Modelling Co-movements and Tail Dependency in the International Stock Market via Copulae

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Abstract

This paper examines international equity market co-movements using time-varying copulae. We examine distributions from the class of *Symmetric Generalized Hyperbolic* (SGH) distributions for modelling univariate marginals of equity index returns. We show based on the goodness-of-fit testing that the SGH class outperforms the normal distribution, and that the Student-t assumption on marginals leads to the best performance, and thus, can be used to fit multivariate copula for the joint distribution of equity index returns. We show in our study that the Student-t copula is not only superior to the Gaussian copula, where the dependence structure relates to the multivariate normal distribution, but also outperforms some alternative mixture copula models which allow to reflect asymmetric dependencies in the tails of the distribution. The Student-t copula with Student-t marginals allows to model realistically simultaneous co-movements and to capture tail dependency in the equity index returns. From the point of view of risk management, it is a good candidate for modelling the returns arising in an international equity index portfolio where the extreme losses are known to have a tendency to occur simultaneously. We apply copulae to the estimation of the Value-at-Risk and the Expected Shortfall, and show that the Student-t copula with Student-t marginals is superior to the alternative copula models investigated, as well the *Riskmetics* approach.

Keywords: International equity market indices, Student-t distribution, symmetric generalized hyperbolic distribution, time-varying copula, Value-at-Risk, world stock index.

JEL: C13, C15, C16, C32, C52

AMS Subject Classification: 62F03, 62P20, 91B28

1 Introduction

Risk modelling requires the understanding of the evolution of portfolio returns where underlying assets often exhibit extreme movements simultaneously. The source of co-movements can be interpreted as the volatility-in-correlation and the tail dependence which is present in the returns of the single stocks, see Andersen, Bollerslev, Diebold and Ebens (2001), as well as in the equity index returns, see Solnik, Boucrelle and Le (1996). Studying correlation and co-movements in the international stock market is important for risk diversification of an international portfolio, see Embrechts, McNeil and Straumann (2001) and Sun, Rachev, Fabobozzi and Petko (2006). Studying the dependency suggests analysis of correlations within a single regional market as well as correlation between regional markets.

When dealing with a single market, one should take into account certain characteristics and stylized facts of the data which include non-Gaussianity, skewness and fat tails in the return distribution, as well as volatility clustering and the leverage effect. Therefore, an underlying realistic assumption about the distributional form of marginals is a crucial first step. Time series of financial data are high-dimensional and have typically a non-Gaussian behavior. Empirical studies have failed to support the assumption that return data follow a normal distribution due to its failure to capture the apparent heavy tails and excess kurtosis in the data, see Granger (2003). Alternative distributions should be used in order to capture characteristics of financial return data. We use several important cases of the *Symmetric Generalized Hyperbolic* (SGH) distribution family discussed in Platen and Rendek (2008) and Wenbo and Kercheval (2008). These include *Student-t*, *Normal Inverse Gaussian* (NIG), *Hyperbolic* (HYP) and *Variance Gamma* (VG) distributions. We show that they allow us to better capture stylized facts of the observed index returns. In particular, the Student-t distribution turns out to be preferable to the normal distribution and other distributions from the SGH family when dealing with equity risk management applications.

To study the dependency among international equity markets, linear correlation is not an appropriate measure of dependency. It fails to measure non-linear dependencies which arise when one works with models other than the multivariate normal one, and it often appears far too low when taking into account that extreme events occur simultaneously, see Embrechts, McNeil and Straumann (2001) and Solnik et al. (1996). A better understanding of the time-varying multivariate (conditional) distribution is vital to many applications in finance, including portfolio selection, option pricing, asset pricing, Value-at-Risk estimation etc. Copulae can be used to describe the dependencies among random variables. They allow to separate the modelling of the marginal distributions from the modelling of the dependence structure. We will use copulae to analyze the non-linear non-Gaussian dependencies between regional* markets, including the modelling of dependent extreme values.

When studying the dependency in international stock markets, we use some regional market indices of each country as a proxy for the market movements of this country. More precisely, we consider the following regional indices: S&P 500 for the USA, Dow Jones EURO STOXX 50 for continental Europe, FTSE 100 for the UK, and TOPIX for Japan. For modelling the world

*We will use the terminologies regional and country interchangeably.

equity market movements we use an equally weighted index EWI104s constructed as a weighted average of 104 world industry sector indices. While regional indices describe market movements of a specific country, the EWI104s describes general market movements, that is, fluctuations of the world stock market as a whole.

In our analysis we proceed as follows: In the first stage we specify marginal distributions for modelling equity market index returns of each country, as well as the world stock index returns. In the second stage we characterize the dependence structure among these by identifying the copula function which fits data best. It will turn out, based on goodness-of-fit testing, that among all distributions from the SGH family, the Student-t distribution provides the best fit. Furthermore, among all copula models investigated in our analysis the Student-t copula allows to model realistically co-movements of log-returns and to capture the tail dependence in the regional equity market indices. Thus, from the point of view of risk management, the Student-t copula with Student-t marginals is a good candidate for modelling returns in an international equity index portfolio since it captures well the extreme losses that have a tendency to occur simultaneously.

The paper is organized as follows: Section 2 discusses specifications of the marginal distributions, which include a brief overview of the SGH family and goodness-of-fit testing for the fitted distributions. Section 3 deals with modelling dependencies by specifying an appropriate copula family. It briefly reviews some facts on copulae, followed by fitting static and time-varying copulae to the data. Finally, applications in risk management are discussed in Section 4. Section 5 summarizes the findings.

2 Specification of Marginal Distributions

Empirical studies have failed to support the assumption that equity index return data follow a normal distribution, since it exhibits much heavier tails and excess kurtosis. We use as alternative distributions several important cases from the family of *Symmetric Generalized Hyperbolic* (SGH) distributions discussed in Hurst and Platen (1997), Platen and Rendek (2008) and Wenbo and Kercheval (2008). These include the *Student-t*, *Normal Inverse Gaussian* (NIG), *Hyperbolic* (HYP) and *Variance Gamma* (VG) distributions.

2.1 Symmetric Generalized Hyperbolic Distributions

Barndorff-Nielsen (1977) has introduced the family of generalized hyperbolic distributions, which has been discussed in its general form in Jørgensen (1982), Barndorff-Nielsen and Stelzer (2004), McNeil, Frey and Embrechts (2005). For the purpose of our research, we will concentrate on the symmetric representation, that is, when the location of the distribution and the skewness parameter are set equal to zero. Therefore, we consider the SGH density in the form:

$$f_X(x) = \frac{1}{\delta\sigma K_\lambda(\bar{\alpha})}\sqrt{\frac{\bar{\alpha}}{2\pi}}\left(1+\frac{x^2}{(\delta\sigma)^2}\right)^{\frac{1}{2}(\lambda-\frac{1}{2})} K_{\lambda-\frac{1}{2}}\left(\bar{\alpha}\sqrt{1+\frac{x^2}{(\delta\sigma)^2}}\right) \tag{1}$$

where $\alpha \neq 0$ if $\lambda \geq 0$ and $\delta \neq 0$ if $\lambda \leq 0$. $K_\lambda(\cdot)$ denotes a modified *Bessel function* of the third kind with index λ, see Abramowitz and Stegun (1972). The parameters λ and $\bar{\alpha}$ are invariant under scale transformations and can be interpreted as the *shape parameters* for the tails of the distribution. Varying λ and $\bar{\alpha}$ allows to specify special cases of the SGH distribution. In particular, we will investigate the following important special cases: the *Variance Gamma* distribution is obtained by setting $\bar{\alpha} = 0$ and $\lambda > 0$, see Madan and Seneta (1990); the *Student-t* distribution assumes $\bar{\alpha} = 0$ and $\lambda < 0$, see Praetz (1972); the *Hyperbolic* distribution is specified when $\lambda = 1$, see Eberlein and Keller (1995); and the *Normal Inverse Gaussian* distribution is obtained by setting $\lambda = -0.5$, see Barndorff-Nielsen (1995). The first two special cases are limiting cases and can be described by taking into account the limiting behavior of the Bessel function involved, see Platen and Heath (2006) for details. The symmetric Variance Gamma density is a two parameter density where λ is a shape parameter. Smaller values of λ indicate increasingly heavier tails. When $\lambda \to \infty$ the Variance Gamma density asymptotically approaches the Gaussian density. For the Student-t density we consider $\lambda \leq -1$ in which case the number of degrees of freedom equals $\nu = -2\lambda \geq 2$. We do not consider the case when $-1 < \lambda < 0$, since it corresponds to $\nu < 2$ for which the normalization constant diverges, see Platen and Rendek (2008), and it is not relevant to the financial applications that we have in mind. Note that in the Student-t case the parameter σ is not the standard deviation of the random variable X, which is $\sigma_X = \sigma\sqrt{\frac{\nu}{\nu-2}}$. When the number of degrees of freedom ν decreases, we observe an increase in the tail heaviness of the density, which implies a larger probability of extreme values. Additionally, with an increase of the degrees of freedom $\nu \to \infty$, the Student-t density converges asymptotically to the Gaussian density. Further details on the representation of the density functions can be found in Platen and Rendek (2008).

2.2 Empirical Methodology: ARMA-GARCH Model for the Marginals

To capture distributional characteristics of index returns which will be used to identify marginals for the joint distribution modelled via copulae, we implement an ARMA-GARCH model as proposed in Dias (2004), McNeil et al. (2005) and Sun et al. (2006). Consider the sequence of i.i.d. random variables $(u_t)_{t\geq 0}$ with zero mean and unit variance. We assume that the log-return process $(X_t)_{t\geq 0}$ follows the ARMA(p_1,q_1)-GARCH(p_2,q_2) model, that is, it satisfies:

$$X_t = \mu_t + \varepsilon_t \text{ with } \varepsilon_t = \sigma_t u_t. \tag{2}$$

Here the process for the conditional mean equation is defined as follows:

$$\mu_t = \mu + \sum_{i=1}^{p_1} a_i(X_{t-i} - \mu) + \sum_{j=1}^{q_1} b_j \varepsilon_{t-j}. \tag{3}$$

The innovations ε_t have by definition a conditional variance $Var(\varepsilon_t|\mathcal{F}_t) = \sigma_t^2$ in the form

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p_2} \alpha_i \sigma_{t-i}^2 + \sum_{j=1}^{q_2} \beta_j \varepsilon_{t-j}^2, \tag{4}$$

with $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, and u_t is independent of $(X_s)_{s\leq t}$. We fit univariate ARMA(1,1)-GARCH(1,1) models via maximum likelihood for each marginal series assuming that the fitted residuals

$$\widehat{u}_t = \frac{X_t - \widehat{\mu}_t}{\widehat{\sigma}_t} = \frac{\widehat{\varepsilon}_t}{\widehat{\sigma}_t} \tag{5}$$

follow certain distributional assumptions, and are approximately i.i.d. We assume that the residuals come from either the normal distribution, or one of the SGH distributions: Student-t, NIG, HYP or VG. McNeil et al. (2005) argue that the GARCH(1,1) model with Student-t innovations is sufficient to remove dependencies in the return series. Moreover, the ARMA term is usually not necessary to generate filtered i.i.d. observations, see Dias (2004) and Wenbo and Kercheval (2008), which allows to set $p_1 = 0$ and $q_1 = 0$, leading to an assumption on a constant conditional mean: $\mu_t = \mu$. After fitting univariate time series by maximum likelihood, we will use a goodness-of-fit test described below, to select the best model for the marginals, and proceed with modelling dependencies via copulae.

2.3 Data

We consider the following market capitalization weighted regional market indices to capture the equity market movements of a specific country: the Standard & Poor's equity index (S&P 500) for the USA, the Eurozone Dow Jones EURO STOXX 50 (DJ EURO STOXX 50) for continental Europe, the Financial Times and London Stock Exchange Index (FTSE 100) for the UK, the Tokyo Stock Price index (TOPIX) for Japan. The data, covering the time span from 01 January 1987 to 10 March 2006 is available from *Datastream Thomson Financial*. The world stock index we use proxies the world stock market movements and is the equally weighted index EWI104s which was constructed in Lee and Platen (2006) and Platen and Rendek (2008) based on 104 world industry sector indices as constituents provided by *Datastream Thomson Financial*. The EWI104s is an almost ideally diversified index over all $d = 104$ industries where all fractions are set equal to $\pi_{\delta_{EWI},t} = 1/d$ with $\sum_{j=1}^{d} \pi_{\delta_{EWI},t} = 1$, $j \in \{1, 2, ..., d\}$. The raw index data for the EWI104s, as well as the four regional equity indices are plotted in Figure 1 for the time span from 01 January 1987 to 10 March 2006. The log-returns and the fitted annualized volatilities obtained using equations (2)-(4) are plotted in Figures 3 and 4, respectively. From these figures we observe that the returns are concurrently correlated and that the regional indices appear to be more volatile than the EWI104s.

2.4 Analysis of Log-returns of the Marginals

First, in order to get a visual impression of the shape of the log-returns, we assume constant volatility and standardize the data to get a sample mean of zero and a sample variance of one. Figure 2 represents the histogram for the pooled data taken from all regional indices for the period from 01 January 1987 to 10 March 2006 displayed in log-scale vs. the normal density (in the left panel) and the Student-t density (in the right panel). We observe already visually an excellent fit of the log-returns to the Student-t density compared to a poor fit to the normal density.

We fit univariate returns of the EWI104s, S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX using equations (2)-(4), assuming either normal, or SGH innovations. To assure that the fitted residuals can be interpreted as being approximately i.i.d., we examine serial correlation in the time-series using sample autocorrelograms and *Ljung-Box statistics*, introduced by Ljung and Box (1978) for the univariate case. The test statistic for testing the null hypothesis of no serial correlation in time series is the modification of the Portmanteau statistic, see Box and Pierce (1970), Hosking (1980), Hosking (1981). With T denoting the number of observations, the test statistic for no serial correlation in lag $m \geq 0$ in the univariate time series has the form:

$$Q(m) = T(T+2)\sum_{l=1}^{m} \frac{\widehat{\rho}_l^2}{T-l}, \tag{6}$$

where ρ_l is a sample correlation at lag $l \geq 0$. Under the null hypothesis of no serial correlation the test statistic is Chi-squared distributed with m degrees of freedom: $Q(m) \sim \chi^2_m$. Using Student-t innovations[†], the results for the test statistics up to lag five are summarized in Table 2, with the p-values given in parentheses. We observe that the values of the test statistics for the absolute and the squared fitted residuals are not significant at the 1% level for the EWI104s, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX, indicating that we cannot reject the hypothesis of no serial correlation in these time series. For the S&P 500 we reject the null hypothesis at lag one, two and three for the absolute fitted residuals at the 1% significance level. However, the autocorrelogram in Figure 5 shows that there is no evidence of serial dependence of the absolute and squared residuals for the higher lags for all time series. Thus, since the absolute and the squared filtered returns show little evidence of serial correlation, they can be used for calibrating parameters of various distributions, as well as for performing goodness-of-fit testing for the best fitting distribution.

