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Abstract. This paper presents a mathematical model for basketball free throws. It is intended to be a supplement to an existing calculus course and could easily be used as a basis for a calculus project. Students will learn how to apply calculus to model an interesting real-world problem, from problem identification all the way through to interpretation and verification. Along the way we will introduce topics such as optimization (univariate and multiobjective), numerical methods, and differential equations.

Key words. basketball, mathematical modeling, calculus projects

AMS subject classifications. 00-01, 00A71, 26A06

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1. Introduction.

In these days of superstar basketball players, you would think that shooting free throws should be as much a formality, and just as exciting, as the extra point in professional football. Not so. Take for example Shaquille O'Neal, the subject of our first model, who as of the end of the 2004–2005 regular season had a career free throw percentage of 53.1%. His troubles seemed to increase during the playoffs, where he shot around 45% from the line. Shaquille is not alone in his free throw shooting troubles. In fact nearly one-third of all NBA players shoot less than 70% from the foul line.

When a basketball player steps up to shoot a free throw he does not usually think (unless he also happens to be a mathematician), "I wonder if my free throw shooting percentage would improve if I changed my initial shooting angle," or "I wonder how air resistance affects the trajectory of my shot," or even "Should I be aiming for the back rim, front rim, or the middle of the basket?" We present here a calculus-based model for basketball free throws to show that they should address some of these musings. We begin by conjecturing that some players shoot poorly from the line because they are shooting the ball at the wrong angle. Therefore, the focus of our model will be the release angle, a simple place to start, and we will extend it later. Some of the more interesting facts that we'll discover by refining and interpreting our model are:

1. The best way to shoot free throws depends upon the person shooting. The two most important factors are their height and

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how consistent they are in controlling both the release angle and the release velocity.

2. In general, the taller you are, the lower your release angle should be. We'll actually see that taller players are allowed more error in both their release angles and release velocities and thus they *should* have an easier time shooting free throws than shorter players.
3. It is much more important to consistently use the right release velocity than the right release angle.
4. The best shot does not pass through the center of the hoop. The best trajectories pass through the hoop somewhere between the center and the back rim. Taller players should shoot closer to the center while shorter players should aim more towards the back rim.

2. Mathematical Modeling. Before we jump into modeling a basketball free throw, it would help for us to tell you exactly what we mean by mathematical modeling:

> Mathematical modeling is the process of formulating real world situations in mathematical terms.

Less formally, mathematical modeling takes observed real-world behaviors or phenomena and describes them using mathematical formulae or equations. All the formulae you see in your physics, chemistry, and biology classes are mathematical models. Mathematical models can be found everywhere, not only in science, but also in the social sciences and even in business. For instance, there are people who get paid very well to model the stock market. By constructing mathematical models, we can often explain real-world behavior, predict how sensitive real-world situations are to certain changes, and even predict future behavior (very useful for the people who model the stock market). The following is a summary of the standard steps for constructing a mathematical model:

1. Identify the problem. What do you want to find out?
2. Derive the model. Identify the constants and variables involved. Make assumptions about which variables to include in the model. Determine the interrelationships between the variables.
3. Solve the equations and interpret the model.
4. Verify the model. Does it answer the original problem? Does it match up to real-world data?
5. Refine the model. If the model is not satisfactory, refine it by removing some of your earlier assumptions.

We'll discuss these steps in greater detail as we use them to model basketball free throws.

3. Our First Model: The Best Angle. It is true for most models, including ours, that trying to include every possible physical effect immediately is rather ambitious, especially if you want to be able to solve the model. The modeling process typically begins with the construction of very simple models which are easy to solve. Models are then refined to make them more realistic, which in turn requires the introduction of more powerful mathematics in order to solve them. In the end the model should

Table 3.1 *The physical constants of the problem.*

Physical constant	Symbol	Value
Rim diameter	D_r	1.5 ft
Ball diameter	D_b	0.8 ft
Horizontal distance traversed	l	13 ft 6.5 in
Vertical distance traversed	h	1 ft $1\frac{3}{4}$ in
Acceleration due to gravity	g	$-32\,\text{ft}\,\text{s}^{-2}$

be refined enough to describe reality as closely as possible while still being solvable. You'll see this refinement process in action as we go through the modeling procedure.

3.1. Problem Definition. When watching basketball players shoot free throws we notice that sometimes they make small errors and still make the basket. It seems reasonable that the amount of error that the player can make and still have the shot go in depends on the initial angle that the ball was thrown. We'll therefore begin by defining the problem as follows:

Given a basketball player of a certain height,
what is the *best angle* for him to shoot a free throw.

3.2. Deriving Our First Model: Identify the Constants and Variables. The physical constants (the diameter of the rim, etc.) that we shall use to derive the equations of motion that govern the flight of the ball can be obtained from various sources, including the Internet, books [14, 17], and actual (tape measure) measurements. The diameter of the rim, D_r, is 1.5 ft. The diameter of the ball, D_b, is taken to be 0.8 ft.[1]

It has been observed [7] that free throws are shot from a few inches in front of the free throw line. We thus take the horizontal distance traversed, l, to be 13'6.5" rather than the total distance, 14', from the free throw line to the center of the hoop. It has also been observed [4, 8, 19] that shooters release the ball, on average, from a height of approximately 1.25 times the shooter's own height. For example, a 7'1" tall player releases the ball, on average, at a height of approximately 8'10$\frac{1}{4}$". Thus, for a 7'1" tall player, we would take the net vertical distance traversed, h, to be 1'1$\frac{3}{4}$". See Table 3.1 and Figure 3.1.

3.3. Deriving Our First Model: Simplifying Assumptions. We make the following simplifying assumptions for our first model:

1. *Allow only "nearly nothing but net" shots.* By this we mean, allow only trajectories that either (a) go directly in (nothing but net), or (b) hit the back of the rim and then go directly in. We do this to account for a large range of successful trajectories while keeping things fairly simple. To make sure that the ball actually goes in, and does not bounce out, after hitting the back of the rim, we shall consider only trajectories where the center of the ball is at or below the height of the rim when the ball hits it. See Figure 3.2.
2. *Ignore air resistance.* The effect of air resistance is minor compared to the mathematical complexity it adds to the model.

[1] The actual diameter of a basketball, D_b, can vary legally from approximately 0.78 ft to 0.81 ft. We assumed an average diameter of 0.8 ft.

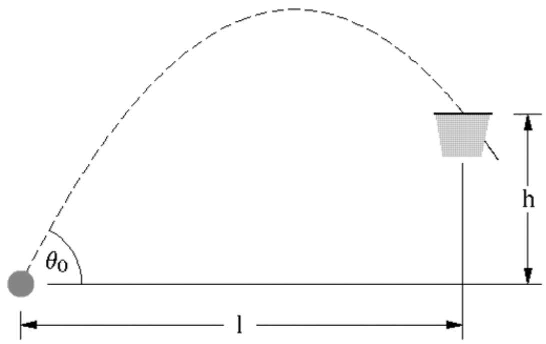

Fig. 3.1 *The conceptualization of the free throw.*

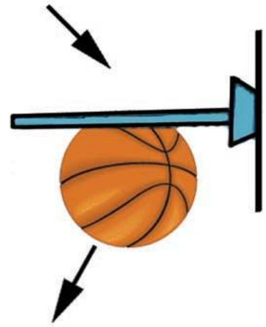

Fig. 3.2 *Ball going in off the back of the rim.*

3. *Ignore any spin the ball may have.* Spin becomes important if we allow the ball to bounce before it goes in. Since we are only allowing nearly nothing but net shots, and ignoring air resistance, we'll also ignore spin.
4. *There is no sideways error in the trajectory.* If you want to be a good free throw shooter, you really ought to shoot straight. The benefit we get from assuming the shooter always shoots straight is that the model will be two-dimensional (constrained to a plane). If transverse error were to be included, the model would be a more realistic, but harder to solve, three-dimensional one.
5. *There is no error in the initial shooting velocity.* We are assuming that some basketball players have problems shooting free throws because they are shooting at the wrong angle. Therefore, our first model concentrates on errors in the release angle only.
6. *The best shot is one that goes through the center of the hoop.* That is, the model will be one in which the initial velocity is the velocity that would drop the center of the ball through the center of the hoop. Some coaches encourage this by placing an insert into the ring that makes the aperture smaller.
7. *The shooter is* 7'1" *tall.* After we find the best angle for Shaq, we will quickly remove this assumption and find the best angle for people of a more diminutive stature.

