2	2
Profile	22111	22112	22121	22122	22211	22212	22221	22222
C_{i-1}	-5	-5	-5	-5	-10	-10	-10	-10
C_i	-11	-11	-20	-20	-5	-5	-10	-10
C_{i+1}	-2	-11	0	-5	-11	-20	-5	-10
f	1	2	2	2	2	2	2	2

Then it is easy to figure out that

$$x_i(t+1)=L_5\ltimes_{j=-2}^{2}x_{i+j}(t), \tag{32}$$

where the structure matrix

$$L_5=\delta_2[1\ 1\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2]. \tag{33}$$

Now we have

$$
\begin{aligned}
x_1(t+1) &= L_5x_4(t)x_5(t)x_1(t)x_2(t)x_3(t) = L_5W_{[2^3,2^2]}x(t)\\
x_2(t+1) &= L_5x_5(t)x_1(t)x_2(t)x_3(t)x_4(t) = L_5W_{[2^4,2]}x(t)\\
x_3(t+1) &= L_5x_1(t)x_2(t)x_3(t)x_4(t)x_5(t) = L_5x(t)\\
x_4(t+1) &= L_5x_2(t)x_3(t)x_4(t)x_5(t)x_1(t) = L_5W_{[2,2^4]}x(t)\\
x_5(t+1) &= L_5x_3(t)x_4(t)x_5(t)x_1(t)x_2(t) = L_5W_{[2^2,2^3]}x(t),
\end{aligned}
\tag{34}
$$

where $x(t) = \ltimes_{j=1}^5 x_j(t)$. *Finally, we have the evolutionary dynamic equation as*

$$
x(t+1) = M_5x(t), \tag{35}
$$

where the strategy transition matrix M_5 *is*

$$
\begin{aligned}
M_5 &= \left(L_5W_{[2^3,2^2]}\right) * \left(L_5W_{[2^4,2]}\right) * L_5 * \left(L_5W_{[2,2^4]}\right) * \left(L_5W_{[2^2,2^3]}\right)\\
&= \delta_{32}[1\ 20\ 8\ 4\ 15\ 32\ 7\ 32\ 29\ 32\ 32\ 32\ 13\ 32\ 32\ 32\\
&\qquad 26\ 18\ 32\ 32\ 32\ 32\ 32\ 32\ 25\ 32\ 32\ 32\ 32\ 32\ 32\ 32].
\end{aligned}
\tag{36}
$$

We call (31) the fundamental evolutionary equation because it can be used to calculate not only the strategy evolutionary equation for S_5, but also for any S_n, $n > 2$. When $n = 3$ or $n = 4$, the constructing process is exactly the same as for $n = 5$. Then the evolutionary dynamic properties can be found via the corresponding transition matrix. We are more interested in large n.

Assume $n > 5$. We consider x_1 first. Using Lemma IV.1, we have

$$
\begin{aligned}
x_1(t+1) &= L_5x_{n-1}(t)x_n(t)x_1(t)x_2(t)x_3(t)\\
&= L_5D_f^{2^{n-2},2^2}x_{n-1}(t)x_n(t)x_1(t)x_2(t)\cdots x_{n-3}(t)\\
&= L_5D_f^{2^{n-2},2^2}W_{[2^{n-2},2^2]}x_1(t)\cdots x_n(t)\\
&:= H_1x(t).
\end{aligned}
$$

Similarly, we have a general expression as

$$
x_i(t+1) = H_ix(t), \quad i = 1,2,\cdots,n, \tag{37}
$$

where

$$
H_i = L_5D_f^{2^5,2^{n-5}}W_{[2^{\alpha(i)},2^{n-\alpha(i)}]}, \quad i = 1,\cdots,n,
$$

where

$$
\alpha(i) = \begin{cases} i-3, & i \geq 3\\ i-3+n, & i < 3. \end{cases}
$$

Finally, the strategy transition matrix can be calculated by

$$
M_n = H_1 * H_2 * \cdots * H_n. \tag{38}
$$

When n is relatively small, say $n < 20$, it is easy to calculate M_n (via laptop) and use it to investigate the properties of the networked game as we did in previous section. But our purpose in this section is to explore general cases where n could be very large. The following is an interesting global result via local property. We first need a lemma.

Define a set of projection matrices $\pi_i \in \mathcal{L}_{k\times k^n}$ as

$$
\begin{aligned}
&\pi_1 := \delta_k\left[\mathbf{1}_{k^{n-1}}^T\ \ 2\times\mathbf{1}_{k^{n-1}}^T\ \ \cdots\ \ k\times\mathbf{1}_{k^{n-1}}^T\right],\\
&\pi_2 := \delta_k\left[\mathbf{1}_{k^{n-2}}^T\ \ 2\times\mathbf{1}_{k^{n-2}}^T\ \ \cdots\ k\times\mathbf{1}_{k^{n-2}}^T\ \ \mathbf{1}_{k^{n-2}}^T\ \ 2\times\mathbf{1}_{k^{n-2}}^T\ \cdots\ \ k\times\mathbf{1}_{k^{n-2}}^T\right],\\
&\vdots\\
&\pi_n = \delta_k\left[1\ \ 2\ \ \cdots\ \ k\ \ 1\ \ 2\ \ \cdots\ \ k\ \ \cdots\ \ 1\ \ 2\ \ \cdots\ \ k\right].
\end{aligned}
\tag{39}
$$

The following projection result is straightforward verifiable.