For goodness-of-fit testing for the residual assumption we use the *Anderson-Darling* (AD) distance and the *Kolmogorov-Simirnov* (KS) distance proposed by Rachev and Mittnik (2000) and Sun et al. (2006):

$$AD = \frac{\sup_{x\in\mathcal{R}} |F_s(x) - \hat{F}(x)|}{\sqrt{\hat{F}(x)(1-\hat{F}(x))}}, \tag{7}$$

$$KS = \sup_{x\in\mathcal{R}} |F_s(x) - \hat{F}(x)|, \tag{8}$$

where $F_s(x)$ denotes the empirical sample distribution and $\hat{F}(x)$ is the estimated distribution. The drawback of the KS statistics is that it is sensitive close to the center of the distribution and not at the tails. The AD statistics allows to overcome this drawback. It captures both, the deviations around the median of the distribution, as well as the discrepancies in the tails. Table 1 summarizes the results of the Anderson-Darling distance for log-returns of the indices using different marginal distributions estimated from the entire time series of the period from 01 January 1987 to 10 March 2006, assuming *constant* volatility. We observe that all distributions from the SGH family provide a considerably better fit than the normal distribution. The Student-t assumption on the marginals leads to the smallest mean values and the smallest dispersion for the AD statistics providing the best fit to the return data.

However, modelling univariate marginals in this way, we assume constant volatility, and thereby, ignore time dependence of serial volatility. Therefore, in the following we apply the ARMA(1,1)-GARCH(1,1) model described by equations (2)-(4) to estimate *time-varying* volatility $\widehat{\sigma}_t$. We apply goodness-of-fit testing to the filtered series $\widehat{u}_t$ which are approximately i.i.d. The results are summarized in Table 3, and Figure 6 shows box-plots for the AD distance for the normal

[†]We do not report the results for other fitted distributions from the SGH family, since they perform similarly.

distribution and all distributions from the SGH family. As can be observed from the tables and the box-plots, the ARMA(1,1)-GARCH(1,1) model with Student-t assumption exhibits smaller mean value for the AD distance and less outliers than the normal distribution and the other distributions from the SGH family. This demonstrates again, as already in the case of constant volatility, the superiority of this fit compared to those of all other models.

Altogether, we obtain that the ARMA(1,1)-GARCH(1,1) model with Student-t residuals provides the best fit to the index data. Thus, assuming Student-t marginals for returns of each regional equity index, we estimate below the dependence structure by fitting multivariate copulae to the data.

3 Specification of the Copula Family

Copulae provide a natural way for measuring the dependence structure between d random variables. These are multivariate distribution functions on the unit cube $[0,1]^d$. They allow to connect their one-dimensional uniform-(0,1) marginals to the joint cumulative distribution function. The formal definition can be found in Nelsen (1998).

Sklar's theorem, see Joe (1997) for a proof, shows that every distribution function can be decomposed into its marginal distribution and a copula, and every distribution function can be obtained by coupling marginal distributions with the dependence structure given by a copula. More precisely, Sklar's theorem states that if F is a d-dimensional distribution function with marginals $F_1 \ldots, F_d$, then there exists a copula C with

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\} \tag{9}$$

for every $x_1, \ldots, x_d \in \overline{\mathbb{R}}$. If $F_1, \ldots, F_d$ are continuous, then C is unique. On the other hand, if C is a copula and $F_1, \ldots, F_d$ are distribution functions, then the function F defined in (9) is a joint distribution function with marginals $F_1, \ldots, F_d$.

Thus, if $X = (X_1, \ldots, X_d)^\top$ is a random vector with distribution $X \sim F_X$ and continuous marginals $X_j \sim F_{X_j}$ $(j = 1, \ldots d)$, then the copula of X is the distribution function C_X of $u = (u_1, \ldots, u_d)^\top \in [0,1]^d$, where $u_j = F_{X_j}(x_j)$:

$$C_X(u_1, \ldots, u_d) = F_X\{F_{X_1}^{-1}(u_1), \ldots, F_{X_d}^{-1}(u_d)\}. \tag{10}$$

For an absolutely continuous copula C we can define the copula density as

$$c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}. \tag{11}$$

Given a random variable $X = (X_1, \ldots, X_d)^\top$, with absolute continuous distribution function F and copula C_X, the density c_X is obtained by differentiating C_X in (10):

$$c_X(u_1, \ldots, u_d) = \frac{f\{F_{X_1}^{-1}(u_1), \ldots, F_{X_d}^{-1}(u_d)\}}{\prod_{j=1}^d f_j\{F_{X_j}^{-1}(u_j)\}} \tag{12}$$

where f is the joint density of F_X and f_j is the density of F_{X_j}. The density function of X is then given by

$$f(x_1, \ldots, x_d) = c_X(u_1, \ldots, u_d) \cdot \prod_{j=1}^d f_j(x_j)$$

with $x_j = F_{X_j}^{-1}(u_j)$.

We will use in our analysis the following notion of a *survival copula*. If F is the distribution function of the random vector $X = (X_1, \ldots, X_d)^\top$ with marginals $F_1 \ldots, F_d$, then there exist a copula C^* with

$$\overline{F}(x_1, \ldots, x_d) = C^*\{\overline{F}_1(x_1), \ldots, \overline{F}_d(x_d)\} \tag{13}$$

where $\overline{F}(x_1, \ldots, x_d) = P(X_1 > x_1, \ldots X_d > x_d)$. C^* is the so-called *survival copula* corresponding to C. In particular, for the bivariate case the survival copula can be defined as follows:

$$C^*(u_1, u_2) = 1 - u_1 - u_2 + C(1 - u_1, 1 - u_2), \tag{14}$$

see Nelsen (1998).

3.1 Copulae Examples

Throughout the paper we will concentrate mostly on two popular copula families: the *elliptical copulae family*, which include the Gaussian copula and the Student-t copula, and the *Archimedean copulae family*, which has e.g. Frank, Gumbel and Clayton copulae as special members. Furthermore, we consider for completeness some mixture models of Archimedean copulae where the distribution function has the form of a convex combination of two or more copulae. These d-dimensional parametric copulae are presented below. A copula parameter controls the degree of dependence. Further copula models, in particular, Ali-Mikhail-Haq, Plakett copulae etc. can be found e.g. in Joe (1993) and Nelsen (1998).

3.1.1 Elliptical Copulae

Elliptical copulae have a dependence structure generated by the elliptical distributions, see e.g. Lindskog, McNeil and Schmock (2001). These include normal and Student-t distributions, as well as the stable distribution class discussed in e.g. Rachev and Mittnik (2000) and Rachev and Han (2000). The modelling of dependency using elliptical distributions can be found in Hult and Lindskog (2001), Fang, Fang and Kotz (2002) and Frahm, Junker and Szimayer (2003). Its applications in finance and risk management are discussed, for instance, in Breymann, Dias and Embrechts (2003), McNeil et al. (2005) and Dias and Embrechts (2008). The Gaussian copula and Student-t copula are presented below.

Gaussian Copula

The Gaussian copula expresses the dependence structure of the multivariate normal distribution, i.e. normal marginal distributions are combined with a Gaussian copula to form a multivariate normal distribution. If $Y_j \sim N(0, 1)$ and $Y = (Y_1, \ldots, Y_d)^\top \sim N_d(0, \Psi)$, where Ψ denotes a correlation matrix, an explicit expression for the Gaussian copula is given by

$$C_\Psi^{Ga}(u_1, \ldots, u_d) = F_Y\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\} \tag{15}$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} 2\pi^{-\frac{d}{2}} \mid \Psi \mid^{-\frac{1}{2}} \exp\left(-\frac{1}{2} r^\top \Psi^{-1} r\right) dr_1 \ldots dr_d.$$

Defining $\zeta_j = \Phi^{-1}(u_j)$, $\zeta = (\zeta_1, \ldots, \zeta_d)^\top$, the density of the Gaussian copula can be written as

$$c_\Psi^{Ga}(u_1, \ldots, u_d) = \mid \Psi \mid^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^\top(\Psi^{-1} - \mathcal{I}_d)\zeta\right\}. \quad (16)$$

Student-t Copula

The Student-t copula expresses the dependence structure from the multivariate Student-t distribution. Let $X = (X_1, \ldots, X_d)^\top \sim t_d(\nu, \mu, \Sigma)$ have a multivariate Student-t distribution with ν degrees of freedom, mean vector μ and positive-definite dispersion or scatter matrix Σ. The copula remains invariant under a standardization of the marginal distributions[‡]. This means that the copula of a $t_d(\nu, \mu, \Sigma)$ distribution is identical to that of a $t_d(\nu, 0, \Psi)$ distribution where Ψ is the correlation matrix associated with Σ. The unique copula is the Student-t copula $C_X = C_{\nu,\Psi}^t$. For $u = (u_1, \ldots, u_d)^\top \in [0, 1]^d$, the Student-t copula is given by

$$C_{\nu,\Psi}^t(u_1, \ldots, u_d) = t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}, \quad (17)$$

where t_ν^{-1} is the quantile function from the univariate t-distribution. The density of the t-copula is given by

$$c_{\nu,\Psi}^t(u_1, \ldots, u_d) = \frac{t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\}}{\prod_{j=1}^d t_{\nu,\Psi}\{t_\nu^{-1}(u_j)\}}. \quad (18)$$

With $\zeta_j = t_\nu^{-1}(u_j)$ the density of the t-copula can be expressed as:

$$c_{\nu,\Psi}^t(u_1, \ldots, u_d) = \mid \Psi \mid^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu+d}{2}) \left\{\Gamma(\frac{\nu}{2})\right\}^{d-1} \left(1 + \frac{1}{\nu}\zeta^\top\Psi^{-1}\zeta\right)^{-\frac{\nu+d}{2}}}{\left\{\Gamma(\frac{\nu+1}{2})\right\}^d \prod_{j=1}^d \left(1 + \frac{1}{\nu}\zeta_j^2\right)^{-\frac{\nu+1}{2}}}. \quad (19)$$

3.1.2 Archimedean Copulae

Gumbel, Clayton and Frank copulae belong to the family of so-called Archimedean copulae which have been studied in relation with modelling portfolio credit risk in McNeil et al. (2005), Dias (2004) and Wu, Valdez and Sherris (2006). These copulae have a simple closed form and are briefly reviewed below.

Clayton Copula

The Clayton copula with the dependence parameter $\theta \in (0, \infty)$ is defined by

$$C_\theta(u_1, \ldots, u_d) = \left\{\left(\sum_{j=1}^d u_j^{-\theta}\right) - d + 1\right\}^{-1/\theta} \quad (20)$$

with the density given by:

$$c_\theta(u_1, \ldots, u_d) = \prod_{j=1}^d \{1 + (j-1)\theta\} u_j^{-(\theta+1)} \left(\sum_{j=1}^d u_j^{-\theta} - d + 1\right)^{-(1/\theta+d)}. \quad (21)$$

[‡]In fact, it remains invariant under any series of strictly increasing transformations of the components of the random vector X, see Nelsen (1998).

As the copula parameter θ tends to infinity, the dependence becomes maximal and as θ tends to zero, we have independence. The Clayton copula can mimic lower tail dependence but no upper tail dependence.

Gumbel Copula

The Gumbel copula with the dependence parameter $\theta \in [1, \infty)$ is given by

$$C_\theta(u_1, \ldots, u_d) = \exp\left[-\left\{\sum_{j=1}^{d}(-\log u_j)^\theta\right\}^{1/\theta}\right]. \tag{22}$$

For $\theta > 1$ this copula generates an upper tail dependence, while for $\theta = 1$ it reduces to the product copula (i.e. independence): $C_\theta(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j$. Maximal dependence is achieved when θ tends to infinity.

3.1.3 Mixture Copula Models

Mixture models as introduced in Joe (1993) can be obtained by building convex combinations of two or more copulae. Denoting C^A and C^B copulae with dependence parameters θ_1 and θ_2, respectively, the mixture model has a form:

$$C_X(u_1, \ldots, u_d, \theta) = \theta_3 C_X^A(u_1, \ldots, u_d, \theta_1) + (1 - \theta_3) C_X^B(u_1, \ldots, u_d, \theta_2). \tag{23}$$

In the following, we will consider four mixture models studied in Dias (2004) when modelling dependencies between FX rates. Angel Canela and Pedreira Collazo (2006) and Hu (2006) study mixture models for modelling dependencies across international financial markets.

3.2 Dependence and Tail Dependence

Before dealing with the tail dependence coefficient measured via copulae, we briefly recall the concept of correlation used to measure the degree of dependence among random variables. Well-known measures of dependence include the Pearson correlation coefficient r, Spearmann's ρ and Kendall's τ. While Pearson's linear correlation depends on the distribution of the univariate marginals (i.e., keeping the dependence structure constant, different marginals might lead to different values for the joint distribution, see Dias (2004)), the other two rank correlations are independent of the univariate marginal distributions. For properties of the dependence measures we refer to Embrechts, McNeil and Straumann (2001). The estimates obtained for these coefficients using log-returns of the sector indices S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX are summarized in Table 4. We observe high values for dependence measures between the Dow Jones EURO STOXX 50 and the FTSE 100, and low dependence between the Japanese TOPIX and the other indices.

3.2.1 Tail Dependence

While Pearson's correlation coefficient measures linear dependence among random variables, *tail dependence* coefficients allow to measure the extreme dependence in the tails of the multivariate distribution. These appear to be particularly useful in insurance and risk management when modelling the joint (dependent) risk, see e.g. Wang (1997) and Embrechts, McNeil and Straumann (2001). The concepts of lower and upper tail dependence refer to the study of the dependence between extreme values in the lower and in the upper tails. The notion of tail dependence in relation to copulae first appeared in Joe (1997). For the bivariate case, the upper and the lower tail dependence coefficients are presented below.

Let (U_1, U_2) be a pair of uniform variables on the unit square $[0,1]^2$, the upper tail dependence coefficient $\lambda_u \in [0,1]$ is defined as

$$\lambda_u = \lim_{u \to 1-} P(U_1 > u | U_2 > u) = \lim_{u \to 1-} \frac{C^*(u,u)}{1-u}. \tag{24}$$

Similarly, the lower tail dependence coefficient $\lambda_l \in [0,1]$ is defined as

$$\lambda_l = \lim_{u \to 0+} P(U_1 \leq u | U_2 \leq u) = \lim_{u \to 0+} \frac{C(u,u)}{u}. \tag{25}$$

If the coefficient of upper tail dependence $\lambda_u \in (0,1]$, then U_1 and U_2 are said to be asymptotically dependent in the upper tail, and if $\lambda_u = 0$, then U_1 and U_2 are said to be asymptotically independent in the upper tail. Similarly, if $\lambda_l \in (0,1]$ or $\lambda_l = 0$, then U_1 and U_2 are said to be asymptotically dependent, or independent, respectively, in the lower tail. For properties of the lower and the upper tail dependence coefficients we refer to Embrechts, Lindskog and McNeil (2001) and Embrechts, McNeil and Straumann (2001). Hu (2006) reviews dependence and tail dependence measures for the mixture copula models. The following result for the Student-t copula, as well the derivations for other copula models can be found in Embrechts, McNeil and Straumann (2001) and McNeil et al. (2005).

