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Modeling anomalous diffusion by a subordinated fractional Lévy-stable process

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Abstract. Two phenomena that can be discovered in systems with anomalous diffusion are long-range dependence and trapping events. The first effect concerns events that are arbitrarily distant but still influence each other exceptionally strongly, which is characteristic for anomalous regimes. The second corresponds to the presence of constant values of the underlying process. Motivated by the relatively poor class of models that can cover these two phenomena, we introduce subordinated fractional Lévy-stable motion with tempered stable waiting times. We present in detail its main properties, propose a simulation scheme and give an estimation procedure for its parameters. The last part of the paper is a presentation, via the Monte Carlo approach, of the effectiveness of the estimation of the parameters.

Keywords: driven diffusive systems (theory), stochastic particle dynamics (theory), fluctuations (theory), stochastic processes (theory)

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Contents

1. Introduction

Anomalous diffusion systems are characterized by nonlinear mean-square-displacement (MSD): $\langle X^2(t)\rangle = t^\beta$ with $\beta \neq 1, \beta > 0$. For $\beta < 1$ the underlying process exhibits subdiffusive dynamics, while in the case $\beta > 1$ it exhibits superdiffusive dynamics [1]. The class of processes used to model super- or subdiffusive behavior is very rich and is growing rapidly. We should mention here the Lévy walk [2], the continuous-time random walk (CTRW) [3, 4] and the correlated continuous-time random walk [5]; for other examples see also [6]–[9]. Apart from the mentioned systems a universal process that can be useful for modeling super- and subdiffusive behavior is the fractional Brownian motion (FBM) [10], which is closely related to fractional Langevin equation motion [11] and is a generalization of classical Brownian motion. Moreover, the FBM is the most classical process commonly considered as a system with long-range dependence (called also long-memory). One way of characterizing long- and short-range dependence of second order processes is in terms of their autocovariance functions. For short-range dependent processes, the relation between values at different times decreases rapidly as the time difference increases. Either the autocovariance drops to zero after a certain time lag, or it eventually has an exponential decay. For processes with long-range dependence there is much stronger relation. It is worth mentioning that the slowly decaying autocovariance function is typical for processes with long-range dependence; however, we observe the same effect for CTRW [12], which has no long-memory property. The phenomenon of long-range dependence has been observed in a variety of systems. In recent years scientists have investigated the long-memory property in different fields such as finance, econometrics, network modeling, hydrology, climate studies, astronomy, linguistics and DNA sequencing [13]–[16]. This phenomenon is also widespread in the physical and natural sciences [17, 18]. For more applications see [19, 20].

One of the extensions of the FBM is the fractional Lévy-stable motion (FLSM), which is constructed in analogy to the relation between Brownian motion and its generalization, Lévy-stable motion [21]. The FLSM is α-stable distributed and, similarly to the FBM, it exhibits a long-memory property. Due to the fact that a second moment of the process does not exist, the long-range dependence in this case is expressed by means of dependence measures adequate for an α-stable distribution [22]. Similarly to the FBM, the FLSM is also useful in modeling anomalous diffusive phenomena and is also a universal model that under some restrictions exhibits super- or subdiffusive behavior. This issue is discussed in the following sections.

Apart from the long-memory property and non-Gaussian behavior, many real data exhibit characteristics not adequate for pure FLSM. More precisely, for some real phenomena we observe constant time periods (called trapping events), i.e. periods for which the observations stay on the same level. This behavior is typical for finance, especially in emerging markets, where the number of participants and in consequence the number of transactions is rather low, and one can observe periods when the prices of assets do not change [23]. This phenomenon is also strongly visible in biophysical studies [24, 25], charge carrier transport in amorphous semiconductors [26], single molecule spectroscopy [27], financial and indoor air quality data [28]–[31].

The classical approach that provides a description of such typical phenomena with observable trapping events is based on the subordination technique. The idea of subordination was introduced by Bochner in 1949 in [32] and explored further by many authors, see for instance [33]. It is based on the replacement of real time in the considered process (called an external process) by some non-decreasing process with stationary independent increments (called a subordinator), which plays the role of random operational time. In recent years the number of papers in this field has grown rapidly since the invention of the technique of subordination through inverse subordinators. More precisely, instead of introducing the appropriate non-decreasing process as an operational time its inverse is treated as the subordinator. A detailed explanation of the subordination concept and the exact definition of the inverse subordinator are presented in the following sections. Processes driven by inverse subordinators are strongly related to the CTRW [3]. Let us mention the case when the external process is classical Brownian motion. In this case the subordinated process can be treated as a limit in the distribution of the appropriate CTRW [34], and therefore its probability density function can be described by the appropriate fractional Fokker–Planck equation, see e.g. [35, 36]. In the literature, apart from the mentioned Brownian motion, there are several other processes considered as external processes within the subordination framework, for example arithmetic Brownian motion [28], geometric Brownian motion [29] and the α-stable Ornstein–Uhlenbeck process [37].

