LA-UR- 11-04014

Approved for public release; distribution is unlimited.

Title: Modeling of Nonlinear Metamaterials

Author(s): P.L. Colestock
M.T. Reiten
J.F. O'Hara

Intended for: SPIE Optics + Photonics
San Diego, CA
Aug 21-25, 2011

Los Alamos NATIONAL LABORATORY
EST. 1943

Modeling of Nonlinear Metamaterials

P. L. Colestock, M. Reiten and J. O'Hara*

Los Alamos National Laboratory

I. Introduction

The potential of engineered materials that possess novel dielectric properties is well known and has been intensively studied in recent years. Much attention has been focused on the properties associated with two- or three-dimensional arrays of infinitesimal unit cells comprised of electromagnetic resonators. These so-called meta-materials comprise a burgeoning and potentially valuable research area. [1-5] However, comparatively little attention has been given to the case when the resonators themselves possess significant nonlinear properties. Such a situation is akin to that of a lasing medium [6] or nonlinear transmission line [7], which have been found to display a wide variety of frequency and amplitude dependent phenomena.

Moreover, an array of coupled nonlinear oscillators was the basis of the famous Fermi, Pasta, Ulam system [8] that was first studied numerically and found to contain unexpected effects, namely a repeated recurrence of a given initial perturbation, and persistent frequency mixing phenomena. In addition, very simple nonlinear resonator circuits were shown to exhibit chaos in an isolated context [9], however it is not clear how these complex phenomena might manifest themselves in an array. It is this situation that has compelled us to study nonlinear meta-materials in particular, with an eye to developing novel applications that can make use of these effects. In particular, we are interested in the frequency agility of nonlinear meta-materials, and in identifying the forms of nonlinear waves that may occur in such systems.

To this end we have undertaken both an experimental and theoretical study of nonlinear resonant circuits in the microwave frequency range. In our case the specific nonlinearity is introduced by the insertion of a biased tunnel diode into the resonator circuit, and the fabrication of prototype circuits is described elsewhere. We report here the results of numerical studies of prototype elementary circuits in both frequency and time domains. Our experimental results will be described in a companion paper. [10] We show the existence of both frequency mixing and chaotic effects, and investigate the properties associated with arrays of such resonators.

In the second section, we describe the stability of the nonlinear circuit of interest, and discuss the frequency and steady-state behavior of the system. In Section 3, we outline the onset of chaotic phenomena, and describe numerical tools to determine its dependence on circuit parameters. In Section 4, we outline the nature of nonlinear waves that can exist in a 1-dimensional array of such resonators, and in Section 5, we discuss future directions of our research, as well as the development of potential applications.

II. Stability and Harmonic Behavior

Many elementary nonlinear resonant circuits have been shown to exhibit complex behavior, such as Chua's circuit [9], among others. We will choose the simple circuit model that is motivated by our experimental configuration, shown in Fig. 1. The physical circuit element consists of a resonator in series with a tunnel diode. Losses are represented by resistor R_i and the circuit is excited by an external source V_0 coupled through impedance R_s.

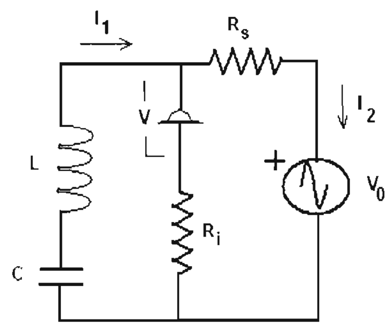

Fig. 1 Lumped element circuit model. An RLC circuit includes a series tunnel diode. An external source V_0 is used to drive the system though coupling impedance R_s.

It is well-known that such circuits can exhibit self-oscillations when biased in the negative resistance portion of the tunnel diode characteristic. To examine the dynamics of such a circuit, we use Kirchhoff's voltage and current laws to express the dynamical equations in the following form

$$\frac{dp}{d\tau} = -\left(\frac{1}{Q} + QR_i\left[b_1 + 2b_2 + 3b_3q^2\right]\right)p - \left(1 + R_i\frac{b_0 + b_1q + b_2q^2 + b_3q^3}{q}\right)q = V_0(\tau)$$

