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Modeling of a folded plate

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Abstract. It is shown that the solution of a three-dimensional linear elasticity problem in a thin folded plate converges strongly in H^1 to a solution of a two-dimensional model as the thickness goes to 0. This model consists of two plate equations coupled through their common edge.

0 Introduction

So far, problems of junctions between 3d, 2d and 1d-elastic structures do not seem to have been investigated from the mathematical viewpoint in spite of their practical importance. The mathematical theories for 3d-elastic bodies, for plates and for rods are each fairly well developed (see for example Wang and Truesdell (1975), Marsden and Hughes (1983) or Ciarlet (1988 a) for 3d-elasticity, Ciarlet and Destuynder (1979), Ciarlet (1980, 1987), Destuynder (1986) for plates, Aganovič and Tutek (1986), Bermudez and Viaño (1984), Rigolot (1976), Cimetière, Geymonat, Le Dret, Raoult and Tutek (1988) for rods and all the references therein) but the question of knowing how these different structures can be attached to one another seems to be mathematically quite open (see Colson (1984) to get an idea of the complexity of the full problem from the engineer's point of view). We present here an approach, based on some of Ciarlet and Destuynder's ideas, for the study of a "folded" plate, i.e. two plates of thickness ε attached along one of their edges at a right angle, (Fig. 1). The bodies are assumed to be linearly elastic. This assumption allows us to derive a limit 2d-model as ε goes to 0 and, at the same time, to obtain strong convergence results for the displacements (also as a byproduct for the stresses, although we do not emphasize this aspect which easily follows from our analysis). However, we do not use the asymptotic expansion method, unlike Ciarlet and Destuynder (1979), but rather we pass to the limit in the variational equations and then identify this limit. The crucial idea for treating this junction problem, also used by Ciarlet, Le Dret & Nzengwa (1987) for a 3d-2d junction and Ciarlet (1988 b) for a 2d-1d junction, seems to be both of interest and of wide applicability. This idea consists in scaling the different parts of the bodies under consideration independently of each other – each in the same way as is usually done in plate and rod theories – but counting the junction region twice, once in each separate scaled part. The scaled displacements are defined on two separate domains and contain the information about the junction twice. The relations expressing that they actually correspond to the same global displacement of the whole structure yield the conditions that the limit displacements must satisfy.

More specifically, we consider a family of homogeneous isotropic linearly elastic 3d-bodies, as depicted in Fig. 1, consisting of two plates of thickness ε perpendicular to each other. The bodies are made of elastic materials with Lamé moduli $\varepsilon^{-3}(\mu, \lambda)$, i.e., which are more and more rigid as ε goes to 0. Appropriate dead loads are assumed to act on the bodies so that finite flexural displacements are expected in the limit. The bodies are assumed to be clamped on parts of the edges of both plates. Then we perform a scaling as indicated above and consider a new scaled unknown, which consists of the pair of scaled (defined as usual) displacements on each plate. Due to the clamping condition, an H^1-bound independent of ε is derived for this unknown. After extraction of a subsequence, the scaled displacements therefore converge weakly to some limit displacement. It is first shown that this displacement is of Kirchhoff-Love type in each plate. The continuity relations in the junction region are then used to derive boundary conditions for these Kirchhoff-Love displacements on the common edge. It is thus proved that the flexural displacements are zero on the common edge, that the angle between the deformed plates is always $\pi/2$ (this is a condition on normal derivatives of the flexural displacements) and that the membrane displacements in the direction of the edge are transmitted. No condition is obtained for the membrane displacements perpendicular to the edge. The conditions above define a closed subspace in H^1 of Kirchhoff-Love displacements and we show then that the limit displacement satisfies well-posed variational equations on this space. This is done by approximating arbitrary test-functions of this space by displacements that satisfy the continuity relations at the junction for $\varepsilon > 0$. The construction of these approximations is rather tricky and involves the edge conditions above in a crucial way. The approximate test-functions are then used in the original variational equations for $\varepsilon > 0$, all the singular terms can be controlled and the limit problem is obtained by computing the limit of the equations as $\varepsilon \to 0$. This limit variational problem can be interpreted in the sense of distributions as a system of coupled plate equations for the flexural displacements – the coupling being indicated above – and as a system of coupled membrane equations with the edge condition above and free perpendicular edge displacements. The solution of the limit problem exists and is unique and therefore the preceding analysis shows that the whole family of scaled displacements converges weakly to this solution. Once this fact is established it is not hard to see that the convergence is actually strong, since we were essentially dealing with minimizers of uniformly strictly convex functionals.

Finally, we list a few extensions and limitations of the present method. Let us emphasize that, in our opinion, the independent scaling idea might prove to be quite useful for treating a variety of junction problems.

Notation. Let Ω be an open subset of $\mathbb{R}^n$ and let m be a positive integer. We denote by
$\mathscr{D}(\Omega)$ the space of C^∞-functions with compact support in Ω,
$L^2(\Omega)$ the space of (classes) of measurable square-integrable real functions on Ω,
$H^m(\Omega)$ the space of functions of $L^2(\Omega)$ whose distributional derivatives up to the order m belong to $L^2(\Omega)$,
$H^1_0(\Omega)$ the closure of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$ and $H^{-1}(\Omega)$ its topological dual.

More generally $H^m_\Gamma(\Omega)$ is the space of H^m-functions whose traces vanish on a part Γ of the boundary of Ω.

Finally, it X is a Hilbert space, $L^2(0,1;X)$ is the space of measurable functions from $]0,1[$ into X such that $\int_0^1 \|u(t)\|_X^2 \mathrm{d}t < +\infty$, and $H^m(0,1;X)$ is the space of functions of $L^2(0,1;X)$ such that all their distributional derivatives with respect to t up to the order m belong $L^2(0,1;X)$.

We refer to Adams (1975), Lions and Magenes (1968a, 1968b) for the general properties of these spaces.

1 The three-dimensional problem

1.1 We consider a family of three-dimensional isotropic homogeneous linearly elastic bodies whose reference configurations are the sets Ω_ε defined for $\varepsilon > 0$ as:

$$\Omega_\varepsilon := \Omega'_\varepsilon \cup \Omega''_\varepsilon,$$

where: $\Omega'_\varepsilon := \{x \in \mathbb{R}^3, 0 < x_1, x_3 < 1, 0 < x_2 < \varepsilon\}$, $\quad \Omega''_\varepsilon := \{x \in \mathbb{R}^3, 0 < x_2, x_3 < 1, 0 < x_1 < \varepsilon\}$.

Let: $\Gamma'_\varepsilon := \{x \in \mathbb{R}^3, x_1 = 1, 0 < x_3 < 1, 0 < x_2 < \varepsilon\}$, $\quad \Gamma''_\varepsilon := \{x \in \mathbb{R}^3, x_2 = 1, 0 < x_3 < 1, 0 < x_1 < \varepsilon\}$.

See Fig. 1.
We introduce the following index convention: Latin indices range in $\{1,2,3\}$, primed Greek indices in $\{1,3\}$ and double-primed Greek indices in $\{2,3\}$.
We assume that the Lamé moduli of the bodies satisfy:

$$(\mu_\varepsilon, \lambda_\varepsilon) = \varepsilon^{-3}(\mu, \lambda) \tag{1.1}$$

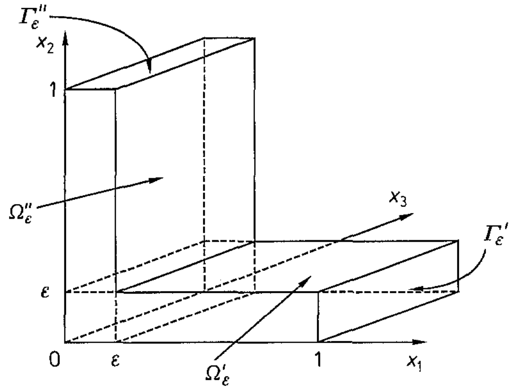


Fig. 1. Elastic 3d-bodies

for some strictly positive constants μ and λ. Since we are in the framework of linearized elasticity, the choice of such exponents in plate theory is merely a question of taste. It is always possible to multiply all data by a "hanging factor" ε^r without modifying the resulting limit equations. The ε^{-3} factor is however the appropriate one for a nonlinear plate to sustain its own weight if its density does not depend on ε, see Ciarlet (1980, 1987). In the present linear case, we can even consider transversal body forces of order ε^{-1} and surface tractions of order 1, and get finite flexural displacements in the limit as $\varepsilon \to 0$. Actually, for simplicity we will consider only body forces. Extension of our results to more general loading is straightforward. The bodies are thus subjected to loads f_i^ε in $L^2(\Omega_\varepsilon)^3$ of the form:

$$f_{\alpha'}^\varepsilon(x_1,x_2,x_3)=\varepsilon^{-2}f_{\alpha'}'\left(x_1,\frac{x_2}{\varepsilon},x_3\right),\quad f_2^\varepsilon(x_1,x_2,x_3)=\varepsilon^{-1}f_2'\left(x_1,\frac{x_2}{\varepsilon},x_3\right),\tag{1.2'}$$

on Ω_ε' and

$$f_{\alpha''}^\varepsilon(x_1,x_2,x_3)=\varepsilon^{-2}f_{\alpha''}''\left(\frac{x_1}{\varepsilon},x_2,x_3\right),\quad f_1^\varepsilon(x_1,x_2,x_3)=\varepsilon^{-1}f_1''\left(\frac{x_1}{\varepsilon},x_2,x_3\right),\tag{1.2''}$$

on $\Omega_\varepsilon''\backslash\Omega_\varepsilon'$, for some functions f_i' and f_i'' in $L^2(\mathbb{R}^3)$. A few words of comments are in order here. We are considering a family of materials that become more and more rigid as $\varepsilon \to 0$. It is thus not too unreasonable to assume that their density goes to infinity at the same time – although this is not so clear when speaking of modern materials. This accounts for the ε^{-1} factor for transversal forces which can be interpreted as the weight of the materials (assume that the bodies are tilted about the vertical). There is no such physical interpretation for the factor ε^{-2} for the in-plane loads. This choice is mathematically coherent with the choice of transversal forces – a plate is indeed much more rigid in the directions of its own plane than in the transversal direction – and leads to $O(\varepsilon)$ membrane displacements. Smaller orders, like the weight in the case above, would lead to smaller order membrane displacements, i.e. in our setting this would be equivalent to having zero in-plane loads. The explicit form (1.2′)–(1.2″) is assumed for convenience only because it yields constant forces after rescaling. What we actually need to complete the following proofs is that the forces be weakly convergent, e.g. in L^2, after rescaling. Therefore, more realistic loads could be assumed at no extra cost. Note that (1.2″) only holds on $\Omega_\varepsilon''\backslash\Omega_\varepsilon'$ to avoid inconsistencies in the junction. This is only due to the specific form (1.2′)–(1.2″), see remark above.

As regards the boundary conditions, we assume that the bodies are clamped on $\Gamma_\varepsilon'\cup\Gamma_\varepsilon''$ and traction-free on the rest of the boundary. Then, the equilibrium equations are naturally expressed in variational form as:

Find u^ε in V^ε such that:

$$\int_{\Omega_\varepsilon} A^\varepsilon e(u^\varepsilon):e(v)-\int_{\Omega_\varepsilon} f^\varepsilon\cdot v=0,\tag{1.3}$$

for all v in V^ε, where:

$$V^\varepsilon:=\{v\in H^1(\Omega_\varepsilon)^3,\quad v=0\quad\text{on}\quad \Gamma_\varepsilon'\cup\Gamma_\varepsilon''\},\tag{1.4}$$

and A^ε is the elasticity tensor with Lamé moduli $(\mu_\varepsilon, \lambda_\varepsilon)$, i.e.:

$$(A^\varepsilon \tau)_{ij} = 2\mu_\varepsilon \tau_{ij} + \lambda_\varepsilon \tau_{ll} \delta_{ij} \quad \text{for all symmetric } \tau. \tag{1.5}$$

An immediate consequence of formula (1.1) and of Korn's inequality is:

Proposition 1.1. For all $\varepsilon > 0$, there exists a unique u^ε solution to problem (1.3)–(1.4).

2 The scaled problems

2.1 We use essentially the same idea as in Ciarlet, Le Dret and Nzengwa (1987) which is to scale the different parts of the bodies independently of each other but counting the junction between these parts twice. To achieve this, let us introduce two different copies of $\mathbb{R}^3$, $(\mathbb{R}^3)'$ and $(\mathbb{R}^3)''$ (we will very soon forget about all that and identify these two copies to everybody's ordinary $\mathbb{R}^3$) and set:

$$\Omega' := \{x \in (\mathbb{R}^3)', 0 < x_i < 1\}, \qquad \Omega'' := \{x \in (\mathbb{R}^3)'', 0 < x_i < 1\}.$$

Let us also define:

$$\omega' := \bar{\Omega}' \cap \{x_2 = 0\}, \quad \Gamma' := \bar{\Omega}' \cap \{x_1 = 1\}, \quad \omega'' := \bar{\Omega}'' \cap \{x_1 = 0\}, \quad \Gamma'' := \bar{\Omega}'' \cap \{x_2 = 1\},$$

$$J'_\varepsilon := \Omega' \cap \{0 < x_1 < \varepsilon\}, \qquad J''_\varepsilon := \Omega'' \cap \{0 < x_2 < \varepsilon\}.$$

We introduce the scaling mapping:

$$\phi^\varepsilon : \Omega' \cup \Omega'' \to \Omega_\varepsilon,$$

$$x \mapsto \begin{cases} (x_1, \varepsilon x_2, x_3) & \text{if } x \in \Omega', \\ (\varepsilon x_1, x_2, x_3) & \text{if } x \in \Omega''. \end{cases} \tag{2.1}$$

Now, the junction between the two plates in Ω_ε is actually counted twice by ϕ^ε, once in Ω' by J'_ε and another time in Ω'' by J''_ε (Fig. 2).
Let: $V := H^1_{\Gamma'}(\Omega')^3 \times H^1_{\Gamma''}(\Omega'')^3$

(this will take the clamping condition into account). Next we define an operator:

$$\begin{aligned} &\Theta^\varepsilon : V^\varepsilon \to V \\ &\quad u \mapsto ((\varepsilon^{-1} u_1, u_2, \varepsilon^{-1} u_3) \circ \phi^\varepsilon, (u_1, \varepsilon^{-1} u_2, \varepsilon^{-1} u_3) \circ \phi^\varepsilon) \end{aligned} \tag{2.2}$$

The operator Θ^ε is not onto. Its range $\Theta^\varepsilon V^\varepsilon$ is a closed subspace of V consisting of pairs (v', v'') satisfying:

$$\begin{cases} \varepsilon v'_1(\varepsilon x_1, x_2, x_3) = v''_1(x_1, \varepsilon x_2, x_3) \\ v'_2(\varepsilon x_1, x_2, x_3) = \varepsilon v''_2(x_1, \varepsilon x_2, x_3) \\ v'_3(\varepsilon x_1, x_2, x_3) = v''_3(x_1, \varepsilon x_2, x_3) \end{cases} \tag{2.3}$$

for almost all $(x_1, x_2, x_3) \in]0, 1[^3$. We set

$$u(\varepsilon) = \Theta^\varepsilon u^\varepsilon \tag{2.4}$$

to be the new scaled unknown whose behavior as $\varepsilon \to 0$ we want to study. Let us introduce the quadratic forms:

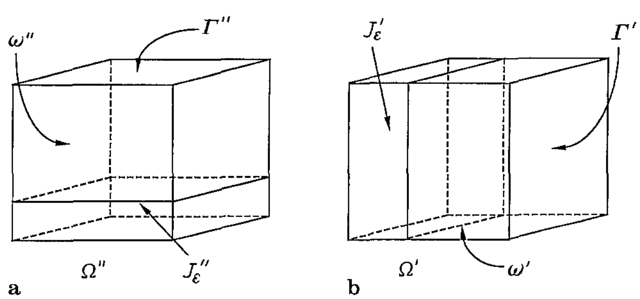


Fig. 2 a and b

$$
\begin{aligned}
B'_\varepsilon(u,v) := {} & 2\mu e_{\alpha'\beta'}(u)\, e_{\alpha'\beta'}(v) + \lambda e_{\alpha'\alpha'}(u)\, e_{\beta'\beta'}(v) \\
& + \varepsilon^{-2}\left(4\mu e_{\alpha'2}(u) e_{\alpha'2}(v) + \lambda\left(e_{\alpha'\alpha'}(u) e_{22}(v) + e_{\alpha'\alpha'}(v) e_{22}(u)\right)\right) \\
& + \varepsilon^{-4}(2\mu+\lambda)\, e_{22}(u) e_{22}(v),
\end{aligned} \tag{2.5}
$$

and let $B''_\varepsilon(u,v)$ denote its double-primed counterpart. Then replacing formulas (1.2) and (2.4) into (1.3) and performing the changes of variables in the integrals, we obtain the following variational equations for $u(\varepsilon)$:

$$
\int_{\Omega'\setminus J'_\varepsilon} B'_\varepsilon(u(\varepsilon),v) + \frac{1}{2}\int_{J'_\varepsilon} (B'_\varepsilon(u(\varepsilon),v) + \int_{\Omega''\setminus J''_\varepsilon} B''_\varepsilon(u(\varepsilon),v) + \frac{1}{2}\int_{J''_\varepsilon} B''_\varepsilon(u(\varepsilon),v) - \int_{\Omega'} f\cdot v - \int_{\Omega''\setminus J''_\varepsilon} f\cdot v = 0 \tag{2.6}
$$

for all v in $\Theta^\varepsilon V^\varepsilon$.