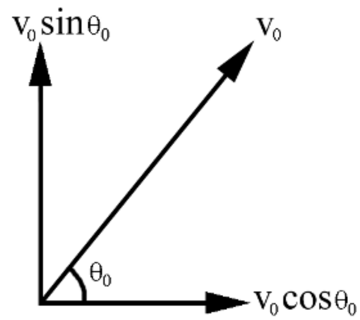

Fig. 3.3 *Resolving the initial velocity into horizontal and vertical components.*

These assumptions may seem very stringent. For example, not everyone is as tall as Shaq, and basketball is not usually played in a vacuum. Remember, though, that to begin with, the model should be a simple one—one that is easy to solve and interpret. Later, in the refinement stage, the model will become more realistic and some of theses assumptions will be removed.

3.4. Deriving Our First Model: Mathematical Interrelationships between the Variables. The goal of this section is to derive a mathematical formula that expresses the amount of error a player can make in the release angle in terms of the other variables identified above. We'll do this by taking standard projectile motion equations that are derived from Newton's second law of motion. A more in-depth discussion of these "projectile motion" equations than presented here can be found in any basic physics book [6]. Instead of finding one long formula for the amount of error that the player can make before missing the basket, it is better to break down the equation into separate parts (called *submodels*) and put things back together later. We begin by resolving the initial velocity v_0 into horizontal and vertical components,

$$v_{\mathrm{H}} = v_0 \cos(\theta_0) \tag{3.1}$$

and

$$v_{\mathrm{V}} = v_0 \sin(\theta_0), \tag{3.2}$$

respectively, where θ_0 is the initial release angle. See Figure 3.3. Using a subscript 0 to identify initial values of variables is a common convention in mathematical modeling. Horizontally, there is no acceleration due to gravity or, by assumption, any air resistance. Thus the horizontal equation of motion is

$$x(t) = vt, \tag{3.3}$$

where $x(t)$ stands for distance, v for velocity, and t for time. Substituting our initial horizontal velocity into this equation we obtain

$$x(t) = v_0 \cos(\theta_0)t. \tag{3.4}$$

Using l as the horizontal distance to the center of the basket and letting T be the time it takes to get there, we substitute $x(T) = l$ into (3.4) and obtain for our model

$$l = v_0 \cos(\theta_0)T. \tag{3.5}$$

Similarly, the vertical equation of motion is given by

$$y(t) = vt + \frac{1}{2}gt^2 = v_0 \sin(\theta_0)t + \frac{1}{2}gt^2, \tag{3.6}$$

where $g = -32\,\mathrm{ft\,s^{-2}}(-9.8\,\mathrm{m\,s^{-2}})$ is the acceleration due to gravity. Substituting $y(T) = h$, the vertical distance to the center of the basket, into the above equation, we obtain for our model

$$h = v_0 \sin(\theta_0)T + \frac{1}{2}gT^2, \tag{3.7}$$

where h is the vertical distance to the center of the basket. Solving (3.5) for T,

$$T = \frac{l}{\cos(\theta_0)v_0}, \tag{3.8}$$

and substituting it into (3.7), we find the initial velocity v_0 needed, for a given initial angle θ_0, so that the basketball goes through the middle of the hoop:

$$v_0 = \frac{l}{\cos\theta_0}\sqrt{\frac{-g}{2\left(l\tan\left(\theta_0\right) - h\right)}}. \tag{3.9}$$

We note that this formula gives us sensible answers only for a limited range of θ_0. Not only does it physically make sense to restrict initial angles to ones that result in forward motion, i.e., $0 < \theta_0 < 90°$, but also notice that the formula gives real values only for $l\tan(\theta_0) - h > 0$ (remember g is negative). Physically this inequality corresponds to the ball having a sufficiently high initial release angle to reach the height of the rim. Thus we take the range of initial release angles to be $\tan^{-1}\left(\frac{h}{l}\right) < \theta_0 < 90°$.

> In modeling, it is always good practice to note the range that your parameters can take. Otherwise you may unwittingly attain solutions which turn out to be nonphysical.

We make special note of the physical range of θ_0 here, because it is important for the numerical methods used to find solutions later in this paper. Furthermore, note that we assume that the ball will not hit the front of the rim on trajectories where the ball passes through the center of the hoop. We will show later that this might not be true, especially for shorter players.

With the equations of motion modeled, we now derive the equations for the amount of error that can be made in the initial angle θ_0 and still have the ball go directly into the basket. Keeping the initial velocity fixed, allowing the initial release angle to vary (this corresponds to our basketball player making an error in his release angle θ_0^{oops}), and replacing l by x in (3.7) and (3.8), we work out the new horizontal position of the ball as it comes back down to the basket height,

$$x = \frac{v_0\cos\left(\theta_0^{\mathrm{oops}}\right)}{-g}\left(v_0\sin\left(\theta_0^{\mathrm{oops}}\right) + \sqrt{v_0^2\sin^2\left(\theta_0^{\mathrm{oops}}\right) + 2gh}\right). \tag{3.10}$$

In the above equation θ_0^{oops} corresponds to a larger (or smaller) release angle (due to player error) than the ideal initial angle θ_0 where the ball passes through the center of the hoop; see Figure 3.4.

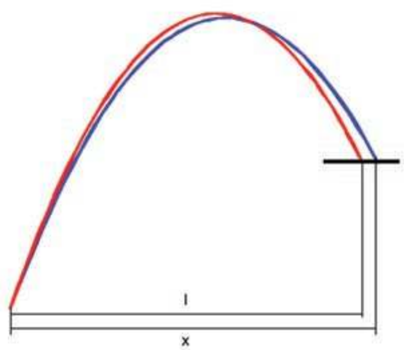

Fig. 3.4 *Comparison of the ideal trajectory that passes through the center on the hoop* (v_0, θ_0) *(red) and the trajectory with an error in the release angle* $(v_0, \theta_0^{\text{oops}})$ *(blue).*

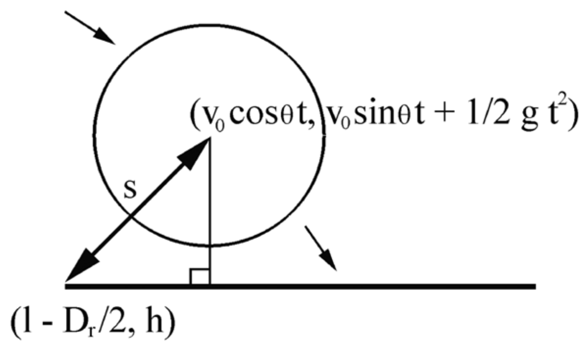

Fig. 3.5 *The distance* s *between the front of the rim and the center of the ball.*

We now derive the following two criteria for the basketball to still go in the net:

1. To avoid contact with the front of the rim, the distance s between the rim and the center of the ball must remain greater than the radius of the ball throughout its trajectory, i.e., for all times t such that $0 < t < T$; see Figure 3.5.[2] Using for convenience the square distance, we have the following criterion for the ball not to hit the front of the rim:

$$s^2 = (x(t) - (l - D_r/2))^2 + (y(t) - h)^2 > (D_b/2)^2. \tag{3.11}$$

2. From (3.10), we note that $x + D_b/2$ is the horizontal distance to the rightmost part of the ball when the center of the ball is level with the basket. And since $l + D_r/2$ is the horizontal distance to the back of the rim, the criterion for having the ball hit the back of the rim as the center of the ball passes through the basket is

$$x + D_b/2 = l + D_r/2. \tag{3.12}$$

[2]We note that this range of t, when the ball is near the rim, can be sharpened to $\frac{l-D_r}{v_0\cos(\theta_0^{\text{oops}})} \le t \le -\frac{1}{g}(v_0\sin(\theta_0^{\text{oops}}) + \sqrt{v_0^2\sin^2(\theta_0^{\text{oops}}) + 2gh})^{-1}$.