Lemma V.2. *Let* $x_i \in \Upsilon_k$, $i = 1, 2, \cdots, n$. $x = \ltimes_{i=1}^{n} x_i$. *Then*

$$\pi_i x = x_i, \quad i = 1, 2, \cdots, n. \tag{40}$$

Assume N is a large scale homogeneous NEG, and a point $x^* \in N$. Denote the j-neighborhood of x^* by U_j, and let M_j be the transition matrix of the dynamics of sub-game U_r, $\alpha = |U_{2r}|$. Moreover, Let β be the label of x^* in the sub-game U_r. Then the initial values of the nodes in sub-game U_r are $(x_1(0), \cdots, x_\beta(0) = x^*(0), \cdots x_\alpha(0))$. Using these notations, we have

Theorem V.3. *Assume there exists an* $r > 0$ *such that*

$$M^{r+1}_{U_{2(r+1)}} = M^{r}_{U_{2(r+1)}}. \tag{41}$$

Then node x^* *converges to*

$$x^*(\infty) = \pi_\beta M^{r}_{U_{2r}} x^*(0), \tag{42}$$

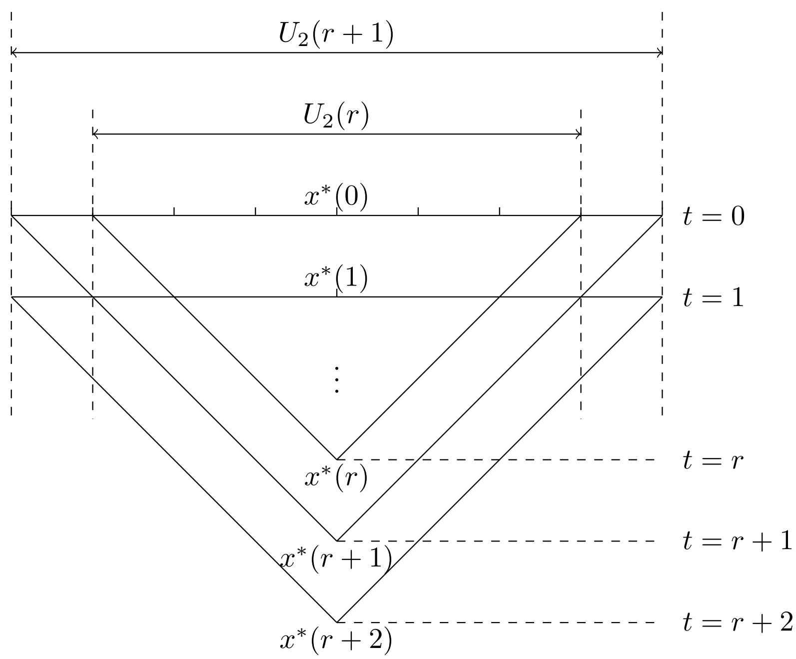

Fig. 3: Mapping for x^*

Proof: Refer to Fig. 3. We set $V_r(t)$ the vector consists of $\{x(t) | x \in U_r\}$ ordered by the sub-game U_r labels. According to (9), $x^*(r)$ is uniquely determined by $V_{2r}(0)$. Hereafter, we denote the element set of a vector $V = (v_1, \cdots, v_s)$ by $\{V\} := \{v_1, \cdots, v_s\}$. Note that $\{V_{2r}(0)\} \subset \{V_{2(r+1)}(0)\}$, using (41), we have

$$\begin{aligned} x^*(r) &= \pi_\beta M^{r}_{U_{2(r)}} V_{2r}(0) \\ &= \pi_\beta M^{r}_{U_{2(r+1)}} V_{2r}(0) \\ &= \pi_\beta M^{r+1}_{U_{2(r+1)}} V_{2(r+1)}(0) = x^*(r+1). \end{aligned}$$

Next, considering $x^*(r+2)$, since

$$\{V_{2r}(1)\} \subset \left\{ M_{2(r+1)} V_{2(r+1)}(0) \right\},$$

we have

$$\begin{aligned} x^*(r+2) &= \pi_\beta M^{r}_{U_{2(r+1)}} V_{2(r+1)}(1) \\ &= \pi_\beta M^{r}_{U_{2(r)}} V_{2r}(1) \\ &= \pi_\beta M^{r+1}_{U_{2(r+1)}} V_{2(r+1)}(0) = x^*(r+1). \end{aligned}$$

Continuing this procedure, we finally have

$$x^*(\infty) = x^*(r).$$

(42) follows immediately. ∎

Example V.4. *Recall Example V.1.*

(i) *Consider the evolutionary game with network S_n or R_∞. Assume n is large enough. Using (37)-(38), we can calculate M_{13} and prove that*

$$M_{13}^3 = M_{13}^2. \tag{43}$$

In one dimensional case, consider the sequential 13 points, which form the U_6 for its central point. (43) verifies (41) with $r = 2$. Hence Theorem V.3 claims that $M_{U_4} = M_9$ determined the final states of the network.