3.2.2 Example on Student-t Tail Dependence

The Student-t copula generates symmetric tail dependence. The tail dependence coefficients are defined by

$$\lambda_u = \lambda_l = 2t_{\nu+1}\sqrt{(\nu+1)(1-\rho)/(1+\rho)}, \tag{26}$$

where t_ν denotes the Student-t distribution function, ν is the number of degrees of freedom, and ρ is the correlation coefficient.

Table 5 shows estimated coefficients of lower and upper tail dependence $\lambda_u = \lambda_l = \lambda$, the estimated number of degrees of freedom ν and the estimated correlation ρ for the Student-t copula of the pairs (S&P 500, DJ EURO STOXX 50), (S&P 500, FTSE 100), (S&P 500, TOPIX) and (FTSE 100, TOPIX), as well as a 3-dimensional copula of (S&P 500, DJ EURO STOXX 50, FTSE 100), and a 4-dimensional copula of (S&P 500, DJ EURO STOXX 50, FTSE 100, TOPIX). We observe that the tail dependence coefficient λ decreases as ν increases. For example, for all combinations, in which the TOPIX is not present, the estimated ν ranges between 3.6 and 4.4, and λ lies between 0.18 and 0.39. However, when the TOPIX is included, we observe an increase in the estimated number of degrees of freedom ν, which is maximal for the pair (S&P 500, TOPIX) where it reaches 11.26, in which case the tail dependence coefficient λ is nearly zero. We will use this result later when testing different copula models to select the best performing copula family.

3.3 Copula Estimation

Consider a vector of random variables: $X = (X_1, ..., X_d)^\top$ with parametric univariate marginal distributions $F_{X_j}(x_j, \delta_j)$, $j = 1, ..., d$. Furthermore, let a copula belong to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. From Sklar's Theorem the distribution of X can be expressed as

$$F_X(x_1, \dots, x_d) = C\{F_{X_1}(x_1; \delta_1), \dots, F_{X_d}(x_d; \delta_d); \theta\} \tag{27}$$

and its density as

$$f(x_1, \dots, x_d; \delta_1, \dots, \delta_d, \theta) = c\{F_{X_1}(x_1; \delta_1), \dots, F_{X_d}(x_d; \delta_d); \theta\} \prod_{j=1}^{d} f_j(x_j; \delta_j), \tag{28}$$

where

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d} \tag{29}$$

is a copula density. For a sample of observations $\{x_t\}_{t=1}^T$ where $x_t = (x_{1,t}, \dots, x_{d,t})^\top$, and a vector of parameters $\alpha = (\delta_1, \dots, \delta_d, \theta)^\top \in \mathbb{R}^{d+1}$ the likelihood function is given by

$$L(\alpha; x_1, \dots, x_T) = \prod_{t=1}^{T} f(x_{1,t}, \dots, x_{d,t}; \delta_1, \dots, \delta_d, \theta) \tag{30}$$

and the corresponding log-likelihood function by

$$\ell(\alpha; x_1, \dots, x_T) = \sum_{t=1}^{T} \ln\left[c\{F_{X_1}(x_{1,t}; \delta_1), \dots, F_{X_d}(x_{d,t}; \delta_d); \theta\}\right] + \sum_{t=1}^{T}\sum_{j=1}^{d} \ln\left[f_j(x_{j,t}; \delta_j)\right]. \tag{31}$$

Our objective is to maximize this log-likelihood numerically. The estimation can be performed, for instance, in the following three different ways, employing the *exact maximum likelihood* (EML), the *inference for marginals* (IFM) and the *canonical maximum likelihood* (CML) method.

3.3.1 Exact Maximum Likelihood (EML)

The exact maximum likelihood (EML) method is straightforward, it estimates the parameter α in one step through

$$\tilde{\alpha}_{EML} = \arg\max_{\alpha} \ell(\alpha). \tag{32}$$

The estimates $\tilde{\alpha}_{EML} = (\tilde{\delta}_1, \dots, \tilde{\delta}_d, \tilde{\theta})^\top$ solve the first order condition

$$(\partial\ell/\partial\delta_1, \dots, \partial\ell/\partial\delta_d, \partial\ell/\partial\theta) = 0. \tag{33}$$

The drawback of the EML method is that with an increasing scale of the problem the algorithm becomes computationally challenging.

3.3.2 Inference for Marginals (IFM)

In the inference for marginals (IFM) method parameters for marginals and copulae are estimated separately, which represents a sequential two-step maximum likelihood method, see McLeish and Small (1988) and Joe (1997). The parameters δ_j from the marginal distributions are estimated in the first step and the dependence parameter θ is estimated in the second step after the estimated marginal distributions have been substituted into the copula. For $j = 1, \ldots, d$ the log-likelihood function for each of the marginal distributions is given by

$$\ell_j(\delta_j) = \sum_{t=1}^{T} \ln f_j \left[x_{j,t}; \delta_j\right] \tag{34}$$

and the estimated marginal parameter is given by

$$\hat{\delta}_j = \arg\max_{\delta} \ell_j(\delta_j). \tag{35}$$

The pseudo log-likelihood function

$$\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) = \sum_{t=1}^{T} \ln \left[c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \ldots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}\right] \tag{36}$$

is maximized over θ to obtain the estimator $\hat{\theta}$ for the dependence parameter θ. The estimates $\hat{\alpha}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve the first order condition

$$(\partial\ell_1/\partial\delta_1, \ldots, \partial\ell_d/\partial\delta_d, \partial\ell/\partial\theta) = 0. \tag{37}$$

3.3.3 Canonical Maximum Likelihood (CML)

In contrast to the EML and IFM methods, where we have to make an assumption about the parametric form of the marginal distributions, the canonical maximum likelihood (CML) method maximizes the pseudo log-likelihood function with empirical marginal distributions:

$$\ell(\theta) = \sum_{t=1}^{T} \ln \left[c\{\hat{F}_{X_1}(x_{1,t}), \ldots, \hat{F}_{X_d}(x_{d,t}); \theta\}\right]. \tag{38}$$

Here the empirical marginal cumulative distribution function is given by

$$\hat{F}_{X_j}(x) = \frac{1}{T+1} \sum_{t=1}^{T} \mathbf{1}_{\{X_{j,t} \le x\}}, \tag{39}$$

see Genest and Rivest (2002). Using this method, the parameter can be estimated in one step by using the estimate

$$\hat{\theta}_{CML} = \arg\max_{\theta} \ell(\theta). \tag{40}$$

3.4 Fitting Static and Time-Varying Copulae

In the following, we aim to fit a parametric copula, that is, to estimate the copula dependence parameter, assuming that the unknown marginals are Student-t with the fixed number of degrees of freedom corresponding to 3.62 for the S&P 500, 4.22 for the Dow Jones EURO STOXX 50, 4.15 for FTSE 100, and 4.31 for TOPIX. To estimate the dependence parameter of the copula, we proceed as follows. First, we transform the original data to the "copula scale", we apply a probability integral transform to obtain uniformly $[0,1]$-distributed values. Then, we apply the IFM method to estimate different copulae in a static, as well as, time-varying setting.

A static copula is assumed to estimate the global (average) dependence parameter using log-return data from the time interval covering the whole time span from 01 January 1987 to 10 March 2006. Tables 6 and 7 show estimated copula parameters using different one-parameter families of copulae, as well as some mixture copula models for the pairs formed by the S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX. The standard errors are reported in parenthesis. Table 8 summarizes the results for a 3-dimensional copula of (S&P 500, DJ EURO STOXX 50, FTSE 100) and a 4-dimensional copula of (S&P 500, DJ EURO STOXX 50, FTSE 100, TOPIX). In the case of the mixture models, the parameters θ_1 and θ_2 represent the dependence parameters for the first and second terms of the mixture, respectively, and θ_3 is the mixture parameter which gives the proportion of the first term. From these tables we observe that the strength of the dependence decreases when we increase the dimension of the problem by adding an additional risk factor in the analysis.

3.4.1 Model Selection

To judge on the performance of each model fitted, we provide the *Akaike Information Criterion* (AIC) introduced in Akaike (1974):

$$AIC = -2l(\alpha; x_1, \ldots, x_T) + 2q. \tag{41}$$

In (41) $l(\alpha; x_1, \ldots, x_T)$ denotes the maximized value of the log-likelihood and q is the number of parameters of the family of distributions fitted. Since the AIC is defined as *minus* twice the log-likelihood plus the penalty term which accounts for the effective number of estimated parameters, smaller values of the AIC indicate a better data fit.

In the last column of Tables 6, 7 and 8 all models are ranked by their AIC (the model ranking is given in parentheses). We observe that for the copulae of pairs, the Student-t copula is ranked first in four out of six cases, followed by the mixture model Gumbel & survival Gumbel for those copulae of pairs where the TOPIX is not present in the analysis. The same copula models are favored by the AIC for a 3-dimensional copula of (S&P 500, DJ EURO STOXX 50, FTSE 100). This indicates that modelling dependency in the tails of a joint distribution is crucial for some portfolio constructed of S&P 500, Dow Jones EURO STOXX 50 and FTSE 100. Furthermore, when including the TOPIX in a portfolio, we observe that the AIC favors another model, the Clayton & Gumbel mixture, where the Clayton term obtains higher weight ranging between 0.85 for the copula of (FTSE 100, TOPIX) and the four-constituent copula of (S&P 500, DJ EURO STOXX 50, FTSE 100, TOPIX) to 0.93 for the copula of (S&P 500, TOPIX). Recall that the Clayton copula can generate lower tail dependence, but not upper tail dependence, and that from Table 5 we know that including the TOPIX in the portfolio leads to smaller numbers for the tail dependence coefficient.

3.4.2 Time-Varying Copulae

The results described above are applied to the estimation of the dependence parameter in a time-varying context. For these purposes we estimate the dependence parameter by using subsets of size n of log-returns, that is, a moving window of size n, $\{\widehat{X}_t\}_{t=s-n+1}^{s}$ scrolling in time for $s = n, ..., T$. This generates a time-series for the dependence parameter $\{\widehat{\theta}_t\}_{t=n}^{T}$. The static case, on the contrary, estimates the dependence parameter at once, based on the entire series of observations.

Figure 7 shows the dependence parameter $\hat{\theta}$ estimated for a three-constituent portfolio constructed of (S&P 500, DJ EURO STOXX 50, FTSE 100) in the upper panel, and a four-constituent portfolio constructed of (S&P 500, DJ EURO STOXX 50, FTSE 100, TOPIX) in the lower panel, using the Student-t copula with Student-t marginals. The dashed line corresponds to the static case, estimated from the whole time interval, and a solid line represents the time-varying dependence parameter estimated using log-returns corresponding to a moving window with a fixed size of $n = 250$ days. Figures 8 and 9 show estimated time-varying parameters for a three- and a four-constituent portfolio estimated using mixture Gumbel & survival Gumbel and mixture Clayton & Gumbel, respectively.

4 Applications in Risk Management

Based on the AIC model selection criterion, we have favored three different models providing the best fit to the data. These are the Student-t copula vs. mixture Gumbel & survival Gumbel model for a 3-constituent portfolio of (S&P 500, DJ EURO STOXX 50, FTSE 100), and the Student-t copula vs. mixture Clayton & Gumbel model for a 4-constituent portfolio of (S&P 500, DJ EURO STOXX 50, FTSE 100, TOPIX). These copula models are employed for the Value-at-Risk and the Expected Shortfall estimation of the respective three- and four-constituent portfolios. The estimation technique and details of the results are described below.

4.1 Value-at-Risk Methodology

The dependency over time between the index returns is of particular importance in risk management since the resulting *profit and loss* (P&L) function is closely linked to the *Value-at-Risk* (VaR) of an international index portfolio. The VaR of a portfolio is determined by the multivariate distribution of risk factor increments. If $w = (w_1, \ldots, w_d)^\top \in \mathbb{R}^d$ denotes a portfolio of positions on d assets and $S_t = (S_{1,t}, \ldots, S_{d,t})^\top$ is a non-negative random vector representing the asset (index) price at time t, the value V_t of the portfolio w is given by:

$$V_t = \sum_{j=1}^{d} w_j S_{j,t}. \tag{42}$$

The random variable

$$L_t = (V_t - V_{t-1}) \tag{43}$$

is called *profit and loss* (P&L) function, it expresses daily changes in the portfolio value. Defining the log-returns as $X_t = \log S_t - \log S_{t-1}$, equation (43) can be written as

$$L_t = \sum_{j=1}^{d} w_j S_{j,t-1} \{\exp(X_{j,t}) - 1\} . \tag{44}$$

If the distribution function of L_t is given by $F_{L_t}(x) = P(L_t \leq x)$, then the VaR of a portfolio with strategy w at time t and level α is defined as the α-quantile from the distribution of the P&L:

$$VaR_t(\alpha) = F_{L_t}^{-1}(\alpha). \tag{45}$$

The *Expected Shortfall* (ES) at time t can be computed as

$$ES_t(\alpha) = \frac{1}{N_{t+1}} \sum_{i=1}^{N_{t+1}} \widehat{L}_{t+1,i} \mathbf{1}_{\{\widehat{L}_{t+1,i} \leq VaR_t(\alpha)\}}, \tag{46}$$

where N_{t+1} denotes the number of simulated portfolio returns with value less or equal than $VaR_t(\alpha)$ and $\widehat{L}_{t+1,i}$ is the i^{th} outcome of the N_{t+1} samples. Obviously $ES_t(\alpha) \geq VaR_t(\alpha)$. Note that the ES has been proposed in several ways as a remedy for the deficiencies of VaR which is in general not a coherent risk measure, see Föllmer and Schied (2004) and McNeil et al. (2005). It follows from (44) and (45) that F_{L_t} depends on the specification of the d-dimensional distribution of the risk factors X_t. Thus, modelling their distribution over time is essential to obtain the quantiles in (45).

4.1.1 RiskMetrics vs. Copula-Based Approach

The *RiskMetrics* technique, a widely used methodology for VaR estimation, see Morgan/Reuters (1996), assumes that the log-returns follow a multivariate normal distribution: $X_t \sim N(0, \Sigma_t)$. The elements of a covariance matrix Σ_t can be estimated using the exponential smoothing technique:

$$(\widehat{\sigma}_t)^2 = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} (X_s)^2. \tag{47}$$

The parameter $\lambda \in (0, 1)$ is a so-called exponential moving average decay factor. Morgan/Reuters (1996) show that the value of 0.05 for λ provides good backtesting results.

