MODELLING DYNAMIC ECONOMIC
PROBLEMS ON THE ANALOG COMPUTER

by
Alfred Engel
and
John J. Kennedy

ABSTRACT

The basic application of analog computer methods to the solution (modelling) of dynamic economic problems is discussed. Consideration is given, especially, to both the necessity and feasibility of utilizing for this purpose the built-in expansion equipment available in modern analog computers. In addition, the solution to several fundamental problems of economic modelling are presented and discussed in their relation to dynamic computation in the broad area of economic analysis.

INTRODUCTION

During the past several decades, considerable effort has been devoted to the mathematical analysis and description of economic concepts, and to the construction of analytical economic models. Fairly good results have been obtained, for example, in determining parameters from experimentally-found statistics, and in predicting behavior in finite difference models. To the latter, it might be noted, the analytic work done by Frisch -- phenomenon of collinearity, (1) -- and Haavelmo -- *"the statistical method used* must *be derived from a model that specifies the relations among the jointly dependent variables,"* (2) -- contributed significantly in the methodology of economic modelling.

Along this same line, the pinpointing of the "prankster phenomenon" by Koopmans (3) eliminated much ambiguity from data correlation: *"the hypothetical model specifying the relations among the variables should be such as to allow for identification of parameters."*

Quite naturally, however, despite the development of analytical techniques such as these, there has been little work done in dynamic modelling, *per se.* As with any 'young' science, with new vistas of research, it is essential that thorough study proceed from idealized 'sample' systems toward the more realistic -- complex and continuous -- systems of description.

Recent developments in analog computer technology -- improved dynamic accuracy in operational amplifiers, higher resolution in attenuators, high-speed repetitive operation capability ... all the way to the built-in expansion equipment usually provided -- now make possible a serious attempt to meet this goal of true economic modelling. This goal, set up by Leontief in his "Studies in the Structure of the American Economy" (4), indeed, seems to set the guide line for future development by its contrasting of static vs dynamic theory.

STATIC AND DYNAMIC THEORY

The major difference between static (sequence) models (implemented on purely digital computers) and dynamic models (implemented on analog or, if more involved, on hybrid computers) is stated clearly in the following definition by Leontieff: *"A static theory derives the changes in the variables of a given system from the observed changes in the underlying structural relationship; dynamic theory goes further and shows how certain changes in the variables can be explained on the basis of fixed (i.e. invariant) structural characteristics of the system".*

From the 1953 edition (4, above), he continues: *"Dynamic theory thus enables us to derive the empirical law of change of a particular economy from information obtained through the observation of its structural characteristics at one single point of time. This possibility, methodologically rather obvious and practically very*

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*important, has unfortunately been obscured by the fact
that most of the recent attempts to determine the structural
characteristics of actual economic systems have been
based on some kind of statistical time—series analysis,
thus giving rise to the erroneous impression that the
empirical laws of change necessarily must be derived
from direct observations of past development."*

The implication here is that finite difference equations of sequence models are to be replaced by sets of differential and difference-differential equations. Differential equations occur, for example, when the "lag" of an input variable can be considered as an improper one, such as with extended payments for _sold_ products or 'gambling', say, with a tax refund sure to come within the next year. Difference-differential equations, on the other hand, are a necessity in the event of a _genuine_ lag which is almost always the result of a behavioral phenomenon due to people in business. In more involved systems, indeed, this lag feature acts as a moderator and stabilizer and usually influences system characteristics considerably.

It is worthwhile, here, to provide some general clarification to Leontieff's definition. Given a difference-differential equation equivalent to a genuine lag relation, and known initial conditions, it is possible to achieve a quasi-continuous solution, x(t), for the system being modelled. However, the same equation written in terms of a finite-differenced sequence model -- with one step equivalent to a whole unit of time -- gives rise only to an 'approximate guess' for _one_ point of time ... it drops all samples in between and lacks a criterion of convergence depending on how fine the time grid is with regard to the input variations.

Granted, the dynamic model sometimes requires an entire function stored among its initial conditions -- but only _once_ in the entire simulation -- but it provides another function for _any_ time, t, in the solution, thus displaying the dynamic characteristics of the simulated system. The sequence model yields only a datum, or set of data, whose mutual connection and continuation are unknown and, actually, undecidable as long as the grid violates Shannon's Sampling Theorem*(5). The result is a framework of static points in problem space which deviate more from the accurate solution as the number of successive time steps increases. This source of error is one of the major

causes for inaccurate economic predictions from sequence models.

MODEL BUILDING AND INTERPRETATION

The feasibility of dynamic economic modelling is enhanced by the following features of presently available computational accessories, over and above the familiar summing and integrating amplifiers which provide the firm basis on which all analog model building is grounded:

a. Logic branching of dynamic quantities can be accomplished at computed points of time by comparators (COMP) ... which compare desired and attained values and make 'decisions' by error signal outputs ... or arbitrarily in time by function switches (FS) ... which permit random inputs to be made manually.

b. Genuine lags can be mechanized by track-and-store amplifiers (TS) -- These amplifiers can, if desired, be used also to smooth out unrealistic discontinuities from their sampled inputs.

c. Exogenous variables whose analytic form is unknown or only approximated can be simulated continuously and with high accuracy as experimental functions on variable diode function generators (VDFG).

d. Sensitivity and vulnerability can be tested with scaled and normalized step inputs--such occurrences as the injection or rejection of money from a system, for example -- at computed times by comparators or arbitrarily by function switches.