(ii) *Since $|U_4(x^*)| = 9$, and $\beta = 5$, According to (42), we define*

$$\begin{array}{l}
\pi_\infty := \pi_5 M_9^2 = \delta_{512} \\
\begin{array}{ccccccccccccccccccc}
[1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \\
2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2].
\end{array}
\end{array}$$

(iii) *Consider R_∞. Setting randomly initial values as:*

$$\begin{array}{l}
\cdots,\ x_{-6}(0),\ x_{-5}(0),\ \cdots,\ x_6(0),\ \cdots \\
= \cdots,\ 1,\ 2,\ 2,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 1,\ 1,\ 2,\ 1, \cdots.
\end{array}$$

Using

$$\begin{array}{l}(x_{-6}(0),\ x_{-5}(0),\ \cdots,\ x_2(0))\\ =(\ 1,\ 2,\ 2,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2)\sim\delta_{512}^{200}.\end{array}$$

Then we have

$$x_{-2}(\infty)=\pi_\infty\delta_{512}^{200}=\delta_2^1.$$

Similarly, we have

$$\begin{array}{ll}x_{-1}(\infty)=\pi_\infty\delta_{512}^{399}=\delta_2^1; & x_0(\infty)=\pi_\infty\delta_{512}^{285}=\delta_2^2\\ x_1(\infty)=\pi_\infty\delta_{512}^{58}=\delta_2^2; & x_2(\infty)=\pi_\infty\delta_{512}^{115}=\delta_2^2.\end{array}$$

(iv) *Since in π_∞ there are 72 entries equal to δ_2^1 and $512-72=440$ entries equal to δ_2^2, if on $\mathbb{R}$ the initial strategies for $i\in\mathbb{Z}$ are distributed with equal probability, then the final stable distribution is with 14.0625% strategy 1 and 85.9372% strategy 2.*

Secondly, we consider how the payoffs affect the evolutionary dynamics. First, from the above argument we have the following observation.

Proposition V.5. *Consider a networked evolutionary game with a given FNG, G. Assume the parameters in payoff bi-matrix is adjustable. Two sets of parameters lead the same evolutionary dynamics, if and only if they lead to same fundamental evolutionary equation (9).*

Example V.6. *Recall Example V.4. All the results obtained there are based on the payoff parameters: $R=-1$, $S=-10$, $T=0$, $P=-5$. A careful verification via Table V shows that as long as the parameters satisfy*

$$\begin{cases}T>R>S,\\ T>P,\\ 2R>T+P,\\ T+P>2S,\\ T+P>R+S,\end{cases}\tag{44}$$

the fundamental evolutionary equation (32) is unchanged. So the results in Example V.4 remain true. In fact, it is verifiable that (44) is also necessary for the L_5 to be as in (33).

To demonstrate how much the payoffs will affect the dynamic behaviors of the networked game, we consider Prisoner's Dilemma again with another set of parameters.

Example V.7. *Consider the same problem as in Example V.4, but with the payoff parameters as: $R=-3$, $S=-20$, $T=0$, $P=-5$. Then it is easy to verify that the fundamental evolutionary equation (32) remain the same but with the parameter matrix (33) being replaced by*

$$L_5=\delta_{32}[1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2].\tag{45}$$

It is also ready to check that this L_5 remains correct, if and only if the parameters satisfy the following (46):

$$\begin{cases}T>R,\\ T>S,\\ P+T>R+S,\\ P+T>2R,\\ P+T>2S.\end{cases}\tag{46}$$

We explore some dynamic properties of the NDG under such payment parameters. Say, choosing $n = 7$, then the evolutionary transition matrix is

$$
\begin{array}{l}
M_G = \delta_{128}[\\
\begin{array}{ccccccccccccccc}
1 & 68 & 8 & 72 & 15 & 80 & 16 & 80 & 29 & 96 & 32 & 96 & 31 & 96 & 32 \\
96 & 57 & 124 & 64 & 128 & 63 & 128 & 64 & 128 & 61 & 128 & 64 & 128 & 63 & 128 \\
64 & 128 & 113 & 116 & 120 & 120 & 127 & 128 & 128 & 128 & 125 & 128 & 128 & 128 & 127 \\
128 & 128 & 128 & 121 & 124 & 128 & 128 & 127 & 128 & 128 & 128 & 125 & 128 & 128 & 128 \\
127 & 128 & 128 & 128 & 98 & 100 & 104 & 104 & 112 & 112 & 112 & 112 & 126 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 122 & 124 & 128 & 128 & 128 & 128 & 128 & 128 & 126 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 114 & 116 & 120 & 120 & 128 & 128 & 128 & 128 & 126 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 122 & 124 & 128 & 128 & 128 & 128 & 128 & 128 \\
126 & 128 & 128 & 128 & 128 & 128 & 128 & 128]
\end{array}
\end{array}
$$

It is easy to calculate that

$$
\begin{array}{l}
M_G^k = \delta_{128}[\\
\begin{array}{ccccccccccccccc}
1 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\
128 & 128 & 128 & 128 & 128 & 128 & 128 & 128]
\end{array}
\end{array}
$$

where $k \geq 3$.

It is clear that unless $x(0) = \delta_{128}^1$, which leads to $x(\infty) = x(3) = \delta_{128}^1 \sim (1,1,1,1,1,1,1)$, any other initial states converge to $\delta_{128}^{128} \sim (2,2,2,2,2,2,2)$.

The above dynamic behavior is interesting: All trajectories will converge to one fixed point unless it started from the other fixed point. We may ask when this will happen. In fact, for homogeneous networked games with $|N| = n$ it is clear that there are at least k fixed points: $(\delta_k^i)^n$, $i = 1, \cdots, k$. Say, $k = 2$, and if there are only two attractors (no cycle), then each trajectory will converge to one of these two fixed points. Then we have the following result.