The key issue in the framework of the subordination technique is the distribution of subordinator increments (which belong to the infinitely divisible class) corresponding to observed constant time periods. Among the rich class of infinitely divisible distributions the tempered stable ones are the most appropriate to model trapping events [38]–[40]. Tempered stable distributions possess many interesting properties, i.e. they have finite moments of all orders and they stay close to the purely α-stable distributions at the same time [41]. The popularity of tempered stable distributions has resulted in many applications, for instance in physics to describe anomalous diffusion [38, 39], biology [42], plasma physics [43] and finance [44].

In this paper we propose a model that combines subordinated fractional Lévy-stable motion and an inverse tempered stable subordinator. This process exhibits long-memory and non-Gaussian behavior (which is a result of properties of the external process) and

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constant time periods (which follow from the introduction of the inverse subordinator). We present the main characteristics of the examined process and propose its simulation scheme and estimation procedure. Moreover, by using Monte Carlo simulations, we check the efficiency of the proposed method.

The paper is structured as follows. In section 2 we present the definition and properties of fractional Lévy-stable motion. In section 3 we introduce fractional Lévy-stable motion subordinated by an inverse tempered stable subordinator and give its analytical properties. Sections 4 and 5 present a numerical approximation scheme and the parameter estimation procedure of the proposed model, respectively. Section 6 summarizes the results and discusses possible extensions of the considered model.

2. The fractional Lévy-stable motion

Before we introduce the general class of fractional Lévy-stable motion we present a special example of this process, namely the fractional Brownian motion. The fractional Brownian motion (FBM) is a mean zero Gaussian process $\{B_H(t)\}$ defined as follows [10]:

$$B_H(t) = \int_{-\infty}^{\infty} \left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right) \mathrm{d}B(u), \qquad t \geq 0, \tag{1}$$

where $\{B(t)\}$ is the classical Brownian motion and $(x)_+ = \max(x, 0)$. The second moment of the process $\{B_H(t)\}$ is expressed as $\langle B_H(t)^2\rangle = \sigma^2 t^{2H}$ (for some parameter $\sigma > 0$). Therefore, the process exhibits subdiffusive dynamics for $H < 1/2$ and superdiffusive dynamics for $H > 1/2$ [21].

For each t the random variable $B_H(t)$ has a Gaussian distribution with mean equal to zero and variance equal to $\sigma^2 t^{2H}$, then its probability density function (PDF) takes the form

$$f_{B_H(t)}(x) = \frac{1}{\sqrt{2\pi}\sigma t^H} \mathrm{e}^{-x^2/2(\sigma t^H)^2}, \qquad x \in R. \tag{2}$$

It is worth mentioning that the above marginal probability density function formula is true for unbiased and unbounded motion. One should remember that the increment process of the FBM, so called fractional Gaussian noise defined as $b_H(n) = B_H(n+1) - B_H(n)$ for $n = 0, 1, \ldots$, is a time-correlated stationary process with covariance function

$$CV_{b_H}(n) = \langle b_H(0) b_H(n)\rangle = \frac{\sigma^2}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right).$$

Moreover, the Laplace transform of $B_H(t)$ is given by

$$\langle \mathrm{e}^{-zB_H(t)}\rangle = \mathrm{e}^{-1/2\left(z\sigma t^H\right)^2}, \qquad t \geq 0, \tag{3}$$

while its Fourier transform is the following:

$$\langle \mathrm{e}^{\mathrm{i}zB_H(t)}\rangle = \mathrm{e}^{-1/2\left(z\sigma t^H\right)^2}, \qquad t \geq 0. \tag{4}$$

In analogy to the relation between Brownian motion and its generalization, Lévy-stable motion, we can also extend the FBM to fractional Lévy-stable motion (FLSM)

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[15, 21, 45, 46]. Namely, for any $0 < H < 1$ the FLSM $\{Z_H^\alpha(t)\}$ is defined in the following way:

$$Z_H^\alpha(t) = \int_{-\infty}^{\infty} \left((t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}\right) \mathrm{d}L_\alpha(u), \quad t \geq 0, \tag{5}$$

where $\{L_\alpha(t)\}$ is the symmetric Lévy-stable motion with index of stability $\alpha \in (0,2)$ with Fourier transform $\langle \mathrm{e}^{\mathrm{i}zL_\alpha(t)}\rangle = \mathrm{e}^{-t|z|^\alpha}$. It is worth mentioning that $\{Z_H^\alpha(t)\}$ is an H-self-similar, stationary-increment process [15] and, unlike in the Gaussian case, an ensemble second moment of FLSM does not exist. The diffusion type is controlled as in FBM by the parameter H, namely for $H < 1/\alpha$ FLSM is subdiffusive and for $H > 1/\alpha$ it exhibits superdiffusive behavior. The diffusion properties of FLSM can also be described using time averaged mean-square-displacement, denoted as sample MSD. Namely, for N observations $\{X_1, X_2, \ldots, X_N\}$, the sample MSD is defined in the following way:

$$M_N(\tau) = \frac{1}{N-\tau} \sum_{i=1}^{N-\tau} (X_{i+\tau} - X_i)^2 \qquad \text{for } \tau = 1, 2, \ldots. \tag{6}$$

It appears that if the sample comes from an H-self-similar Lévy-stable process with stationary increments then for large N/τ sample MSD asymptotically behaves in distribution as a power law

$$M_N(\tau) \overset{D}{\sim} \tau^{2d+1}, \tag{7}$$

where $d = H - 1/\alpha$ and $\overset{D}{\sim}$ denotes similarity in distribution. Therefore, by estimating the power-law parameter $2d+1$ of $M_N(\tau)$ for large N/τ we obtain information about the diffusive characteristics of FLSM. When $d = H - 1/\alpha < 0$ then the process is subdiffusive, when $d = H - 1/\alpha > 0$ it is superdiffusive.