In the copulae-based approach we first correct the contemporaneous volatility in the log-returns process using the t-GARCH(1,1) model described in Section 2.2. We employ the IFM technique to estimate parametric copulae. After fitting time-varying volatilities (plotted in Figure 4) of the marginals using t-GARCH(1,1) we estimate a time-varying copula dependence parameter $\{\widehat{\theta}_t\}_{t=n}^{T}$ using a moving window of size $n = 250$. The estimated parameter is then used to generate Monte-Carlo samples of the P&L using the specified copula. Their quantiles at different confidence levels give estimates of the time-varying $\{\widehat{VaR}_t\}_{t=n}^{T}$, and averaging over the worst $100 \cdot \alpha\%$ cases leads to the estimated $\{\widehat{ES}_t\}_{t=n}^{T}$.

4.1.2 Backtesting

Backtesting is applied to evaluate the performance of the procedure. It uses part of the information available to estimate the one period ahead risk measures and afterwards compares them with the values actually observed. It compares the estimated values of VaR with the true realizations $\{L_t\}$ of the P&L function by computing the exceedance ratio $\widehat{\alpha}$:

$$\widehat{\alpha} = \frac{1}{T-n} \sum_{t=n}^{T} \mathbf{1}_{\{L_t < \widehat{VaR}_t(\alpha)\}}. \tag{48}$$

Whenever $L_t < \widehat{VaR}_t$, we say that a violation of the VaR has occurred. By definition VaR is a number such that the loss cannot occur with probability larger than α. Therefore, the ratio of the number of such violations to the total number of observations in the backtesting period should be close to a target level α.

4.2 Results

Based on $N = 10000$ simulations we estimate the VaR and the ES for confidence levels $\alpha \in \{0.1, 0.05, 0.01, 0.005, 0.001\}$. Average one-day estimates for a 3-constituent portfolio of (S&P 500, DJ EURO STOXX 50, FTSE 100) and a 4-constituent portfolio of (S&P 500, DJ EURO STOXX 50, FTSE 100, TOPIX) are reported in Tables 9 and 11, respectively, with standard errors reported in parentheses. Table 9 shows the results for VaR and ES, estimated using the Student-t copula and a mixture Gumbel & survival Gumbel copula, compared to the *Riskmetrics* approach. Table 11 employs a Clayton & Gumbel mixture copula instead. Recall, that the marginals are assumed to be Student-t and with fixed number of degrees of freedom for all models. From the tables we observe that the dependence structure given by the copula function influences the VaR and the ES numbers. For the lower significance levels, e.g. 10.0%, 5.0%, we observe high VaR numbers in the case of *Riskmetrics.* For increasing significance levels to e.g. 0.5%, 0.1%, the effect of fat tails becomes much stronger, therefore, the VaR will be underestimated if one assumes normality. This leads to the smaller VaR and ES numbers for the *Riskmetrics* case compared to the Student-t copula case. VaR numbers for the mixture models lie roughly in the range between the estimated VaR for the *Riskmetrics* and the Student-t copula VaR, indicating that it outperforms the *Riskmetrics* approach but underperforms the Student-t copula method. Not surprisingly, errors increase as the estimated risk measures go further into the tail.

To confirm the outperformance of the Student-t copula over two other models we report the exceedance ratios for all confidence levels in Tables 10 and 12 for a three- and a four-constituent portfolio, respectively. We compare the performances of the methods by checking how close is the percentage of VaR violations to the targets of $10.0\%, 5.0\%, 1.0\%, 0.5\%, 0.1\%$. For all copula models we observe that the estimated exceedance ratios $\hat{\alpha}$ are consistent with a specified target confidence level α. For example, for our three-constituent portfolio of (S&P 500, DJ EURO STOXX 50, FTSE 100) the estimated $1.0\%, 0.5\%, 0.1\%$ *Riskmetrics* VaR is violated by $2.4\%, 1.7\%, 0.9\%$ of observations, respectively, that is, the model underestimates the VaR. Choosing the mixture Gumbel & survival Gumbel model improves the performance, leading to less VaR violations corresponding to $2.2\%, 1.1\%, 0.2\%$, respectively. However, the mixture model still underestimates VaR. Student-t models perform significantly better, leading to the percentage of VaR violations close to the target levels of α. The overall performance is summarized by computing a sum of relative squared errors $\sum_{\alpha}((\alpha - \widehat{\alpha})/\alpha)^2$ and a sum of relative absolute errors $\sum_\alpha |\alpha - \widehat{\alpha}|/\alpha$ reported in the last column of Tables 10 and 12. We observe that the Student-t copula with Student-t marginals leads to the smallest errors for the three-dimensional case and, thus, the best backtesting results. Note that the TOPIX does not enter the 3-constituent portfolio. This is in line with the results on the AIC model selection reported in Table 8 where the Student-t copula model was ranked first followed by the mixture model. In case of a four-constituent portfolio with TOPIX included in the portfolio, the mixture Clayton & Gumbel model performs overall slightly better, leading to the sum of squared (absolute) errors of 1.06 (2.06) for the mixture model compared to 1.49 (2.28) for the Student-t model. However, the mixture model underestimates VaR at e.g. $10.0\%, 5.0\%$ and 0.5% significance levels.

Figures 10 and 11 show P&L outcomes and VaR estimated at different confidence levels of 10.0%, 5.0%, 1.0% together with exceedances computed for the 1.0% significance level. The upper panel corresponds to the Student-t copula, and the lower panel refers to the respective mixture model for the three- and a four-constituent portfolio.

Overall, we conclude that the Student-t copula and the mixture models are clearly superior to the *Riskmetrics* approach. Furthermore, the Student-t copula with Student-t marginal performs significantly better than the mixture model in terms of backtesting when modelling VaR for our three-constituent portfolio, and is slightly outperformed by the mixture model in our four-constituent case.

5 Conclusions

This paper examines time-varying copulae to model the dependency of regional equity indices on the international equity market. We use portfolios constructed of the following indices: S&P 500 for the USA, Dow Jones EURO STOXX 50 for continental Europe, FTSE 100 for the UK, and TOPIX for Japan.

Based on the Anderson-Darling statistics we show that the Student-t assumption on the marginals provides the best fit to the index data across the alternative distributions from the class of Symmetric Generalized Hyperbolic distributions, as well as the normal distribution.

We use Student-t marginals with about four degrees of freedom and t-GARCH(1,1)-fitted time-varying volatilities for each series of index returns to generate the dependence structure by fitting multivariate copulae to the data. We consider different copula specifications which include some mixture models allowing to generate asymmetric dependencies in the tails of the distribution. The Akaike Information Criteria indicate that the Student-t copula with Student-t marginals allows for a realistic modelling of co-movements and captures well the tail dependence in the index returns of an international portfolio.

When modelling Value-at-Risk and Expected Shortfall using copulae, we show that the Student-t copula and the mixture copula models outperform the *Riskmetrics* approach. Furthermore, the Student-t copula with Student-t marginals performs significantly better than some alternative mixture copula models in terms of VaR backtesting when modelling VaR for a portfolio with the three constituents S&P 500, Dow Jones EURO STOXX 50 and FTSE 100. However, on average it is slightly outperformed by the mixture model in the case with the four constituents S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX.

Altogether, the Student-t copula with Student-t marginals allows for flexible modelling of the joint distribution by splitting marginals from the dependence structure, and capturing a non-linear behavior and extreme values arising in the distribution of the log-returns. It is a strong candidate model from the point of view of risk management which allows the investors to allocate their capital in a most effective way when dealing with the modelling of potential extreme losses that have a tendency to occur simultaneously in regional equity indices.

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Table 1: Anderson-Darling distance and Kolmogorov-Smirnov distance for EWI104s, S&P 500 and Dow Jones EURO STOXX 50, FTSE 100 and TOPIX log-returns, using different marginal distributions. Estimated from the whole time series of the period from 01 January 1987 to 10 March 2006 assuming constant volatility.

			EWI104s		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	1.616788	0.031107	0.063653	0.429391	0.440249
AD_{median}	0.161642	0.026724	0.054528	0.079064	0.084363
AD_{std}	4.310154	0.024337	0.064946	0.935552	0.950782
AD_{max}	31.81653	0.321230	1.044265	6.219692	6.196912
AD_{min}	0.000232	0.000212	0.000017	0.000458	0.000060
AD_{range}	31.81630	0.321019	1.044248	6.219234	6.196852
			S&P 500		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	1.356158	0.030634	0.054019	0.411616	0.430305
AD_{median}	0.173919	0.016333	0.037722	0.042945	0.041857
AD_{std}	3.769661	0.052518	0.061608	1.02216	1.059168
AD_{max}	30.24927	0.813711	0.821604	6.178514	6.225368
AD_{min}	0.002185	0.000039	0.000004	0.000079	0.000052
AD_{range}	30.24708	0.813672	0.821599	6.178436	6.225316
			Dow Jones EURO STOXX 50		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	1.351050	0.014727	0.087296	0.049472	0.473320
AD_{median}	0.211767	0.009762	0.055306	0.048157	0.060959
AD_{std}	2.815063	0.013300	0.137143	0.029320	1.428817
AD_{max}	18.96239	0.095215	2.393155	0.333732	21.72984
AD_{min}	0.000286	0.000024	0.000020	0.000190	0.000333
AD_{range}	18.96210	0.095192	2.393135	0.333542	21.72950
			FTSE 100		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	1.149534	0.038767	0.108343	0.438483	0.494560
AD_{median}	0.153101	0.029698	0.058666	0.043228	0.045778
AD_{std}	2.589437	0.035602	0.215906	1.090280	1.260014
AD_{max}	18.25937	0.439793	3.768648	6.753976	8.177442
AD_{min}	0.000240	0.000059	0.000047	0.000043	0.000003
AD_{range}	18.25913	0.439735	3.768601	6.753933	8.177439
			TOPIX		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	1.298336	0.015972	0.076726	0.178283	0.068037
AD_{median}	0.157526	0.013549	0.033476	0.031368	0.026075
AD_{std}	2.832192	0.016437	0.161895	0.505222	0.176657
AD_{max}	18.16919	0.193644	2.734778	6.000118	2.919006
AD_{min}	0.000020	0.000022	0.000031	0.000095	0.000030
AD_{range}	18.16917	0.193623	2.734748	6.000024	2.918976

Table 2: Ljung-Box statistic at different lags for the absolute and the squared residuals of the log-returns of EWI104s, S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX; p-values are given in parenthesis.

	Ljung-Box statistic for absolute residuals $\widehat{u}_t$				
Index	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5
EWI104s	0.2425(0.6224)	2.8489(0.2406)	2.8490(0.4155)	6.5394(0.1623)	6.5435(0.2569)
S&P 500	11.166(0.0008)	11.703(0.0029)	12.041(0.0072)	12.371(0.0148)	13.208(0.0215)
Dow Jones EURO STOXX 50	5.1755(0.0229)	7.5279(0.0232)	7.5502(0.0563)	11.771(0.0191)	13.438(0.0196)
FTSE 100	3.7456(0.0529)	3.7630(0.1524)	3.7669(0.2878)	5.6265(0.2288)	5.6347(0.3434)
TOPIX	0.5664(0.4517)	0.5746(0.7503)	2.1702(0.5378)	2.5681(0.6325)	6.0607(0.3004)
	Ljung-Box statistic for squared residuals $\widehat{u}_t^2$				
	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5
EWI104s	2.9339(0.0867)	3.6474(0.1614)	4.0320(0.2580)	5.5038(0.2394)	5.6971(0.3368)
S&P 500	0.9515(0.3293)	1.1087(0.5744)	1.1106(0.7745)	1.5022(0.8263)	1.6266(0.8980)
Dow Jones EURO STOXX 50	0.2160(0.6421)	0.5003(0.7787)	0.5204(0.9144)	1.3759(0.8484)	1.3853(0.9260)
FTSE 100	3.8682(0.0492)	3.9675(0.1375)	4.0459(0.2565)	4.6542(0.3247)	4.7160(0.4515)
TOPIX	2.0752(0.1497)	2.2070(0.3317)	2.5445(0.4673)	2.9157(0.5720)	3.4072(0.6375)

Table 3: Anderson-Darling distance and Kolmogorov-Smirnov distance for EWI104s, S&P 500 and Dow Jones EURO STOXX 50, FTSE 100 and TOPIX log-returns, using different marginal distributions. Estimated from the whole time series of the period from 01 January 1987 to 10 March 2006 assuming time-varying volatility estimated using ARMA(1,1)-GARCH(1,1) model.

			EWI104s		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	1.490917	0.037894	0.069538	0.221952	0.224156
AD_{median}	0.090005	0.033707	0.053669	0.066937	0.076008
AD_{std}	4.640579	0.026619	0.077276	0.416044	0.483405
AD_{max}	35.91549	0.329346	1.046693	5.285221	7.127305
AD_{min}	0.000650	0.000059	0.000123	0.000162	0.000008
AD_{range}	35.91484	0.329287	1.046569	5.285060	7.127298
			S&P 500		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	0.705431	0.025874	0.052369	0.166808	0.180805
AD_{median}	0.099477	0.021449	0.030704	0.036524	0.038781
AD_{std}	1.917500	0.020885	0.073779	0.384024	0.403709
AD_{max}	12.34245	0.228615	0.818680	5.502823	5.178977
AD_{min}	0.000025	0.000005	0.000059	0.000006	0.000013
AD_{range}	12.34242	0.228611	0.818621	5.502815	5.178964
			Dow Jones EURO STOXX 50		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	0.698899	0.034206	0.086242	0.497675	0.318385
AD_{median}	0.081210	0.027996	0.040001	0.042892	0.049475
AD_{std}	2.151991	0.032731	0.156050	1.497084	0.831028
AD_{max}	17.95773	0.534882	2.386785	20.53927	12.25963
AD_{min}	0.000201	0.000033	0.000126	0.000111	0.000026
AD_{range}	17.95753	0.534849	2.386659	20.53916	12.25960
			FTSE 100		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	0.430330	0.035574	0.105875	1.218255	1.158709
AD_{median}	0.051894	0.027707	0.046036	0.043556	0.049943
AD_{std}	0.944762	0.037833	0.245016	4.380829	4.296439
AD_{max}	5.985886	0.622037	3.777301	62.70089	69.01680
AD_{min}	0.000627	0.000020	0.000152	0.000158	0.000046
AD_{range}	5.985259	0.622017	3.777150	62.70073	69.01676
			TOPIX		
Anderson-Darling	Normal	Student-t	NIG	HYP	VG
AD_{mean}	0.499650	0.023266	0.070841	0.558110	0.289041
AD_{median}	0.095646	0.015207	0.022977	0.023640	0.026182
AD_{std}	0.956378	0.033897	0.178793	1.971034	0.983859
AD_{max}	6.192678	0.582280	2.727558	27.67034	15.26800
AD_{min}	0.001471	0.000016	0.000029	0.000189	0.000032
AD_{range}	6.191207	0.582264	2.727529	27.67016	15.26796

Table 4: Pearson correlation r, Spearmann's ρ and Kendall's τ for the log-returns of EWI104s, S&P 500, Dow Jones EURO STOXX 50, FTSE100 and TOPIX from 01 January 1987 to 10 March 2006.