Proposition V.8. *Assume the dynamic equation of a NEG on S_n has only two attractors, which are fixed points $(\delta_2^1)^n$ and $(\delta_2^2)^n$. Moreover, if for the fundamental evolutionary equation*

$$
\mathrm{Col}_j(M_5) \neq \delta_{2^5}^1, \quad j \neq 1, \quad \left(or \quad \mathrm{Col}_j(M_5) \neq \delta_{2^5}^{2^5}, \quad j \neq 2^5 \right), \tag{47}
$$

then

$$
x_j(\infty) = \delta_2^2, \quad j \neq 1, \quad \left(corresp. \;\; x_j(\infty) = \delta_2^1, j \neq 2^n \right). \tag{48}
$$

Proof: Consider the first case, (47) means if in a sequential 5 elements there is a δ_2^2 then their images contain at least one δ_2^2. Hence δ_2^2 will not disappear. But eventually, $x(t)$ will converge to an attractor, then it will converge to $\delta_{2^n}^1$. (The alternative case in parentheses is similar.) ∎

Example V.9. *Consider Example V.7 again. M_5 can be calculated easily as*

$$
\begin{aligned}
M_5 = \delta_{32}[&1, 20, 8, 24, 15, 32, 16, 32, 29, 32, 32, 32, 31, 32, 32, 32, \\
&26, 28, 32, 32, 32, 32, 32, 32, 30, 32, 32, 32, 32, 32, 32, 32].
\end{aligned}
$$

It is clear that

$$\mathrm{Col}_j(M_5) \neq \delta_{32}^1, \quad j \neq 1.$$

It is easy to check that under this set of parameters M_ℓ, $\ell = 5, 6, 7, 8, 9, 10, 11, 12$, the corresponding M_ℓ has only two attractors, as fixed points $\delta_{2^\ell}^1$ and $\delta_{2^\ell}^{2^\ell}$. According to Proposition V.8, as long as $x(0) \neq \delta_{2^\ell}^1$, all the states converge to δ_2^2.

Our conjecture is: this is true for all $\ell \geq 5$.

VI. Control of NEGs

Definition VI.1. *Let $((N,E),G,\Pi)$ be an NEG, $\{X,W\}$ be a partition of N,* i.e., *$X \cap W = \emptyset$ and $N = X \cup W$. Then $((X \cup W, E), G, \Pi)$ is called a control NEG, if the strategies for nodes in W, denoted by $w_j \in W$, $j = 1, \cdots, |W|$, at each moment $t \geq 0$, can be assigned. Moreover, $x \in X$ is called a state and $w \in W$ is called a control.*

Definition VI.2.

1) *A state x_d is said to be $T > 0$ step reachable from $x(0) = x_0$, if there exists a sequence of controls $w_0, \cdots, w_{T-1}$ such that $x(T) = x_d$. The set of T step reachable states is denoted as $R_T(x_0)$;*
2) *The reachable set from x_0 is defined as*

$$R(x_0) := \cup_{t=1}^{\infty} R_t(x_0).$$

3) *A state x_e is said to be stabilizable from x_0, if there exist a control sequence $w_0, \cdots, w_\infty$ and a $T > 0$, such that the trajectory from x_0 converges to x_e, precisely, $x(t) = x_e$, $t \geq T$. x_e is stabilizable, if it is stabilizable from $\forall x_0 \in \mathcal{D}_k^n$.*

Next, we consider the dynamics of a control NEG. Assume $X = \{x_1, \cdots, x_n\}$ and $W = \{w_1, \cdots, w_m\}$, and we set $x = \ltimes_{i=1}^n x_i$ and $w = \ltimes_{j=1}^m w_j$, where $x_i,\ w_j \in \Delta_k \sim \mathcal{D}_k$ and $k = |S_0|$. Then for each $w \in \Delta_{k^m}$ we can have a strategy transition matrix M_w, which is called the control-depending strategy transition matrix. As a convention, we define

$$M\left(w = \delta_{k^m}^i\right) := M_i, \quad i = 1, 2, \cdots, k^m. \tag{49}$$

The set of control-depending strategy transition matrices is denoted by $\mathcal{M}_W$. Let $x(0)$ be the initial state. Driven by control sequence

$$w(0) = \delta_{k^m}^{i_0},\ w(1) = \delta_{k^m}^{i_1},\ w(2) = \delta_{k^m}^{i_2},\ \cdots,$$

the trajectory will be

$$x(1) = M_{i_0}x(0),\ x(2) = M_{i_1}M_{i_0}x(0),\ x(3) = M_{i_2}M_{i_1}M_{i_0}x(0),\ \cdots.$$

From this observation, the following result is obvious.