The marginal distribution of $Z_H^\alpha(t)$ is described by a Lévy-stable random variable with stability index α and scale parameter ct^H for some positive $c > 0$. Unlike in the FBM case, the FLSM density function cannot be expressed using a closed-form formula, therefore its distribution is often characterized by the Fourier transform [46]

$$\langle \mathrm{e}^{\mathrm{i}zZ_H^\alpha(t)}\rangle = \mathrm{e}^{-\left(|z|ct^H\right)^\alpha}, \qquad t \geq 0. \tag{8}$$

The probability density function of FLSM can be obtained by calculating the inverse Fourier transform of the characteristic function. That results in the following integral formula:

$$f_{Z_H^\alpha(t)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}zx} \mathrm{e}^{-\left(|z|ct^H\right)^\alpha} \,\mathrm{d}z. \tag{9}$$

Since the correlation cannot be considered as a main measure of dependence of the FLSM (the second moment does not exist for $\alpha < 2$), then other measures of dependence have to be introduced. One of the measures of dependence defined for systems with infinite variance is the codifference, which for the stationary process $\{X(t)\}$ takes the following form [46]:

$$CD_X(t) = \log\left\langle \mathrm{e}^{\mathrm{i}(X(t)-X(0))}\right\rangle - \log\left\langle \mathrm{e}^{\mathrm{i}X(t)}\right\rangle - \log\left\langle \mathrm{e}^{-\mathrm{i}X(0)}\right\rangle, \qquad t \geq 0. \tag{10}$$

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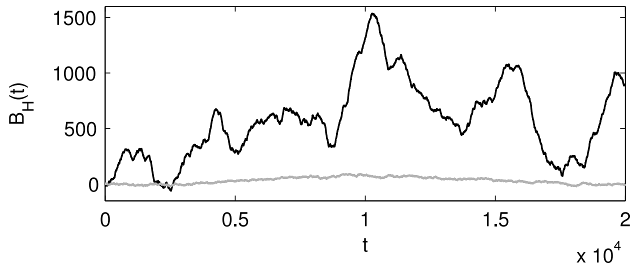

Figure 1. Two exemplary trajectories of the process $\{B_H(t)\}$ with $\sigma = 1$, $H = 0.4$ (gray line) and $H = 0.9$ (black line).

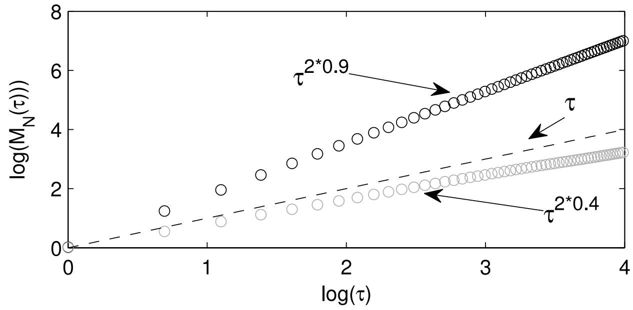

Figure 2. The sample MSDs of the process $\{B_H(t)\}$ (in log–log scale) with $\sigma = 1$, $H = 0.4$ (gray circles) and $H = 0.9$ (black circles). The black dashed line represents the sample MSD of the normal diffusive scenario.

One can find the main properties of codifference in, for instance, [46]–[49]. We only mention here that the codifference carries enough information to detect the ergodic properties of the examined process and is useful in testing the long-memory property for infinite variance systems [22]. For the increments of the FLSM, so called fractional Lévy-stable noise defined as $z_H^\alpha(n) = Z_H^\alpha(n+1) - Z_H^\alpha(n)$ for $n = 0, 1, 2, \ldots$, the codifference for large n can be approximated by the power function [50], i.e. if either $0 < \alpha \leq 1$, $0 < H < 1$ or $1 < \alpha < 2$, $1 - 1/\alpha(\alpha - 1) < H < 1$, $H \neq 1/\alpha$, then for $n \to \infty$

$$CD_{z_H^\alpha}(n) \sim Cn^{\alpha H - \alpha}.$$

If $1 < \alpha < 2$ and $0 < H < 1 - 1/\alpha(\alpha - 1)$, then for $n \to \infty$

$$CD_{z_H^\alpha}(n) \sim Dn^{H - 1/\alpha - 1},$$

where the constants C and D are independent of n.