$$\frac{dq}{d\tau} = p$$

where q is the normalized tunnel diode voltage and p is its normalized rate of change. We have assumed the usual cubic dependence of the tunnel diode current on voltage, as given in the numerator of the second term on the right hand side of the *dp/dt* expression. Fixed points in the dynamics occur when $p = \frac{dp}{d\tau} = 0$. They represent the locations in the *p-q* plane where steady-state solutions may reside, *if* they are stable. There are one or three such points in the above case, depending on the circuit losses. For sufficiently low loss and for a DC bias in a certain range, one of the fixed points becomes unstable, and the circuit may exhibit self-oscillation. However, due to the fact that the large-amplitude asymptotes of the tunnel diode function are in the first and third quadrants of the phase plane, it can be shown that the oscillations approach a limit cycle and do not grow without bound. We adopt a numerical approach to the integration of the above equations using a standard fourth-order Runge-Kutta method. A typical result for this case is shown in Figure 2. Here the circuit proceeds from any given initial condition to the limit cycle.

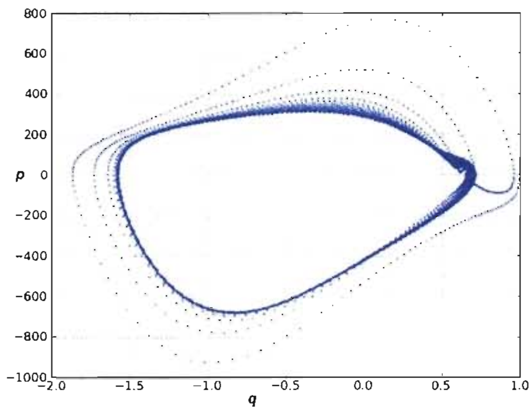

Fig. 2 Limit cycle under conditions of self-oscillation in the *p-q* phase plane.

A corresponding frequency domain picture shows a rich spectrum of harmonics is generated, though the lines remain relatively narrow, as shown in Figs. 3a and 3b, consistent with the limit cycle representation. The detailed spectral response depends on whether the drive frequency is above or below the self-oscillation frequency, reminiscent of frequency sum rules associated with the Manley-Rowe relations [11]. However, those rules were derived for lossless systems, and our system contains significant loss. By direct computation, we find that the powers associated with each harmonic do not exactly obey the Manley-Rowe power laws. Nonetheless, the frequency sum rules appear to apply.

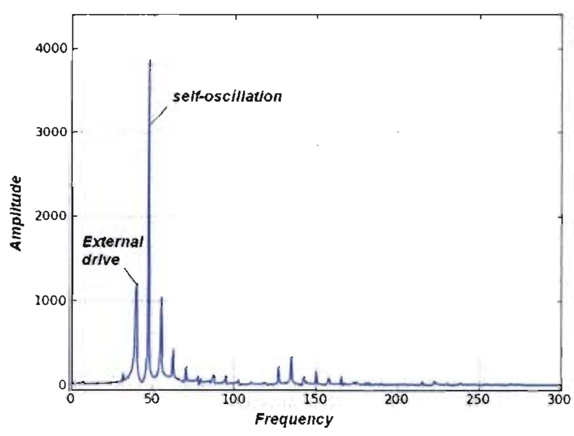

Fig 3a Spectral response for excitation below the self-oscillation frequency. Upper sidebands appear, consistent with frequency sum rules.

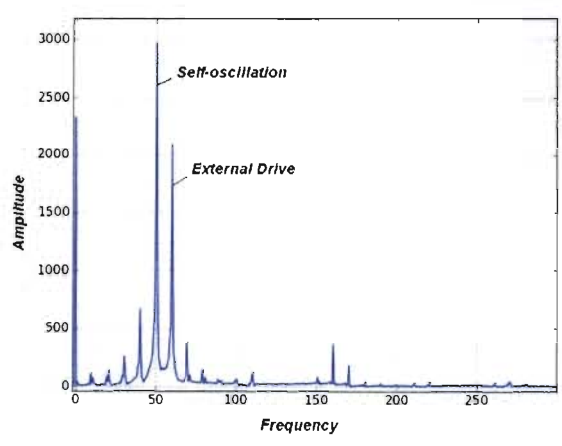

Fig 3b Spectral r4esponse for excitation above the self-oscillation frequency. Lower sidebands are dominant.

In addition, we observe an amplitude dependence of the response such that sufficiently large drive amplitudes tend to lock the response to the drive frequency, suppressing significant harmonic generation, as shown in Fig. 4.