e. The feature of repetitive operation (Rep Op) permits display of solutions on an oscilloscope while parameters are being changed manually, thus providing a truly dynamic system model.

f. In larger systems, master control for the logic states and modes of a program can be executed either by a memory logic group (MLG) ... which directly governs track-and-store modes, the state of AND and OR gates, etc. ... or by a digital operations system (DOS) ... which provides for digital-analog communication and bidirectional conversion, data buffering, and logic control. MLG is a feature of the EAI PACE® 231R-V General Purpose Analog Computing System. A digital operations system (DOS 350) provides programmable logic and communication as an integral part of the EAI

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*This Theorem states that: Sampling must occur at such a rate or density that _no_ information of relevance (i.e. information required because of being characteristic) is lost. In other words, the sampled function must allow for reconstruction of the original function with a _prescribed_ measure of accuracy.

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HYDAC® 2400 Combined Hybrid Computing System.

The class of problems embraced by the equipment listed under Item f is beyond the scope of this paper, and will not be discussed further here.

It can be seen from this brief review of computational features that the analog computer can be an ideal tool for dynamic economic model simulation. In view of the national -- and international -- implications of economic competition and rivalry, such activity in dynamic modelling and vulnerability testing is certainly justified.

There are obstacles, however, in contemporary economic model building, and several examples will be given later showing how these obstacles can be overcome by means of modern analog computer equipment. It will be helpful, though, to discuss these problems briefly at this time especially since they are closely related to model building and interpretation.

1. Policy Decisions: Leontief's definition: *"Questions of policy can have an operational meaning only if one assumes that the structure of certain sectors of the economy can be changed"*, regarding the effect of policy changes on the economic system is already formulated in such a way as to fit into analog computer philosophy. Thus, a "question" regarding the effect of a policy change can be asked of a comparator whose decision, based on predetermined levels of achievement, will continue with existing business and/or economic principles (as simulated) or will switch to an alternative, pre-patched simulation embracing those changes in principle which are assumed necessary in theory to accommodate the policy change.

The mechanization of policy decisions on comparators is always applicable whenever the need or demand for a decision is endogenous within the economic structure. In addition, there might be randomly distributed (and rare) exogenous demands for changes in policy requiring similar kinds of connecting and disconnecting also mechanizable on a comparator. These can include such events as unexpected injection or rejection of money, stocks, or materials, etc., or booms or depressions. Of course, these random inputs also can be scaled so as to represent defined quantities and can be put onto the system manually by function switches at the discretion of the investigator.

Obviously, a function switch also can be used to test stability, sensitivity and vulnerability of a system or whether the system can or cannot stand a sudden, percentagewise change without oscillation or breakdown. Operating a function switch eventually can cause a comparator to change system policy. In this regard, a sufficiently fast comparator to pick up accumulated exogenous demands used in combination with the Repetitive Operation feature provides one of the most powerful tools for the dynamic study of simulated economic systems. Selected portions of a run can be displayed on the oscilloscope, showing several output variables simultaneously, while manual modification of parameter values provide a quick check of the dynamic response of the system.

2. Open Systems: The usage of this term is that a mathematically inhomogeneous system of ordinary differential equations (ODE) is equivalent to an economic system with an exogenous function. That is, the function cannot be derived within the given system; it comes from 'outside' and the system is called 'open'. In mechanizing open systems, the number of 'openings' equals the number of variable diode function generators required to set up these experimental functions. The VDFG can be adjusted with an amazing degree of accuracy in such systems even though the function might appear rather wild in spikes or slopes.

In open systems, the exogenous function remains the same for each run. If it were partially subject to changes due to variables computed in the rest of the system if would be partially derivable within the system and thus not entirely exogenous no matter whether its analytical form were known or not. Of course, it would have to be known how to derive the function in general terms whenever a statement relative to change were made. Otherwise, the system would be unrealistic and, thus, unjustified.

3. Forced Goal Systems: This type of system applies if the final state of an economy is prescribed, such as a national emergency in case of war, survival conditions of vital industries, etc. Mathematically speaking, it represents a boundary value problem but it should be noted that in the field of economics this does not imply by itself that the goal must be reached along an optimized path.

Essentially, there are two techniques available for dealing with Forced Goal Systems. One is simple trial and error with repeated runs, preferably in high speed repetitive operation; it can be mechanized to be fully automatic also and applies to the original system.

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The other technique is an approach with adjoint variables as treated in the theory of differential equations and in the simulation of engineering systems; it implies optimized paths. The increase in required equipment usually is negligible if the adjoint variables are selected carefully, based on a good understanding of the original system.

This so-called "adjoint technique", although not too complicated itself, is beyond the scope of this present paper in which only the basic applications are discussed and will not be treated further.

4. *Irreversibility:* Strict or partial irreversibility of goods in stock, invested capital or real estate bought can give rise to situations calling for policy decisions such as, for example, when the market volume drops. Usually, such a policy change demand pinpoints an error in past decisions and it can create contradictions in small systems, calling for a simulation on a broader basis in order to find out how the fault could be avoided. Leontief's discussion of this problem under the general heading "defects of multi-phase theory" gives a general theorem which proves that surplus stock in these cases is to be tolerated just as entropy is in, say, thermodynamic systems. He argues that "in terms of these principles, the deviation into the alternative path would be...as one actually has reason to believe ...explicitly rejected".