In figure 1 we present two exemplary trajectories of the process $\{B_H(t)\}$ with common parameter $\sigma = 1$ and different self-similarity indices $H = 0.4$ and 0.9, which reflect subdiffusive and superdiffusive dynamics, respectively. In figure 2 the sample MSDs of the trajectories from figure 1 are depicted. In figure 3 the behavior of the covariance function of fractional Gaussian noise with $H = 0.1$ and 0.9 is presented.

In analogy to the FBM case, in figure 4 we present two exemplary trajectories of the process $\{Z_H^\alpha(t)\}$ with common parameters $\alpha = 1.6$, $c = 1$ and different self-similarity indices $H = 0.4$ and 0.9. Figure 5 clearly shows that the sample MSDs of the trajectories from figure 4 exhibit sub- and superdiffusive dynamics, respectively. Figure 6 depicts the

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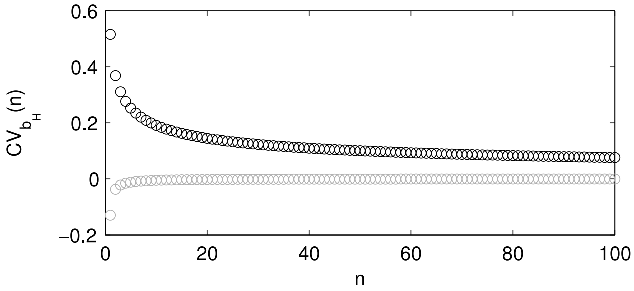

Figure 3. The covariance of the increments of the process $\{B_H(t)\}$ with $\sigma = 1$, $H = 0.4$ (gray circles) and $H = 0.9$ (black circles).

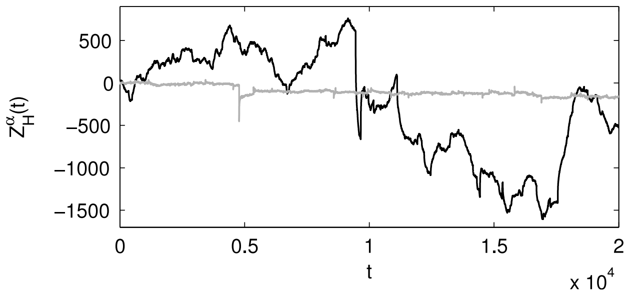

Figure 4. Two exemplary trajectories of the process $\{Z_H^\alpha(t)\}$ with $c = 1$, $\alpha = 1.6$, $H = 0.4$ (gray line) and $H = 0.9$ (black line).

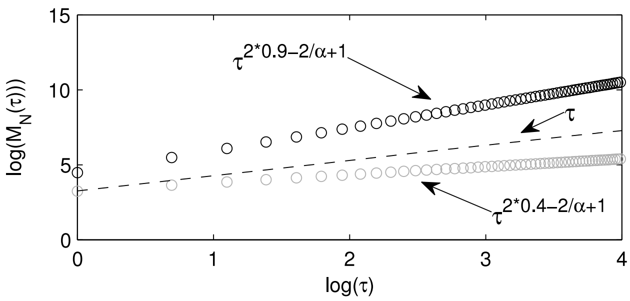

Figure 5. The sample MSDs of the process $\{Z_H^\alpha(t)\}$ (in log–log scale) with $H = 0.4$ (gray circles) and $H = 0.9$ (black circles). The black dashed line represents the sample MSD of the normal diffusive scenario.

asymptotic behavior of the empirical codifference of the fractional Lévy-stable noise. The empirical codifference is an estimator of the theoretical codifference and for the stationary sample $\{X_1, X_2, \ldots, X_N\}$ is defined as follows [51]:

$$ECD(n) = \log\phi(1, -1, n) - \log\phi(1, 0, n) - \log\phi(0, -1, n),$$

where for $n \geq 0$

$$\phi(\theta_1, \theta_2, n) = \frac{1}{N-n}\sum_{j=1}^{N-n} \exp\left(\mathrm{i}(\theta_1 X_{j+n} + \theta_2 X_j)\right).$$

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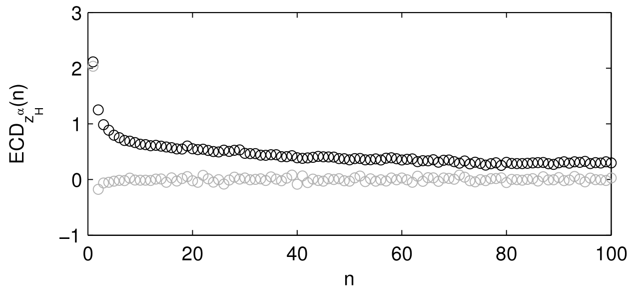

Figure 6. The empirical codifference of the increments of the process $\{Z_H^\alpha(t)\}$ with parameters $\alpha = 1.6$, $c = 1$ and $H = 0.4$ (gray circles) and $H = 0.9$ (black circles).

3. The fractional Lévy-stable motion driven by a tempered stable process

In this section we propose a model that is constructed by subordination of the fractional Lévy-stable motion by a tempered stable inverse subordinator.