3.5. Solving the Equations. To find the error allowed for a given initial angle θ_0, we keep v_0 fixed and solve numerically for the unique release angles $\theta_{\text{low}} < \theta_0$ and $\theta_{\text{high}} > \theta_0$, which are, respectively, the solutions to the following equations:

$$s^2 - (D_b/2)^2 = 0, \tag{3.13}$$

the ball is released at an angle lower than θ_0 and just misses the front of the rim as it goes in, and

$$x - l + \frac{D_b - D_r}{2} = 0, \tag{3.14}$$

the ball is released at a angle higher than θ_0 and hits the back of the rim and goes in. We note here that increasing the initial angle may increase x_h, the distance to the center of the ball as it passes through the center of the hoop, but after a certain point x_h will start to decrease. This can happen before the ball hits the back of the rim. So for certain trajectories, there is no solution to (3.14) and both θ_{low} and θ_{high} are solutions to (3.13). This behavior will be made more apparent in the next few sections. After solving for θ_{low} and θ_{high}, we find the minimum deviation from θ_0,

$$e\,(\theta_0) = \min\{\theta_{\text{high}} - \theta_0, \theta_0 - \theta_{\text{low}}\}. \tag{3.15}$$

The best release angle is the one that maximizes this function.

In calculus, to find a maximum of a function, we usually differentiate it and then find the zero of the derivative. It can be shown that for a differentiable function, the derivative is zero at the maximum. This is a so-called necessary condition. To ensure a maximum, the second derivative must also be negative at that point. This additional condition is called a sufficient condition. So we are tempted to solve the equation $e'\,(\theta_0) = 0$. Unfortunately if we tried to do this, we would run into trouble. To see why let's examine Figure 3.6, a plot of the error function $e\,(\theta_0)$ for various values of θ_0. We created Figure 3.6 using linear interpolation (connecting the points with lines). The function is clearly *not differentiable* at the maximum; the left-hand slope is not equal to the right-hand slope. This can be easily explained by recognizing that (3.15) contains the min function, which can introduce nondifferentiability. Most standard optimization methods require at least first differentiability and therefore cannot find the maximum of this function. That is, we can't take the derivative and set it equal to zero as we usually do to find the maximum because the function is not differentiable. So to find the maximum we use numerical methods that work for non-differentiable functions. You may have already seen in your calculus class numerical methods used to approximate function values (tangent line approximations) or to find roots of equations (Newton's method). It is also possible to find maxima and minima numerically. The exact numerical method you use to find the maximum is not important, but the interested reader can find more information on optimization methods in standard texts; see, for example, [10]. Using a computer algebra system's optimization routine,[3] we get a best angle for our simplest model of

$$\theta^*_{\text{center}} \approx 48.18^\circ.$$

To have the ball pass through the center of the hoop for this angle, Shaq would have to release the ball consistently at an initial velocity of

[3]We used MATLAB, but you could use Maple or *Mathematica* if you prefer. You can download our MATLAB code from http://epubs.siam.org/sam-bin/dbq/article/33955.

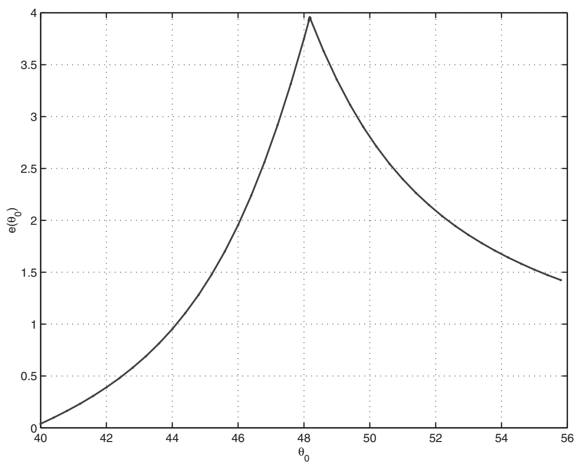

Fig. 3.6 *The error about θ_0 for which the basketball still goes in.*

$$v^*_{\text{center}} \approx 6.62\,\text{m/s}\ (21.7\,\text{ft/s}).$$

Note that labeling an optimal solution with an $*$ is another common convention in mathematical modeling. It may seem silly to imagine Shaq stepping up to the line and thinking "48.18°, 48.18°, I must shoot at 48.18°."[4] For some players though, it may have to start this way. Then, with practice, making the same shot over and over again, it will hopefully become unconscious; see section 9.3.

3.6. Interpreting the Model. It seems that we have found the best angle for Shaq to shoot his free throws. This is assuming Shaq can consistently control his release velocity. Upon further thought, this assumption is not too realistic: a player is quite likely to make errors in both his release velocity and his release angle. We made the assumption in the first place only to make our first model easy to derive and solve. Upon closer inspection, as we'll see in our second model, the above best solution really does require a precise release velocity, especially for shorter players. So should we recommend this angle for Shaq? Probably not, especially if he has trouble releasing the ball with a consistent velocity. It's probably close, but we can do better.

3.7. Refining the Model. When modeling, it is customary to make some assumptions, as we did above, that make solutions easier to find. The more assumptions one makes, the less accurate the model usually becomes. Once a solution has been found using a simple model, it is then usual to try to relax as many conditions as possible, usually one at a time, in order to attain solutions that hopefully better describe reality.

After the interpretation of the solution from our first model, the first assumption that we'll remove is the condition that the ball must pass through the center of the hoop, assumption (5). By doing so we will have to go back to step (2) in our modeling procedure and rederive a more accurate model. The cyclic process of refining and rederiving is standard in real-life modeling. We're showing you the process here explicitly so you'll see how it's done. In most articles you don't see this process, only the end result.

[4] By the end of the paper, we'll want Shaq to shoot at 52.37° not 48.18°.

4. Our Second Model. The Best Trajectory. Let's improve our model by removing the assumption that the best shot is one where the ball goes through the center of the hoop. Still keeping the same equations of motion, and sticking with our 7'1" basketball player, we now let both the initial velocity v_0 and the initial angle θ_0 vary independently and at the same time. Each pair (v_0, θ_0) will give the ball a trajectory that results in either a made basket or a missed basket.

4.1. Is There a Better Target Than the Center? Constructing a Feasible Region. Again, the model function is not differentiable, so simple calculus fails us here, and a numerical investigation is needed:

> The *feasible region* of trajectories is the set of all pairs (v_0, θ_0) that result in a successful free throw (using the assumptions on allowable trajectories).

The boundary of the feasible region can be found by writing a program that uses derivative-free algorithms for finding the minimum of a univariate function; see Figure 4.1. The blue boundary on the left corresponds to solutions of (3.13), the ball grazing the front of the rim as it goes in. The red boundary on the right corresponds to solutions of (3.14), the ball hitting the back of the rim. The green line labeled "center" corresponds to solutions of (3.9), the ball passing through the center of the hoop. The numerical methods that we used are usually discussed in a typical numerical methods class. These methods are interesting in and of themselves, especially with regard to the accuracy of the methods in general. We shall refrain from discussing them here, but you should look them up, or derive a suitable numerical method yourself; see the exercises. We do note that we do not require high accuracy for our particular model.