	Pearson correlation coefficient r				
	EWI104s	S&P 500	DJ ES 50	FTSE 100	TOPIX
EWI104s	1.0000000				
S&P 500	0.3432575	1.0000000			
Dow Jones EURO STOXX 50	0.6529906	0.4169373	1.0000000		
FTSE 100	0.5853366	0.4176875	0.7402107	1.0000000	
TOPIX	0.4184749	0.1112277	0.2635704	0.2626654	1.0000000
	Spearmann's ρ				
	EWI104s	S&P 500	DJ ES 50	FTSE 100	TOPIX
EWI104s	1.0000000				
S&P 500	0.2962977	1.0000000			
Dow Jones EURO STOXX 50	0.5616447	0.3369013	1.0000000		
FTSE 100	0.4920005	0.3578612	0.6642572	1.0000000	
TOPIX	0.3494137	0.1119928	0.2336505	0.2067932	1.0000000
	Kendall's τ				
	EWI104s	S&P 500	DJ ES 50	FTSE 100	TOPIX
EWI104s	1.0000000				
S&P 500	0.2045868	1.00000000			
Dow Jones EURO STOXX 50	0.4040263	0.23556472	1.0000000		
FTSE 100	0.3488939	0.24997723	0.4918589	1.0000000	
TOPIX	0.2405465	0.07586944	0.1614175	0.1418930	1.0000000

Table 5: Estimated tail dependence coefficient $\lambda = \lambda_u = \lambda_l$, the number of degrees of freedom ν and the correlation ρ. Estimated using the Student-t copula for the pairs (S&P 500, DJ ES 50), (S&P 500, FTSE 100), (S&P 500, TOPIX) and (FTSE 100, TOPIX), a 3-dimensional copula of (S&P 500, DJ ES 50, FTSE 100) and a 4-dimensional copula of (S&P 500, DJ ES 50, FTSE 100, TOPIX). Data covers time span from 01 January 1987 to 10 March 2006,

Copula	ν	ρ	$\lambda = \lambda_l = \lambda_u$
(S&P 500, DJ ES 50)	4.401111	0.3987770	0.1838132
(S&P 500, FTSE 100)	3.722864	0.4366287	0.2348772
(S&P 500, TOPIX)	11.26293	0.1181285	0.0088138
(DJ ES 50, FTSE 100)	4.264437	0.7129477	0.3886651
(DJ ES 50, TOPIX)	7.583633	0.2575067	0.0522758
(FTSE 100, TOPIX)	9.547416	0.2480824	0.0292512
(S&P 500, DJ ES 50, FTSE 100)	3.584181	0.5246560	0.2900480
(S&P 500, DJ ES 50, FTSE 100, TOPIX)	5.011470	0.3648167	0.1453324

Table 6: Copula dependence parameter, standard errors, AIC and model ranking (in parenthesis), estimated for different copulae assuming Student-t marginals for the pairs of log-returns of S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX. In the case of mixture models, the parameters θ_1 and θ_2 are the dependence parameters for the first and second terms of the mixture, respectively, and θ_3 is the mixture parameter which gives the proportion of the first term.

(S&P 500, Dow Jones EURO STOXX 50)				
Copula family	$\widehat{\theta}_1(\hat{s.e.})$	$\widehat{\theta}_2(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
AMH	0.830857(0.000242)	-	-	-685.10(10)
Clayton	0.506535(0.000564)	-	-	-616.88(11)
Frank	2.623623(0.009578)	-	-	-691.61(9)
Gaussian	0.404432(0.000139)	-	-	-732.23(8)
Gumbel	1.358234(0.000249)	-	-	-793.55(6)
Plakett	3.773934(0.025373)	-	-	-771.09(7)
Student-t	0.398777(0.000206)	-	-	**-910.34(1)**
Clayton & surv. Clayton	0.485517(0.071340)	1.099605(0.198363)	0.583701(0.051689)	-828.65(5)
Clayton & Gumbel	0.228705(0.040673)	1.919703(0.111136)	0.554277(0.042995)	-875.41(3)
surv. Clayton & surv. Gumbel	0.308214(0.050013)	1.884531(0.123729)	0.554277(0.042995)	-848.62(4)
Gumbel & surv. Gumbel	1.208340(0.025353)	2.065686(0.170736)	0.697168(0.044822)	**-876.61(2)**
(S&P 500, FTSE 100)				
Copula family	$\widehat{\theta}_1(\hat{s.e.})$	$\widehat{\theta}_2(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
AMH	0.877309(0.000199)	-	-	-758.11(10)
Clayton	0.628307(0.000703)	-	-	-718.23(11)
Frank	2.894370(0.010148)	-	-	-775.50(9)
Gaussian	0.453370(0.000133)	-	-	-816.32(8)
Gumbel	1.420639(0.000297)	-	-	-873.99(6)
Plakett	4.155037(0.030489)	-	-	-853.45(7)
Student-t	0.436629(0.000217)	-	-	**-1012.52(1)**
Clayton & surv. Clayton	1.162593(0.218111)	0.669177(0.092488)	0.418331(0.051208)	-917.39(5)
Clayton & Gumbel	0.318263(0.050861)	1.952291(0.115171)	0.532836(0.046807)	-950.31(3)
surv. Clayton & surv. Gumbel	0.363507(0.059322)	1.883106(0.103689)	0.519942(0.045951)	-948.69(4)
Gumbel & surv. Gumbel	1.243717(0.031112)	2.005463(0.137576)	0.636167(0.050293)	**-967.70(2)**
(S&P 500, TOPIX)				
Copula family	$\widehat{\theta}_1(\hat{s.e.})$	$\widehat{\theta}_2(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
AMH	0.343811(0.001485)	-	-	-58.86(7)
Clayton	0.150366(0.000388)	-	-	-68.48(4)
Frank	0.697033(0.008801)	-	-	-53.04(9)
Gaussian	0.115939(0.000249)	-	-	-49.93(10)
Gumbel	1.068007(0.000129)	-	-	-42.33(11)
Plakett	1.430615(0.004499)	-	-	-54.96(8)
Student-t	0.118128(0.000273)	-	-	**-74.61(1)**
Clayton & surv. Clayton	0.132904(0.022524)	1.193569(1.185655)	0.947824(0.043568)	-68.58(3)
Clayton & Gumbel	0.115115(0.025006)	1.806004(0.537352)	0.932927(0.040120)	**-70.77(2)**
surv. Clayton & surv. Gumbel	2.674385(4.760467)	1.072005(0.013107)	0.021055(0.032403)	-64.92(6)
Gumbel & surv. Gumbel	1.785085(0.576285)	1.061260(0.015450)	0.057097(0.040120)	-66.37(5)

Table 7: Copula dependence parameter, standard errors, AIC and model ranking (in parenthesis), estimated for different copulae assuming Student-t marginals for the pairs of log-returns of S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX. In the case of mixture models, the parameters θ_1 and θ_2 are the dependence parameters for the first and second terms of the mixture, respectively, and θ_3 is the mixture parameter which gives the proportion of the first term.

	(Dow Jones EURO STOXX 50, FTSE 100)			
Copula family	$\widehat{\theta_1}(\hat{s.e.})$	$\widehat{\theta_2}(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
AMH	1.000000(0.005396)	-	-	-2495.47(11)
Clayton	1.406544(0.001137)	-	-	-2670.43(10)
Frank	5.951390(0.012649)	-	-	-3002.77(9)
Gaussian	0.707104(0.000035)	-	-	-3095.10(7)
Gumbel	1.964733(0.000532)	-	-	-3046.28(8)
Plakett	12.12717(0.185987)	-	-	-3221.17(5)
Student-t	0.712948(0.000057)	-	-	**-3364.03(1)**
Clayton & surv. Clayton	1.664937(0.099117)	2.200696(0.167507)	0.564240(0.024139)	-3150.47(6)
Clayton & Gumbel	1.200580(0.128810)	2.418690(0.106660)	0.394510(0.028700)	-3279.39(3)
surv. Clayton & surv. Gumbel	3.403908(0.444728)	1.892394(0.034519)	0.220560(0.024715)	-3270.04(4)
Gumbel & surv. Gumbel	2.736944(0.134834)	1.723420(0.043646)	0.419230(0.035792)	**-3324.06(2)**
	(Dow Jones EURO STOXX 50, TOPIX)			
Copula family	$\widehat{\theta_1}(\hat{s.e.})$	$\widehat{\theta_2}(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
AMH	0.655199(0.000614)	-	-	-328.52(7)
Clayton	0.335801(0.000412)	-	-	-371.72(6)
Frank	1.559018(0.008083)	-	-	-297.57(10)
Gaussian	0.260018(0.000173)	-	-	-326.10(8)
Gumbel	1.178222(0.000158)	-	-	-268.86(11)
Plakett	2.232172(0.009187)	-	-	-317.06(9)
Student-t	0.257507(0.000213)	-	-	-390.59(5)
Clayton & surv. Clayton	0.310304(0.025021)	1.200672(0.410199)	0.878077(0.037735)	-392.22(4)
Clayton & Gumbel	0.266833(0.026465)	1.898542(0.271677)	0.860732(0.037052)	**-398.67(2)**
surv. Clayton & surv. Gumbel	1.667922(0.728685)	1.171986(0.013894)	0.061270(0.029161)	-396.17(3)
Gumbel & surv. Gumbel	1.955947(0.331888)	1.154098(0.016293)	0.096550(0.036831)	**-406.20(1)**
	(FTSE 100, TOPIX)			
Copula family	$\widehat{\theta_1}(\hat{s.e.})$	$\widehat{\theta_2}(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
AMH	0.335559(0.044385)	-	-	-260.80(8)
Clayton	0.331820(0.000473)	-	-	-297.38(5)
Frank	1.425567(0.008566)	-	-	-232.98(10)
Gaussian	0.255971(0.000203)	-	-	-265.21(7)
Gumbel	1.170043(0.000172)	-	-	-213.47(11)
Plakett	2.061494(0.008352)	-	-	-243.22(9)
Student-t	0.248082(0.000239)	-	-	-293.19(6)
Clayton & surv. Clayton	0.335559(0.044385)	0.585374(0.365508)	0.835962(0.090107)	-305.28(3)
Clayton & Gumbel	0.292111(0.043747)	1.514145(0.448117)	0.849749(0.097071)	**-313.45(1)**
surv. Clayton & surv. Gumbel	54.59107(16.94812)	1.175092(0.012844)	0.010116(0.003896)	**-310.48(2)**
Gumbel & surv. Gumbel	1.749433(0.770825)	1.161687(0.018259)	0.069130(0.062588)	-303.83(4)

Table 8: Copula dependence parameter, standard errors, AIC and model ranking (in parenthesis), estimated for different copulae assuming Student-t marginals for the three and four-dimensional data of log-returns of S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX. In the case of mixture models, the parameters θ_1 and θ_2 are the dependence parameters for the first and second terms of the mixture, respectively, and θ_3 is the mixture parameter which gives the proportion of the first term.

	3-dim portfolio (S&P 500, Dow Jones EURO STOXX 50, FTSE 100)			
Copula family	$\widehat{\theta}_1(\hat{s.e.})$	$\widehat{\theta}_2(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
Clayton	0.761801(0.000302)	-	-	-2811.53(9)
Frank	3.492155(0.004482)	-	-	-2950.77(8)
Gaussian	0.531821(0.000045)	-	-	-3208.50(6)
Gumbel	1.509687(0.000149)	-	-	-3043.15(7)
Student-t	0.524656(0.000008)	-	-	**-3928.61(1)**
Clayton & surv. Clayton	0.922227(0.051830)	1.176395(0.067929)	0.526103(0.022631)	-3482.41(4)
Clayton & Gumbel	0.444848(0.027397)	2.261876(0.063801)	0.544275(0.023184)	-3533.11(3)
surv. Clayton & surv. Gumbel	0.511552(0.034746)	2.042400(0.055737)	0.495830(0.025025)	-3423.92(5)
Gumbel & surv. Gumbel	1.894318(0.064543)	1.411731(0.035420)	0.481504(0.023499)	**-3612.18(2)**
	4-dim portfolio (S&P 500, Dow Jones EURO STOXX 50, FTSE 100, TOPIX)			
Copula family	$\widehat{\theta}_1(\hat{s.e.})$	$\widehat{\theta}_2(\hat{s.e.})$	$\hat{\theta}_3(\hat{s.e.})$	AIC
Clayton	0.460034(0.000130)	-	-	-2495.15(6)
Frank	2.136350(0.002528)	-	-	-2236.07(7)
Gaussian	0.378284(0.000048)	-	-	-2645.79(5)
Gumbel	1.290089(0.000006)	-	-	-2193.28(8)
Student-t	0.364817(0.000073)	-	-	**-3325.99(2)**
Clayton & surv. Clayton	0.614358(0.032137)	0.678867(0.045221)	0.570954(0.022230)	-3040.83(4)
Clayton & Gumbel	0.347367(0.013310)	2.298225(0.058947)	0.851707(0.007111)	**-3693.53(1)**
surv. Clayton & surv. Gumbel	0.342768(0.013719)	2.226239(0.056340)	0.838576(0.007696)	-3046.57(3)
Gumbel & surv. Gumbel	1.509240(0.017013)	1.460484(0.016137)	0.472377(0.008150)	-928.27(9)

Table 9: Average one-day VaR and ES (standard errors reported in parentheses) obtained for portfolio $w = (1, 1, 1)^\top$ and different confidence levels α. Estimated for a portfolio constructed of S&P 500, Dow Jones EURO STOXX 50 and FTSE 100 assuming Student-t marginals.

Copula	α	$VaR_t(s.e.)$	$ES_t(s.e)$
Student-t	0.1	8.367720 (5.936709)	13.63373 (9.600081)
	0.05	11.63965 (8.228813)	17.45316 (12.25592)
	0.01	20.42360 (14.35879)	28.32472 (19.81916)
	0.005	25.01291 (17.53015)	34.20856 (23.88475)
	0.001	38.52563 (27.15569)	51.44901 (36.55230)
Gumbel & surv. Gumbel	0.1	6.889122 (4.778975)	10.17219 (7.019050)
	0.05	9.123218 (6.317515)	12.45905 (8.587481)
	0.01	14.29932 (9.837058)	18.19985 (12.46410)
	0.005	16.69643 (11.47290)	21.03255 (14.41506)
	0.001	23.12079 (15.83239)	28.89332 (19.80532)
Riskmetrics	0.1	8.846485 (7.587636)	8.846485 (7.587636)
	0.05	11.34009 (9.713624)	11.34009 (9.713624)
	0.01	15.99587 (13.68190)	15.99587 (13.68190)
	0.005	17.66844 (15.10453)	17.66844 (15.10453)
	0.001	21.08556 (17.99120)	21.08556 (17.99120)

Table 10: Exceedances ratio $\hat{\alpha}$ for portfolio $w = (1,1,1)^\top$ and different confidence levels α, estimated for a portfolio constructed of S&P 500, Dow Jones EURO STOXX 50 and FTSE 100 assuming Student-t marginals.