Firstly, we elaborate the concept of subordination within the well known CTRW process framework. The CTRW process was introduced by Montroll and Weiss in 1965 [3]. It is a generalization of the classical random walk, which assumes that consecutive random jumps are separated by time intervals of random lengths called waiting times. Namely, a CTRW process is generated by a sequence of independent and identically distributed vectors $\{J_i, W_i\}_{i\geq 1}$, where J_i denotes the ith random jump and the positive random variable W_i denotes the waiting time between the $(i-1)$th and ith jumps. If we introduce the counting process

$$N(t) = \max\{n \geq 0 : W_1 + W_2 + \cdots + W_n \leq t\}$$

then the following process defines the CTRW:

$$R(t) = \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0.$$

It appears, see theorem 3.1 in [52], that if the following scaling limit[1] holds:

$$\left\{\left(\frac{J_1 + J_2 + \cdots + J_{[nt]}}{b_1(n)}, \frac{W_1 + W_2 + \cdots + W_{[nt]}}{b_2(n)}\right)\right\}_{t\geq 0} \Longrightarrow \{(A(t), D(t))\}_{t\geq 0}, \tag{11}$$

as $n \to \infty$ and $\{D(t)\}$ is a strictly increasing process then the scaling limit of the CTRW is the following:

$$\left\{\frac{R([nt])}{b_3(n)}\right\}_{t\geq 0} \Longrightarrow \left\{\left(A^-\left((D^{-1})^-(t)\right)\right)^+\right\}_{t\geq 0}, \tag{12}$$

as $n \to \infty$, where $\{A^-(t)\}$ and $\{(D^{-1})^-(t)\}$ are the left-continuous versions of processes $\{A(t)\}$ and $\{D^{-1}(t)\}$ from the limit in (11), $\{(D^{-1})(t)\}$ defined as $\{(D^{-1})(t)\} = \inf\{x : D(x) > t\}$ is the inverse process to $\{D(t)\}$, $b_1(c)$, $b_2(c)$, $b_3(c)$ are appropriate functions (for

[1] In fact by '$\Longrightarrow$' we denote here the weak functional convergence in the Skorokhod J_1 topology (see [53] for more details).

details see [52]) and $[x]$ is the integer part of x. The scaling limit $\left\{(A^-((D^{-1})^-(t)))^+\right\}$ in (12) is the right-continuous version of process $\{A^-((D^{-1})^-(t))\}$ and is a subordinated process, where $\{A^-(t)\}$ is called an external process, while $\{(D^{-1})^-(t)\}$ is called an inverse subordinator. Processes $\{A^-(t)\}$ and $\{D^-(t)\}$ appear as the scaling limits of scaled partial sums of jumps and waiting times of the CTRW scheme (11), respectively. In the following part of this section we present the model of the subordinated process, for which the external process is fractional Lévy-stable motion with a tempered stable inverse subordinator. The proposed model is defined as follows:

$$Y(t) = Z_H^\alpha(S(t)), \qquad t \geq 0, \tag{13}$$

where $\{Z_H^\alpha(t)\}$ is a fractional Lévy-stable motion defined in (5) and $\{S(t)\}$ is the inverse tempered stable subordinator defined as

$$S(t) = \inf\{\tau > 0 : T(\tau) > t\}, \qquad t \geq 0, \tag{14}$$

with $\{T(\tau)\}$ being a strictly increasing tempered stable Lévy process with Laplace transform of the following form:

$$E(\mathrm{e}^{-uT(\tau)}) = \mathrm{e}^{-\tau((u+\lambda)^\gamma - \lambda^\gamma)}. \tag{15}$$

The parameter $\lambda > 0$ is called the tempering index (or tempering parameter), while $0 < \gamma < 1$ is the stability index. It is worth noting that a CTRW scheme that leads in the scaling limit to fractional Lévy-stable motion subordinated by a tempered stable inverse subordinator defined in (13) can be constructed by combining the results presented in [54, 55], which cover the problem of CTRW processes resulting in the scaling limit in fractional Lévy-stable motion and a tempered stable subordinator, respectively.

There is a strong relation between the tempered stable Lévy process and a strictly increasing γ-stable process, namely the PDF of $T(\tau)$ can be formulated as

$$f_{T(\tau)}(x) = \mathrm{e}^{-\lambda x + \lambda^\gamma \tau} f_{U(\tau)}(x), \tag{16}$$

where $f_{U(\tau)}(\cdot)$ is the PDF of the stable Lévy motion with index of stability γ, scale parameter $\sigma = (\cos \pi\gamma/2)^{1/\gamma}$, skewness $\beta = 1$ and shift $\eta = 0$. One observes that for $\lambda = 0$ the tempered stable Lévy process defined by its Laplace transform in (15) is a strictly increasing γ-stable process $\{U(\tau)\}$.