Note the crucial trick of splitting the elastic energy in the junction in two, so as to make clear that the functional defined on V by formula (2.6) is coercive.

3 The limit problem as $\varepsilon \to 0$

3.1 Let us derive an estimate for $u(\varepsilon)$. Let $\|v\|_V^2 = \|\nabla v'\|^2_{L^2(\Omega')^9} + \|\nabla v''\|^2_{L^2(\Omega'')^9}$ for all $v = (v', v'')$ in V. As in Ciarlet, Le Dret and Nzengwa (1987) we set:

$$
\varkappa_{\alpha'\beta'}(\varepsilon) := e_{\alpha'\beta'}(u(\varepsilon)), \qquad \varkappa_{\alpha'2}(\varepsilon) := \frac{1}{\varepsilon} e_{\alpha'2}(u(\varepsilon)), \qquad \varkappa_{22}(\varepsilon) := \frac{1}{\varepsilon^2} e_{22}(u(\varepsilon)) \tag{3.1}
$$

on Ω', and we define analogous quantities on Ω''. Then we have:

Proposition 3.1. There exists a constant C independent of ε such that

$$
\|u(\varepsilon)\|_V \le C. \tag{3.2}
$$

Proof: Let $v = u(\varepsilon)$ in the variational Eqs. (2.6). This yields, by the positivity of the elasticity tensor:

$$
\begin{aligned}
\mu\|\varkappa(\varepsilon)\|^2_{L^2(\Omega'\cup\Omega'')^9} &\le \frac{1}{2}\int_{\Omega'} B'_\varepsilon(u(\varepsilon), u(\varepsilon)) + \frac{1}{2}\int_{\Omega''} B''_\varepsilon(u(\varepsilon), u(\varepsilon)) \\
&\le \int_{\Omega'\setminus J'_\varepsilon} B'_\varepsilon(u(\varepsilon), u(\varepsilon), + \frac{1}{2}\int_{J'_\varepsilon} B'_\varepsilon(u(\varepsilon), u(\varepsilon)) + \int_{\Omega''\setminus J''_\varepsilon} B''_\varepsilon(u(\varepsilon), u(\varepsilon)) + \frac{1}{2}\int_{J''_\varepsilon} B''_\varepsilon(u(\varepsilon), u(\varepsilon)) \\
&= \int_{\Omega'} f\cdot u(\varepsilon) + \int_{\Omega''\setminus J''_\varepsilon} f\cdot u(\varepsilon) \\
&\le c_1\|u(\varepsilon)\|^2_{L^2(\Omega'\cup\Omega'')^3}
\end{aligned} \tag{3.3}
$$

But, for $\varepsilon \le 1$ and by Korn's inequality (recall that clamping holds on Γ' and Γ''),

$$
\|\varkappa(\varepsilon)\|^2_{L^2(\Omega'\cup\Omega'')^9} \ge \|e(u(\varepsilon))\|^2_{L^2(\Omega'\cup\Omega'')^9} \ge c_2\|u(\varepsilon)\|_V^2, \tag{3.4}
$$

which implies, together with inequality (3.3), the desired result. □

We can therefore extract a subsequence $u(\varepsilon_n)$ such that:

Corollary 3.1. $u(\varepsilon_n) \rightharpoonup u(0)$ weakly in V as $\varepsilon_n \to 0$.

As the limit $u(0)$ will turn out to be unique, the whole family $u(\varepsilon)$ will actually converge weakly to $u(0)$. We will thus denote the subsequence ε_n simply as ε for the sake of brevity.

3.2 Let us now proceed to identify the limit $u(0)$. To begin with, we state the:

Proposition 3.2. The limit displacement $u(0)$ is of Kirchhoff-Love type in Ω' and Ω'', i.e. there exist six functions $\zeta'_{\alpha'} \in H^1(\omega')$, $\zeta'_2 \in H^2(\omega')$, $\zeta''_{\alpha''} \in H^1(\omega'')$, $\zeta''_1 \in H^2(\omega'')$ such that:

$$
u(0)(x) = \left(\zeta'_1(x_1,x_3) - \left(x_2 - \frac{1}{2}\right)\partial_1\zeta'_2(x_1,x_3), \zeta'_2(x_1,x_3), \zeta'_3(x_1,x_3) - \left(x_2 - \frac{1}{2}\right)\partial_3\zeta'_2(x_1,x_3)\right) \tag{3.5}
$$

in Ω' and:

$$u(0)(x) = \left(\zeta_1''(x_2,x_3), \zeta_2''(x_2,x_3) - \left(x_1 - \frac{1}{2}\right)\partial_2\zeta_1''(x_2,x_3), \zeta_3''(x_2,x_3) - \left(x_1 - \frac{1}{2}\right)\partial_3\zeta_1''(x_2,x_3)\right) \quad (3.6)$$

in Ω''. Moreover

$$\zeta_i'(1,x_3) = \partial_1\zeta_2'(1,x_3) = \zeta_i''(1,x_3) = \partial_2\zeta_1''(1,x_3) = 0\,. \quad (3.7)$$

Proof: As $u(\varepsilon)$ is bounded in V, we deduce from inequality (3.3) that $\varkappa(\varepsilon)$ is bounded in $L^2(\Omega' \cup \Omega'')^9$. Therefore, it follows from the definition of $\varkappa(\varepsilon)$, (3.1), that:

$$\|e_{22}(u(\varepsilon))\|_{L^2(\Omega')} = O(\varepsilon^2), \quad \|e_{\alpha'2}(u(\varepsilon))\|_{L^2(\Omega')} = O(\varepsilon)\,, \quad (3.8)$$

$$\|e_{11}(u(\varepsilon))\|_{L^2(\Omega'')} = O(\varepsilon^2), \quad \|e_{\alpha''1}(u(\varepsilon))\|_{L^2(\Omega'')} = O(\varepsilon)\,, \quad (3.9)$$

which by weak lower semicontinuity of the norm implies:

$$\begin{cases} e_{22}(u(0)) = e_{\alpha'2}(u(0)) = 0 & \text{in} \quad \Omega'\,, \\ e_{11}(u(0)) = e_{\alpha''1}(u(0)) = 0 & \text{in} \quad \Omega''\,. \end{cases} \quad (3.10)$$

It is known that formulas (3.10) are equivalent to the representation formulas (3.5)–(3.6) with the indicated regularity for the ζ's, see Ciarlet and Destuynder (1979). Formulas (3.7) then follow from the clamping condition on $\Gamma' \cup \Gamma''$. □

3.3 It is a priori clear that the functions ζ must satisfy usual plate and membrane equations inside ω' and ω''. The main problem is to determine which conditions they satisfy on their common boundary $\gamma := \{x_1 = x_2 = 0\}$. Such conditions indeed come from the continuity relations (2.3) for $\varepsilon > 0$. Let us first derive the easiest ones. We will make repeated use of the following lemma:

Lemma 3.1. Let X be a Hilbert space and let u belong to $H^1(0,1;X)$. Then u belongs to $C([0,1];X)$ and we have:

$$\|u(x+h) - u(x)\|_X \le Ch^{1/2}\|u\|_{H^1(0,1;X)}\,, \quad (3.11)$$

whenever x and $x+h$ belong to $[0,1]$.

Proof: This is essentially Lemma 2.1, p. 17 of Lions and Magenes (1968 b).

Proposition 3.3. We have:

$$\begin{cases} \zeta_2'(0,x_3) = 0, \quad \zeta_3'(0,x_3) = \zeta_3''(0,x_3)\,, \\ \zeta_1''(0,x_3) = 0, \end{cases} \quad (3.12),\ (3.13)$$

for all x_3 in $]0,1[$.

Remarks: (1) Formulas (3.12) mean that the flexural displacements of each plate are zero on γ. This is natural, once an a priori bound is known to hold, since the flexural displacement of one plate on γ is a membrane displacement of the other plate which is much smaller indeed. *Thus, the fold stiffens the plates.*

(2) Formula (3.13) means that the fold does not affect directly the membrane displacements in its own direction. Such displacements are transmitted through the fold. The fold however influences those displacements globally, since as we will see, the displacements orthogonal to γ are independent of each other and thus the equations for membrane displacements are not reducible to those of a single plane elastic membrane.

