

# **Modelling Methods for Energy in Buildings**

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University of Northumbria at Newcastle

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# Preface

Buildings consume between 40% and 60% of all energy use in most developed economies throughout the world, and an increasing proportion in many developing and emerging economies. Correspondingly, they are responsible for a similar proportion of humankind's carbon dioxide emissions with a consequential impact on global warming. Indeed, the International Council for Research and Innovation in Building and Construction (CIB) in 2003 established a Working Commission to stimulate and coordinate international research cooperation on the impact that buildings have on climate change as well as the impact that climate change will have on buildings in the future. Whilst the precise causes and scale of these impacts will remain uncertain and will be the subject of debate for some time to come, what is not in doubt is that our current dependency on carbon-based fossil fuels for providing comfort and amenity in buildings cannot be sustained and that alternative strategies require to be explored, designed, implemented, tested and demonstrated in the immediate years ahead. At the heart of this is the need to both extend and improve our knowledge of the processes of energy exchange in buildings whether they be the 'high tech' commercial buildings in the centres of our cities or the simpler and often naturally-ventilated buildings and residential habitats that surround them. Whilst many of the methods involved in our knowledge of these processes are grounded in well-established physical laws, in the past 30 years there have been dramatic improvements in the computational resources available for the treatment of these often very complex and detailed methods. This book concerns itself with a review and description of these methods with particular reference to heat and fluid flow problems in buildings and their related systems, plant and control.

Throughout most of the twentieth century, calculations of the thermal response of buildings have made a convenient and yet thoroughly erroneous assumption – that the boundary conditions within which the problem is cast are static. The external air temperature and moisture content, the wind speed and direction, the internal air temperature and moisture content were all treated as constants whilst the building was assumed to be unoccupied. Thus a fictitious steady-state problem could be defined which enabled a 'worst case' heating load on the building to be trivially calculated. In the later part of the twentieth century, the impact of the user and a wider range of external climate variables were introduced by assuming uniformly cyclic internal heat gains and solar irradiances transmitting through glazing and acting on internal surfaces. The impact of the thermal capacity of the building envelope material began to be considered as having an important influence on energy exchange during different times of

the usage cycle, and simple analytical methods were developed and introduced to deal with the lagging effect of solar heat transfer through opaque elements in a ‘quasi-static’ manner, together with a more thorough treatment of the geometry of solar heat transmission through glazing and related shading systems. In the 1970s, developments in mainframe digital computers enabled a more thorough treatment of the underlying theory to be made for the first time, free from the assumption of static boundary conditions, and, with it, a more intuitive understanding of the time response of building energy flows without the need to resort to empiricism. During this time, building energy modelling methods came to be classified according to the manner in which conduction heat transfer through fabric was treated; the main approaches being the numerical solution of the governing Fourier equation using finite difference methods and a less computationally expensive approach using response factors derived analytically from the time series response to a unit pulse excitation at the surfaces of an element. Practice in North America tended to adopt the latter approach whilst European practice mainly focused on the former. In the 1980s personal computers became available and the first commercial codes for design calculations using steady-state methods of energy in buildings became available. Initially these codes addressed heat loss calculations and quasi-steady-state heat gain and peak internal summer temperature calculations mostly based on either the Response Factor Method or another analytical method based on the frequency response of building elements – the Admittance Method. Later in the 1980s, the first commercial codes for fluid flow in building services became available, enabling computer-aided pipework and ductwork designs to be done. Research progress meanwhile continued to refine thermal simulations mainly based on Response Factor and Finite-difference methods, and program codes capable of representing multiple interacting zones in a building began to emerge largely based on mainframe computers or the then UNIX workstations.