There is one solution, however, which can be simulated very nicely on an analog computer. Provided that market volume is governed by a variable which is independent of the endangered (irreversible) quantity -- governed, say, by advertising or by changes in another (competitive) product -- then, instead of changing the phase of simulation, it is possible to decrease an otherwise constant parameter in the precarious equation (mechanized on a servopotentiometer) while increasing market support until market volume increases again. (Cure the cause instead of submitting to the results.) In larger systems, of course, the very loose relationship existing between different products will avoid bottlenecks of this kind.

5. *Individual vs. General Models:* It is of importance here that the greater relative value of individual models as opposed to general ones be stressed. The purpose of economic modeling in the first place is to analyze the economic structure of a single entity, not to examine that entity's profile against the overall or general economic structure of the market place, the financial community, or even the nation as a whole.

Accordingly, the most time-consuming part of economic model-building is to satisfy Haavelmo's principle concerning the establishing of a truly individual model. Above all, the only proper beginning is from precise definitions of qualitative and quantitative variables. A single gap in the chain of definitions can, and usually will, void all further analysis.

Next, by following, say, an incoming order through every detail between its receipt and completion (i.e., receiving payment), a number of individual structures and parameters will be revealed about a company. (It is these details which count in competition, not the general framework which is, after all, the common mold.) The numerical value of these parameters can be obtained by assigning a departmental 'work factor' for production and clerical operations based on average handling time, average cost per job, efficiency under various workload conditions. etc.

Finally, superimposed on this structure of internal workload processing are factors reflecting the general policy of the management, the internal and external economics of plant management, the company's accomodation to the environment (the market itself, suppliers, competitors, taxes, etc.) and the individual emergency decision lines (personnel, price relations, weighting of advertising, etc.).

Such analysis by itself, if carefully done, can tell an amazingly accurate story of what is really going on in a company businesswise. If these data and structures are investigated in their dynamics...run on the computer, tested in various forms, changed to account for hypothetical situations, etc. ...there can hardly be a loss to a company in any business competition.

THE MECHANIZATION OF SYSTEM EQUATIONS

This section will deal with the actual mechanization of specific economic models. Here, interpretation is of the essence. For example, in sequence models, functions are always encountered in terms of differenced arguments, such as $X(t-1)$ or X_{-1}. Sometimes the function itself is differenced; for example, $X(t) - X(t-1)$ or ∇X_0, $X(t-1) - X(t-2)$ or ∇X_{-1}. Beside the fact that differences are derivable from differentials and from differential quotients, there still remain here alternative interpretations of these symbolic notations depending upon the intention of their authors.

That is, it seems natural in economics to approximate differential quotients only by backward differences (∇) instead of by central (δ) or forward (∇) differences*. Thus...

$$\frac{d}{dt} X(t) = \dot{X}(t) \text{ corresponds to } \nabla X =$$

$$\text{Def } X(t) - X(t-1) \text{ or } X_0 - X_{-1} \tag{1}$$

$$\frac{d^2}{dt^2} X(t) = \ddot{X}(t) \text{ corresponds to } \nabla(\nabla X) = \Delta^2 X = \tag{2}$$

$$X(t) - 2X(t-1) + X(t-2) \text{ or } X_0 - 2X_{-1} + X_{-2}\cdots$$

However, it is with these approximations where the alternative in interpretation arises. The validity of equations (1) and (2) depends, quite naturally, upon whether the relationship

$$\frac{\nabla X}{X} \ll 1 \tag{3}$$

is satisfied or not. The following sections will show the implications in economics of this interpretive choice.

1. Non Genuine Lags: Samuelson's Second Interaction Model is described by Beach (2) with the following set of equations:

$$Y(t) = C(t) + I(t) \qquad \text{denoting Income} \tag{4}$$

$$C(t) = aY(t-1) \qquad \text{denoting Consumption} \tag{5}$$

$$I(t) = b\left[C(t) - C(t-1)\right] + d \quad \text{denoting Investment} \tag{6}$$

$$Y(O) = Y_0, \ Y(1) = Y_1 \quad \text{denoting Initial Conditions} \tag{7}$$

By substitution

$$Y(t) = aY(t-1) + ab\left[Y(t-1) - Y(t-2)\right] + d \tag{8}$$

Applying equations of type (1) and (2) to Equation (8) is equivalent to denying genuine lags in Samuelson's Model. The unit of time (t=1), say, would be equivalent to one month and all variables (Y, C, and I) would have to satisfy Equation (3). Then and only then would his solutions be correct [refer to Equations (111), (112) and (113) in Reference (5)] .

On this basis, it is justified to transform Equation (8) into an ordinary differential equation and, since the most-shifted argument is (t-2), a second-order ODE must be expected. A general and simple scheme for this transformation is:

$$
\begin{aligned}
d = {}& Y(t) - a(1-b)\,Y(t-1) + ab\,Y(t-2) \\
& + 2ab\,Y(t-1) - 2ab\,Y(t-2) \\
& -abY(t) + ab\,Y(t) \\
& + a(1-3b)\,Y(t) \\
& -a(1-3b)\,Y(t)
\end{aligned}
$$

$\left.\begin{array}{c}\\ \\ \\ \end{array}\right\}$ to account for $\ddot{Y}$

$\left.\begin{array}{c}\\ \\ \end{array}\right\}$ to account for $\dot{Y}$

$$d = (1-a+2ab)\,Y(t) + a(1-3b)\underbrace{\left[Y(t) - Y(t-1)\right]}_{\dot{Y}} + $$

$$ab\underbrace{\left[Y(t) - 2Y(t-1) + Y(t-2)\right]}_{\ddot{Y}}$$

The final equation is

$$\ddot{Y}(t) = -\left(\frac{1}{b} - 3\right)\dot{Y}(t) - \left(2 - \frac{1}{b} + \frac{1}{ab}\right)Y(t) + d \tag{9}$$