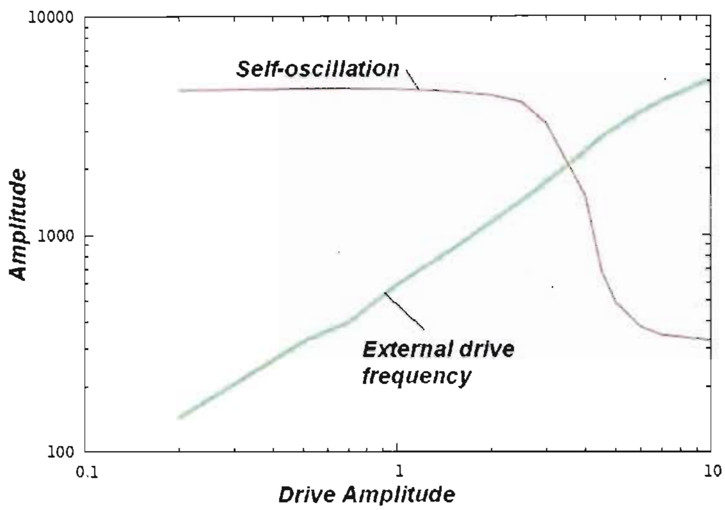

Fig. 4 Frequency locking as a function of drive amplitude.

We note that all of this behavior has been qualitatively confirmed in the experiments [10].

III. Onset of Chaos

When the circuit is driven by an external source V_0 in the transition region of Fig. 4, then the two oscillators may interact to produce a chaotic phase behavior, as shown in Fig. 5. However, it is not obvious from such a representation that the circuit behavior is truly chaotic. In this case it is helpful to use a Poincare Surface-of-Section to display the circuit behavior. The voltage is sampled once per cycle of the drive voltage and displayed over many cycles of the response. When the circuit is locked to the drive, the resulting display is confined to a small number of lines representing the harmonic periods present in the response. Chaos may cause randomness in both phase and amplitude, which is clearly shown to be confined to a specific operating band, as shown in Fig. 6.

A further corroboration of the chaotic behavior may be obtained by the well-known method of finding the largest Lyapunov exponent. This is done numerically by tracking the growth of neighboring phase-space trajectories. Formally, the *maximum* Lyapunov exponent is defined as the limit

$$\lambda_{\max} = \lim_{\substack{t \to \infty \\ R_0 \to 0}} \frac{1}{t} \log\left(\frac{\Delta R}{\Delta R_0}\right)$$

where ΔR is the maximum separation of adjacent trajectories that are initially separated by ΔR_0.

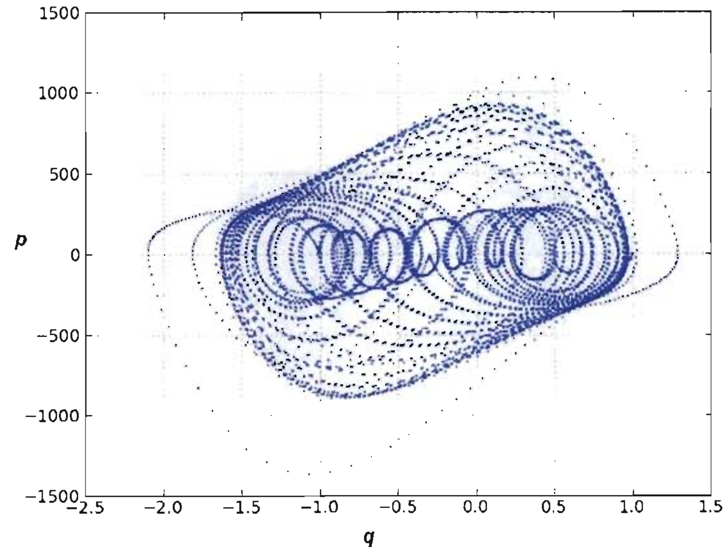

Fig. 5 Phase plane representation of chaotic behavior.