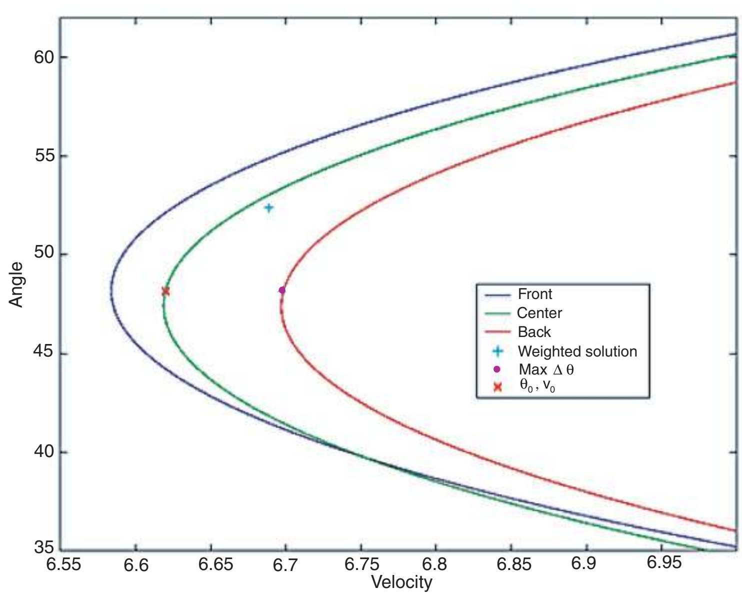

Fig. 4.1 *The feasible region for our model with locations of suggested optimal trajectories.*

Looking at the previous solution location in the feasible region, the red $\times$ on Figure 4.1, we see that requiring the ball to pass through the center of the hoop may not be best. There is much more room to overshoot than to undershoot. This trajectory would be good for a person who, when they missed, missed by consistently overshooting, but for someone who consistently undershoots when they miss this trajectory is definitely not best. What we have discovered here is that

> What we should consider best depends upon
> the individual and the way they shoot.

What pairs (v_0, θ_0) allow for the largest error in angle? If we let the ball go through the hoop at any position and maximize the allowed error in initial angle, we obtain the point marked with a dot on Figure 4.1. We see that this candidate optimal trajectory corresponds to a maximum allowable error in θ_0 but no allowable error in v_0; see also [2]. The combination that maximizes the amount of allowable error in the velocity is not as easy to find. It turns out that increasing the release angle also increases the allowable error in velocity. This continues until the release angle to hit the front and back of the rim for a given velocity are equal; i.e., this *rainbow shot* has no allowable error in the release angle; see Figure 4.2. Such a high angle, high velocity shot may not be physically possible to shoot. We may not be strong enough or we may hit the ceiling! The third point marked with a blue + on Figure 4.1 will be discussed later.

As we pointed out earlier, formula (3.9) is valid only in a certain range. Figure 4.1 shows that at a low release angle (i.e., an angle below the intersection point of the green and blue curves), the velocity given by (3.9) would lead to trajectories where the ball would pass through the rim before reaching the center of the hoop. This means

> If you have a shallow shot, don't aim for the center of the basket
> because you'll hit the front rim and probably miss.

Figures 4.2 and 4.3 show the percentage error allowed in the release angle and release velocity when shooting a free throw. To create Figures 4.2 and 4.3, we calculated the allowed error in velocity and angle for a grid of 400×400 points. The figures show the contours of the errors in percent. Figure 4.2, the graph for error in release angle, is nearly symmetric around the optimal release angle $\theta^*_{\text{center}} \approx 48.18°$, located in the deepest red region. So if we do not require the ball to go through the center of the hoop, any free throw that is made with a velocity between about 6.595 m/s and 6.7 m/s will lie in the deepest red region that allows for the maximum error in release angle. Figure 4.3 shows the maximum allowable percentage error in velocity. It is clear that this graph is not symmetric, but instead seems to favor larger release angles.

Summarizing, to answer the question of what is best, we really need to consider both the release angle and release velocity simultaneously. This is because

> 1. The trajectory that maximizes the allowed error in release angle is also the trajectory that allows no error in release velocity.
> 2. The trajectory that maximizes the allowed error in release velocity is also the trajectory that allows no error in release angle.
> 3. The best angle from our first model allows little room to undershoot, a problem a lot of players have. So the best shot for any given player is unlikely to pass through the center.

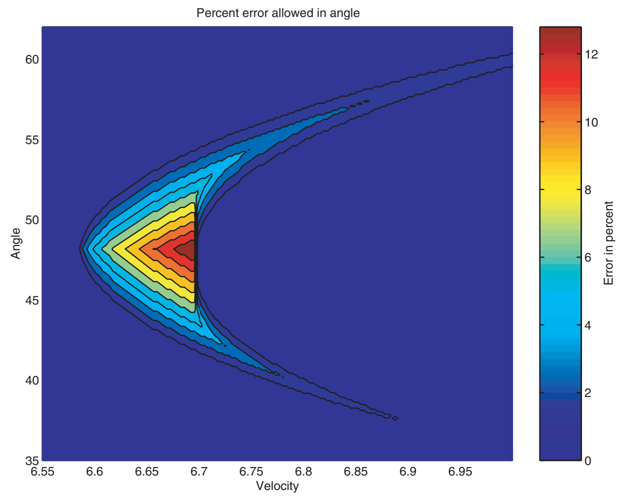

Fig. 4.2 *The percentage error in angle allowed.*

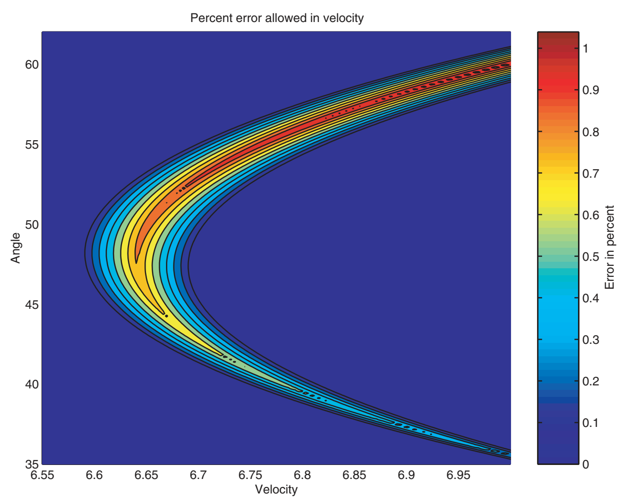

Fig. 4.3 *The percentage error in velocity allowed.*

So we were right to be unsatisfied with the optimal trajectory from the first model, especially if our player makes roughly symmetric errors in his initial velocity as well as in his initial angle when shooting. To analyze things a little more deeply (more refining), we used numerical methods to construct regions of percent error in both angle and velocity.

4.2. A Redefinition of the Problem. We have learned that we should not focus on just the release angle. So we need to redefine our problem slightly, changing "best angle" to "best trajectory." The new problem then becomes

Given a basketball player of a certain height,
what is the *best trajectory* for him to shoot a free throw?

Since the units of degrees and feet per second (or meters per second) are not directly relatable, the question of what we mean by "best trajectory" must also be clarified. One way to allow a comparison between the two different errors is by looking at the percentage error. That is, relative to the velocity and angle with which the player wants to throw the ball, how much error can he make percentagewise and still make the free throw? Figures 4.2 and 4.3 already show the error in this measure. From this it is obvious that the percent error in angle can be much larger than the percent error in velocity. Another conclusion we can therefore make is that

It is much more important to use the right velocity
as compared to the right angle.