Copula	0.1	0.05	0.01	0.005	0.001	$\sum_\alpha((\alpha-\widehat{\alpha})/\alpha)^2$	$\sum_\alpha \lvert\alpha-\widehat{\alpha}\rvert/\alpha$
Student-t	0.094180	0.043981	0.004442	0.001111	0.000444	1.242913	2.076854
Gumbel & surv. Gumbel	0.125499	0.077521	0.021990	0.011106	0.002221	4.788473	4.446913
Riskmetrics	0.106525	0.063471	0.024190	0.016866	0.009099	73.31631	12.22592

Table 11: Average one-day VaR and ES (standard errors reported in parentheses) obtained for portfolio $w = (1,1,1,1)^\top$ and different confidence levels α. Estimated for a portfolio constructed of S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX assuming Student-t marginals.

Copula	α	$VaR_t(s.e.)$	$ES_t(s.e)$
Student-t	0.1	8.694056 (5.325734)	15.03824 (9.449525)
	0.05	12.66563 (7.928426)	19.63659 (12.42850)
	0.01	23.19557 (14.75000)	32.60920 (20.83317)
	0.005	28.67367 (18.33681)	39.65182 (25.52060)
	0.001	44.66574 (28.97034)	60.07442 (39.44106)
Clayton & Gumbel	0.1	7.282619 (4.238482)	12.88768 (7.603326)
	0.05	10.80745 (6.397911)	16.93568 (10.03881)
	0.01	20.09178 (11.91436)	28.25327 (16.70725)
	0.005	24.89026 (14.75315)	34.35621 (20.35394)
	0.001	38.77440 (23.12767)	52.32670 (31.74362)
Riskmetrics	0.1	9.166406 (6.858449)	9.166406 (6.858449)
	0.05	12.46968 (9.471922)	12.46968 (9.471922)
	0.01	18.96584 (14.50667)	18.96584 (14.50667)
	0.005	21.43761 (16.40011)	21.43761 (16.40011)
	0.001	26.53300 (20.25335)	26.53300 (20.25335)

Table 12: Exceedances ratio $\hat{\alpha}$ for portfolio $w = (1,1,1,1)^\top$ and different confidence levels α, estimated for a portfolio constructed of S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX assuming Student-t marginals.

Copula	0.1	0.05	0.01	0.005	0.001	$\sum_\alpha((\alpha-\widehat{\alpha})/\alpha)^2$	$\sum_\alpha \lvert\alpha-\widehat{\alpha}\rvert/\alpha$
Student-t	0.096179	0.041315	0.003110	0.000888	0.000444	1.491430	2.278987
Clayton & Gumbel	0.127277	0.062639	0.007552	0.024621	0.000222	1.064764	2.059529
Riskmetrics	0.102308	0.058145	0.017310	0.010874	0.004882	17.01467	5.974256

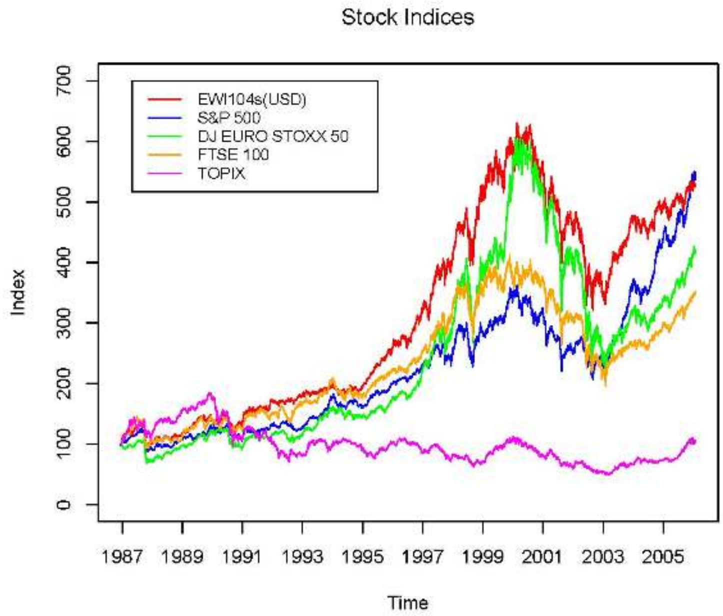

Figure 1: World Stock Equally Weighted Index (EWI104s), Standard & Poor's value weighted index (S&P 500), Eurozone Dow Jones EURO STOXX 50, Financial Times and London Stock Exchange Index (FTSE100), Tokyo Stock Price index (TOPIX) from 01 January 1987 to 10 March 2006.

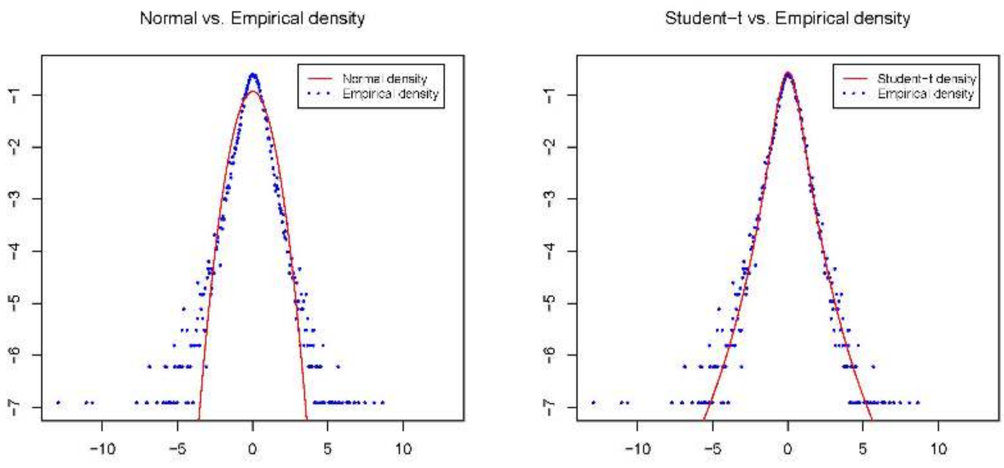

Figure 2: Logarithm of the histogram for the pooled data vs. normal density (left panel) and Student-t density (right panel). Pooled data is taken for indices S&P 500, Dow Jones EURO STOXX 50, FTSE100, TOPIX from 01 January 1987 to 10 March 2006. Estimated number of degrees of freedom for the Student-t distribution is $\nu = 3.15$.

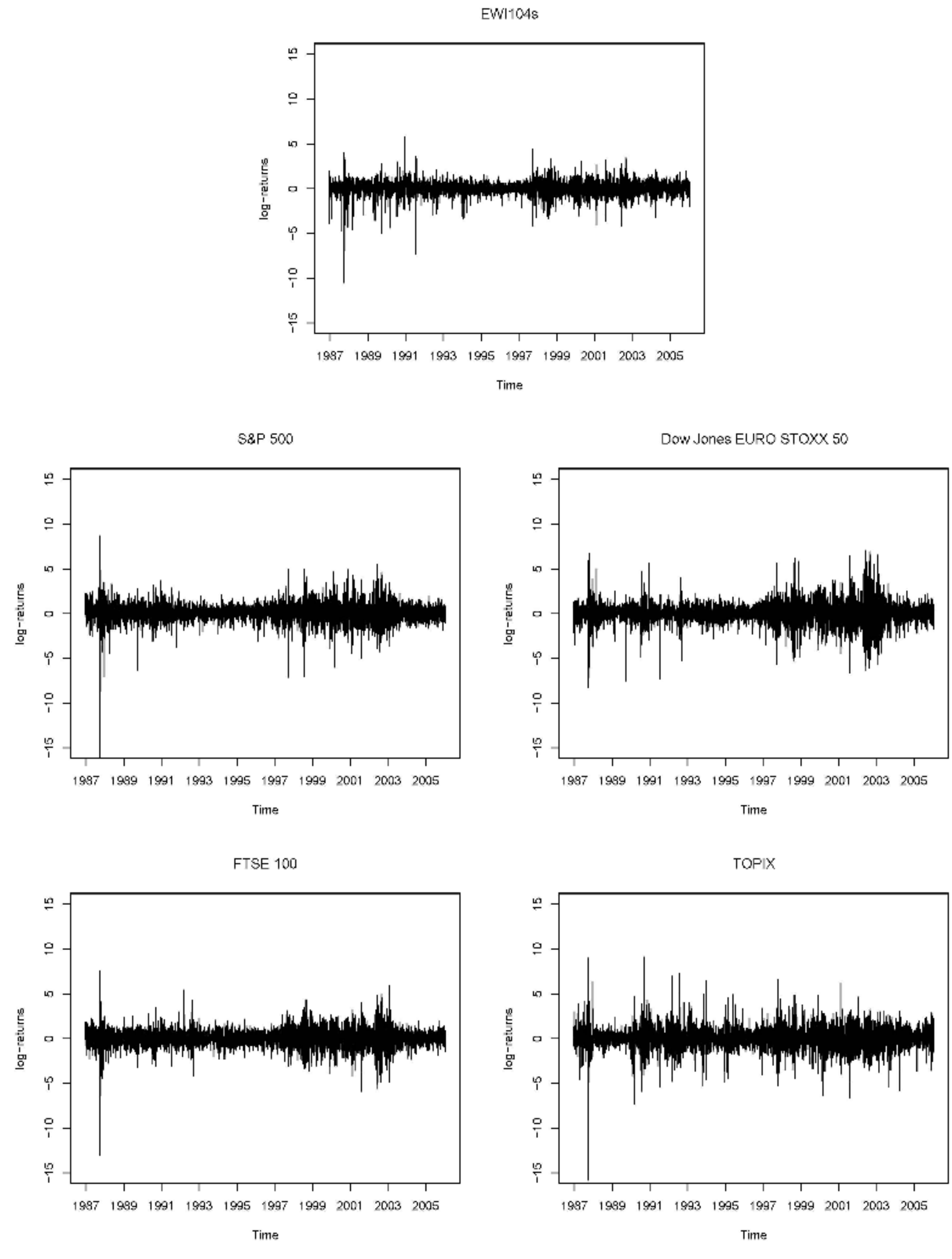

Figure 3: Daily log-returns in percentages for the EWI104s, S&P 500, Dow Jones EURO STOXX 50, FTSE100, TOPIX from 01 January 1987 to 10 March 2006.

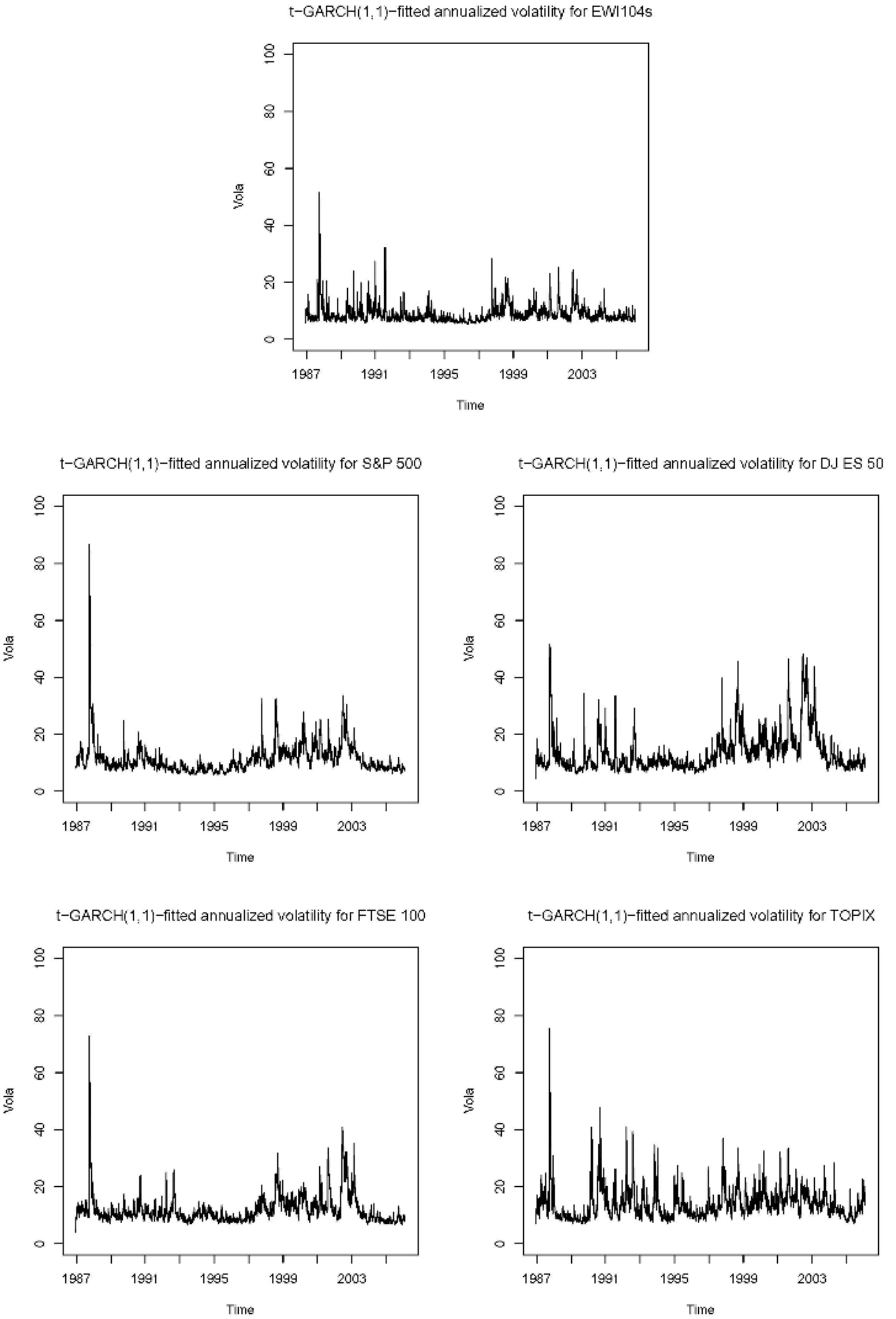

Figure 4: t-GARCH(1,1)-fitted annualized volatilities in percentages for the log-returns of EWI104s, S&P 500, Dow Jones EURO STOXX 50, FTSE100, TOPIX from 01 January 1987 to 10 March 2006.