Another development started to take place in the 1980s – the introduction of models of heating and air conditioning plant and control systems. Initially, the simplest of these amounted to nothing more than control functions enabling plant energy flows to be both switched on or off and to track idealised set-points, but other developments began to take place in which detailed steady-state descriptions of plant were defined enabling the comparative design of system component options to be conducted for the first time. Later, dynamic descriptions of plant became available, though in a more bespoke context generally for the specific analysis and design of control systems. The steady-state approach to plant modelling has more or less remained the dominant method due to its computational efficiency together with the recognition that for a majority of building energy modelling problems, interest lies in the longer time-horizon situation over a complete season for which detailed dynamics of the plant are neither necessary nor of interest. Progress was also made during this period on the treatment of moisture sorption in building fabric elements though this has tended to remain as an option in the very few codes that have included it for application to the relatively few cases when it is needed.

By the mid 1990s, sophisticated and robust commercial codes were widely available on personal computers for a wide variety of building energy design and analysis problems. Research progress through the early 1990s mainly focused on the (then) neglected problem of fluid flow in and around buildings and the first computational fluid dynamics (CFD) codes for the special case dealing with the low Reynolds number flow conditions associated with building ventilation became available. Much of the later work – through to current times – on these methods (leading to commercial codes in some cases) has tended to focus on ways of treating

transitional laminar-turbulent flow for the difficult problem frequently encountered in building ventilation – that of conjugate heat and fluid flow at low Reynolds number. However, the high computational demand of most CFD problems has meant that a seamless or truly conjugate and universally applicable model of energy and fluid flow in buildings, including plant and control, moisture sorption, and the stochastic influences of the user, remains elusive. On the other hand, many would argue that such an embracing and comprehensive description would in any case be unnecessary given that many building energy analyses require to address one, or a small subset, of problems. So long as the boundary conditions of the problem of interest can be sufficiently defined for an accurate solution of that immediate problem, then the method selected need be no more comprehensive than is needed to address the problem itself. And so many of the methodological developments have tended to take place in isolation which means that ‘hand-shaking’ between methods and codes is at best primitive at present and the issue of methodological interoperability is more or less where we are now. Much of the recent research concerning energy modelling in buildings has been attempting to develop ways of enabling methods described in a variety of program codes using a variety of data management protocols and requiring different levels of user skill and expertise to integrate with one another robustly and efficiently.

This book concerns itself with a review and descriptions of the various state-of-the-art methods for modelling energy and fluid flow problems in buildings. It is hoped that it offers something for the researcher, the scholar and the practitioner in equal measure. It is aimed at a wide readership of those engaged in the physical science of energy in buildings: building physicists, architectural scientists, building and building services engineers, HVAC engineers and building technologists.

Chapter 1 deals with methods for treating heat exchange through building elements. The transient heat conduction problem is defined and the various analytical methods for the solution of the governing partial differential equations are described and compared: conventional time domain methods; the use of Laplace transforms; approaches using unit impulse response factors (and the more computationally-efficient form of this using operator- $z$  transfer functions); the frequency-response or ‘admittance’ method; first- and higher-order lumped capacitance methods. Chapter 1 concludes with a review and description of methods for dealing with absorption, transmission and reflection of solar radiation through glass.

In Chapter 2 the ‘building blocks’ describing conduction heat exchange through building elements are linked with procedures for dealing with the surface effects of convection and radiation leading to a closure of methods for treating the entire room space. As a lead-in to this, an alternative *numerical* approach for treating element conduction is described as a preferred choice over the analytical methods of Chapter 1 due to the flexibility and ease with which this sits alongside other numerical methods applicable to building energy. Issues concerning the explicit, implicit and semi-implicit numerical solution of the governing equations together with considerations for dealing with numerical instability are explored. Heat balances due to convection and radiation on the external surfaces of elements are then dealt with followed by methods for treating the radiation interchange between bounding internal surfaces of a room space, as well as methods for dealing with solar radiation at both external opaque surfaces and at internal surfaces after direct and diffuse transmission through glazing. The chapter concludes with a synthesis of each method for dealing with conduction through building elements with the overall building space heat balance including some simplified contemporary approaches for calculating design loads on spaces.