This equation is the ODE-equivalent of Equation (8); written in this manner, with the highest derivative on the L.H.S., it is possible to construct the unscaled circuit diagram of Figure 1 using the "bootstrap method" of analog computation. (Figure 2 describes the symbols used in the diagram.)
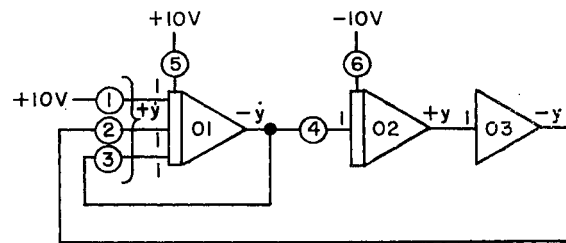

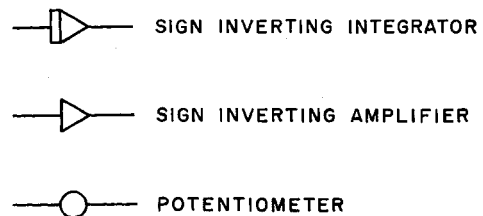Figure 1. Mechanization of equation (9)

—▷— SIGN INVERTING INTEGRATOR

—▷— SIGN INVERTING AMPLIFIER

—◯— POTENTIOMETER

Figure 2. Symbols used in Analog Computer Diagram

Contrary to mathematical usage, most authors use the forward difference symbol, Δ, when actually backward difference, ∇, is meant.

Potentiometers 1, 2 and 3 should be adjusted, according to Equation (9), so as to represent the values of d, (2 - 1/b + 1/ab), and (1/b - 3) respectively, assuming all of these quantities to be positive. (It should be noted that positive coefficients are almost always assured if the feedback loops contain an odd number of amplifiers (negative feedback) and the system equations have a realistic meaning. In this case, there are two loops, one with integrator 01, and one with integrators 01 and 02 and amplifier 03, thus assuring positive potentiometer settings. Generally, due to sign inversion, negative feedback stabilizes the solution while positive feedback offers the explosive models and noisy solutions worthy of dynamic simulation.)

Scaling: Depending on problem complexity or on the objectives of a particular simulation, it is necessary to 'scale' the equations to fit properly on the computer. Thus, even though the three potentiometers described above represent all of the problem coefficients, an additional potentiometer, 04, was used to enter integrator 02. The reason for this is as follows:

In computer time, one second corresponds to one year of problem time (based on the original set of equations). In the simulation, however, it is desired to have one second on the computer correspond to one month of problem time. Accordingly, it is necessary to reduce the inputs to the integrators to 1/12 of their previous values by adjusting potentiometers 01, 02 and 03 appropriately, and by adding potentiometer 04 to the circuit set at 1/12 = 0.0833.

In addition to time scaling, there is amplitude scaling by which it is assured that the maximum problem value is represented by the maximum voltage available on the computer. Thus, to achieve maximum resolution, the variables of interest here, viz Y, $\dot{Y}$ and $\ddot{Y}$, would be represented on the EAI TR-20 General Purpose Analog Computer by the full ±10 volts available on the machine.

It is beyond the scope of this general presentation to go into the details of time and amplitude scaling. The reader is referred to the Primer on Analog Computation (Application Study: 1.1.2a, Bulletin No. ALAC 64002-1) from the EAI Applications Reference Library for a complete description of these subjects.

Initial Conditions: The remaining potentiometers, 05 and 06, supply initial condition (I.C.) values for the two integrators. Thus, potentiometer 05

supplies $\dot{Y}(O) = \dot{Y}_O$ and potentiometer 06 supplies $Y(O) = Y_O$.

It can be seen now that Equation (7) of the model actually states a boundary value problem to be met at time $t = 1$. This, however, is an incremental value; otherwise, the ODE approach would not be valid. This apparent contradiction would be revealed, on detailed examination, to be inherent in the static conception of sequence models and, in fact, to be the basis on which such models are built.

The dynamic concept, on the other hand, escapes such difficulties by employing the I.C.'s as required by the theory of ODE's. Thus, if there were a real boundary value problem, then $t = T$ at the boundary could not, of course, be allowed to be the unit step in time.

Continuing, then, on the basis of this dynamic concept, it is necessary to mechanize the remaining system equations, viz, (5) and (6), to complete the model. It can be derived easily that the following equivalent equations for them are valid:

$$\dot{C}(t) = -\frac{1}{b} C(t) + \frac{1}{b} Y(t) - \frac{d}{b} \qquad (10)$$

$$I(t) = Y(t) - C(t) \qquad (11)$$

Expanding the circuitry of Figure 1 to include these equations, as shown in Figure 3, the mechanization of Samuelson's Second Interaction Model is complete through the simultaneous solution of Equations (9), (10), and (11). Here, potentiometers 07 and 09 refer to 1/b, and 08 to d/b. A third initial condition, $C(o) = C_O$, is needed which, due to the finite difference assumptions, cannot be derived from Equation (5). Potentiometers 11 and 12
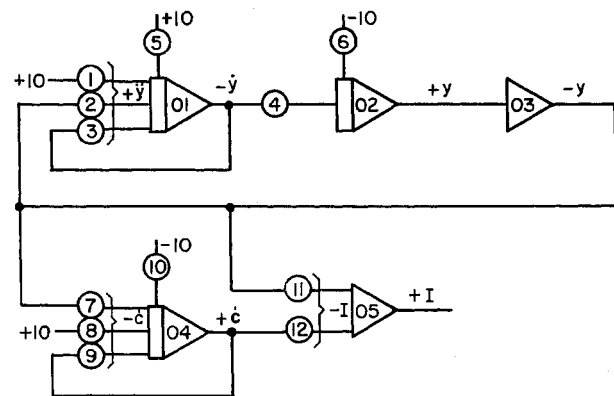