How do we find the optimal solution when we have two different measures we want to minimize, and the two of them oppose each other? Problems of this kind are called multiobjective optimization problems, and there are many different ways to solve them.[5]

One way to solve the multiobjective problem is by fixing the angle that allows the largest error for many velocities and then maximizing the error in velocity. This results in the point marked with a dot in Figure 4.1. Note again that this worked only because we could decouple the two optimizations.

Another, more widely applicable method for solving multiobjective problems is to introduce weights for each objective, and optimize a weighted combination of objectives. Figure 4.4 shows an example of this. In this example we combine the two weighted objectives by taking the minimum of the percent error in angle plus five times the percent error in velocity. This means we have made the following definition:

The best trajectory is one that puts five times as much emphasis
on the error in velocity than on the error in angle.

The optimum for this function is marked in Figure 4.1 with a blue +. Note that this combined function is not differentiable, and therefore many conventional optimization methods will not work for this function.

Formulating the combined objective function for a multiobjective optimization requires great care, but also allows great flexibility. For example, if a player consistently shoots too short, a greater weight can be placed on a lower than necessary velocity.[6]

By minimizing the percent error in angle, and five times the percent error in velocity, we find an optimal release angle and velocity for Shaq of

$$\theta^*_{\text{percent}} \approx 52.37^\circ \text{ and } v^*_{\text{percent}} \approx 6.7\,\text{m/s}.$$

We see that both the initial angle and the initial velocity are important to control to a high degree of accuracy when shooting free throws. We stress here that the above result is for our definition of best. For example, if you are more worried about the consistency of your release angle than the consistency of your release velocity, then

[5]Multiobjective optimization is an exciting area of research where much work still needs to be done.

[6]More advanced methods for multiobjective optimization try to remove the requirement of defining a weighting function, and instead return a set of points that would be optimal for a certain combination. These points are called "Pareto optima." See [9] for a description of one of these methods.

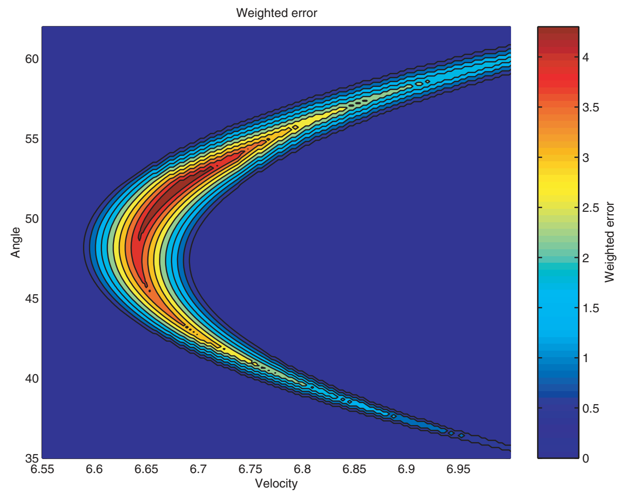

Fig. 4.4 *The weighted error.*

less weight should be placed on error in velocity as compared to error in angle. This would result in a slightly lower $\theta^*_{\text{percent}}$ (by a degree or two). Similarly someone who has serious trouble shooting with a consistent release velocity should release the ball at a slightly higher angle than the average shooter.

In fact it is clearly evident that the optimal trajectory should be decided player by player according to whether the player consistently has more trouble controlling his initial velocity or his initial angle. Even players who, when they miss, miss consistently by shooting too short (like Shaq) can be accounted for.[7]

5. Our Third Model: Including Air Resistance. So far our model has excluded air resistance. This is because free throws have relatively low velocities and short travel times. It is also hard to do. The effect of air resistance, though, still needs to be considered in any full treatment. Hamilton and Reinschmidt [7] qualitatively discussed the inclusion of air resistance and suggested that any derived optimal release angles would be lower by about 2°. This turns out to be an overestimate, as we'll soon see.

5.1. Air Resistance. Motion with air resistance is usually discussed extensively in an introductory differential equations class. For a particularly good module see [3].

As the basketball travels through the air as it heads towards the basket, a force (viscous drag) that opposes the direction of motion arises due to air resistance. This force is proportional to the velocity of the ball at each instant:

$$F_{\text{viscous}} = kv = 3\pi\mu D_b v, \tag{5.1}$$

where $\mu = 1.2165 * 10^{-5}\,\text{lb}/(\text{ft s})$ is the viscosity of air at 20°C and 1 atm, and $D_b = 0.8\,\text{ft}$ is the diameter of the basketball. This gives a value of $k = 9.172 * 10^{-5}\,\text{lb/s}$ for our model. We are making assumptions here about temperature, pressure, and humidity. If we find out that air resistance is very important, we will have to be more precise here in order to better match reality. Think Denver Nuggets.

[7] In our opinion Shaq misses because he shoots at too low a release angle and has erratic velocity control; see the last exercise.

5.2. Our Third Model: Revised Equations of Motion. We now go back to step (2) in our modeling procedure and rederive the equations of motion. Assuming the ball is released with initial velocity v_0 and release angle θ_0, we use Newton's second law, $F = ma$, to derive the equations of motion in both the horizontal,

$$m\frac{dv_{\mathrm{H}}}{dt} = -kv_{\mathrm{H}}, \tag{5.2}$$

and vertical,

$$m\frac{dv_{\mathrm{V}}}{dt} = mg - kv_{\mathrm{V}}, \tag{5.3}$$

directions, where kv_{H} and kv_{V} are viscous drag terms; see (5.1). These differential equations are both *separable*. This means we can rearrange (separate) the equations with one variable on the left and the other variable of the right and then integrate both sides to find the solution (see the appendix for details).

Upon separating and integrating we find the horizontal equation of motion,

$$x(t) = \frac{mv_0}{k}\cos(\theta_0)\left(1 - e^{-\frac{k}{m}t}\right). \tag{5.4}$$

Similarly the vertical equation of motion is given by

$$y(t) = \frac{mg}{k}t - \frac{m}{k}\left(v_0\sin(\theta_0) - \frac{mg}{k}\right)e^{-\frac{k}{m}t} + \frac{m}{k}\left(v_0\sin(\theta_0) - \frac{mg}{k}\right). \tag{5.5}$$

Allowing both the initial angle and initial velocity to vary at the same time, we numerically find the region in the angle-velocity plane for which a free throw is made and compare it to the same region using our model without air resistance; see Figure 5.1. Again the boundary on the left corresponds to the ball grazing the front of the rim as it goes in, and the boundary on the right corresponds to the ball hitting the back of the rim.

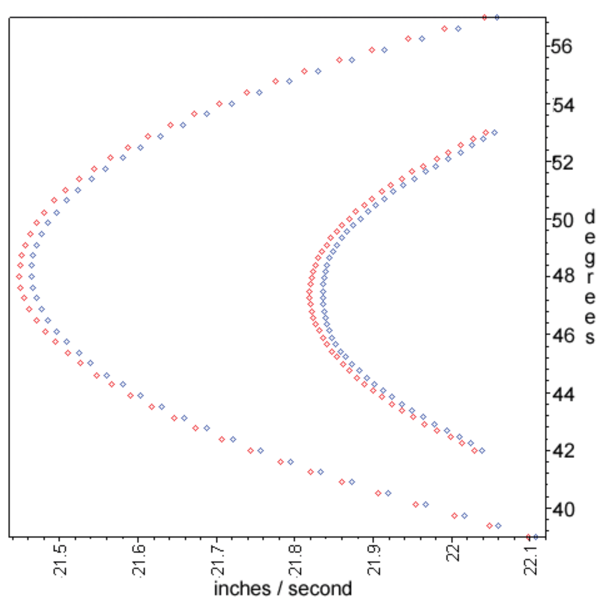

Fig. 5.1 *The feasible regions for models with (blue) and without (red) air resistance.*

5.3. Interpreting the Model. As we can clearly see from Figure 5.1, air resistance plays a very small part in the trajectory of a free throw. Specifically, including air resistance increases the optimal trajectory's initial velocity by approximately 0.0381 cm/s for our 7'1" player.