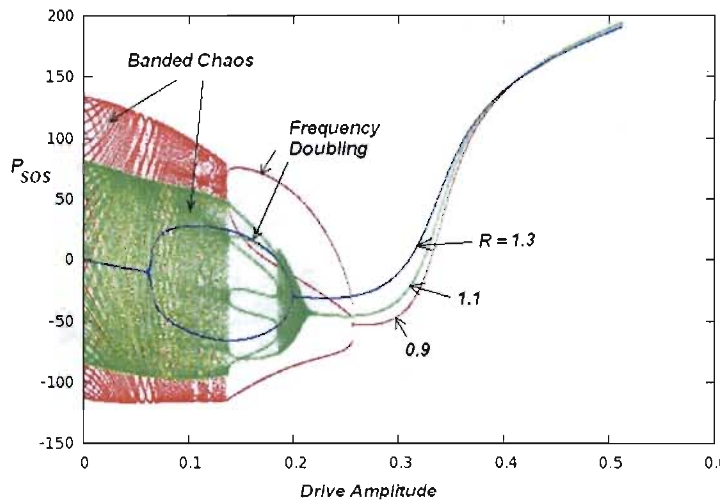

Fig. 6 Poincare Surface-of-Section. Note abrupt changes in period and chaotic Regions as a function of drive amplitude and circuit losses.

The results for the above case are shown in Fig. 7, indicating the Poincare SOS shows chaotic behavior just where the Lyapunov exponents become positive. It is worthwhile to note that self-oscillation can lead to chaotic behavior in conjunction with external excitation, when the internal and external frequencies are not quite locked, namely in the transition region shown in Fig. 4.

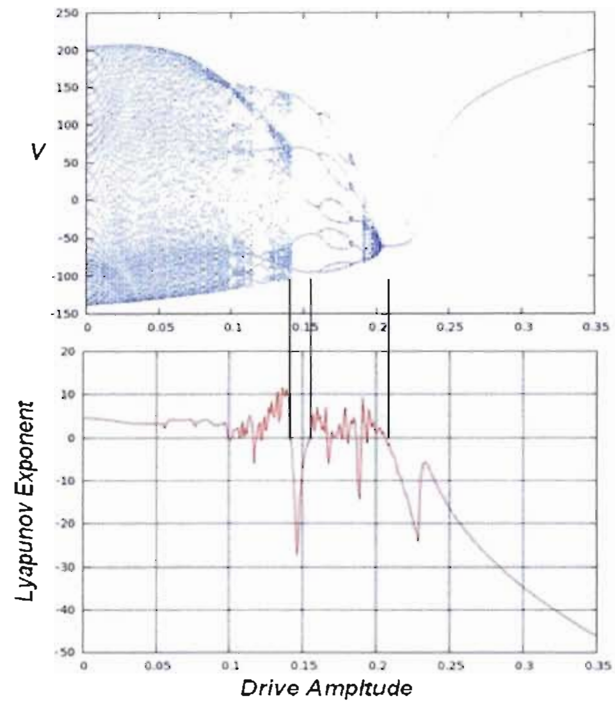

Fig. 7 Lyapunov exponents as a function of external drive amplitude. A positive exponent implies the existence of chaotic orbits caused by the interplay of internal and external oscillators. The chaotic behavior disappears for sufficiently large drive amplitude, locking the response to the drive frequency.

IV. Nonlinear Waves

It is worthwhile to study the properties of arrays of such nonlinear resonators. As a first attempt to understand the complex behavior of such systems, it is useful to restrict our attention to one-dimensional arrays, namely a nonlinear transmission line such as that shown in Fig. 8. These systems have received a great deal of study in recent years because of their propensity to produce solitons and rich harmonic generation [7].

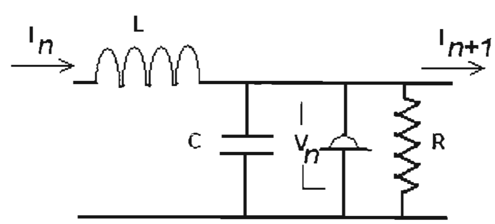

Fig. 8 Single node of an infinitely long transmission line loaded with tunnel diodes.

To find the nonlinear waves in this system, we use a direct approach of integrating a high-order system of coupled circuit equations where we introduce a pulse as an initial condition and observe its propagation through the system. Our results show that solitons can indeed exist on such arrays, but the nonlinear or soliton-like behavior is strongly determined by losses. In fact we observe a continuous transformation of the pulse propagation from that of a linear dispersive response to one in which the pulse shape and amplitude remain nearly unchanged as the pulse propagates, depending on the loss-per-cell. Hence we are evidently in the realm of *dissipative solitons* [12], which have been observed in many other systems, as shown in Fig. 9.