Figure 3. Samuelson's Second Interaction Model

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are required only if I, C and Y have different amplitude scaling factors. Of course, the time scaling factor, say $\beta = 12$, enters potentiometers 07, 08, and 09 if it is contained in the setting of the other potentiometers since time scaling must be consistent throughout the whole circuitry.

Indeed, if there were 10 or more differential equations describing the system, they would lead to circuits similar to that shown in Figure 3 where, for instance, one function $(-Y)$ enters into $-\dot{C}$ of another function to establish all the mutual connections required by the system equations. The only limitation, in fact, is the availability of integrators, amplifiers and potentiometers in the equipment complement.

Example: The effect of Policy Decisions on the Economy:

In order to give a simple example of the effect of policy decisions on the economy, it should be recalled that Samuelson's Second Interaction Model contains savings as a part of the investment, i.e., there is only a partial irreversibility. Furthermore, investment as an input must be a positive quantity. Moreover, people will stop investing money whenever consumption exceeds a specified fraction, F, of their income, i.e., when $C \geq FY$. (At this point, in a modelled situation, a comparator in the computer would throw in order to change the circuit connections.) They will even withdraw savings from investments gradually when this occurs (partial reversibility). As a result, two different phases are created in the system:

PHASE I

$$\ddot{Y} = -A\dot{Y} - BY + D \quad [\text{as in (9)}] \; (12a)$$

$$\dot{C} = -EC + EY - G \; [\text{as in (10)}] (12b) \quad \Big\} \; \text{if } C > FY \; (12d)$$

$$I = Y - C \qquad [\text{as in (11)}] \; (12c)$$

PHASE II

$$\dot{Y} = -A\dot{Y} - BY + D \qquad (13a)$$

$$\dot{C} = -EC + EY - G + EKI \qquad (13b) \quad \Big\} \; \text{if } C \geq FY \; (13d)$$

$$I = (I-F) \, Y_T - K^* \, (t-T) =$$
$$\qquad I_T - K^* \, (t-T) \qquad (13c)$$

The withdrawal of KI and adding it to income has no effect, of course, on the income cycle; compare (12a) and (13a). However, it does add to income

in Equation (13b) where consumption is concerned. Because of the positivity of EKI, in fact, it will increase $+\dot{C}$ and thus $+C$ as opposed to Equation (12d). Hence, this is no cure for the problem of decreased investments but only makes it worse by increasing consumption. Consequently, there is <u>no</u> return from Phase II into Phase I without an exogenous correction <u>not yet in the system.</u> (This is one point criticized by Leontief.)

Of course, lowering the consumption parameter, E, is a cure but this obviously is exogenous, implying, as it does, control of the price index. The system, therefore, is not self-regulating as suggested (ideal business cycle); the coefficients of the problem must be subjected to control. From the computational point of view it is the servo-set potentiometer that quarantees dynamic equilibrium under the law of policy decisions.

By mechanizing the sets of equations, (12) and (13), it can be shown how an unregulated economy breaks down (provided E is large enough) even though all the money saved previously (reversible money) should be spent. That the extraction of K_t from I could be terminated if the simulation proceeded toward a minimum I_0 (controlled by a second comparator) is immaterial with this phenomenon.

Because of Equation (13c), it is necessary to hold the value of I beginning from the instant Phase II takes over. Therefore, it is desireable to mechanize $\dot{I} = \dot{Y} - \dot{C}$ and have the integral I remain constant after its inputs are removed.

Figure 4 is seen to be the proper modification to Figure 3 to effect this mechanization. Here, the derivative $-\dot{Y}$ is available as it was in Figure 3; however, to make $-\dot{C}$ available it was necessary to add summer 06. In order to hold $I(T) = I_T$, amplifier 05 had to be patched as an integrator whose I.C. on potentiometer 16 has already been defined. Amplifier 07 is required to provide the proper sign of C for $-\dot{C}$ as prescribed by Equations (12b) and (13b).