Proof of Proposition 3.3.: For all v in $H^1(0,1;L^2(\omega''))$, let us define:

$$T_\varepsilon'(v) := \frac{1}{\varepsilon}\int_0^\varepsilon v(s,x_2,x_3)\,ds \in L^2(\omega'') \quad (3.14)$$

and T_ε'' by switching primes and coordinates. As the function $u(\varepsilon)_2'$ belongs to the space

$H'(\Omega') \subset H^1(0,1;L^2(\omega''))$, we can apply Lemma 3.1, with $X = L^2(\omega'')$. It follows then from formula (3.14) that:

$$\|T'_\varepsilon(u(\varepsilon)'_2) - u(\varepsilon)'_{2|x_1=0}\|_{L^2(\omega'')} \leq C\varepsilon^{1/2} \tag{3.15}$$

As $u(\varepsilon)'_{2|x_1=0} \rightharpoonup \zeta'_{2|x_1=0}$ in $H^{1/2}(\omega'')$, we deduce that:

$$\zeta'_{2|x_1=0} = \lim_{\varepsilon\to 0} T'_\varepsilon(u(\varepsilon)'_2) \quad \text{in} \quad L^2(\omega''). \tag{3.16}$$

But, using formula (2.3), we get:

$$\begin{aligned} T'_\varepsilon(u(\varepsilon)'_2) &= \int_0^\varepsilon u(\varepsilon)''_2\left(\frac{s}{\varepsilon}, \varepsilon x_2, x_3\right) ds \\ &= \varepsilon \int_0^1 u(\varepsilon)''_2(t, \varepsilon x_2, x_3)\, dt \end{aligned} \tag{3.17}$$

and therefore:

$$\begin{aligned} \| T'_\varepsilon(u(\varepsilon)'_2)\|''_{L^2(\omega'')} &= \varepsilon^2 \int_0^1\int_0^1 \left(\int_0^1 u(\varepsilon)''_2(t,\varepsilon x_2, x_3)\, dt\right)^2 dx_2\, dx_3 \\ &\leq \varepsilon^2 \int_0^1\int_0^1\int_0^1 u(\varepsilon)''_2(t,\varepsilon x_2,x_3)^2\, dt\, dx_2\, dx_3 \\ &\leq \varepsilon \| u(\varepsilon)''_2\|^2_{L^2(\Omega'')} \leq C\varepsilon. \end{aligned} \tag{3.18}$$

by formula (3.2). Then $\zeta'_2(0,x_3) = 0$ and the same proof also works for ζ''_1. Let us turn to formula (3.13). We perform the same kind of computation on the third equation in formula (2.3).

$$\begin{aligned} T'_\varepsilon(u(\varepsilon)'_3) &= \int_0^1 u(\varepsilon)'_3(\varepsilon s, x_2, x_3)\, ds \\ &= \int_0^1 u(\varepsilon)''_3(t, \varepsilon x_2, x_3)\, dt \end{aligned} \tag{3.19}$$

Integrating (3.19) with respect to x_2, we obtain:

$$\int_0^1 T'_\varepsilon(u(\varepsilon)'_3)\,(x_2,x_3)\, dx_2 = \int_0^1 T''_\varepsilon(u(\varepsilon)''_3)\,(x_1,x_3)\, dx_1\,. \tag{3.20}$$

Since the operator:

$$\begin{aligned} &L^2([0,1]^2) \to L^2([0,1]) \\ &\qquad v \mapsto \int_0^1 v(t,x)\, dt \end{aligned} \tag{3.21}$$

is continuous, equation (3.20) holds true in $L^2(\gamma)$, and we can pass to the limit in (3.20), by using the equivalents of formula (3.16). This yields:

$$\zeta'_3(0,x_3) = \zeta''_3(0,x_3) \tag{3.22}$$

since $\partial_3\zeta'_2(0,x_3) = \partial_3\zeta''_1(0,x_3) = 0$ by formula (3.12). □

Proposition 3.3. is actually easy to guess from the continuity condition (2.3). Here comes a more subtle condition on normal derivatives.

Proposition 3.4. We have:

$$\partial_1\zeta'_2(0,x_3) = -\,\partial_2\zeta''_1(0,x_3) \tag{3.23}$$

for all x_3 in $]0,1[$.

Remark: If we go back to the geometrical meaning of displacements, we see that formula (3.23) expresses the fact that the two plates stay perpendicular to each other in their deformed configuration.

Proof of Proposition 3.4. First of all, let us differentiate the first equation in formula (2.3) with respect to x_2:

$$\partial_2 u(\varepsilon)'_1(\varepsilon x_1, x_2, x_3) = \partial_2 u(\varepsilon)''_1(x_1, \varepsilon x_2, x_3). \tag{3.24}$$

Now, as was pointed out to the author by F. Murat, we remark that:

$$\begin{cases} \partial_2 u(\varepsilon)'_1(x_1) \in L^2(0,1; L^2(\omega'')) \quad \text{and} \\ \partial_1(\partial_2 u(\varepsilon)'_1)(x_1) = \partial_2(\partial_1 u(\varepsilon)'_1)(x_1) \in L^2(0,1; H^{-1}(\omega'')). \end{cases} \tag{3.25}$$

Therefore, we can apply Lemma 3.1 to $\partial_2 u(\varepsilon)'_1$ with $X = H^{-1}(\omega'')$. Similarly:

$$\partial_1 u(\varepsilon)''_2(x_2) \in H^1(0,1; H^{-1}(\omega')). \tag{3.26}$$

In particular, $\partial_2 u(\varepsilon)'_1$ has a trace at $x_1 = 0$ in $H^{-1}(\omega'')$. Moreover, since $u(\varepsilon)'_1 \rightharpoonup u(0)'_1$ in $H^1(\Omega')$, $\partial_2 u(\varepsilon)'_1 \rightharpoonup \partial_2 u(0)'_1$ in $H^1(0,1; H^{-1}(\omega''))$, and:

$$\partial_2 u(\varepsilon)'_{1|x_1=0} \rightharpoonup -\partial_1 \zeta'_{2|x_1=0} \quad \text{in} \quad H^{-1}(\omega''). \tag{3.27}$$

Similarly:

$$\partial_1 u(\varepsilon')'_{2|x_2=0} \rightharpoonup -\partial_2 \zeta''_{1|x_2=0} \quad \text{in} \quad H^{-1}(\omega'). \tag{3.28}$$

Let us choose three functions φ_i in $\mathscr{D}(]0,1[)$, multiply equation (3.24) by $\prod_{i=1}^{3} \varphi_i(x_i)$, and integrate:

$$\overbrace{\int_{\Omega'} \partial_2 u(\varepsilon)'_1(\varepsilon x_1, x_2, x_3) \prod_{i=1}^{3} \varphi_i(x_i)\,dx}^{I_1^\varepsilon} = \overbrace{\int_{\Omega''} \partial_2 u(\varepsilon)''_1(x_1, \varepsilon x_2, x_3) \prod_{i=1}^{3} \varphi_i(x_i)\,dx}^{I_2^\varepsilon}. \tag{3.29}$$

Now, setting $g(\varepsilon) = \varkappa_{12}(\varepsilon)$ we have:

$$\partial_2 u(\varepsilon)''_1(x) = \varepsilon g(\varepsilon)(x) - \partial_1 u(\varepsilon)''_2(x), \tag{3.30}$$

with $g(\varepsilon)$ bounded in $L^2(\Omega'')$ (this follows from formula (3.9)). Therefore, the second integral can be rewritten as:

$$I_2^\varepsilon = \overbrace{\varepsilon \int_{\Omega''} g(\varepsilon)(x_1, \varepsilon x_2, x_3) \prod_{i=1}^{3} \varphi_i(x_i)\,dx}^{J^\varepsilon} - \int_{\Omega''} \partial_1 u(\varepsilon)''_2(x_1, \varepsilon x_2, x_3) \prod_{i=1}^{3} \varphi_i(x_i)\,dx. \tag{3.31}$$