Chapter 3 is entirely devoted to methods for dealing with fluid flow within the built environment with a focus on air movement for ventilation. A treatment of the modelling of moisture flow through building elements is first dealt with as a linked heat and mass transfer problem. The conservation equations for mass, momentum and energy are then dealt with as a basis for the development of a computational fluid dynamics (CFD) approach to ventilation problems in buildings, including a review of the various approaches for modelling turbulence especially when linked with heat transfer at solid boundaries. Methods for discretising the governing fluid flow equations are reviewed together with common approaches to solving the resulting discretised equation sets. Chapter 3 concludes with a discussion of simplified methods of modelling ventilation flow paths using flow networks and zonal approaches.

Chapters 4 and 5 move on to look at methods for the treatment of systems, plant and control. Chapter 4 deals with two approaches to modelling plant in the steady-state – the use of curve fitting to experimental or manufacturers' data and the use of a rigorous theoretical approach as an alternative. The latter is based on the treatment of sensible and latent cooling in the air cooling process typified by an air conditioner. This represents one of the most detailed heat exchange problems encountered in building energy plant but, by relaxing the modelling assumptions, approaches to modelling other less complex sub-systems also emerge. The chapter concludes with a treatment of flow network modelling applicable to fan and pump systems. In Chapter 5, plant-modelling methods focusing on dynamic approaches with main applicability to the analysis of control systems are dealt with. Again, the sensible and latent cooling process is first treated as a generic exemplar and simplifications for less complex heat exchange processes are drawn out. There is also a treatment of the various approaches to model order reduction and model linearisation in order to improve computational efficiency without compromising accuracy. Methods for reducing problems to linearised transfer functions are dealt with for those interested in classical block diagram methods. Chapter 5 also deals with methods for the representation of control elements (i.e. control valves, dampers, drive systems) and concludes with principles for modelling conventional controllers as well as approaches to a number of 'smart' control strategies.

Chapters 6 and 7 deal with building energy modelling in practice. Chapter 6 starts by outlining the key stages in the analysis of building energy and environment problems using model-based methods and then draws out the various methods described elsewhere in the book as a methodological hierarchy. The major part of Chapter 6 then deals with a review based on the literature of some of the most recent developments of the various methods dealt with throughout the book. Chapter 7 then deals with the interrelationships between methods and their integration, again, mainly drawing from examples reported in the literature.

It is hoped that the end product provides a comprehensive text on the subject of the mathematical modelling of energy and environment in buildings both for the treatment of individual problems, as well as for the appreciation of the wider range of issues with which the individual problem might be expected to interact. To date, there are in excess of 500 program codes around the world that deal with these problems to a greater or lesser extent but probably fewer than 20 of them have sufficient rigour and detail to treat most building energy simulations with a sufficient degree of comprehensiveness needed in modern building design analysis. This book is intended both for those intending to use an existing program code with a thirst for more detailed insights into both the potential and the flaws of the methods involved as well as for those with a thirst for understanding who are inclined to develop new models from scratch.

## Chapter 1

# Heat Transfer in Building Elements

A key function of the envelope of a building is to act as a passive climate modifier to help maintain an indoor environment that is more suitable for habitation than the outdoors. However, besides providing shelter from stormy weather, the building envelope alone can hardly ensure that the indoor environmental conditions will always be comfortable to the occupants, or be suitable for the intended purposes of the indoor spaces, particularly during periods of unfavourable outdoor conditions, such as in the night or when the outdoor air is stagnant.

The need for building design features that would facilitate use of natural ventilation and daylight has diminished, as active means of environmental control, such as central heating, ventilating and air-conditioning (HVAC) systems and electric lights, can be used instead to maintain adequate indoor thermal and visual environmental conditions, and air quality. This has also helped to remove the restrictions imposed on the design of buildings, particularly to maximisation of the amount of floor area that can be built upon a given piece of land.