In addition, potentiometer 13 provides the value of F from Equations (12d) and (13d), and the comparator throws at $t = T$ to its positive output when $C - FY \geq 0$. Consequently, the comparator relay disconnects or connects the eight contacts as indicated. (Usually, two comparators are required to govern four separate pairs of contacts.) In fact, set (12) now is changed into set (13). Potentiometer 14 with K^* is the sole input into amplifier 05 whose output starts from I_T; amplifier 06 has an additional term on potentiometer 15.
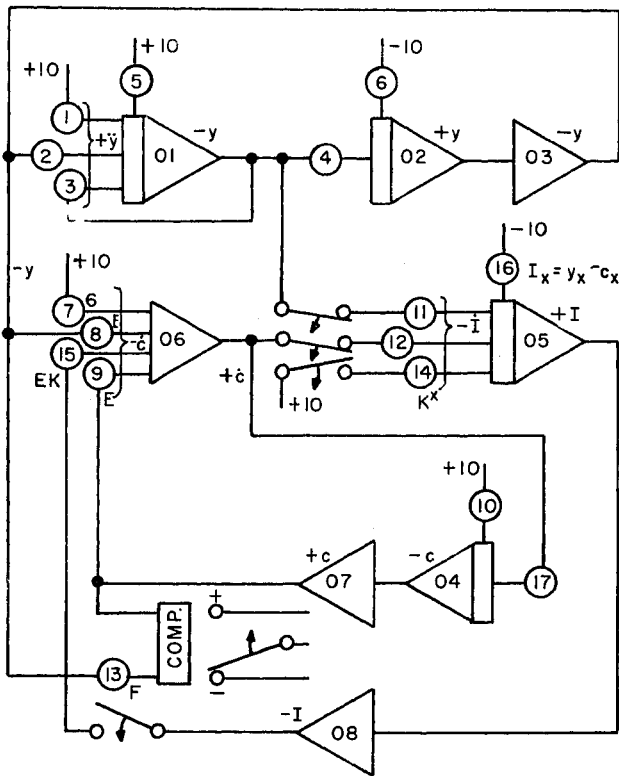
Figure 4. Samuelson's Second Interaction Model with Leontief's Two-phase Policy (all switches down in Phase I)

It should be noted that during Phase II the comparator never can receive a voltage $C - FY < 0$ to throw the system back to minus, i.e. to Phase I.

2. *Genuine Lags:* A genuine lag with regard to variable time, t, occurs if at time, t, a dependent variable, z, is required with its value equal to z $(t-T)$ of time $(t-T)$ and simultaneously there exists a variable, θ, such that

$$ 0 \leq \theta \leq T \quad \text{with} \quad \frac{Z(t) - Z(t-\theta)}{Z(t - \theta/2)} \ll 1 $$

Figure 5 illustrates the implications of this definition. It can be seen in (a), for example, that a system with a genuine lag can be unaffected for long periods of time, whereas in other cases ... the stock market, say ... one-half hour of the day (b) or one day of an otherwise constant month (c) can create definite disturbances or even outright market shock.

Two classes of lagged inputs can be distinguished, as shown in Figure 6. The continuous lag, (a), occurs with the delay of a production line manufacturing very large quantities of a product ... thumbtacks, say, or other small mass-produced
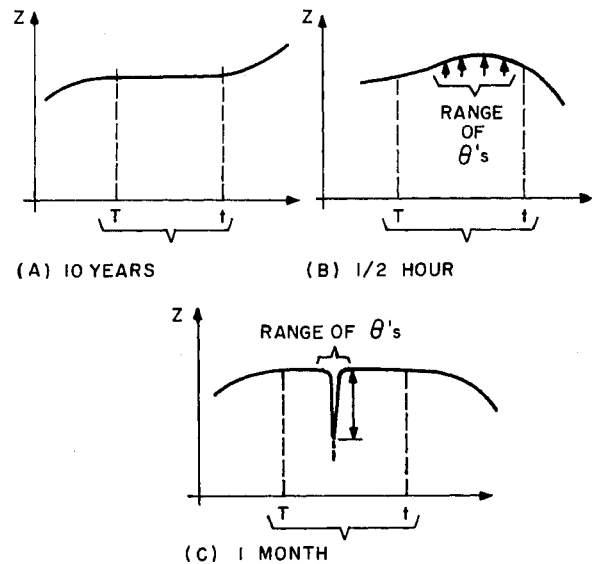

Figure 5. (a) no genuine lag, (b), (c) genuine lags indicated by θ ranges

item. The sampled lag, (b), occurs if the product is large enough for one unit to be a sample of item ... a six-jet airliner, for example.
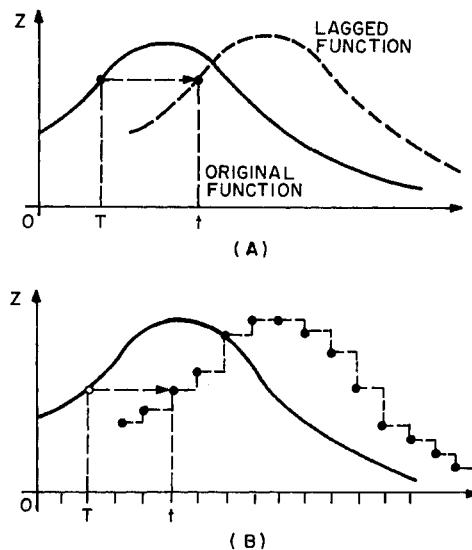

Figure 6. (a) continuous lag, (b) sampled lag

One of the simplest systems with genuine lags is given by W. Leontief:

$$ X_1(t) - b_1 \dot{X}_2(t + T_1) = Y_1(t) \tag{14} $$

$$ X_2(t) - b_2 \dot{X}_1(t + T_2) = Y_2(t) \tag{15} $$

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Differentiation of Equation (15) and substitution of it into Equation (14) establishes a difference-differential equation for X_1; a similar procedure yields the equation for X_2:

$$\ddot{X}_1 (t) = AX_1 (t-T) - AY_1 (t-T) - B_2 \dot{Y}_2 (t-T_2)$$

$$= AX_1 (t-T) + Y_1{}^* (t) \tag{16}$$

$$\ddot{X}_2 (t) = AX_2 (t-T) - AY_2 (t-T) - B_1 \dot{Y}_1 (t-T_2)$$

$$= AX_2 (t-T) + Y_2{}^* (t) \tag{17}$$

where $A = 1/b_1 b_2$, $B_1 = 1/b_1$, $T = T_1 + T_2$, and $Y_1{}^*$, $Y_2{}^*$ as indicated.