Proposition VI.3. *Consider a control NEG $((X \cup W, E), G, \Pi)$, with $|X| = n$, $|W| = m$, $|S_0| = k$.*

1) *x_d is reachable from x_0, if and only if there exists a sequence $\{M_0, M_1, \cdots, M_{T-1}\} \subset \mathcal{M}_W$, $T \leq k^n$, such that*

$$x_d = M_{T-1}M_{T-2}\cdots M_1 M_0 x_0. \tag{50}$$

2) *x_d is stabilizable from x_0, if and only if (i) x_d is reachable from x_0 and there exists at least one $M^* \in \mathcal{M}_W$, such that x_d is a fixed point of M^*.*

An immediate consequence of Proposition VI.3 is the following:

Corollary VI.4. *For any $x_0 \in \mathcal{D}_k^n$, the reachable set satisfies*

$$R(x_0) \subset \cup_{M \in \mathcal{M}_W} \mathrm{Col}(M). \tag{51}$$

We use an example to depict this:

Example VI.5. *Consider a game $((N,E),G,\Pi)$, where (i) $N = (X \cup W)$, with $X = \{x_1, x_2, x_3\}$, $W = \{w\}$, the network graph is shown in Fig. 4; (ii) G is Benoit-Krishna Game [4, 28]; (iii) $\Pi = \Pi - I$. Recall that Benoit-Krishna Game has $S_0 = \{1(D):\ Deny,\ \ 2(W):\ Waffle,\ \ 3(C):\ Confess\}$. The payoff bi-matrix is shown in Table VI.*

TABLE VI: Payoff Table (Benoit-Krishna)

$P_1\backslash P_2$	$D=1$	$W=2$	$C=3$
$D=1$	(10, 10)	(−1, −12)	(−1, 15)
$W=2$	(−12, −1)	(8, 8)	(−1, −1)
$C=3$	(15, −1)	(8, 1)	(0, 0)

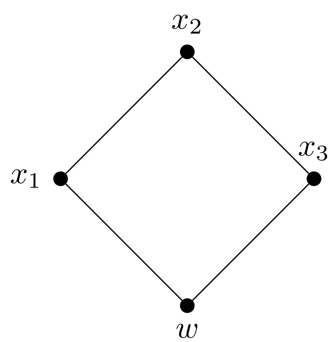

Fig. 4: Control of BK-game

This model can be explained as follows. There is a game of three players $\{x_1, x_2, x_3\}$. x_1 is the head, who is able to contact x_2 and x_3. w is a detective, who sneaked in and is able to contact only x_2 and x_3. The purpose of w is to let all x_i to confess.

First, we calculate the control-depending strategy transition matrix by letting $w = \delta_3^i$, $i = 1, 2, 3$ respectively. Then we have

$$\begin{aligned}
M(w=\delta_3^1) \quad &= M_1 = \quad \delta_{27}[1,1,9,1,1,9,27,27,27,1,1,9,1,14,18,27,7,27,\\
&\qquad\qquad 25,25,27,25,26,27,27,27,27]\\
M(w=\delta_3^2) \quad &= M_2 = \quad \delta_{27}[1,1,9,1,5,3,27,27,27,1,11,18,13,14,14,27,14,14,\\
&\qquad\qquad 25,26,27,19,14,14,27,14,27]\\
M(w=\delta_3^3) \quad &= M_3 = \quad \delta_{27}[21,21,27,21,24,27,27,27,27,21,1,27,24,14,14,27,14,27,\\
&\qquad\qquad 27,27,27,27,14,27,27,27,27].
\end{aligned} \tag{52}$$

Then it is easy to verify the reachable set of x_0. Say, $x_0 = \delta_{27}^1 \sim (1,1,1)$. Using $u(0) = \delta_3^1$ or $u(0) = \delta_3^2$, $x(1) = \delta_{27}^1$. Using $u(0) = \delta_3^3$, $x(1) = \delta_{27}^{21}$. Using any $u(1)$ to δ_{27}^{21} we have $x(2) = \delta_{27}^{27}$, which is a fixed point. We conclude that $R(\delta_{27}^1) = \delta_{27}\{1,21,27\}$. Here, $\delta_n\{\alpha_1, \cdots, \alpha_s\}$ is a shorthand of $\{\delta_n^{\alpha_1}, \cdots, \delta_n^{\alpha_s}\}$. Similarly, we have $R(\delta_{27}^2) = \delta_{27}\{1,21,27\}$, $R(\delta_{27}^3) = \delta_{27}\{9,27\}$, $R(\delta_{27}^4) = \delta_{27}\{1,21,27\}$, $R(\delta_{27}^5) = \delta_{27}\{1,5,14,24,21,27\}$, $R(\delta_{27}^6) = \delta_{27}\{1,3,9,21,27\}$, $R(\delta_{27}^7) = \delta_{27}\{27\}$, $R(\delta_{27}^8) = \delta_{27}\{27\}$, $R(\delta_{27}^9) = \delta_{27}\{27\}$, $R(\delta_{27}^{10}) = \delta_{27}\{1,21,27\}$, $R(\delta_{27}^{11}) = \delta_{27}\{1,11,21,27\}$, $R(\delta_{27}^{12}) = \delta_{27}\{1,9,14,18,21,27\}$, $R(\delta_{27}^{13}) = \delta_{27}\{1,13,14,21,24,27\}$, $R(\delta_{27}^{14}) = \delta_{27}\{14\}$, $R(\delta_{27}^{15}) = \delta_{27}\{14,18,27\}$, $R(\delta_{27}^{16}) = \delta_{27}\{27\}$, $R(\delta_{27}^{17}) = \delta_{27}\{7,14,27\}$, $R(\delta_{27}^{18}) = \delta_{27}\{14,27\}$, $R(\delta_{27}^{19}) = \delta_{27}\{25,27\}$, $R(\delta_{27}^{20}) = \delta_{27}\{14,25,26,27\}$, $R(\delta_{27}^{21}) = \delta_{27}\{27\}$, $R(\delta_{27}^{22}) = \delta_{27}\{19,25,27\}$, $R(\delta_{27}^{23}) = \delta_{27}\{14,26,27\}$, $R(\delta_{27}^{24}) = \delta_{27}\{14,27\}$, $R(\delta_{27}^{25}) = \delta_{27}\{27\}$, $R(\delta_{27}^{26}) = \delta_{27}\{14,27\}$, $R(\delta_{27}^{27}) = \delta_{27}\{27\}$.