The increased reliance on HVAC and lighting systems for active control over the indoor environmental conditions, however, has made buildings the dominant energy consumers in modern cities worldwide. This has not only accelerated the depletion of the limited reserve of fossil fuels on earth; it has also exacerbated global warming and environmental pollution due to the emissions of combustion products resulting from burning of fuels for generating electricity, steam, hot water or chilled water for use in buildings. Buildings also contribute to other environmental problems, such as the use of CFC as refrigerants in HVAC plants and halons as fire extinguishing agents, which are causes of the depletion of the stratospheric ozone layer. Therefore, besides fulfilment of the functional needs and aesthetics, the environmental performance of buildings, which includes energy efficiency, has become an essential attribute of environmentally friendly buildings.

Measures that can be used to enhance the energy efficiency of a building include the adoption of building design features that can help reduce the frequency and intensity of use of the HVAC plants and the lighting installations, and the use of more efficient HVAC and lighting system designs and equipment. For instance, the cooling or heating load due to heat transmission through the building envelope can be reduced through the use of thermal insulation at external walls and roofs, and high performance glazing and shading devices at windows. The use of energy-efficient boilers and chillers, variable speed motor drives in heating and air-conditioning systems, and energy-efficient lamps and electronic ballasts can lead to very significant energy saving.

In the design process, the effectiveness of individual energy efficiency enhancement measures, particularly the possible energy and running cost saving, would need to be quantified. The financial benefit, derived from the difference in the annual energy consumption of the building with and without a particular measure, would be essential input to a financial appraisal for determining whether or not to adopt individual measures, and for selecting the most viable choices.

Quantification of the annual energy use in a building requires prediction of the space cooling loads of individual rooms in the building that would arise at different times in the operating periods throughout the year. This involves determination of the heat and mass transfer through the building envelope that are significant parts of the heat and moisture gains or losses of an indoor space. The other sources of heat and moisture gains include occupants, equipment and appliances present within the air-conditioned spaces, and infiltration.

This chapter reviews the methods for modelling the heat and mass transfer through the building envelope, which is a key starting point in the prediction of the annual energy consumption in a building. In most such analyses, the mass transfer modelled would be limited to the bulk air transport into or out of buildings through infiltration and exfiltration, while the moisture transport through the building fabric elements would be ignored.

## 1.1 Heat and mass transfer processes in buildings

The range of heat and mass transfer processes that would take place in buildings is as illustrated in Fig. 1.1, which shows a perimeter room on an intermediate floor in a multi-storey office building. The room is separated from the outdoors by an external wall and a window, and from adjoining rooms at the sides by internal partitions, and at above and below by a ceiling and a floor slab. The room is equipped with a HVAC system that would supply heating or cooling to the room by circulating air between the room and the air-handling unit via the supply and return air ducts.

As shown in Fig. 1.1, the heat and mass transfer processes that would take place in a building include:

- (a) conduction heat transfer through the building fabric elements, including the external walls, roof, ceiling and floor slabs and internal partitions;
- (b) solar radiation transmission and conduction through window glazing;
- (c) infiltration of outdoor air and air from adjoining rooms;
- (d) heat and moisture dissipation from the lighting, equipment, occupants and other materials inside the room; and
- (e) heating or cooling and humidification or dehumidification provided by the HVAC system.

The conduction heat transfer through an opaque building fabric element, such as an external wall as shown in Fig. 1.2, is the effect of the convective heat that the surface at each side of the element is exchanging with the surrounding air, and the radiant heat exchanges with other surfaces that the surface is exposed to. For an external wall or a roof, the radiant heat exchange at