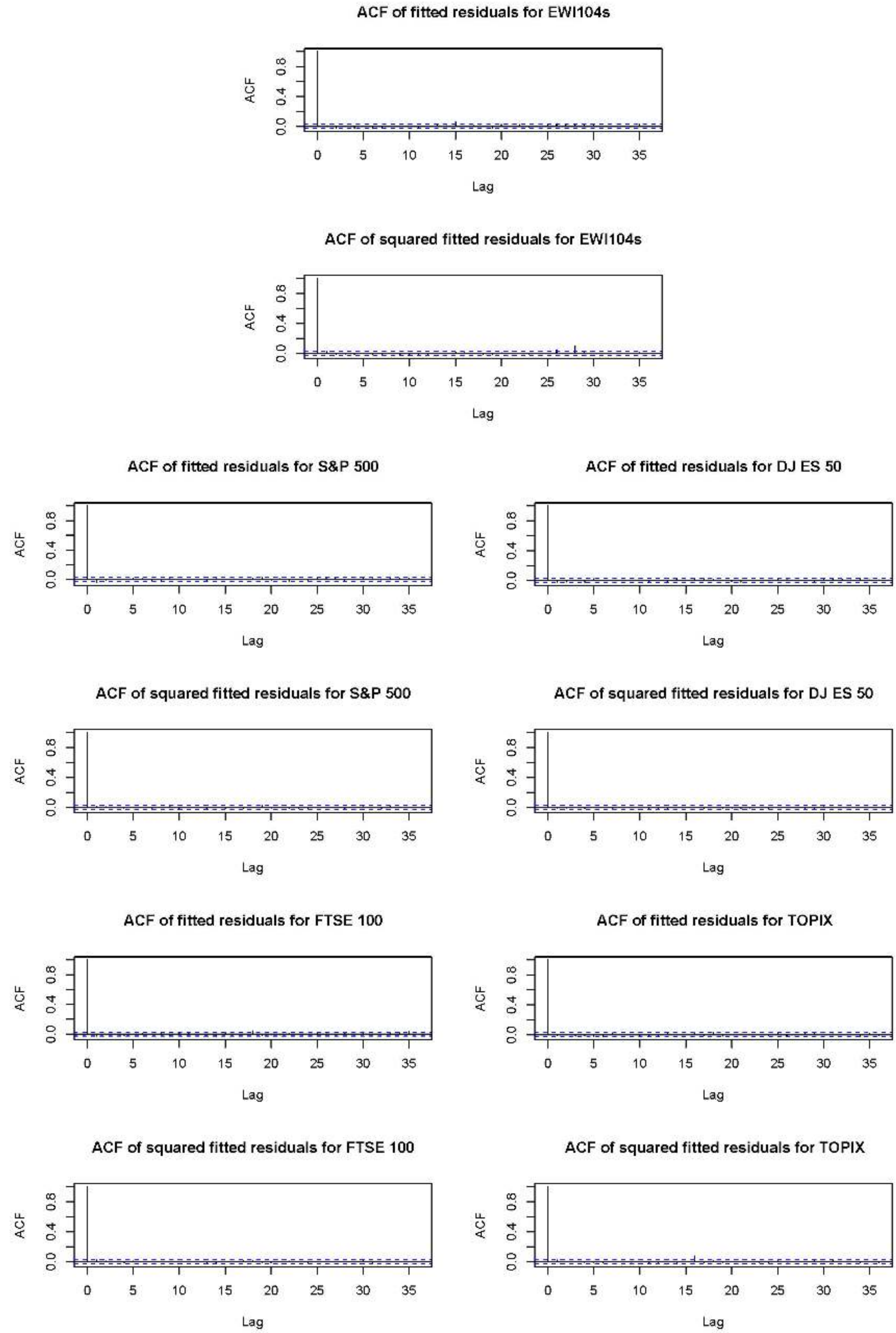

Figure 5: Autocorrelograms for fitted absolute and squared residuals of log-returns of EWI104s, S&P 500, Dow Jones EURO STOXX 50, FTSE100, TOPIX from 01 January 1987 to 10 March 2006.

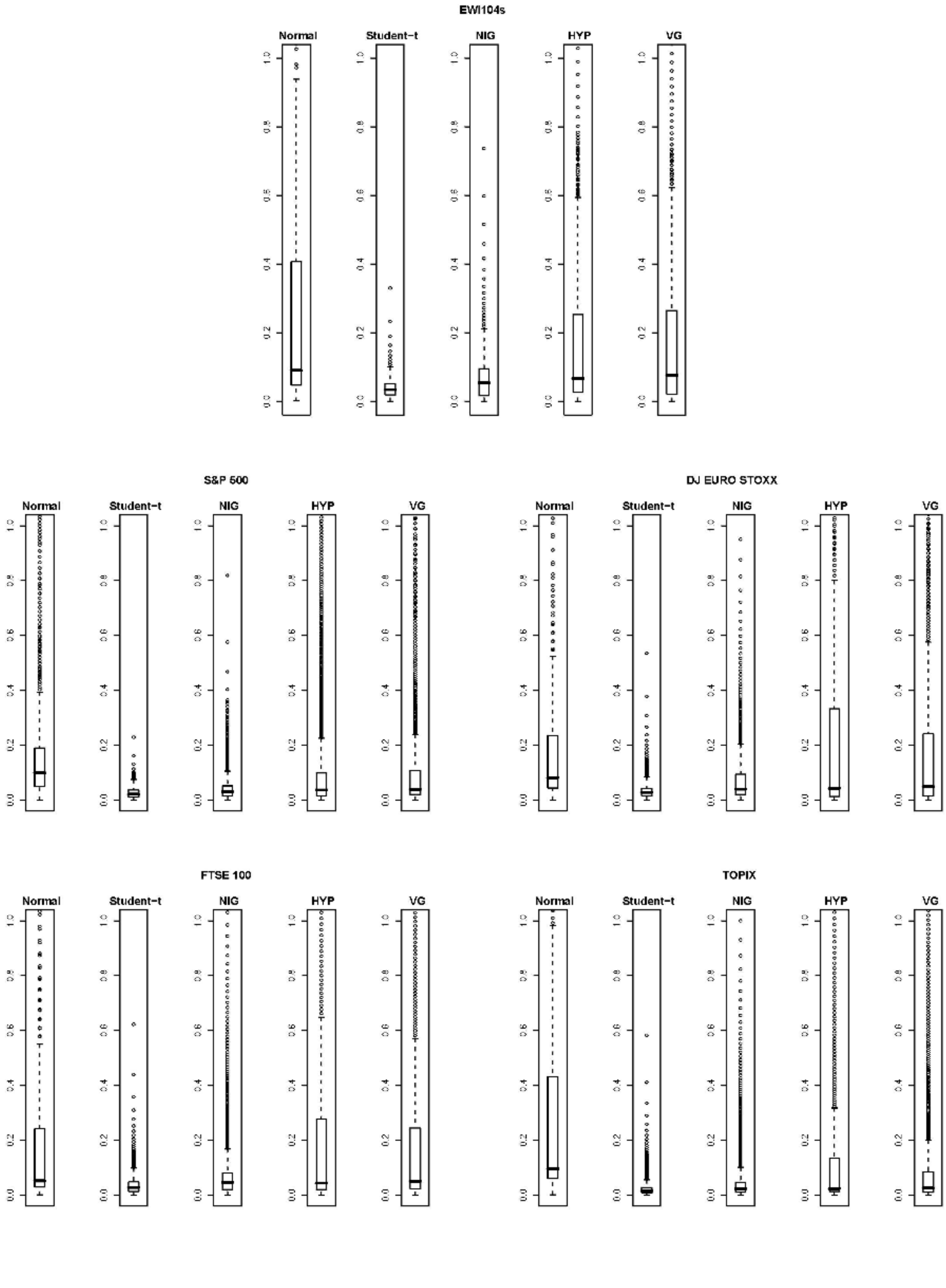

Figure 6: Box-plots for Anderson-Darling distance for modelling marginal distributions of EWI104s, S&P 500, Dow Jones EURO STOXX 50, FTSE100, TOPIX with alternative residual distributions.

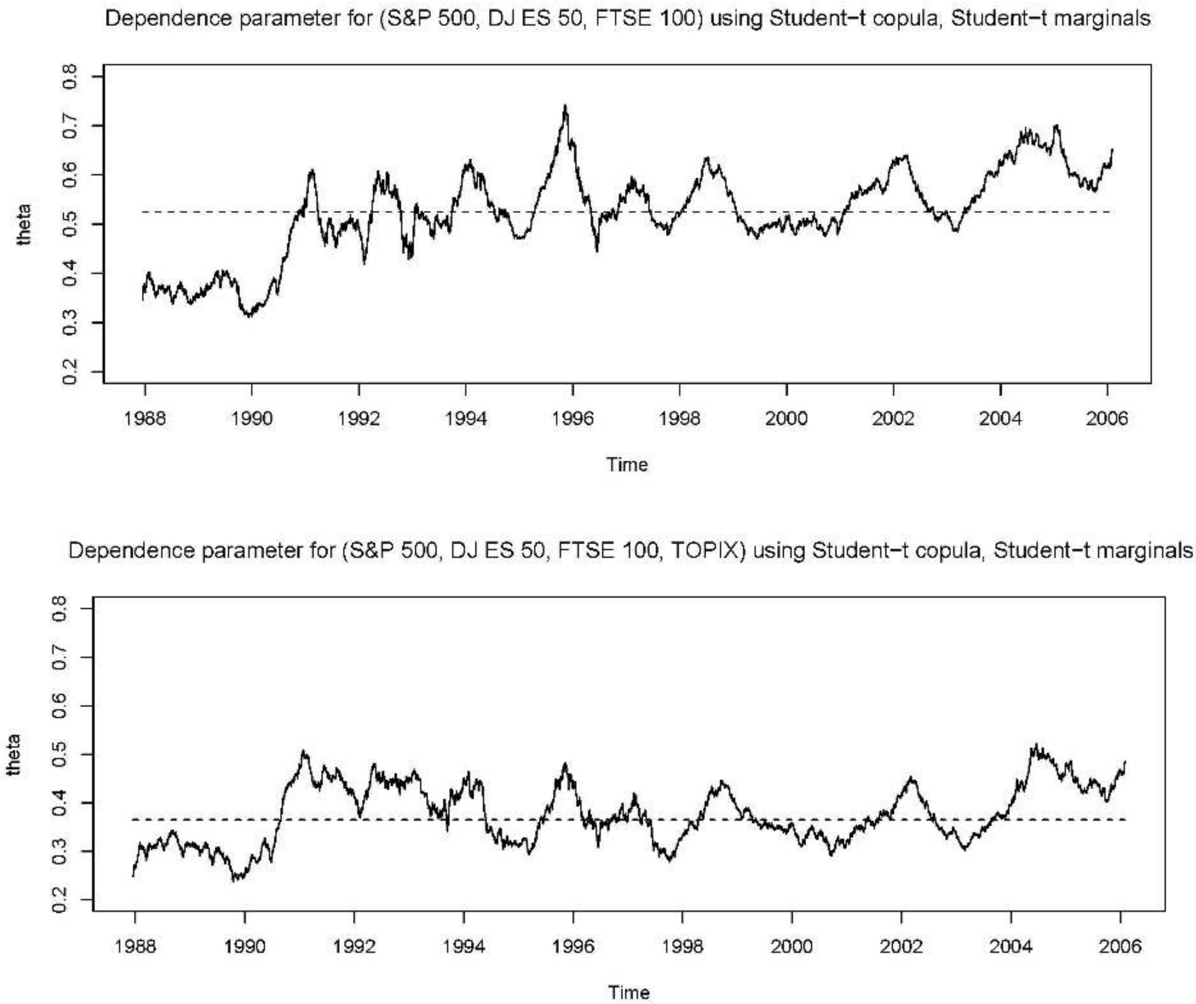

Figure 7: Copula dependence parameter $\hat{\theta}$ estimated for a 3-constituent portfolio constructed of (S&P 500, Dow Jones EURO STOXX 50, FTSE 100) (upper panel) and 4-constituent portfolio constructed of (S&P 500, Dow Jones EURO STOXX 50, FTSE 100, TOPIX) (lower panel) using Student-t copula with Student-t marginals. Time-varying parameter (solid line) is estimated from the moving window of 250 days. Global parameter (dashed line) is estimated from the whole time period from 01 January 1987 to 10 March 2006. The IFM method is applied with t-GARCH(1,1)-fitted volatilities for the marginals, the number of degrees of freedom for the marginals is fixed.

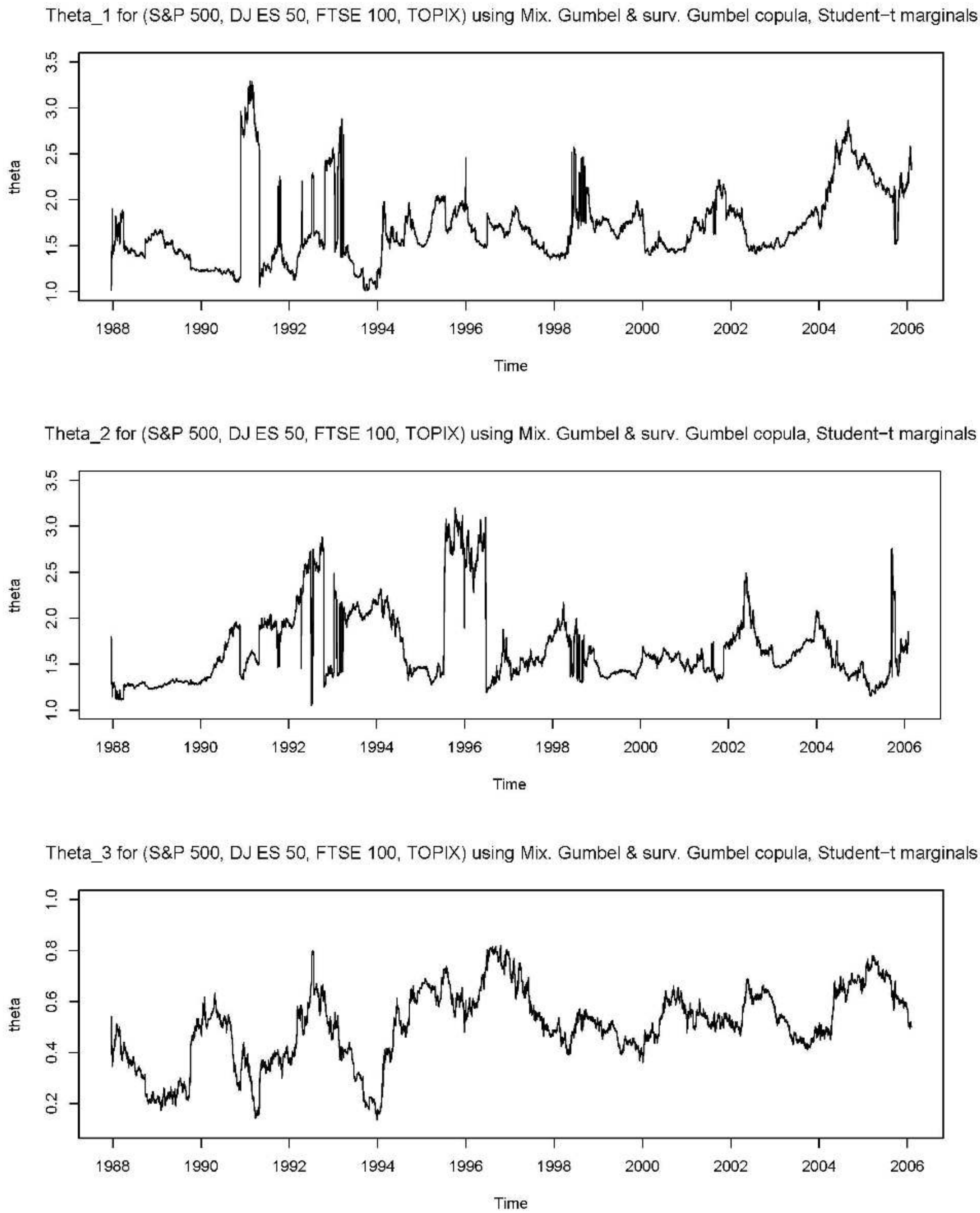

Figure 8: Estimated parameters $\hat{\theta}_1$ (upper panel), $\hat{\theta}_2$ (middle panel), $\hat{\theta}_3$ (lower panel) for the mixture model $C_X(u_1, u_2, u_3, \theta) = \theta_3 C_X^{Gumbel}(u_1, u_2, u_3, \theta_1) + (1 - \theta_3) C_X^{Surv.Gumbel}(u_1, u_2, u_3, \theta_2)$. Estimated for a 3-constituent portfolio constructed of (S&P 500, Dow Jones EURO STOXX 50, FTSE 100) using moving window of 250 days, assuming Student-t marginals.