The relation between tempered stable and inverse tempered stable processes is discussed in detail in [28]. We only mention here the relation between the PDFs of such systems, namely

$$f_{\mathrm{S}(t)}(x) = \frac{1}{\gamma x} t f_{T(x)}(t) + \lambda \frac{1}{\gamma x} \int_0^t u f_{T(x)}(u)\,\mathrm{d}u - \lambda^\gamma \int_0^t f_{T(x)}(u)\,\mathrm{d}u, \tag{17}$$

where $f_{\mathrm{S}(t)}(\cdot)$ and $f_{T(\tau)}(\cdot)$ denote the PDFs of the inverse tempered stable and tempered stable processes, respectively. Moreover, it is shown in [28] that for large and small values of t the above formula for the PDF of the inverse tempered stable subordinator can be approximated by $(t/\gamma x) f_{T(x)}(t)$. Now, using the relation between the strictly increasing tempered stable and γ-stable processes given in (16) we can express the PDF of $S(t)$ for small and large values of t by means of the γ-stable process PDF,

$$f_{\mathrm{S}(t)}(x) \sim \frac{t}{\gamma x} f_{T(x)}(t) = \frac{t}{\gamma x} \mathrm{e}^{-\lambda t + \lambda^\gamma x} f_{U(x)}(t).$$

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Using the self-similarity property of the process $\{U(\tau)\}$ and the relation between this process and its inverse process $\{K(t)\}$ defined in the same way as in (14) but for the γ-stable Lévy process [56] we obtain for small and large values of t

$$f_{\mathrm{S}(t)}(x) \sim \frac{t}{\gamma x} \mathrm{e}^{\lambda t - \lambda^\gamma x} f_{U(x)}(t) = \frac{t}{\gamma x} \mathrm{e}^{-\lambda t + \lambda^\gamma x} \frac{1}{x^{1/\gamma}} f_{U(1)}(t/x^{1/\gamma}) = \frac{1}{t^\gamma} \mathrm{e}^{-\lambda t + \lambda^\gamma x} f_{K(1)}(x/t^\gamma).$$

Since the process $\{K(t)\}$ is self-similar with the self-similarity index equal to γ, then for small and large t we have the following approximation:

$$f_{\mathrm{S}(t)}(x) \sim \mathrm{e}^{-\lambda t + \lambda^\gamma x} f_{K(t)}(x). \tag{18}$$

Using the above formula we can approximate also the Laplace transform for $S(t)$, namely for small and large t we get

$$\left\langle \mathrm{e}^{-zS(t)} \right\rangle \sim \int_0^\infty \mathrm{e}^{-\lambda t - (z - \lambda^\gamma) x} f_{K(t)}(x)\, \mathrm{d}x = \mathrm{e}^{-\lambda t} \left\langle \mathrm{e}^{-(z-\lambda^\gamma)K(t)} \right\rangle.$$

The Laplace transform for $K(t)$ has the following form [56]:

$$\left\langle \mathrm{e}^{-zK(t)} \right\rangle = E_\gamma(-zt^\gamma),$$

where

$$E_\gamma(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\gamma + 1)} \tag{19}$$

is a generalized Mittag-Leffler function with parameter γ and $\Gamma(\cdot)$ is the gamma function.

Therefore, as a final result we get the Laplace transform of $S(t)$ for small and large values of t,

$$\left\langle \mathrm{e}^{-zS(t)} \right\rangle \sim \mathrm{e}^{-\lambda t} E_\gamma(-(z - \lambda^\gamma) t^\gamma).$$

Moreover, the lth moment of $S(t)$ for all $l > 0$ and small and large t is expressed as

$$\langle S(t)^l \rangle \sim \mathrm{e}^{-\lambda t} \int_0^\infty x^l \mathrm{e}^{\lambda^\gamma x} f_{K(t)}(x)\, \mathrm{d}x.$$

Using the above mentioned characteristics of the inverse tempered stable subordinator we can calculate the main properties of the process $\{Y(t)\}$ defined in (13). For each t the PDF of $Y(t)$ is given by

$$f_{Y(t)}(x) = \int_0^\infty f_{Z_H^\alpha(z)}(x) f_{\mathrm{S}(t)}(z)\, \mathrm{d}z, \tag{20}$$

where $f_{Z_H^\alpha(z)}(\cdot)$ is given in (9) while $f_{\mathrm{S}(t)}(\cdot)$ is given in (17). Using the approximation of the PDF for the $\{S(t)\}$ process for small and large values of t given in (18) we obtain

$$f_{Y(t)}(x) \sim \frac{\mathrm{e}^{-\lambda t}}{2\pi} \int_0^\infty \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}ux - (|u| c z^H)^\gamma} \mathrm{e}^{\lambda^\alpha z} f_{K(t)}(z)\, \mathrm{d}u\, \mathrm{d}z,$$

where $f_{K(t)}(\cdot)$ is the PDF of the inverse γ-stable subordinator. It is worth mentioning that for large values of t the $\{Y(t)\}$ process behaves like the external process $\{Z_H^\alpha(t)\}$. This is related to the fact that $S(t) \sim ct$, when $t \to \infty$ and c is some positive constant, see proposition 3 in [57]. Therefore the codifference of the process $\{Y(t)\}$ has the same asymptotic behavior as for the fractional Lévy-stable motion.