6. Everyone Else.

So now we know that if you are 7'1" tall, you should shoot your free throws at an angle of approximately 52.37° with a velocity of 6.7 m/s. Well, unless you happen to be 7'1" tall, this information is not much use to you. Fortunately, there is nothing special about our assumption that the shooter is of this height. Having worked out the optimal trajectory for our 7'1" player it is now fairly straightforward to apply the same techniques for a player of any height; see Table 6.1. To generate Table 6.1, we used the model without air resistance with a weighted objective function of the percent error in angle plus five times the percent error in velocity.

One thing to note here is that the amount of error in both release angle and release velocity increase monotonically with height. This means that

The taller you are, the better free throw shooter you should be.

Table 6.1 *The optimum trajectories for people of various heights.*

Height	Height	Release angle	Release velocity	Max error θ_0	Max error v_0
5'	1.52 m	56.64°	7.34 m/s	2.08°	0.0538 m/s
5'1"	1.55 m	56.47°	7.32 m/s	2.09°	0.0542 m/s
5'2"	1.57 m	56.31°	7.29 m/s	2.11°	0.0547 m/s
5'3"	1.60 m	56.14°	7.26 m/s	2.13°	0.0551 m/s
5'4"	1.63 m	55.97°	7.24 m/s	2.15°	0.0555 m/s
5'5"	1.65 m	55.80°	7.21 m/s	2.17°	0.0560 m/s
5'6"	1.68 m	55.63°	7.18 m/s	2.18°	0.0564 m/s
5'7"	1.70 m	55.45°	7.16 m/s	2.20°	0.0568 m/s
5'8"	1.73 m	55.28°	7.13 m/s	2.22°	0.0573 m/s
5'9"	1.75 m	55.11°	7.10 m/s	2.24°	0.0577 m/s
5'10"	1.78 m	54.94°	7.08 m/s	2.26°	0.0581 m/s
5'11"	1.80 m	54.77°	7.05 m/s	2.27°	0.0585 m/s
6'	1.83 m	54.60°	7.02 m/s	2.29°	0.0590 m/s
6'1"	1.85 m	54.43°	7.00 m/s	2.31°	0.0594 m/s
6'2"	1.88 m	54.25°	6.97 m/s	2.33°	0.0598 m/s
6'3"	1.91 m	54.08°	6.95 m/s	2.35°	0.0602 m/s
6'4"	1.93 m	53.91°	6.92 m/s	2.36°	0.0607 m/s
6'5"	1.96 m	53.74°	6.89 m/s	2.38°	0.0611 m/s
6'6"	1.98 m	53.57°	6.87 m/s	2.40°	0.0615 m/s
6'7"	2.01 m	53.40°	6.84 m/s	2.42°	0.0619 m/s
6'8"	2.03 m	53.22°	6.82 m/s	2.43°	0.0623 m/s
6'9"	2.06 m	53.05°	6.79 m/s	2.45°	0.0627 m/s
6'10"	2.08 m	52.88°	6.76 m/s	2.47°	0.0632 m/s
6'11"	2.11 m	52.71°	6.74 m/s	2.49°	0.0636 m/s
7'	2.13 m	52.54°	6.71 m/s	2.50°	0.0640 m/s
7'1"	2.16 m	52.37°	6.69 m/s	2.52°	0.0644 m/s
7'2"	2.18 m	52.20°	6.66 m/s	2.54°	0.0648 m/s
7'3"	2.21 m	52.02°	6.64 m/s	2.55°	0.0652 m/s

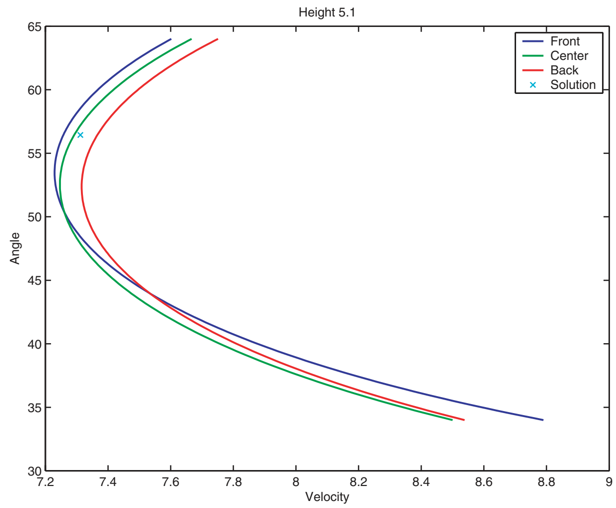

Fig. 6.1 *The feasible region and suggested optimal trajectories for a person* 5'1" *tall.*

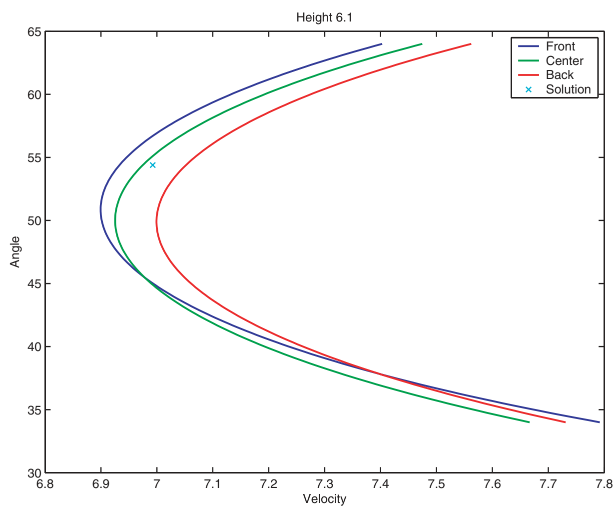

Fig. 6.2 *The feasible region and suggested optimal trajectories for a person* 6'1" *tall.*

Looking at Figures 6.1, 6.2, and 6.3 we see that

> The best shot does not pass directly through the center of the hoop. In fact, the shorter you are, the closer to the back of the hoop you should shoot.

We can see this clearly by examining Figure 6.4. This has interesting consequences for some coaches who use an insert during practice that reduces the aperture of the rim. This insert does seem to help players shoot closer to the center of the hoop but according to our analysis this is something we may not actually want to encourage. Also, by plotting all three together (see Figure 6.5), we clearly see the smaller feasible

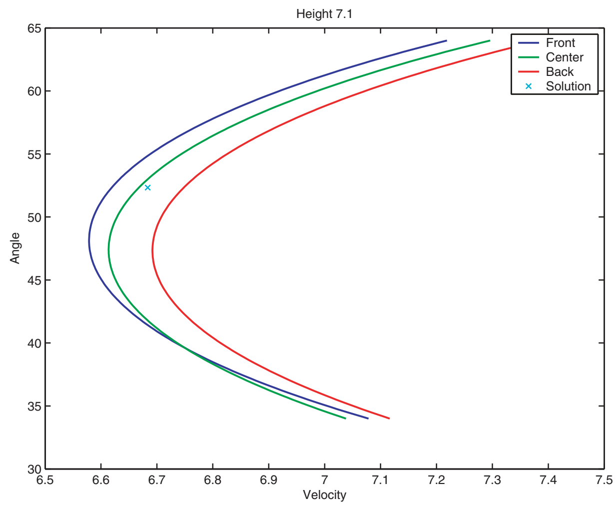

Fig. 6.3 *The feasible region and suggested optimal trajectories for a person* 7'1" *tall.*