There are two common fixed points: $x_e^1 = \delta_{27}^{14}$ and $x_e^2 = \delta_{27}^{27}$, so the overall system is not stabilizable. But any $x(0) \in \Delta_{27}\backslash\{\delta_{27}^{14}\}$, can be stabilized to $x_e^2 = \delta_{27}^{27}$ via a proper control sequence. For example, when $x(0) = \delta_{27}^6$, we can drive it to x_e^2 by any one of the following control sequences:

(i) *$w(0) = \delta_3^3$, then the trajectory will be $x(1) = M_3 x(0) = \delta_{27}^{27}$;*

(ii) *$w(0) = \delta_3^2$, $w(1) = \delta_3^3$, then the trajectory will be $x(1) = M_2 x(0) = \delta_{27}^9, x(2) = M_3 M_2 x(0) = \delta_{27}^{27}$;*

(iii) *$w(0) = \delta_3^1$ and $w(1)$ can choose any one of $\delta_3^1, \delta_3^2, \delta_3^3$, then the trajectory will be $x(1) = M_1 x(0) = \delta_{27}^9$, and $x(2) = M_1 M_1 x(0) = \delta_{27}^{27}$, or $x(2) = M_2 M_1 x(0) = \delta_{27}^{27}$ or $x(2) = M_3 M_1 x(0) = \delta_{27}^{27}$.*

Next, we consider the consensus of control NEGs.

Definition VI.6. *Let $\xi \in \mathcal{D}_k$. An NEG is said to reach a consensus at ξ if it is stabilizable to $x^e = \xi^n$.*

The following observation is obvious.

Proposition VI.7. *An NEG can not reach a consensus, if there are more than one common fixed point for all $M \in \mathcal{M}_W$. If there is a common fixed point x^e, and the NEG can reach a consensus at ξ, then $\xi^n = x^e$.*

Example VI.8. (i) *Consider the Prisoner's Dilemma Game over S_6 with strategy updating rule $\Pi = \Pi - I$, where $\{x_1, x_2, x_3, x_4, x_5\}$ are normal players, called the states, and $x_6 = w$ is the control, connected to x_1 and x_5.*
The control-depending strategy transition matrices are:

$$
\begin{aligned}
M(w=\delta_2^1) = M_1 = \delta_{32}[&17,2,24,4,31,32,32,32,29,32,32,32,31,32,32,32,\\
&17,18,32,32,32,32,32,32,25,32,32,32,32,32,32,32]\\
M(w=\delta_2^2) = M_2 = \delta_{32}[&1,4,8,4,15,16,7,8,29,32,32,32,13,32,15,32,\\
&26,28,32,32,32,32,32,32,26,32,32,32,30,32,32,32].
\end{aligned}
\tag{53}
$$

We can see that there are two common fixed points: $x_1^e = \delta_{32}^4$ and $x_2^e = \delta_{32}^{32}$. Hence the NEG can not reach consensus.

(ii) *Next, we add another control,* i.e., $x_1, x_2, x_3, x_4, x_5, u_1, u_2$ *form an S_7. Then the control-depending strategy transition matrices become:*

$$
\begin{aligned}
M(w_1=\delta_2^1, w_2=\delta_2^1) = M_1 = \delta_{32}[&1,1,8,4,16,16,8,8,29,29,32,32,13,16,16,16,\\
&28,25,32,32,32,32,32,32,28,32,32,32,32,32,32,32]\\
M(w_1=\delta_2^1, w_2=\delta_2^2) = M_2 = \delta_{32}[&1,1,8,4,16,16,16,16,29,29,32,32,29,32,32,32,\\
&20,17,32,32,32,32,32,32,28,32,32,32,32,32,32,32]\\
M(w_1=\delta_2^2, w_2=\delta_2^1) = M_3 = \delta_{32}[&17,17,24,20,32,32,24,24,29,29,32,32,29,32,32,32,\\
&28,25,32,32,32,32,32,32,28,32,32,32,32,32,32,32]\\
M(w_1=\delta_2^2, w_2=\delta_2^2) = M_4 = \delta_{32}[&1,1,8,4,32,32,32,32,29,29,32,32,29,32,32,32,\\
&20,17,32,32,32,32,32,32,28,32,32,32,32,32,32,32].
\end{aligned}
\tag{54}
$$

It is ready to check that there is only one common fixed point: $x_e = \delta_{32}^{32}$, where $x_e = \xi^n$ with $\xi = \delta_2^2$. Moreover, x_e is reachable from any $x(0)$. Therefore, the NEG can reach consensus at $\xi = \delta_2^2$.

VII. Conclusion

The networked evolutionary games were considered. First of all, it was proved that the strategy evolutionary dynamics of an NEG can be expressed as a k-valued logical dynamic network. This expression gives a precise mathematical model for NEGs. Based on this model the emergence properties of NEGs were investigated. Particularly, the homogeneous networked games were studied in detail, and some general properties for arbitrary size networks were revealed. Finally, the control of NEGs was explored, and conditions for consensus were presented.