Let us show that the first integral in (3.31) vanishes in the limit. In fact, we have:

$$|J^\varepsilon| \le \int_0^1\int_0^\varepsilon\int_0^1 |g(\varepsilon)(x_1, x_2, x_3)\,\varphi_1(x_1)\,\varphi_2\left(\frac{x_2}{\varepsilon}\right)\varphi_3(x_3)|\,dx_1\,dx_2\,dx_3 \le C\varepsilon^{1/2}\|g(\varepsilon)\|_{L^2(\Omega'')}, \tag{3.32}$$

by the Cauchy-Schwarz inequality. Therefore, we need only to consider the second integral, which is exactly of the same type as I_1^ε after switching primes and indices. Let us thus turn to studying I_1^ε. Lemma 3.1. applied to $v(\varepsilon)(x_1) := \partial_2 u(\varepsilon)'_1(\varepsilon x_1)$ yields:

$$\|v(\varepsilon)(x_1) - v(\varepsilon)(0)\|_{H^{-1}(\omega'')} \le C\varepsilon^{1/2} x_1^{1/2} \le C\varepsilon^{1/2}, \tag{3.33}$$

whence from (3.27):

$$v(\varepsilon)(\cdot) \rightharpoonup -\partial_1 \zeta'_{2|x_1=0} \quad \text{in} \quad H^{-1}(\omega''), \tag{3.34}$$

uniformly with respect to x_1. Let us define a linear mapping:

$$L_{\varphi_1} : L^2(0,1; H^{-1}(\omega'')) \to H^{-1}(\omega'')$$
$$v \mapsto \int_0^1 v(x_1)\,\varphi_1(x_1)\,dx_1,$$

as

$$< L_{\varphi_1} v, \psi > := \int_0^1 < v(x_1), \psi > \varphi_1(x_1)\, dx_1, \tag{3.36}$$

for all ψ in $H_0^1(\omega'')$ (all $< \cdot , \cdot >$ symbols will denote pairings between $H_0^1(\omega')$ and $H^{-1}(\omega')$ or $H_0^1(\omega'')$ and $H^{-1}(\omega'')$). Clearly, L_{φ_1} is well defined and continuous with norm $\|\varphi_1\|_{L^2(0,1)}$ As $\varphi_2(x_2)\varphi_3(x_3)$ belongs to $H_0^1(\omega'')$, we see from formula (3.34) that:

$$I^\varepsilon = < L_{\varphi_1}(v'(\varepsilon)), \varphi_2 \otimes \varphi_3 > \to - < L_{\varphi_1}(\partial_1 \zeta'_{2|x_1=0}), \varphi_2 \otimes \varphi_3 > \quad \text{as} \quad \varepsilon \to 0 \tag{3.37}$$

but $L_{\varphi_1}(\partial_1 \zeta'_{2|x_1=0}) = (\partial_1 \zeta'_{2|x_1=0}) \times (\int_0^1 \varphi_1(x_1)\, dx_1)$ as is seen from the definition, so that:

$$\begin{aligned} I_1^\varepsilon &\to - < \partial_1 \zeta'_{2|x_1=0} \int_0^1 \varphi_1(x_1)\, dx_1, \varphi_2 \otimes \varphi_3 > \\ &= - \int_{\Omega'} \partial_1 \zeta'_2(0,x_3)\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\, dx \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{3.38}$$

Similarly, from formulas (3.31) and (3.32), we see that

$$I_2^\varepsilon \to \int_{\Omega''} \partial_2 \zeta''_1(0,x_3)\varphi_2(x_2)\varphi_1(x_1)\varphi_3(x_3)\, dx \quad \text{as} \quad \varepsilon \to 0 \tag{3.39}$$

and the conclusion follows, since $\varphi\mathbf{1}$, $\varphi\mathbf{2}$ and $\varphi\mathbf{3}$ are arbitrary in $\mathscr{D}(]0,1[)$. □

Let us sum up the properties of the ζ's found so far in a theorem.

Theorem 3.1. The functions ζ'_i, ζ''_i satisfy:

$$\begin{cases} \zeta'_2(0,x_3) = \zeta''_1(0,x_3) = 0 \\ \partial_1 \zeta'_2(0,x_3) = -\partial_2 \zeta''_1(0,x_3) \\ \zeta'_3(0,x_3) = \zeta''_3(0,x_3) \end{cases} \tag{3.40}$$

for all x_3 in $]0,1[$.

3.4. We are now in a position to completely determine $u(0)$: conditions (3.40) are the only restrictions imposed a priori on the unknowns as we will now prove. The method consists in showing that we can pass to the limit in the variational equations (3.6) with arbitrary test-functions satisfying (3.40). For simplicity, we will treat the flexural displacement and the membrane displacements separately.

We have proved that the pair (ζ'_2, ζ''_1) belongs to the space:

$$\begin{aligned} \mathscr{V}_1 := \{ &(\xi', \xi'') \in H^2(\omega') \times H^2(\omega''), \xi'(0,x_3) = \xi''(0,x_3) = \xi'(1,x_3) = \xi''(1,x_3) = 0, \\ &\partial_1 \xi'(1,x_3) = \partial_2 \xi''(1,x_3) = 0, \partial_1 \xi'(0,x_3) = -\partial_2 \xi''(0,x_3)\}. \end{aligned} \tag{3.41}$$

The space $\mathscr{V}_1$ is indeed closed in $H^2_{\Gamma'}(\omega') \times H^2_{\Gamma''}(\omega'')$ and any functional that is coercive on the latter space defines a well-posed variational problem on the former. Actually we will prove that:

Theorem 3.2. The pair $(\zeta'_2, \zeta''_1) \in \mathscr{V}_1$ is the unique solution of the variational equations:

$$\begin{aligned} &\int_{\omega'} \left(\frac{1}{6} \mu \partial_{\alpha'\beta'} \zeta'_2 \, \partial_{\alpha'\beta'} \xi' + \frac{\mu\lambda}{6(2\mu+\lambda)} \partial_{\alpha'\alpha'} \zeta'_2 \, \partial_{\beta'\beta'} \xi' \right) dx \\ &+ \int_{\omega''} \left(\frac{1}{6} \mu \partial_{\alpha''\beta''} \zeta''_1 \, \partial_{\alpha''\beta''} \xi'' + \frac{\mu\lambda}{6(2\mu+\lambda)} \partial_{\alpha''\alpha''} \zeta''_1 \, \partial_{\beta''\beta''} \xi'' \right) dx \\ &= \int_{\omega'} (-M_{\alpha'} \partial_{\alpha'} \xi' + F'_2 \xi')\, dx + \int_{\omega''} (-M_{\alpha''} \partial_{\alpha''} \xi'' + F''_1 \xi'')\, dx \end{aligned} \tag{3.42}$$

where:

$$M_{\alpha'} = \int_0^1 \left(x_2 - \frac{1}{2}\right) f'_{\alpha'} \mathrm{d}x_2, \quad F'_2 = \int_0^1 f'_2 \mathrm{d}x_2, \quad M_{\alpha''} = \int_0^1 \left(x_1 - \frac{1}{2}\right) f''_{\alpha''} \mathrm{d}x_1, \quad F''_1 = \int_0^1 f''_1 \mathrm{d}x_1 .$$

for all test-functions (ξ', ξ'') in $\mathscr{V}_1$.

Lemma 3.2. The following weak L^2-convergences hold:

$$\begin{cases} \varepsilon^{-1} e_{\alpha' 2}(u'(\varepsilon)) \rightharpoonup 0 \\ \varepsilon^{-2} e_{22}(u'(\varepsilon)) \rightharpoonup -\dfrac{\lambda}{2\mu + \lambda} e_{\alpha'\alpha'}(u'(0)) \end{cases} \tag{3.43'}$$

in Ω' and:

$$\begin{cases} \varepsilon^{-1} e_{\alpha'' 1}(u''(\varepsilon)) \rightharpoonup 0 \\ \varepsilon^{-2} e_{11}(u''(\varepsilon)) \rightharpoonup -\dfrac{\lambda}{2\mu + \lambda} e_{\alpha''\alpha''}(u''(0)) \end{cases} \tag{3.43''}$$

in Ω''.

Proof: The proof is very similar to the one of Ciarlet, Le Dret and Nzengwa (1987), but we include it for completeness. Let us introduce the scaled stresses:

$$\Sigma_{\alpha'\beta'}(\varepsilon) := 2\mu \varkappa_{\alpha'\beta'}(\varepsilon) + \lambda(\varkappa_{\gamma'\gamma'}(\varepsilon) + \varkappa_{22}(\varepsilon))\delta_{\alpha'\beta'}$$
$$\Sigma_{\alpha' 2}(\varepsilon) := 2\mu\varepsilon^{-1}\varkappa_{\alpha' 2}(\varepsilon)$$
$$\Sigma_{22}(\varepsilon) := (2\mu + \lambda)\varepsilon^{-2}\varkappa_{22}(\varepsilon) + \lambda\varepsilon^{-2}\varkappa_{\gamma'\gamma'}(\varepsilon)$$

and the analog on Ω''. It is easy to check that they satisfy: $\partial_j \Sigma_{ij}(\varepsilon) = f_i$ in $\Omega' \cup \Omega''$.

In particular: $\partial_2 \Sigma_{\alpha' 2}(\varepsilon) = f_{\alpha'} - \partial_{\beta'} \Sigma_{\alpha'\beta'}(\varepsilon)$.