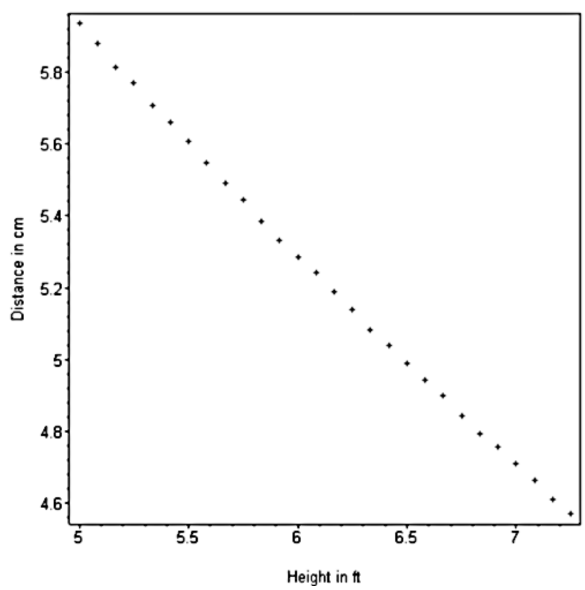

Fig. 6.4 *The optimal distance past the center of the hoop versus height of shooter.*

region available for shorter players, confirming our earlier observation and implying that Shaq really should do better.

7. Comparing Our Models with Real-World Data. A true test of a model's validity lies in its ability to predict real-life behavior. That is, when modeling you need to evaluate the model after you've created it. To see if the predictions made by our model match up to reality, we compared our predicted optimal release angles to observed shooting angles. There have been several biomechanical studies on free

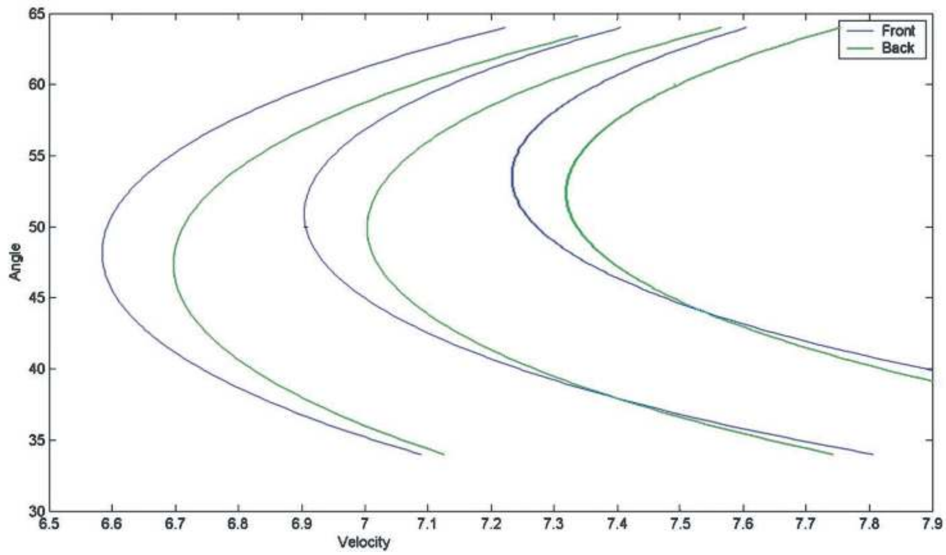

Fig. 6.5 *The feasible regions, from left to right, of players* 7'1", 6'1", *and* 5'1" *tall, respectively.*

throw shooter's release angles; see, e.g., [8, 15, 18, 4]. The four studies reported mean release angles of 52.9°, 52°, 50°, and 51.9°, respectively, which are toward the lower end of our model's optimal release angle range (52.0°–56.6°); see Table 6.1. This may be because the optimal angles at the high end of our range are for shorter people, starting at 5', so maybe the studies didn't include such players. Unfortunately, the position of release is not presented and the number of subjects in the studies was often small, so a reliable comparison of our theoretical calculations with real-world data is currently not possible.

Our model gives a maximum allowable error in initial release angle and initial velocity. Can basketball players shoot consistently and accurately, as the model requires? Do players have greater variabilities in release angles or in release velocities? As mentioned before, the answers to these questions are important for the validity of our definition of best trajectory. The answers to both questions are currently unknown and open to further research.

Our model also predicts that taller players should have a much easier time shooting free throws than shorter players. In reality, the opposite seems to be true,[8] with the shorter guards having the best free throw percentages. That is, in this case our model does not match reality. Does this mean our model is wrong? Not necessarily, it just means that some taller players either are shooting at the wrong angles, have trouble shooting consistently, or both. Why this may be true is not clear and is also open to further research.

8. Summary. We have derived a model for basketball free throw shooting in which we have allowed both the release angle and the release velocity to vary independently in an attempt to answer the question, "What is the best angle to shoot a free throw?" From examination of this model, we have seen that defining what we

[8]With some notable exceptions. Take, for example, Rik Smits (7'4"), who shot a career 77.3% from the line, or Dirk Nowitzki (7'0"), who shoots 85.3% from the line.

mean by best really depends upon the shooter—not only how tall they are, but more importantly how well they can consistently control both the release angle and the release velocity of the ball when they shoot. In general, though, we have reached the following conclusions:

1. The taller you are, the better free throw shooter you should be. This is because taller players have more room to make errors in both release angle and release velocity and still have the ball go in the basket. Tall players who are poor free throw shooters either are shooting at the wrong angle or more likely are inconsistent in their release angle, release velocity, or both.
2. The shorter you are, the larger the release angle should be. This makes sense physically, as shorter people have more vertical distance to cover when shooting. It is good to see that our model confirms this.
3. The best shot does not go through the center of the basket. The trajectories that allow for maximum error pass somewhere between the center of the basket and the back rim. The shorter you are, the closer to the back of the rim you should aim.
4. It is much more important to use the right velocity as compared to the right angle. Players who miss mainly due to their lack of consistency in their release velocities can improve their chances by shooting at a higher angle than someone who can consistently control the release velocity.

For the rest of the paper, we shall present possible avenues for future work, good for class projects of all difficulty levels. Most are accessible to calculus students, but some may need more advanced knowledge from a differential equations or numerical methods class. We have also included a list of exercises for your enjoyment (see section 10). Again the difficulty of these exercises ranges from the comparatively easy to the quite difficult.

9. If You Are Really Serious about Improving Your Free Throw Shooting. Let's recap what we've done so far. We have worked out the best[9] angle to shoot a free throw under the following (remaining) assumptions (see Table 6.1):

1. Allow only "nearly nothing but net" shots.
2. Ignore any spin the ball may have.
3. Assume there is no transverse error in the ball's trajectory.

To further improve our model, these three remaining assumptions should also be removed from our model. We shall not do so here, but rather leave them as open problems, some of which may be difficult. Instead, let's discuss the probable changes in the optimal angle if the assumptions were to be removed.

9.1. Allowing the Ball to Bounce and Spin. If we removed the assumption of "nearly nothing but net" shots, we would then allow the ball to bounce not only off the rim but also off the backboard before it goes in. All players are trained to shoot free throws straight in and not off the backboard, but there is the interesting possibility that the best shot is one that goes in off the backboard. Unfortunately,

[9]By best angle, we mean the angle from the weighted solution used in our model. This definition of best is by no means unique, and others could be considered. We considered three others ourselves before we decided on this one.

the model would become significantly more complicated if this assumption were to be removed.

The way the ball bounces depends on something called the coefficient of restitution of the ball. This is a measure of how "bouncy" the ball is. An official ball dropped from a height of 1.8m (measured from the bottom of the ball) must return to a height of 1.2–1.4m (measured from the top of the ball); see [13]. This is rather a big range of "bounciness," which would need careful consideration in any new model.