Overall, the main contribution of this paper is the presentation of a mathematical model for NEGs. Though NEG is now a hot topic and there are many interesting results, to the authors' best knowledge, previous works are experience-based, (say, mainly via simulation and/or statistics). Our model makes further theoretical study possible. However, our results are very elementary. There are pretty of rooms for further study.

References

[1] A. Adamatzky, On dynamically non-trivial three-valued logics: oscillatory and bifurcatory species, *Chaos Solitons & Fractals*, Vol. 18, 917-936, 2003.

[2] R. Axelrod, W.D. Hamilton, The evolution of cooperation, *Science*, Vol. 211, 1390-1396, 1981.

[3] R. Axelrod, *The Evolution of Cooperation*, Basic Books, New York, 1984.

[4] J.P. Benoit, V. Krishna, Finitely repeated games, *Econometrica*, Vol. 17, No. 4, 317-320, 1985.

[5] J.M. Bilbao, *Cooperative Games on Combinatorial Structures*, Kluwer Acad. Pub., Boston, 2000.

[6] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D. U. Hwang, Complex networks Structure and dynamics, *Phys. Rep.*, Vol. 424, 175, 2006.

[7] R. Branzei, D. Dimitrov, S. Tijs, *Models in Cooperative Game Theory*, 2nd Ed., Springer-Verlag, Berlin, 2008. (X. Liu, J. Liu, Chinese translation, Science Press, 2011.)

[8] D. Cheng, H. Qi, Controllability and observability if Boolean control networks, *Automatica*, Vol. 45. Mo. 7, 1659-1667, 2009.

[9] D. Cheng, H. Qi, Z. Li, *Analysis and Control of Boolean Networks - A Semi-tensor Product Approach*, Springer, London, 2011.

[10] D. Cheng, H. Qi, Y. Zhao, *An Introduction to Semi-tensor Product of Matrices and Its Applications*, World Scientific, Singapore, 2012.

[11] D. Cheng, T. Xu, F. He, A numerical solution to Nash equilibrium, (preprint).

[12] J.G. Gardeñes, M. Romance, R. Criado, D. Vilone, A. Sanchez, Evolutionary games defined at the network mesoscale: The public goods game, *Chaos*, 21, 016113, 2011.

[13] R. Gibbons, *A Primer in Game Theory*, Bell & Bain Ltd., Glasgow, 1992.

[14] G. Hardin, The tragedy of the commons, *Science*, Vol. 162, 1243-1248, 1968.

[15] C. Hauert, M. Doebeli, Spatial structure often inhibits the evolution of cooperation in the snowdrift game, *Nature*, Vol. 428, 643-646, 2004.

[16] P. Holme, J. Saramaki, Temporal networks, *Physics Reports*, Vol. 519, No. 3, 97-125, 2012.

[17] S.A. Kauffman, Metabolic stability and epigenesis in randomly constructed genetic nets, *J. Theoretical Biology*, Vol. 22, 437-467, 1969.

[18] S.A. Kauffman, *The Origins of Order: Self-organization and Selection in Evolution*, Oxford University Press, New York, 1993.

[19] O. Kirchkamp, Simultaneous evolution of learning rules and strategies, *J. Econ. Behav. Organ.*, Vol. 40, 295, 1999.

[20] D. Laschov, M. Margaliot, A maximum principle for single-input Boolean control networks, *IEEE Trans. Aut. Contr.*, Vol. 56, No. 4, 913-917, 2011.

[21] Z. Li, D. Cheng, Algebraic approach to dynamics of multivalued networks, *Int. H. Bifur. Chaos*, Vol. 20, No. 3, 561-582, 2010.

[22] L. Ljung, T. Söderström, *Theory and Practice of Recursive Identification*, MIT Press, 1982.

[23] L. Lu, M. Medo, C.H. Yeung, et al., Recommender systems, *Physics Reports*, Vol. 519, No. 1, 1-49, 2012.

[24] M.A. Nowak, R.M. May, Evolutionary games and spatial chaos, *Nature*, Vol. 359, 826-829, 1992.

[25] M.A. Nowak, R.M. May, The spatial dilemmas of evolution, *Int. J. Bifu. Chaos*, Vol. 3, No. 1, 35-78, 1993.

[26] M.A. Nowak, *Evolutionary Dynamics: Exploring the Equations of Life*, Belknap Press of Harvard Univ., Cambridge, 2006.

[27] D.G. Rand, S. Arbesman, N.A. Christakis, Dynamic social networks promote cooperation in experimens with humans, *Proc. Nat. Acad. Sci. USA*, 19193-19198, 2011.

[28] E. Rasmusen, *Games and Information: An Introduction to Game Theory*, 4th Ed., Basil Blackwell, Oxford, 2007.

[29] C.P. Roca1, J. A. Cuesta1, A. Sanchez, Effect of spatial structure on the evolution of cooperation, *Phys. Rev., E*, Vol. 80, 46106, 2009.

[30] F.C. Santos, M.D. Santos, J.M. Pacheco, Social diversity promotes the emergence of cooperation in public goods games, *Nature*, Vol. 454, 213-216, 2008.

[31] I. Shmulevich, E.R. Dougherty, S.Kim, and W. Zhang, Probabilistic Boolean network, a rule-based uncertainty model for gene regulatory networks, *Bioinformatics*, Vol. 18, No. 2, 261-274, 2002.