Now, each function $\Sigma_{\alpha'\beta'}(\varepsilon)$ is bounded in $L^2(\Omega')$ independently of ε, therefore, $\partial_{\beta'}\Sigma_{\alpha'\beta'}(\varepsilon)$ is bounded in $L^2(0,1;H^{-1}(\omega'))$ and as $F_{\alpha'}$ belongs to $L^2(\Omega')$, we deduce that:

$$\partial_2 \varkappa_{\alpha' 2}(\varepsilon) = \frac{\varepsilon}{2\mu}(f_{\alpha'} - \partial_{\beta'}\Sigma_{\alpha'\beta'}(\varepsilon))$$

converges strongly to 0 in $L^2(0,1;H^{-1}(\omega'))$. As the traces of $\varkappa_{\alpha' 2}(\varepsilon)$ at $x_2 = 0$ (resp. $x_2 = 1$) tend to 0 strongly in $H^{-1/2}(x_2 = 0)$ (resp. $H^{-1/2}(x_2 = 1)$) it follows that $\varkappa_{\alpha' 2}(\varepsilon) \to 0$ strongly in $H^1(0,1;H^{-1}(\omega'))$.

Similarly: $\partial_2 \Sigma_{22}(\varepsilon) = f_2 - \partial_{\beta'}\Sigma_{2\beta'}(\varepsilon)$.

and the same argument as before shows that:

$$\partial_2((2\mu + \lambda)\varkappa_{22}(\varepsilon) + \lambda\varkappa_{\gamma'\gamma'}(\varepsilon)) = \varepsilon^2(f_2 - \partial_{\beta'}\Sigma_{2\beta'}(\varepsilon))$$

converges strongly to 0 in $L^2(0,1;H^{-1}(\omega'))$. As the traces at $x_2 = 0$ and 1 also converge to 0, we see that $(2\mu + \lambda)\varkappa_{22}(\varepsilon) + \lambda\varkappa_{\gamma'\gamma'}(\varepsilon) \to 0$ strongly in $H^1(0,1;H^{-1}(\omega'))$. Formula (3.43′) then follows from the definition of the $\varkappa$'s. We argue in the same fashion on Ω'' and thereby conclude the proof of the lemma. □

Proof of Theorem 3.2: Let us choose (ξ', ξ'') in $V_1 \cap C^\infty$. Then, in a neighborhood of $x_1 = 0$, the following expansions are valid:

$$\begin{aligned} \xi'(x_1, x_3) &= x_1 \partial_1 \xi'(0, x_3) + g'_1(x_1, x_3) \quad \text{with} \quad |g'_1| \le c x_1^2 \\ &= -x_1 \partial_2 \xi''(0, x_3) \overset{2}{+} g'_1(x_1, x_3) \end{aligned} \tag{3.44'}$$

since $\xi'(0,x_3)=0$ and $\partial_1\xi'(0,x_3)=-\partial_2\xi''(0,x_3)$, and:

$$\partial_1\xi'(x_1,x_3)=\partial_1\xi'(0,x_3)+g_2'(x_1,x_3)\quad\text{with}\quad |g_2'|\le c x_1. \tag{3.45'}$$

(actually $g_2'=\partial_1 g_1'$, of course). Similarly, in a neighborhood of $x_2=0$, we have:

$$\begin{aligned}\xi''(x_2,x_3)&=x_2\partial_2\xi''(0,x_3)+g_1''(x_2,x_3)\quad\text{with}\quad |g_1''|\le c x_2^2\\&=-x_2\partial_1\xi'(0,x_3)+g_1''(x_2,x_3)\end{aligned} \tag{3.44''}$$

and

$$\partial_2\xi''(x_2,x_3)=\partial_2\xi''(0,x_3)+g_2''(x_2,x_3)\quad\text{with}\quad |g_2''|\le c x_2. \tag{3.45''}$$

The trick is to define a test-function $v(\varepsilon)$ belonging to $\Theta^\varepsilon V^\varepsilon$ and hence admissible in equation (2.6), in such a way that $v(\varepsilon)$ approximates the Kirchhoff-Love displacement corresponding to (ξ', ξ'') sufficiently closely so as to allow us to pass to the limit in the singular terms of equation (2.6) using only the information of Lemma 3.2.; i.e. using the fact that integrals of a product of a weakly convergent sequence by a strongly convergent sequence converge to the integral of the product of the limits.
Let us call:

$$v=\begin{cases}\left(-\left(x_2-\dfrac{1}{2}\right)\partial_1\xi',\ \xi',\ -\left(x_2-\dfrac{1}{2}\right)\partial_3\xi'\right) & \text{in}\quad \Omega',\\[2ex] \left(\xi'',\ -\left(x_1-\dfrac{1}{2}\right)\partial_2\xi'',\ -\left(x_1-\dfrac{1}{2}\right)\partial_3\xi''\right) & \text{in}\quad \Omega''.\end{cases} \tag{3.46}$$

Then we define $v(\varepsilon)$ as: $v(\varepsilon)=v_1(\varepsilon)+v_2(\varepsilon)$ where $v_1(\varepsilon)$ is such that:

$$v_1'(\varepsilon)=\begin{cases}(\varepsilon^{-1}\xi''(\varepsilon x_2,x_3),\ \xi'(x_1,x_3),0) & \text{for}\quad 0<x_1<\varepsilon,\\[1ex] \left({}^*,\xi'(x_1,x_3),\ -\left(x_2-\dfrac{1}{2}\right)\left(\dfrac{x_1-\varepsilon}{\varepsilon}\right)\partial_3\xi'(x_1,x_3)\right) & \text{for}\quad \varepsilon<x_1<2\varepsilon,\\[2ex] \left(-x_2\partial_1\xi'(x_1,x_3),\ \xi'(x_1,x_3),\ -\left(x_2-\dfrac{1}{2}\right)\partial_3\xi'(x_1,x_3)\right) & \text{for}\quad x_1>2\varepsilon.\end{cases} \tag{3.47'}$$

in Ω' and:

$$v_1''(\varepsilon)=\begin{cases}(\xi''(x_2,x_3),\varepsilon^{-1}\xi'(\varepsilon x_1,x_3),\ 0) & \text{for}\quad 0<x_2<\varepsilon,\\[1ex] \left(\xi''(x_2,x_3),{}^{**},\ -\left(x_1-\dfrac{1}{2}\right)\left(\dfrac{x_2-\varepsilon}{\varepsilon}\right)\partial_3\xi''(x_2,x_3)\right) & \text{for}\quad \varepsilon<x_2<2\varepsilon,\\[2ex] \left(\xi''(x_2,x_3),\ -x_1\partial_2\xi''(x_2,x_3),\ -\left(x_1-\dfrac{1}{2}\right)\partial_3\xi''(x_2,x_3)\right) & \text{for}\quad x_2>2\varepsilon.\end{cases} \tag{3.47''}$$

in Ω'', where (and here is the point where the conditions entering the definition of $\mathscr{V}_1$ are crucial):

$$\begin{cases}{}^*=-x_2\partial_1\xi'(0,x_3)-x_2\left(\dfrac{x_1-\varepsilon}{\varepsilon}\right)g_2'(x_1,x_3)+\dfrac{2\varepsilon-x_1}{\varepsilon^2}g_1''(\varepsilon x_2,x_3)\\[2ex] {}^{**}=-x_1\partial_2\xi''(0,x_3)-x_1\left(\dfrac{x_2-\varepsilon}{\varepsilon}\right)g_2''(x_2,x_3)+\dfrac{2\varepsilon-x_2}{\varepsilon^2}g_1'(\varepsilon x_1,\varepsilon_3)\end{cases} \tag{3.48}$$

The remainder $v_2(\varepsilon)$ is defined as:

$$v_2'(\varepsilon) = \begin{cases} \left(\frac{1}{2}\partial_1\xi'(0,x_3), \frac{\varepsilon}{2}\partial_2\xi''(0,x_3), 0\right) & \text{for } 0 < x_1 < \varepsilon, \\ \left(\frac{1}{2}\partial_1\xi'(2(x_1-\varepsilon),x_3), \quad \frac{\varepsilon}{2}\partial_2\xi''(2(x_1-\varepsilon),x_3), 0\right) & \text{for } \varepsilon < x_1 < 2\varepsilon, \\ \left(\frac{1}{2}\partial_1\xi'(x_1,x_3), \quad \frac{\varepsilon}{2}\partial_2\xi''(x_1,x_3), 0\right) & \text{for } x_1 > 2\varepsilon, \end{cases} \tag{3.49'}$$