Functions Y_1 and Y_2 can be considered as exogenous and, after transformation into $Y_1{}^*$ and $Y_2{}^*$, are set up on two separate VDFG's for $t = +0$ back to $t = -T$. A symbolic circuit diagram would appear as in Figure 7, and applies for each one of the two functions, X_i ($i = 1,2$), separately.

The righthand half of Figure 7 merely simulates a second-order ODE. General methods of mechanizing continuous, stepped-sampled, and smoothed-sampled lags will be discussed later. These are labelled here, for convenience, only as two-part networks "Delay T".
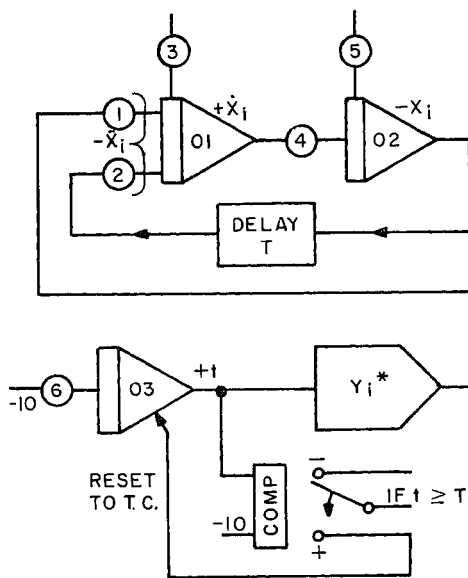

Figure 7. A possible interpretation of equations (16), (17).

The lefthand half of the figure symbolizes the VDFG for $Y_i{}^*$, and a time base integrator, (03), supplies an input voltage whose slope -- speed of problem run -- is determined by potentiometer 06.

Thus, when the output of integrator 03 exceeds +10 volts, maximum time, T_1 has elapsed and the comparator throws to plus, resetting integrator 03 to its initial conditions.

This should be interpreted as equivalent to $Y_i{}^*$'s being periodic in T -- although X_i and $\dot{X}_i$ must not be so -- and unaltered by the function X_i. Consequently, repeated intervals of length T always start with the first value of function $Y_i{}^*$ while the initial conditions of $\dot{X}_i$ and X_i in the nth run are different by their respective final values of run (n-1).

There is another situation possible with this model. Assuming that Y_1, Y_2 ... etc. are not periodic in T, they would have to be generated in some dynamic fashion, a definition of which would have to be provided for in additional system equations. This, of course, would transform their model character from exogenous to endogenous. Then, satisfying Haavelmo's Principle, there would have to be four mutually independent relations in X_1, X_2, Y_1, Y_2 and their derivatives instead of only the two relations shown in Equations (14) and (15).

3. The Mechanization of Lags: Among the three kinds of genuine lags in economic model simulation, the easiest to mechanize is the stepped-sampled lag. This is accomplished with so-called "track-store amplifiers" (TS) ... integrators alternating between their I.C. and HOLD modes. Chains of n TS amplifiers can be patched in order to delay samples by n/2 TS intervals in time.

TS amplifier modes are controlled by a master clock. Figure 8 shows a chain of two TS amplifiers; the first input and last output correspond to the terminals of the "Delay T" box of Figure 7. The performance of TS I and TS II versus time is best explained by the diagram shown in Figure 9.

During the tracking period (TR), the amplifier attains an output voltage equal to the inverted input very quickly; during the store period (ST),
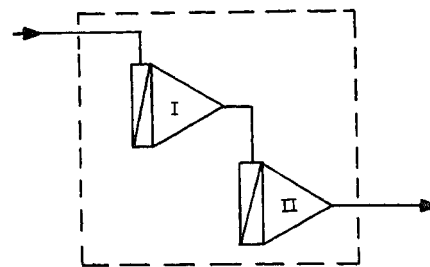

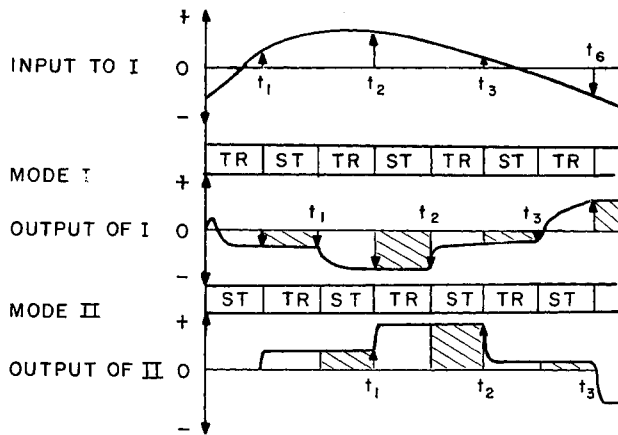Figure 8. Stepped-sampled lag with two TS amplifiers.

Figure 9. Time characteristics of a TS chain.

the amplifier preserves its latest value from the TR period no matter how its I.C. changes. Thus, after the specified "lag" time, the properly delayed input value is returned to the simulation as a 'staircase function'.