4. Numerical simulation scheme

In this section we introduce a simulation scheme of subordinated fractional Lévy-stable motion. The proposed procedure is based on an algorithm used in simulation of subordinated Brownian motion [58]. Its main idea is to simulate independent trajectories of the inverse tempered stable subordinator $\{S(t)\}$ and the fractional Lévy-stable motion $\{Z_H^\alpha(\tau)\}$ and to take their superposition, resulting in the trajectory of the considered process $\{Y(t)\}$ defined in (13).

Firstly, we simulate an approximate trajectory $S_\delta(t)$ of the inverse tempered stable subordinator $\{S(t)\}$ defined in (14). Let us define $S_\delta(t)$ with step length $\delta > 0$ in the following way:

$$S_\delta(t) = (\min\{n \in \mathbb{N} : T(\delta n) > t\} - 1)\,\delta,$$

where $T(\delta n)$ for $n = 1, 2, \ldots$ are the values of the tempered stable subordinator at the points $\delta n, n = 1, 2, \ldots$, which can be simulated by the method presented in [59]. Observe that the trajectory $S_\delta(t)$ has increments of length δ at random time moments governed by the process $\{T(\tau)\}$ and therefore $S_\delta(t)$ is an approximation of the operational time.

Secondly, we simulate the approximate trajectory of the external process—fractional Lévy-stable motion. The method was introduced in [60] and is based on the generation of an approximate path of the stationary increment process $z_H^\alpha(\delta n)$. Its main idea is to approximate the stochastic integral

$$z_H^\alpha(\delta n) = \int_{-\infty}^{\infty} \left((\delta n - u)_+^{H-1/\alpha} - (\delta n - u - 1)_+^{H-1/\alpha}\right) \mathrm{d}L_\alpha(u), \qquad n = 1, 2, \ldots$$

by the Riemann sum

$$Y_{m,M}(\delta n) = \sum_{j=1}^{mM} \left((j/m)_+^{H-1/\alpha} - (j/m - 1)_+^{H-1/\alpha}\right) Z_{\alpha,m}(m\delta n - j), \qquad n = 1, 2, \ldots, \qquad (21)$$

where $Z_{\alpha,m}(j) = L_\alpha((j+1)/m) - L_\alpha(j/m)$. The parameters m and M control the mesh size and the kernel function cut-off, respectively. For large values of these parameters the variables $Y_{m,M}(\delta n)$ approximate well the variables $z_H^\alpha(\delta n)$ [60]. The direct implementation of the sum (21) yields a slow algorithm and cannot be efficiently applied to generate long approximations in real time. Therefore, the method presented in [60] uses the fast Fourier transform algorithm and the technique of discrete Fourier transforms for some circular sequences. The details can be found in [60]. The trajectory of the fractional Lévy-stable motion $Z_H^\alpha(\tau)$ can be obtained as cumulative sums of variables $z_H^\alpha(\delta n)$. Finally, by taking the superposition $Y_\delta(t) = Z_H^\alpha(S_\delta(t))$ we obtain the approximated trajectory of the process $\{Y(t)\}$.

In figure 7 we present two exemplary sample paths of the fractional Lévy-stable motion subordinated by an inverse tempered stable subordinator with common parameters $\gamma = 0.3, \lambda = 0.6, \alpha = 1.6$ and different H parameters: $H = 0.4$ and 0.9.

5. Estimation procedure

The estimation procedure of the parameters of fractional Lévy-stable motion driven by an inverse tempered stable subordinator is divided into two steps.

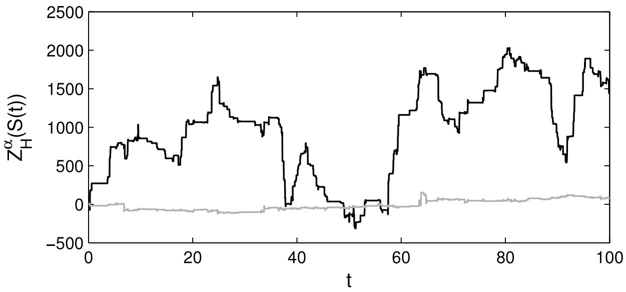

Figure 7. An exemplary sample path of the process $\{Y(t)\}$ with parameters $\gamma = 0.3, \lambda = 0.6, \alpha = 1.6$, $H = 0.4$ (gray line) and $H = 0.9$ (black line). We observe here the constant time periods typical for processes subordinated by inverse subordinators.