(note that $\partial_2\xi''$ (x_1,x_3)) means here the derivative of ξ'' with respect to its first variable x_2 taken at the point (x_1, x_3) and:

$$v_2''(\varepsilon) = \begin{cases} \left(\frac{\varepsilon}{2}\partial_1\xi'(0,x_3), \frac{1}{2}\partial_2\xi''(0,x_3), 0\right) & \text{for } 0 < x_2 < \varepsilon, \\ \left(\frac{\varepsilon}{2}\partial_1\xi'(2(x_2-\varepsilon),x_3), \frac{1}{2}\partial_2\xi''(2(x_2-\varepsilon),x_3), 0\right) & \text{for } \varepsilon < x_2 < 2\varepsilon, \\ \left(\frac{\varepsilon}{2}\partial_1\xi'(x_2,x_3), \frac{1}{2}\partial_2\xi''(x_2,x_3), 0\right) & \text{for } x_2 > 2\varepsilon. \end{cases} \tag{3.49''}$$

Inspection of formulas (3.47)–(3.48) for $x_1 = \varepsilon, 2\varepsilon$ and $x_2 = \varepsilon, 2\varepsilon$ reveals that $v_1(\varepsilon)$ is continous in Ω' and Ω'', by formulas (3.44)–(3.45), and is thus in H^1; so is $v_2(\varepsilon)$. Moreover, it is clear that $v(\varepsilon)$ satisfies the continuity conditions (2.3) by construction, and is thus admissible in equation (2.6). It is also clear that $v(\varepsilon) \to v$ strongly in $L^2(\Omega' \cup \Omega'')$. Let us consider the convergence of the strain tensors. Since the definition of $v(\varepsilon)$ is symmetric with respect to the primes, we can deal with $v'(\varepsilon)$ alone. We have:

$$e_{22}(v'(\varepsilon)) = 0 \tag{3.50}$$

which takes care of the ε^{-4} terms in equation (2.6). Then, for $0 < x_1 < \varepsilon$:

$$\begin{aligned} 2e_{21}(v'(\varepsilon)) &= \partial_2\xi''(\varepsilon x_2, x_3) + \partial_1\xi'(x_1,x_3) \\ &= g_2''(\varepsilon x_2, x_3) + g_2'(x_1,x_3) \end{aligned} \tag{3.51}$$

by formula (3.45). For $\varepsilon < x_1 < 2\varepsilon$:

$$2e_{21}(v'(\varepsilon)) = \frac{2\varepsilon - x_1}{\varepsilon}(g_2''(\varepsilon x_2, x_3) + g_2'(x_1,x_3)) + \varepsilon\partial_{22}\xi''(2(x_1-\varepsilon),x_3) \tag{3.52}$$

and for $x_1 > 2\varepsilon$:

$$2e_{21}(v'(\varepsilon)) = \frac{\varepsilon}{2}\partial_{22}\xi''(x_1,x_3). \tag{3.53}$$

A routine calculation using the estimates on the functions g_i'', g_i'' implies that:

$$\varepsilon^{-1}e_{21}(v'(\varepsilon)) \to \frac{1}{4}\partial_{22}\xi''(x_1,x_3) \quad \text{strongly in} \quad L^2(\Omega'). \tag{3.54}$$

This takes care of the term $\varepsilon^{-2}\int\limits_{\Omega'} e_{21}(u'(\varepsilon))e_{21}(v'(\varepsilon))$, which therefore tend to 0 by Lemma 3.2.

Next, we have for $0 < x_1 < \varepsilon$:

$$2e_{23}(v'(\varepsilon)) = \partial_3\xi'(x_1,x_3) + \frac{\varepsilon}{2}\partial_{23}\xi''(0,x_3) \tag{3.55}$$

for $\varepsilon < x_1 < 2\varepsilon$:

$$2e_{23}(v'(\varepsilon)) = \frac{2\varepsilon - x_1}{\varepsilon}\,\partial_3\xi'(x_1,x_3) + \frac{\varepsilon}{2}\,\partial_{23}\xi''(2(x_1-\varepsilon),x_3), \tag{3.56}$$

and for $x_1 > 2\varepsilon$:

$$2e_{23}(v'(\varepsilon)) = \frac{\varepsilon}{2}\,\partial_{23}\xi''(x_1,x_3). \tag{3.57}$$

It is easy to see from an estimate of the form $|\partial_3\xi'(x_1,x_3)| < cx_1$, that $\varepsilon^{-1}e_{23}(v'(\varepsilon)) \to 1/4\,\partial_{23}\xi''(x_1,x_3)$ strongly in $L^2(\Omega')$. Finally, using again the estimates on the functions g, we see that $e_{\alpha'\beta'}(v'(\varepsilon)) \to e_{\alpha'\beta'}(v')$ strongly in $L^2(\Omega')$. We can thus pass to the limit as $\varepsilon \to 0$ in equation (2.6), which yields equations (3.42) for (ξ', ξ'') very regular, after integration with respect to x_2 in Ω' and x_1 in Ω''. Then a (not so obvious, see remark 2 below) density argument shows that equations (3.42) hold for all (ξ', ξ'') in $\mathscr{V}_1$. Now, equations (3.42) clearly define a well-posed variational problem, having thus a unique solution in $\mathscr{V}_1$, namely (ζ_2', ζ_1''). □

Remarks. (1) We recognize in (3.42) the weak form of equations for two plates coupled through their normal derivatives on their common edge (expressed here with the more familiar material constants E and ν):

$$\begin{cases} \dfrac{E}{12(1-\nu^2)}\Delta^2\zeta_2' = \partial_{\alpha'}M_{\alpha'} + F_2' & \text{in } \omega' \\[2ex] \dfrac{E}{12(1-\nu^2)}\Delta^2\zeta_1'' = \partial_{\alpha''}M_{\alpha''} + F_1'' & \text{in } \omega'' \end{cases} \tag{3.58}$$

$$\begin{cases} \zeta_2'(0,x_3) = \zeta_1''(0,x_3) = 0 \\ \partial_1\zeta_2'(0,x_3) = -\,\partial_2\zeta_1''(0,x_3) \end{cases} \tag{3.59}$$

$$m_{11}'(0,x_3) = m_{22}''(0,x_3) \tag{3.60}$$

plus the conditions coming from the clamping on Γ' and Γ'' and the traction-free condition on the rest of the boundary, where the bending moments m_{ij} are defined by

$$m_{\alpha'\beta'}' := -\frac{E}{12(1-\nu^2)}\left[(1-\nu)\,\partial_{\alpha'\beta'}\zeta' + \nu\,\Delta\zeta'\,\delta_{\alpha'\beta'}\right]$$

$$m_{\alpha''\beta''}'' := -\frac{E}{12(1-\nu^2)}\left[(1-\nu)\,\partial_{\alpha''\beta''}\zeta'' + \nu\,\Delta\zeta''\,\delta_{\alpha''\beta''}\right]$$

Formula (3.60) comes formally from integrations by parts in equations (3.42).

(2) Note that formulas (3.59) imply that the function:

$$\tilde{\zeta}:]-1,1[\times]0,1[\to \mathbb{R}$$

$$(s,t) \mapsto \begin{cases} \zeta_1''(s,t) & \text{if } s<0 \\ \zeta_2'(s,t) & \text{if } s>0 \end{cases}$$

(which sort of "unfolds" the plates) is actually H^2 with respect to (s,t). The density argument of the proof of Theorem 3.2. relies upon the fact that it is possible to approximate in the H^2-norm any $H^2(]-1,1[\times]0,1[)$ function vanishing at $s=0$ by C^∞-functions also vanishing at $s=0$. Let us briefly sketch how this can be done. First of all, due to the boundary conditions at $s=\pm 1$, we can extend any such $\tilde{\zeta}$ by reflexions at $t=0$ and 1 into an H^2-function of $Q =]-1,1[\times]-1,2[$ vanishing in a fixed neighborhood of $t=-1$ and $t=2$. This extension is actually a continuous linear operator. Let v belong to $H^{3/2}(]-1,2[)$ with compact support in $]-1,2[$. Then the function $P(v)$ defined by:

$$\begin{cases} \Delta^2 P(v) = 0 & -1 < s < 0 \quad \text{and} \quad 0 < s < 1, -1 < t < 2 \\ P(v) = v & \text{for } s = 0 \\ P(v) = 0 & \text{on } \partial Q \\ \dfrac{\partial P(v)}{\partial n} = 0 & \text{on } \partial Q \cup \{s = 0\} \end{cases}$$

is clearly $H^2(Q)$, depends continuously on v and if v is C^∞, so is $P(v)$. Now, let ϱ_ε be a smoothing kernel. We approximate ζ by $\zeta_\varepsilon = \varrho_\varepsilon * \zeta - P(\varrho_\varepsilon * \zeta_{|s=0})$.
Note that the function ζ above does not however satisfy (the notation is self-explanatory):

$$\frac{E}{12(1-\nu^2)} \Delta^2 \zeta = \partial_\alpha M_\alpha + F \quad \text{in} \quad]-1, 1[\times]0, 1[.$$