[32] J.M. Smith, G.R. Price, The logic of animal conflict, *Nature*, Vol. 246, 15-18, 1973.

[33] R. Sugden, *The Economics of Rights, Cooperation and Welfre*, Blackwwell, Oxford, 1986.

[34] G. Szabo and C. Toke, Evolutionary prisoner's dilemma game on a square lattice, *Phys. Rev. E*, Vol. 58, 69, 1998.

[35] Pl Taylor, L. Jonker, Evolutionary stable strategies and game dynamics, *Math. Biosci.*, Vol. 16, 76-83, 1978.

[36] H. Tembine, E. Altman, R.E. Azouzi, Y. Hayel, Evolutionary game in wireless networks, *IEEE Trans. Sys, Man, Cyber., Part B: Cyber.*, Vol. 40, No. 3, 634-646, 2010.

[37] A. Traulsen, M.A. Nowak, J.M. Pacheco, Stochastic dynamics of invasion and fixation, *Phys. Rev., E*, Vol. 74, 011909, 2006.

[38] L.G. Volker, M. Conrad, The role of weak interactions in biological systems: the dual dynamic model, *J. Theor. Biol.*, Vol. 193, 287-306, 1998.

[39] Z. Xie, *An Introduction to Game Theory*, Science Press, Beijing, 2010. (in Chinese)

[40] L. Wang, F. Fu, X. Chen, et al., Evolutionary games on complex networks, *CAAI Trans. on Intel. Sys*, Vol. 2, No. 2, 1-9, 2007.

APPENDIX
SEMI-TENSOR PRODUCT OF MATRICES

Semi-tensor product of matrices was proposed by us. It is convenient in dealing with logical functions. We refer to [9, 10] and the references therein for details. In the follows we give a very brief survey.

Definition A.1. *Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote by $t := \operatorname{lcm}(n,p)$ the least common multiple of n and p. Then we define the semi-tensor product (STP) of A and B as*

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \tag{55}$$

Remark A.2.
- *When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.*
- *When $n = rp$, denote it by $A \succ_r B$;*
  when $rn = p$, denote it by $A \prec_r B$.
  These two cases are called the multi-dimensional case, which is particularly important in applications.
- *STP keeps almost all the major properties of the conventional matrix product unchanged.*

We cite some basic properties which are used in this note.

Proposition A.3. 1) *(Associative Low)*

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C. \tag{56}$$

2) *(Distributive Low)*

$$\begin{aligned}(A + B) \ltimes C = A \ltimes C + B \ltimes C.\\ A \ltimes (B + C) = A \ltimes B + A \ltimes C.\end{aligned} \tag{57}$$

3)

$$(A \ltimes B)^T = B^T \ltimes A^T. \tag{58}$$

4) *Assume A and B are invertible, then*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{59}$$

Proposition A.4. *Let $X \in \mathbb{R}^t$ be a column vector. Then for a matrix M*

$$X \ltimes M = (I_t \otimes M) \ltimes X. \tag{60}$$

Definition A.5.

$$W_{[n,m]} := \delta_{mn}\,[1, m+1, 2m+1, \cdots, (n-1)m+1, 2, m+2, 2m+2, \cdots, (n-1)m+2 \\ \cdots, n, m+n, 2m+n, \cdots, mn\,] \in \mathcal{M}_{mn\times mn}, \tag{61}$$

which is called a swap matrix.

Proposition A.6. *Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then*

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \tag{62}$$

Finally, we consider how to express a Boolean function into an algebraic form.

Theorem A.7. *Let $f : \mathcal{B}^n \to \mathcal{B}$ be a Boolean function expressed as*

$$y = f(x_1, \cdots, x_n), \tag{63}$$

where $\mathcal{B} = \{0, 1\}$. Identifying

$$1 \sim \delta_2^1, \quad 0 \sim \delta_2^2. \tag{64}$$

Then there exists a unique logical matrix $M_f \in \mathcal{L}_{2\times 2^n}$, called the structure matrix of f, such that under vector form, by using (64), (63) can be expressed as

$$y = M_f \ltimes_{i=1}^n x_i, \tag{65}$$

which is called the algebraic form of (63).

Assume $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$. We identify

$$\mathcal{D}_k \sim \Delta_k \tag{66}$$

by

$$1 \sim \delta_k^1,\ 2 \sim \delta_k^2,\ \cdots,\ k \sim \delta_k^k. \tag{67}$$

Assume $f(x_1, \cdots, x_n) : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$. Using identification (66)-(67), f becomes a mapping $f(x_1, \cdots, x_n) : \prod_{i=1}^n \Delta_{k_i} \to D_{k_0}$. We call later the vector form of f.

Theorem A.7 can be generalized as

Theorem A.8. (i) *Let $f(x_1, \cdots, x_n) : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$. Using identification (66)-(67), f can be expressed in its vector form as*

$$y = M_f \ltimes_{i=1}^n x_i, \tag{68}$$

where $M_f \in \mathcal{L}_{k_0\times k}$ is unique, and is called the structure matrix of f. Moreover, (68) is called the algebraic form of f.

(ii) *If in (i) $\mathcal{D}_{k_i}$ is replaced by Υ_{k_i}, then the result remains true except that the transition matrix M_f in (68) is replaced by an $M_f \in \Upsilon_{k_0\times k}$.*