Other things to consider when letting the ball bounce are the softness (or stiffness) of the rim and the spin of the ball. In particular, the optimal trajectory would now be one with optimal release angle, release velocity, and release spin, and in reality, a player does have to concentrate on all three of these variables when shooting a free throw. With all these considerations in mind, it is unlikely that the best shot is one where you aim to bounce the ball off something before having it go in. Even so, some serious researchers, based upon qualitative investigations, have suggested the optimal trajectory is a backboard shot; see, e.g., [16].

What does need to be included in any new model (at least as a first step) are trajectories for which the ball hits the backboard and then either (a) goes directly down and in or (b) goes in after hitting the front of the rim. For these trajectories spin is important. Shooting the ball with backspin should broaden the allowed error range $e(\theta_0)$, though this really needs to be investigated quantitatively.

9.2. Including a Sideways Error. Assuming that there is no transverse error in the trajectory seems to us to make good sense. Shooting straight is the first thing you should try to get right when shooting free throws, especially if you want to be a professional. It would be interesting to find out exactly how allowing transverse error would affect the optimal release angle. With the inclusion of transverse error, to avoid contact with the rim, the distance between any part of the rim and the center of the ball must remain greater than the radius of the ball throughout its trajectory. Mathematically this problem becomes very interesting and difficult, but the same ideas apply. Additionally we would also be faced with choosing a new definition of best. We would have to deal with two initial angles, vertical and horizontal, as well as the initial velocity and possibly spin. Challenging, but definitely an area where future research needs to be done.

9.3. The Psychology of Free Throw Shooting. We conclude by discussing some nonmathematical considerations for improving basketball free throw shooting.[10]

Sport psychology has become a serious business. Scores of scientists are out there trying to invent techniques that will enhance sport performance. Popular techniques that seem to work include mental practice, self-affirmation, stress management, and biofeedback. For example, take mental practice, known to sport psychologists as visuo-motor behavior rehearsal. Recent studies [12, 11] claim to show that mental practice enhances free throw shooting performance.

Another technique that seems to help free throw shooting is having a preshot routine [5]. A preshot routine is a set pattern of actions and thoughts performed before every free throw. Indeed, many professional basketball players have preshot routines, some of which are unusual [1]. In short, free throw shooting can be as much a mental task as a physical one. And I thought I was just a bad shot!

[10] Are there really areas where mathematics doesn't apply? Of course not; the mathematics used to do research in the areas discussed in this section we call statistics.

10. Exercises.
Here are some problems the reader may wish to consider:

1. In our first simplifying assumption we allow only nearly nothing but net shots, whereby the ball is allowed to hit the back rim, but only if the center of the ball is at or below the basket height when it does so. How would you redefine "nearly nothing but net" to allow for all trajectories that hit the back rim and then go in (without hitting the backboard)? Using this new definition of "nearly nothing but net," would you expect the optimal angle to be higher, lower, or the same as before? Why?
2. Discuss the effect of spin on the definition of "nearly nothing but net." See exercise 1.
3. Use trigonometry to explain Figure 3.3.
4. Use (3.7) and (3.8) to derive (3.9).
5. Derive (3.10). From your derivation, explain why the ball is moving down, and not up, as it passes through the hoop.
6. Following the beginning of section 3.5, use a computer algebra system to plot $e(\theta_0)$ for someone of your own height. Compare your graph with Figure 3.5.
7. We noted in section 3.5 that for certain (v_0, θ_0) pairs that result in the basketball passing through the center of the hoop, changing the release angle θ_0 (but keeping v_0 fixed) will never cause the ball to hit the back rim. Locate and describe the region where this happens in Figure 4.1. Physically what is happening here?
8. Use numerical methods to construct your own feasible regions for Shaq and for yourself, i.e., for someone of your height. Use numerical integration or some other technique to approximate the areas of the two regions. Are they different? What can you conclude?
9. In section 4.2, we claim that it is "obvious" that it is more important to use the right velocity as compared to the right angle. Give *details* explaining why our claim makes sense for our model?
10. Derive (5.5) by using the technique of separation of variables to solve (5.3). Hint: Look in the appendix.
11. In Figures 6.1–6.3, the feasible regions for players 5'1", 6'1", and 7'1" tall, the left and right boundaries intersect. Estimate the location where this occurs for Shaq. What does this correspond to physically?
12. At the end of section 7, we mentioned that some taller players have poor free throw shooting percentages. Make several conjectures about why this may be true. How would you test your conjectures?
13. From a purely mathematical point of view it is interesting to imagine the ball being a shape other than round. Spin now becomes important. Why?
14. Suppose you have been hired by Shaq to help him with his free throws. Discuss in detail how you would accomplish this.

Appendix. Separation of Variables.
Separation of variables is a mathematical technique used to solve first order separable differential equations. A first order differential equation can be written in the form

$$\frac{dy}{dx} = f(x, y). \tag{A.1}$$

If the function $f(x, y)$ can be written as $f(x, y) = g(x)h(y)$, the equation is called *separable* and can be solved using separation of variables. If your equation is not separable, you have to use more powerful techniques to solve it. These techniques are discussed in a standard differential equations course. Assuming that our equation is separable, we solve it as follows:

1. *Separate.* Rewrite the equation as

$$\frac{dy}{dx} = g(x)h(y) \tag{A.2}$$

and separate (move all the x's on one side and all the y's on the other side), to obtain

$$\frac{1}{h(y)}dy = g(x)dx. \tag{A.3}$$

2. *Integrate.* Now that we have just one variable on each side, we can integrate (antidifferentiate) both sides to obtain a solution:

$$\int \frac{1}{h(y)}dy = \int g(x)dx. \tag{A.4}$$

A.1. Example. We shall use separation of variables to solve (5.2), the first differential equation of the air resistance section. We'll leave the solution of the other differential equation as an exercise. Recall the differential equation for the horizontal motion:

$$m\frac{dv_{\mathrm{h}}}{dt} = -kv_{\mathrm{h}}, \tag{A.5}$$

where m and k are constants. Separating, we obtain

$$\frac{1}{v_{\mathrm{h}}}dv_{\mathrm{h}} = -\frac{k}{m}dt. \tag{A.6}$$

Integrating both sides,

$$\int \frac{1}{v_{\mathrm{h}}}dv_{\mathrm{h}} = -\frac{k}{m}\int 1dt, \tag{A.7}$$

we obtain

$$\ln v_{\mathrm{h}} = -\frac{k}{m}t + c, \tag{A.8}$$

where c is the constant of integration. Exponentiating both side to remove the logarithm, and using the initial condition $v_{\mathrm{h}}(0) = v_0 \cos(\theta_0)$ to find c, we obtain

$$v_{\mathrm{h}} = \frac{dx}{dt} = v_0 \cos(\theta_0)\, e^{-\frac{k}{m}t}. \tag{A.9}$$

This equation, in x and t, is also separable. Separating, we obtain

$$dx = v_0 \cos(\theta_0)\, e^{-\frac{k}{m}t}dt. \tag{A.10}$$

Integrating both sides,

$$\int 1dx = v_0 \cos(\theta_0) \int e^{-\frac{k}{m}t}dt, \tag{A.11}$$

we obtain

$$x = -\frac{mv_0}{k}\cos(\theta_0)\, e^{-\frac{k}{m}t} + c, \tag{A.12}$$

where c is again a constant of integration. Using the initial condition $x(0) = 0$ to find c, we obtain (cf. (5.4))

$$x(t) = \frac{mv_0}{k}\cos(\theta_0)\left(1 - e^{-\frac{k}{m}t}\right). \tag{A.13}$$

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