3.5. Let us now turn to finding the membrane displacements. Since their determination follows closely the lines of the proof of Theorem 3.2., we will only sketch the argument. We have seen that the quadruple $(\zeta'_{\alpha'}, \zeta''_{\alpha''})$ belongs to the space:

$$\mathscr{V}_2 := \{(\xi'_{\alpha'}, \xi''_{\alpha''}) \in H^1(\omega')^2 \times H^1(\omega'')^2, \xi'_{\alpha'}(1, x_3) = \xi''_{\alpha''}(1, x_3) = 0, \xi'_3(0, x_3) = \xi''_3(0, x_3)\} \tag{3.61}$$

We introduce the notation:

$$\boldsymbol{u}^{0\prime} := (\zeta'_1, \zeta'_3) \quad \text{in} \quad \omega', \qquad \boldsymbol{u}^{0\prime\prime} := (\zeta''_2, \zeta''_3) \quad \text{in} \quad \omega''$$

and

$$\boldsymbol{v}' := (\xi'_1, \xi'_3) \quad \text{in} \quad \omega', \qquad \boldsymbol{v}'' := (\xi''_2, \xi''_3) \quad \text{in} \quad \omega''$$

Then, we have:

Theorem 3.3. The membrane displacements $(\zeta'_{\alpha'}, \zeta''_{\alpha''})$ are the unique solution in V_2 of the variational equations:

$$\begin{aligned} &\int_{\omega'} \left(2\mu e_{\alpha'\beta'}(\boldsymbol{u}^{0\prime}) e_{\alpha'\beta'}(\boldsymbol{v}') + \frac{2\mu\lambda}{2\mu+\lambda} e_{\alpha'\alpha'}(\boldsymbol{u}^{0\prime}) e_{\beta'\beta'}(\boldsymbol{v}') \right) \mathrm{d}x \\ &+ \int_{\omega''} \left(2\mu e_{\alpha''\beta''}(\boldsymbol{u}^{0\prime\prime}) e_{\alpha''\beta''}(\boldsymbol{v}'') + \frac{2\mu\lambda}{2\mu+\lambda} e_{\alpha''\alpha''}(\boldsymbol{u}^{0\prime\prime}) e_{\beta''\beta''}(\boldsymbol{v}'') \right) \mathrm{d}x \\ &= \int_{\omega'} F'_{\alpha'} \xi'_{\alpha'} \mathrm{d}x + \int_{\omega''} F''_{\alpha''} \xi''_{\alpha''} \mathrm{d}x \end{aligned} \tag{3.62}$$

where $F'_{\alpha'} = \int_0^1 f'_{\alpha'} \mathrm{d}x_2, \quad F''_{\alpha''} = \int_0^1 f''_{\alpha''} \mathrm{d}x_1, \quad$ for all $\quad (\xi'_{\alpha'}, \xi''_{\alpha''})$ in $\mathscr{V}_2$.

Proof: Given $(\xi'_{\alpha'}, \xi''_{\alpha''})$ in $\mathscr{V}_2$, we define a displacement $v(\varepsilon)$ in $\Theta^\varepsilon\, \boldsymbol{V}^\varepsilon$ much as in the proof of Theorem 3.2.:

$$v'(\varepsilon) = \begin{cases} (\xi'_1(0, x_3), \varepsilon\xi''_2(0, x_3), \xi'_3(0, x_3)) & \text{for} \quad 0 < x_1 < \varepsilon, \\ (\xi'_1(2(x_1-\varepsilon), x_3), \varepsilon\xi''_2(2(x_1-\varepsilon), x_3), \xi'_3(2(x_1-\varepsilon), x_3)) & \text{for} \quad \varepsilon < x_1 < 2\varepsilon, \\ (\xi'_1(x_1, x_3), \varepsilon\xi''_2(x_1, x_3), \xi'_3(x_1, x_3)) & \text{for} \quad x_1 > 2\varepsilon, \end{cases}$$

$$v''(\varepsilon) = \begin{cases} (\varepsilon\xi'_1(0, x_3), \xi''_2(0, x_3), \xi''_3(0, x_3)) & \text{for} \quad 0 < x_2 < \varepsilon, \\ (\varepsilon\xi'_1(2(x_2-\varepsilon), x_3), \xi''_2(2(x_2-\varepsilon), x_3), \xi''_3(2(x_2-\varepsilon), x_3)) & \text{for} \quad \varepsilon < x_2 < 2\varepsilon, \\ (\varepsilon\xi'_1(x_2, x_3), \xi''_2(x_2, x_3), \xi''_3(x_2, x_3)) & \text{for} \quad x_2 > 2\varepsilon. \end{cases}$$

Then it is easy to check that $e_{22}(v'(\varepsilon)) = e_{11}(v''(\varepsilon)) = 0$, that $\varepsilon^{-1} e_{\alpha' 2}(v'(\varepsilon))$ and $\varepsilon^{-1} e_{\alpha'' 1}(v''(\varepsilon))$ converge strongly in L^2 and that $e_{\alpha'\beta'}(v'(\varepsilon))$ and $e_{\alpha''\beta''}(v''(\varepsilon))$ also converge strongly in L^2, and we conclude as in the proof of Theorem 3.2. □

Let us summarize all the results of Section 3.

Theorem 3.4. The family $u(\varepsilon)$ converges weakly in $H^1(\Omega')^3 \times H^1(\Omega'')^3$ to a Kirchhoff-Love displacement $u(0)$ defined by equations (3.5)–(3.6)–(3.42)–(3.62).

Once this weak convergence is established, it is almost immediate, albeit tedious, to show that:

Theorem 3.5. $u(\varepsilon) \to u(0)$ strongly in $H^1(\Omega')^3 \times H^1(\Omega'')^3$.

The argument is as in Ciarlet, Le Dret and Nzengwa (1987). Therefore we will omit this last proof.

4 Extensions and open problems

4.1. The most immediate and straightforward extension apart from more general loadings, is to consider arbitrary shaped ω' and ω'', with a common straight edge. Another easy one is to deal with a "T-structure" (Fig. 3): with a clamping condition on some part of each of the three plates. The result is the obvious one obtained by patching our previous results and depicted in Fig. 4.

One could also consider the corner structure of Fig. 5.

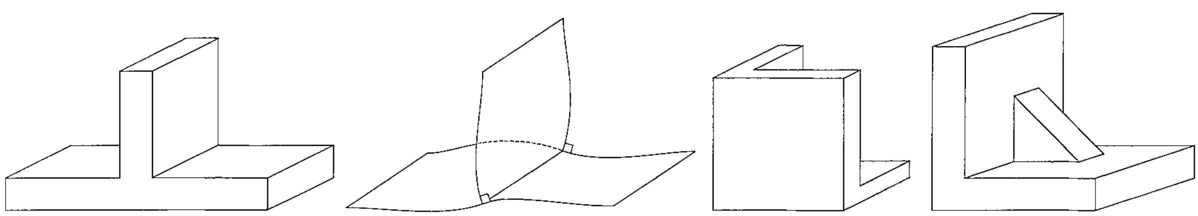
The author's conjecture is that the result would also be obtained by "patching the three folds". It is clear that the limit solution must satisfy the analogs of Theorem 3.1. on each fold. Proving along the preceding lines that these conditions alone determine the limit problem requires the construction of rather intricate test-functions. This question will be addressed in a forthcoming paper.

Another interesting extension is to consider a fold of arbitrary angle. This extension does not seem to be entirely straightforward.

4.2. It is open problem to find out what happens if only one of the two plates is clamped somewhere on its boundary. The method presented here fails from the start, since in this case the bound (3.4) clearly cannot hold. However, there certainly must exist some 2d-model for this structure. In the same spirit, one would like to be able to consider a fold with a reinforcement: but this cannot be done in the same fashion as before if no clamping is assumed on the small triangle (Fig. 6).

Another open problem is the extension to nonlinear elasticity. Since no existence theorem similar to Proposition 1.1. is known in this case, our method cannot be immediately adapted. However, we feel that the trick of counting the junction twice is to be kept in mind when approaching this problem.

Finally, we must acknowledge that the terminology "fold" we have used throughout this paper is somewhat inaccurate. A real fold model should involve plasticity phenomena, loss of homogeneity and isotropy in the folded region and should not be posed in the rectangular geometry we assumed, but rather in a smoothly curved domain. Some of the ideas exposed in this paper might be useful, however, in the study of these more general situations.



3 4 5 6

Figs. 3–6

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