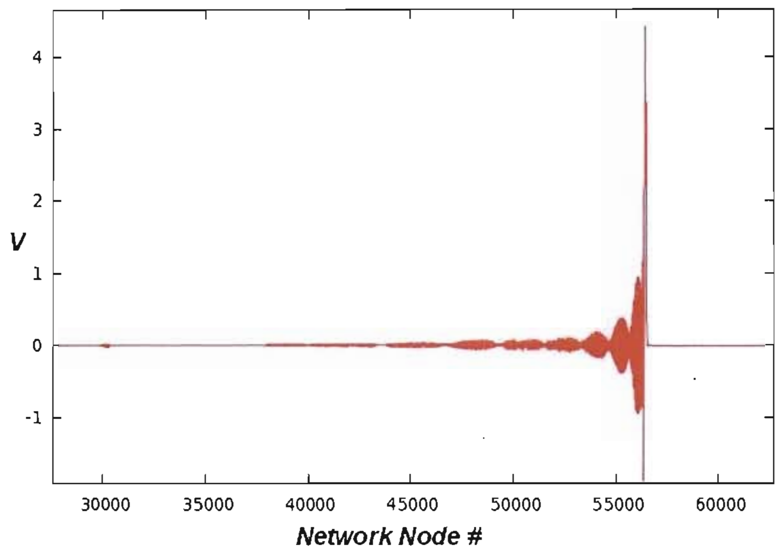

Fig. 9 Time domain representation of a disspative soliton on a tunnel-diode loaded transmission line. Note the existence of potentially chaotic phase in the tails of the pulse .

It is interesting to note that chaotic phase behavior appears to occur in the tails of the soliton, but not for the solitary pulse itself. Presumably the amplitude of the soliton edge causes frequency locking which suppresses much of the harmonic generation. It is conjectured that Fermi, Pasta Ulam Recurrence may be related to this phenomenon.

V. Discussion and Future Directions

In this work, we have explored numerically the phenomena that can be expected in single nonlinear cells that are being considered as meta-material building blocks. We have found rich harmonic generation and solitary wave generation are possible in such systems. Moreover, chaotic behavior can occur in specific bands determined by circuit parameters and bias settings. Qualitatively, our numerical results reproduce many of the features that have been observed experimentally for such cells.

We note that these results represent a fairly exotic taxonomy of effects which have yet to yield specific applications. However, there are allusions in the literature to possibilities including highly frequency-agile antenna systems, non-reciprocal (hence low detectability) receivers and encryption, among others. At the very least we hope to shed light on the connections between FPU Recurrence, dissipative solitons and chaos.

In future work we plan to extend these models to fully two- and even three-dimensional arrays of elementary cells. We will focus on questions related to specific harmonic generation, nonlinear wave propagation and overall system efficiency.

* Current Address: University of Oklahoma

The authors wish to acknowledge fruitful and helpful discussions with Peter Milonni and Stuart Trugman. Research supported by Los Alamos National Laboratory LDRD 20110027DR.

References

1. D. Schurig et al., Science **314**, 977 (2006).

2. A. Erentok et al., IEEE T. Ant. Prop. **56**, 691 (2008).

3. J. O'Hara et al.,Opt. Express **16**, 1786 (2008).

4. W. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).

5. J. O'Hara et al., Nat. Photonics **2**, 295 (2008)

6. P. Milonni, M-L Shih and J. Ackerhalt, Chaos in laser-matter interactions, World Scientific Lecture Notes in Physics, vol. 6, World Scientific, (1987)

7. M. Lin and W. Duan, Chaos, Solitons and Fractals, 24, p. 191 (2005)

8. E. Fermi, J. Pasta, S. Ulam, LASL Report. 1940, May (1955)

9. J. C. Sprott, Chaos and Time Series Analysis, Oxford University Press, (2003)

10. J. O'Hara, et. al. This proceedings

11. J. M. Manley and H. E. Rowe, Proceedings of the Institute of Radio Engineers, 44:904–913 (1956)

12. N. Akhmediev, A. Ankiewicz, Dissipative Solitons, Lecture Notes in Physics, Springer, (2005)