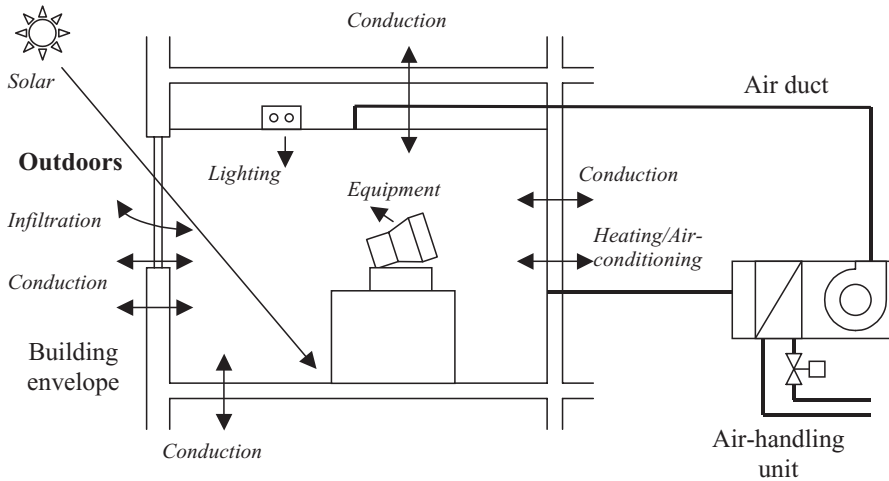


Fig. 1.1 Heat and mass transfer processes involved in building energy simulation.

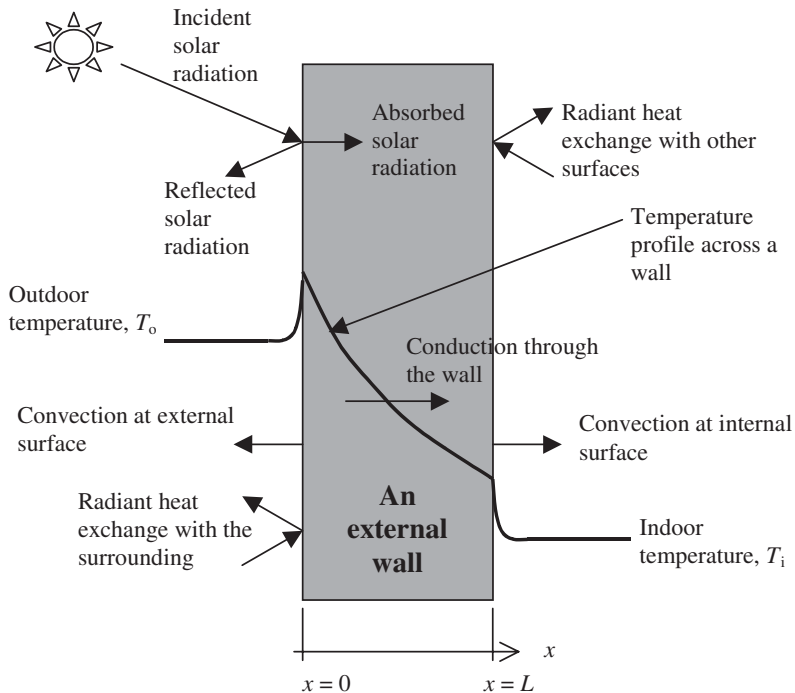
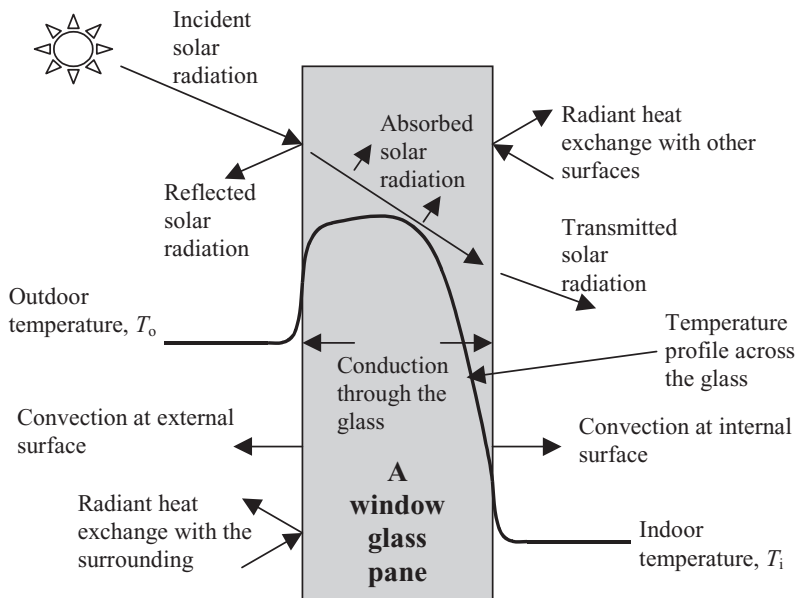


Fig. 1.2 Heat transfer at an external wall.

the external side includes the absorbed solar radiation, including both direct and diffuse radiation.