In some cases, however, because system requirements are opposed to such an essential staircase function of abruptly varying height, it is necessary to smooth out this final delayed output -- smoothed-sampled lag. Here, the final stage is terminated by an ordinary integrator with a summing feedback, as shown in Figure 10, thus producing a continuous lagged output as close to the original function as possible within the limited density of sampling (sampling rate). The objective here, of course, is to obey Shannon's Sampling Theorem as well as possible by loosing only a minimum of vital "information". Accordingly, the intimate details of any delays are not shown; only their cumulative effect on the function itself is produced.

A comprehensive, non-mathematical explanation of the performance of this circuit is as follows: assume that TS I has just tracked for sample f_n of the original function, f(t), and has gone into store while the integrator slowly builds up to the value of the preceeding sample, f_{n-1}. Then, summing amplifier, A, forms their difference and supplies
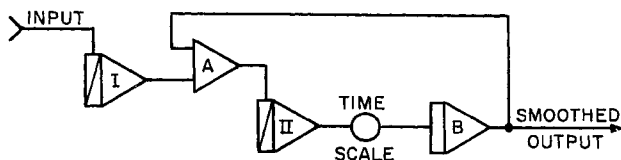

Figure 10. Smoothed sampled lag with two TS amplifiers.

it to TS II which tracks while TS I stores. On the next tracking period of TS I, amplifier TS II will store this difference, thus causing integrator B to integrate at precisely this rated voltage. If the integrator potentiometer is adjusted properly, the output of B attains f_n always at TS seconds later than if occurred at the input of TS I. Through this repetitive process, the delayed function between f_n and f_{n-1} is approximated by smoothed samples in a straight line segment.

It can be seen from Figure 10 that equipment requirements soon would become excessive should the delay problem occur in several places in a system. With such systems, where the delay could be judged as a continuous lag, the use of a hybrid system is justified. This would mean that the data to be delayed would be converted from analog to digital (AD), stored digitally for the required period of time -- which can even be a varying and computed T -- and then converted back digital to analog (DA) and fed into the analog circuit.

The hybrid approach permits the simulation of tremendous systems, handles multiple channels of information easily, and can be programmed to obey Shannon's Theorem at less expense than a purely analog simulation of a large economic system would be. Thus, delayed functions as shown in Figure 5a can be achieved very easily and in great number.

4. Further Improvements Toward Reality: In the interest of keeping simulated models within the bounds of reality at all times, there are several important features of the analog computer -- even in small- and medium-sized installations -- which are very helpful in this regard.

The function switch is one of these features. As a manual switch, operable at random, it provides not only for alternating between different phases of the simulation but also (and of most importance) for the sudden injection and rejection of quantities such as can occur in even the most sophisticated of economies. It is this simple instrument which tests such important dynamic characteristics as sensitivity and vulnerability of systems. Indeed, the beginning of fluctuations due to injection can be determined as easily as can be the limited amount of capital needed to be withdrawn from a system in order to force it into phase change. This type of analysis can be especially gratifying when system stability cannot be determined with paper and pencil, i.e. when due to nonlinear phenomena, experimental tests become a necessity.

The much discussed question of the *ceiling problem* is another dynamic characteristic which requires realistic analysis. Warranted and artificial disturbances in business (fluctuations, depressions, booms, etc.) can cause this dynamic problem to occur whenever ... as a result ... a "quantity of merit" (e.g. effectiveness, rentability, gross national product, etc.) approaches its absolute maximum in a non-asymptotic way. Assuming that some relationship defining this maximum value either statistically or dynamically is known, the question is: How can asymptotic merging be enforced?

A comparator (the automatic analagon of an FS) together with a servo-controlled limiter can be used to mechanize the necessary control function. (It should be noted that in a free economic system a "limiter" is nothing more or less than the free and conscientious decision of a responsible executive -- a human being voluntarily serving the common good if vital objects are at stake.)

Given the difference between the "ceiling", Y_m (which may be dynamic), and the present value of the quantity, Y, in question, this difference (Y_m-Y) can be used as a proportionate measure for the first derivative, $\dot{Y}$, of said quantity. (As a dynamic quantity, $\dot{Y}$ always is available somewhere in the system of ODE's, in contrast to sequence models.) Thus, as (Y_m-Y) approaches zero, $\dot{Y}$ can be forced to approach zero simultaneously at some prescribed rate.

Whenever $\dot{Y}$ should exceed the allowed margin, K (Y_m-Y), a comparator throws in order to feed $\dot{Y}$ via a limiter with adjustable servo potentiometer, and this potentiometer is driven by the value K (Y_m-Y). After limitation of $\dot{Y}$, of course, its integral, Y, will change more slowly and Y can meet Y_m only asymptotically as desired. No transients can occur. A record of time when the comparator throws and of the quantity $\dot{Y} - K$ (Y_m-Y) is equivalent to a bookkeeping record of when and how much control is to be enforced.

Finally, it should be noted that extremely large logical and memory capabilities are an integral part of the new and sophisticated hybrid computers. These computers can proceed with digital subroutines and data processing simultaneously with analog simulations, and all are under the master control of a digital operations system.

The simulation of the dynamics of a national economy, say, incorporating 25 ODE's with 10 genuine lags, each one requiring 100 samples per unit of delay, and 6 analog decisions to be instrumented at digital speed, plus provision for manual or digital computed switching among three separate phases is well within the capabilities of modern hybrid computers.

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