- In the first step we divide the analyzed time series $Y_1, Y_2, \ldots, Y_n$ into two vectors. The first one (vector T) represents lengths of constant time periods observed in the data, which means the number of consecutive observations that are on the same level. According to the idea of constructing the inverse subordinators, the vector T constitutes a sample of independent identically distributed (i.i.d.) random variables from the same distribution as the subordinator, i.e. in our case from the tempered stable distribution defined by the Laplace transform in (15). The second vector (vector Z) arises after removing the constant time periods. This scheme is a standard procedure in the analysis of the processes subordinated by inverse subordinators and has been used for many applications. For instance, external processes such as Brownian motion with drift and tempered stable subordinator were examined in [28, 30], while the geometric Brownian motion subordinated by an inverse tempered stable process was discussed in [29] in the context of financial data. A stable subordinator has been considered for many external processes, see [23, 37]. Comparison of different subordinators can be found in [23].
- In the second step we separately analyze the vectors T and Z. On the basis of constant time periods we estimate the parameters γ and λ that describe the tempered stable distribution. We propose here to use the method presented in [29] where the parameters were fitted by using the form of the right tail of the tempered stable distribution, i.e. the truncated power function $\mathrm{e}^{-\lambda x} x^{-\gamma}$ was fitted to the empirical right tail. One of the possible methods of fitting is the least squares method [29]. The other parameters of the subordinated fractional Lévy-stable motion (i.e. α and H) defined in (13) are estimated on the basis of the vector Z. The α parameter is the index of stability of the increments of the external process, therefore we can estimate it by using the regression method [61]–[63] applied to the differenced series Z. The parameter $H = d + 1/\alpha$ can be estimated by using the sample MSD, which for the fractional Lévy-stable motion asymptotically behaves in distribution as a power law, see equation (7). There are also other methods that can be useful in the problem of the estimation of the H parameter. One of them is the absolute value method [64]. The known methods for estimation of the memory parameter d are the variance method [65], the variance of residuals method [66], the R/S method [67] and its modified version [64], the periodogram method [68] and the Whittle method [69]. However, in this paper we do not use them.

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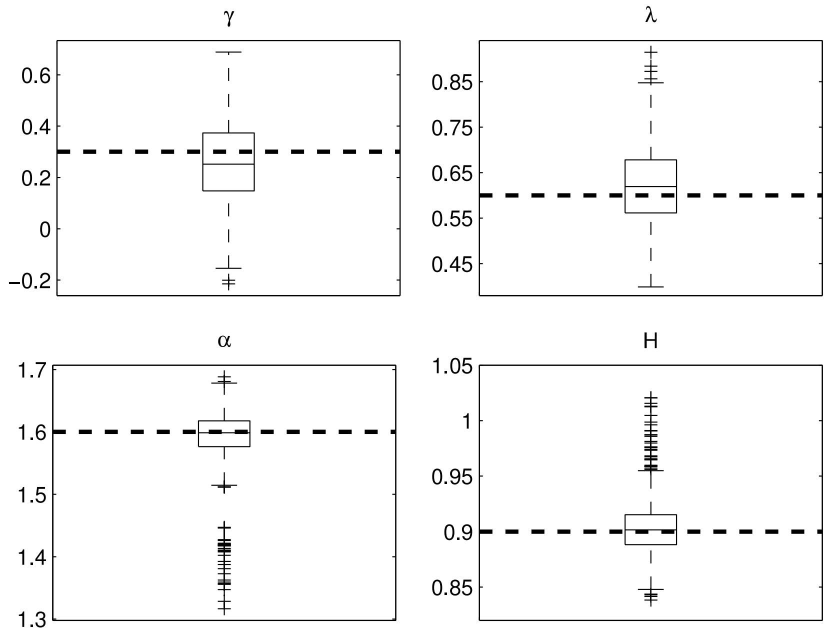

Figure 8. The box plots of the estimated parameters calculated on the basis of 1000 Monte Carlo simulations of the fractional Lévy-stable motion subordinated by an inverse tempered stable process. The black dashed lines correspond to the true values of the parameters: $\gamma = 0.3, \lambda = 0.6, \alpha = 1.6$ and $H = 0.9$.

In order to show the effectiveness of the described scheme we simulate 1000 trajectories of the fractional Lévy-stable motion subordinated by an inverse tempered stable process with one set of parameters $\gamma = 0.3, \lambda = 0.6, \alpha = 1.6$ and $H = 0.9$. Each trajectory is of length 10 000. Then, on the basis of simulated samples we estimate the parameters γ, λ, α and H and construct the box plots. The results are presented in figure 8. As we can observe, the obtained values are close to the theoretical ones.

6. Conclusions

In this paper we have introduced a model of fractional Lévy-stable motion subordinated by an inverse tempered stable subordinator. The new model can be used as a universal tool for modeling super- and subdiffusive processes. We have shown its properties such as Fourier transform, integral form of the probability density function and its asymptotic behavior. Moreover, we have proposed a numerical simulation scheme and estimation procedure of the considered process.

The new process is constructed by subordination of the classical fractional Lévy-stable motion by an inverse tempered stable subordinator. Therefore, it exhibits some properties of external processes like long-range dependence and the possibility of modeling anomalous diffusion. However, subordination by an inverse tempered stable subordinator results in some important differences in comparison to the classical fractional Lévy-stable motion. Namely, for short time scale the new process has non-stationary increments and is not α-stable distributed. Moreover, due to subordination the trajectories of the considered process are piecewise constant. On the other hand, at large time scale the fractional Lévy-

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stable motion subordinated by an inverse tempered stable subordinator has stationary increments and is α-stable distributed as the external process.

The considered model can be generalized by replacing the inverse tempered stable subordinator by other subordinators having their origins in the wide class of infinitely divisible processes.

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