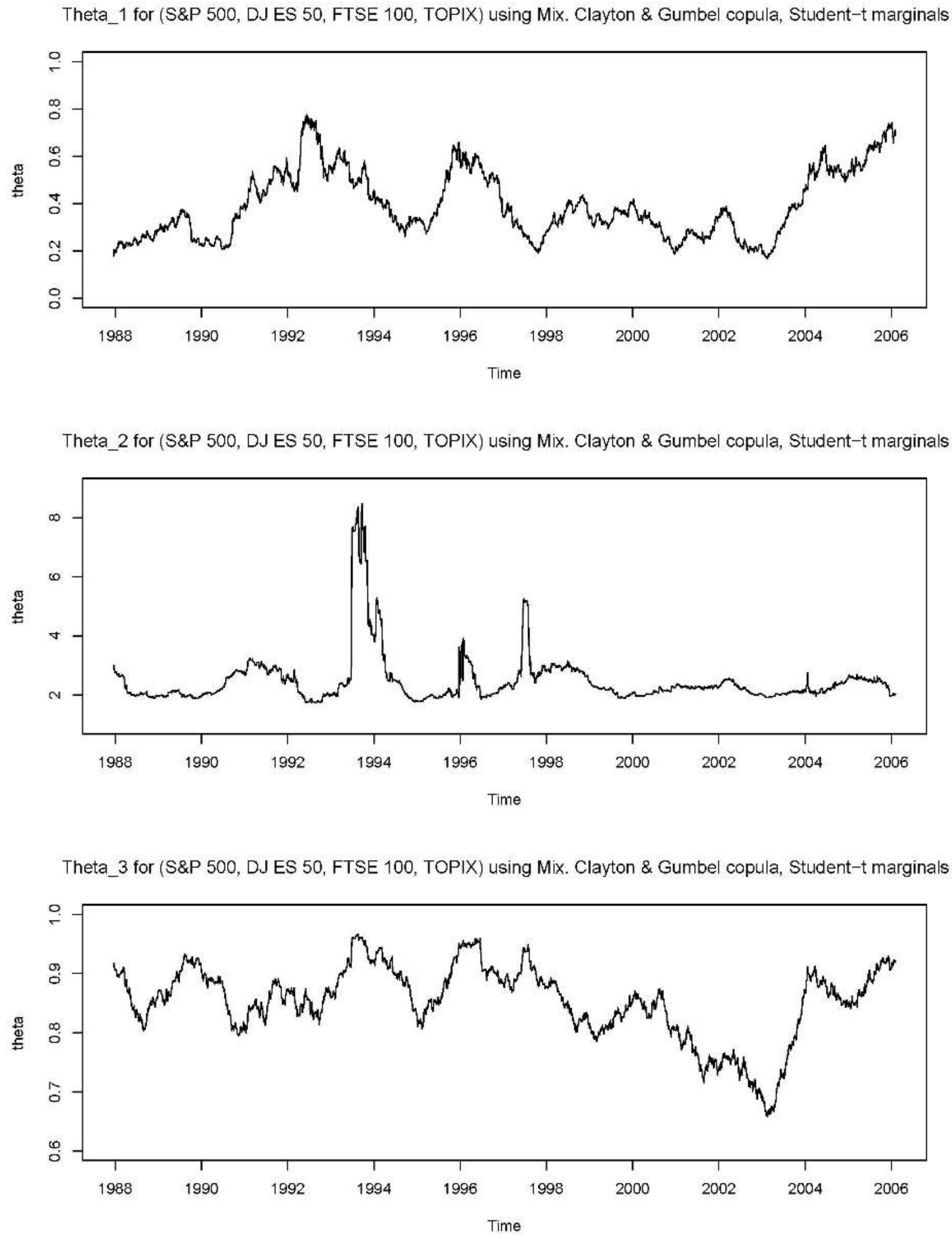

Figure 9: Estimated parameters $\hat{\theta}_1$ (upper panel), $\hat{\theta}_2$ (middle panel), $\hat{\theta}_3$ (lower panel) for the mixture model $C_X(u_1, u_2, u_3, u_4, \theta) = \theta_3 C_X^{Clayton}(u_1, u_2, u_3, u_4, \theta_1) + (1-\theta_3) C_X^{Gumbel}(u_1, u_2, u_3, u_4, \theta_2)$. Estimated for a 4-constituent portfolio constructed of (S&P 500, Dow Jones EURO STOXX 50, FTSE 100, TOPIX) using moving window of 250 days, assuming Student-t marginals.

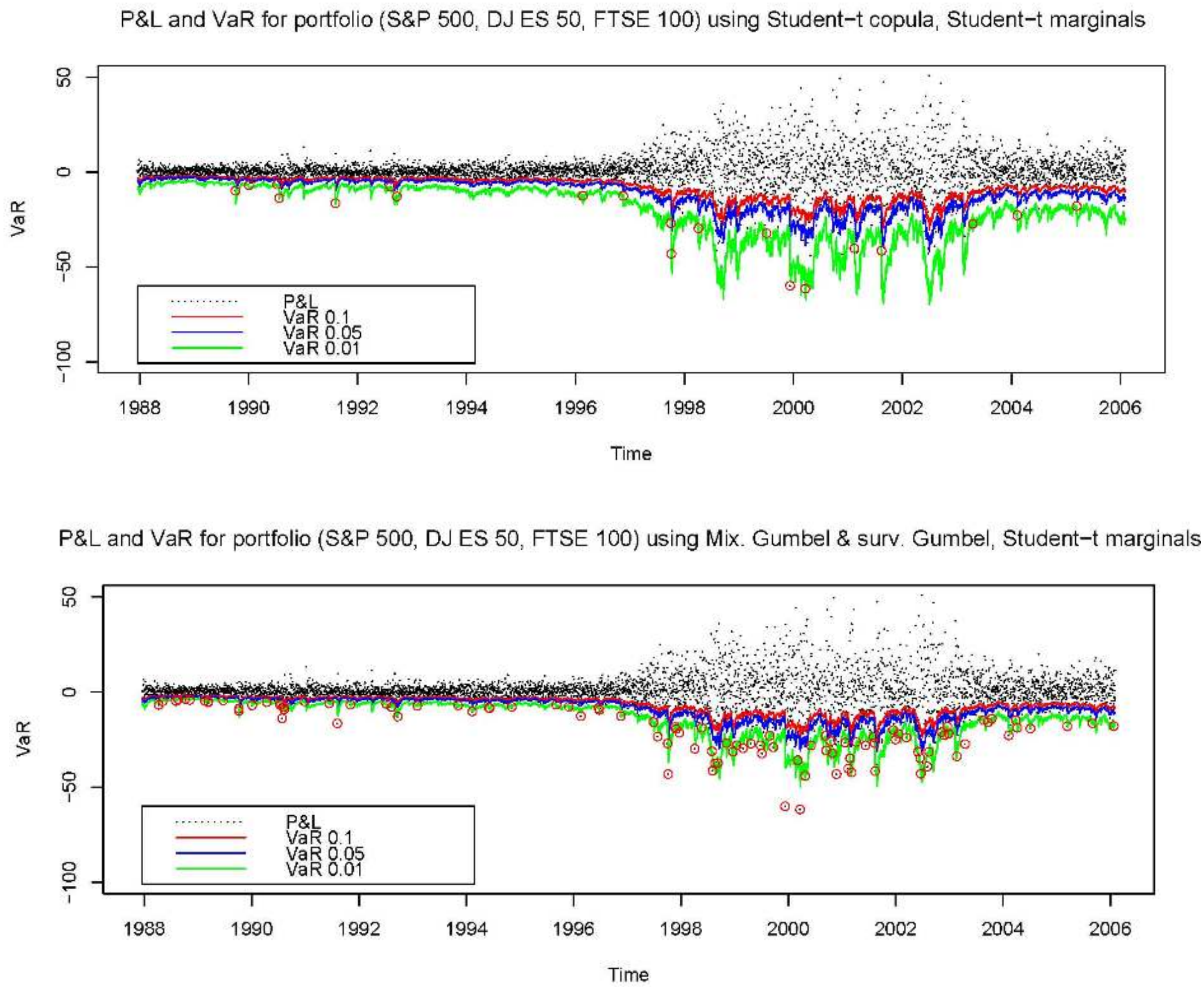

Figure 10: P&L, VaR estimated at different confidence levels $\alpha \in \{0.1, 0.05, 0.01, 0.005, 0.001\}$ using Student-t copula with Student-t marginals (upper panel) and mixture model Gumbel & survival Gumbel with Student-t marginals (lower panel) for a 3-constituent portfolio $w = (1, 1, 1)^\top$ constructed of S&P 500, Dow Jones EURO STOXX 50 and FTSE 100. Exceedances are plotted at level $\alpha = 0.01$.

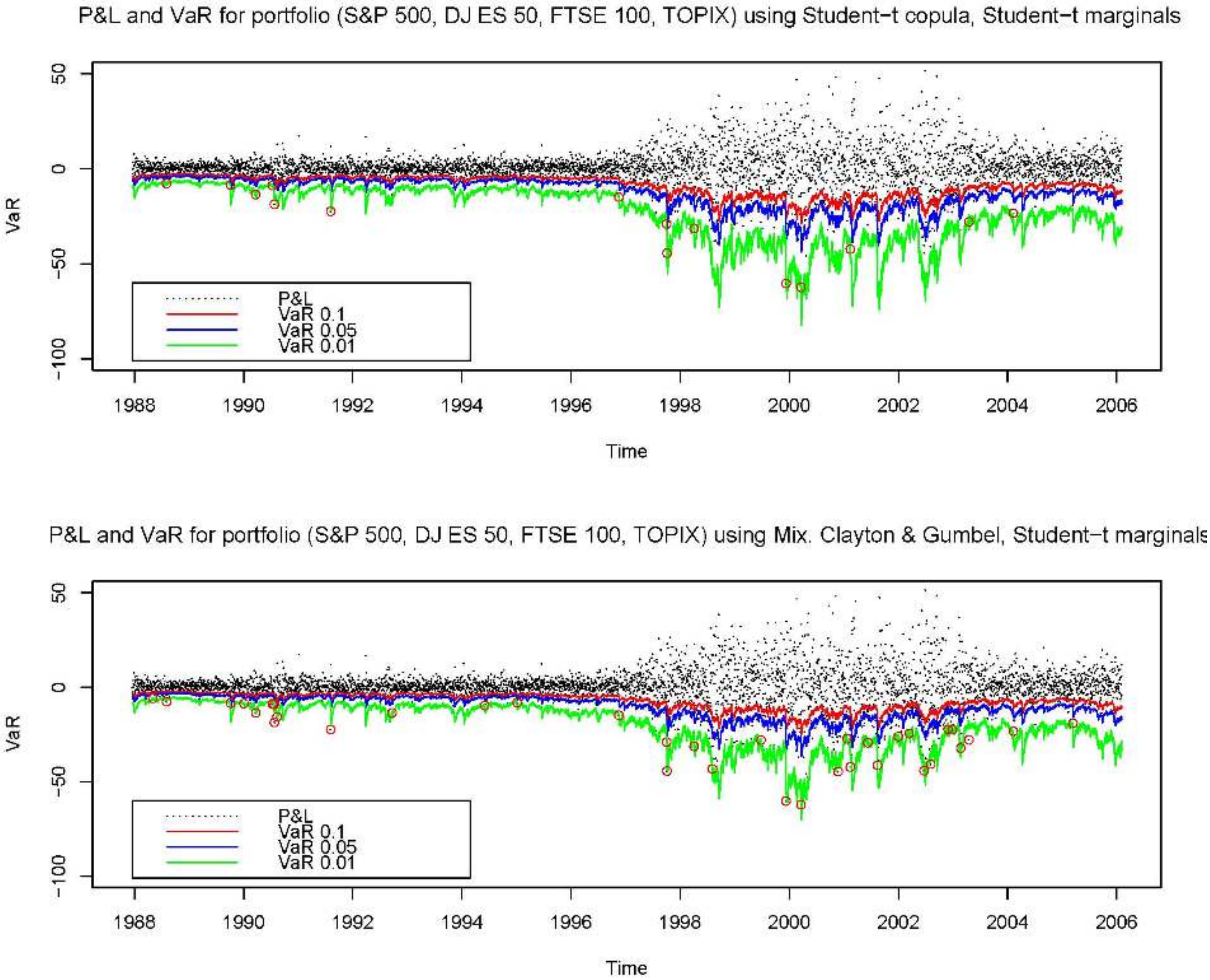

Figure 11: P&L, VaR estimated at different confidence levels $\alpha \in \{0.1, 0.05, 0.01, 0.005, 0.001\}$ using Student-t copula with Student-t marginals (upper panel) and mixture model Clayton & Gumbel with Student-t marginals (lower panel) for a 4-constituent portfolio $w = (1, 1, 1, 1)^\top$ constructed of S&P 500, Dow Jones EURO STOXX 50, FTSE 100 and TOPIX . Exceedances are plotted at level $\alpha = 0.01$.