The heat transfer through a window is shown in Fig. 1.3. The window glass will transmit part of the incident solar radiation into the indoor space. While the solar radiation penetrates





**Fig. 1.3** Heat transfer at a window glass pane.

the glass pane, some of the energy will be absorbed by the glass, leading to an increase in the glass temperature, which, in turn, will cause heat to flow in both the indoor and the outdoor directions, first by conduction within the glass and then by convection and radiation at the surfaces at both sides.

The heat flows through a building fabric element resulting from the absorbed solar radiation and the outdoor to indoor temperature difference are often treated together through the use of an equivalent outdoor air temperature, called ‘sol-air temperature’, that will, in the absence of the radiant heat exchange, cause the same amount of conduction and convection heat flow through the element. A similar parameter, called ‘environmental temperature’, is used to account for the combined effects of the convective heat transfer from the internal surface to the room air and the radiant energy gain at the surface.

The transmitted solar radiation from the windows will be imparted to the indoor air and become cooling load only after it has been absorbed by the internal surfaces. Consequently, the temperature at such surfaces will rise, leading to convective heat flow from the surfaces into the room air. It is this convective heat flow that will affect the indoor air temperature and constitutes a component of the space cooling load. This cooling load component will differ in magnitude and in the time of occurrence of its peak value from those of the radiant heat gain, as shown in Fig. 1.4, which is the result of the thermal capacitance of the fabric elements or furniture materials that are subject to thermal irradiation. Besides the short wave radiation from the sun, radiant heat gains from the lighting and equipment and the long wave radiation exchange among the internal surfaces within the space will need to undergo a similar process to become a cooling load.

When there are air movements into and out of an indoor space, heat and moisture will be brought into or out of the room if the airs that enter the space are at thermodynamic states



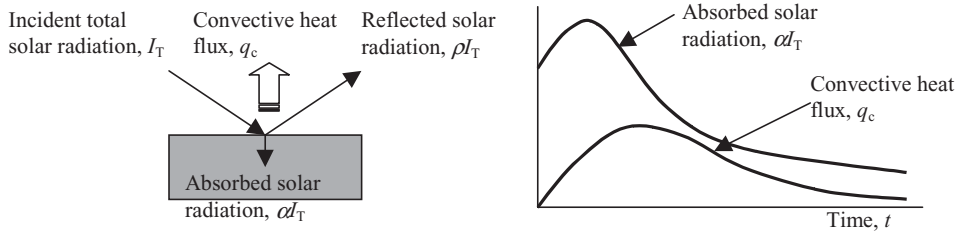


Fig. 1.4 Radiant heat gain and the resultant cooling load.

different from that of the indoor air. The air movements can be set up by pressure differences between the room and the adjoining rooms and the outdoor, due to wind or stack effect, or imbalance in the supply and extract flow rates maintained by the ventilation system.

The thermodynamic state of the air within the room would vary with the net heat and moisture gains experienced by the room air, resulting from heat and moisture exchanges with the enclosure surfaces, air transport into or out of the room bringing with it heat and moisture, heat and moisture gains from sources present within the room, and heating, cooling, humidification or dehumidification provided by the HVAC system serving the room. These heat and moisture transfer processes would need to be modelled for the prediction of the indoor air condition or the rate of heating or cooling, and humidification or dehumidification required for maintaining the indoor air condition at the set point state.

## 1.2 Heat transfer through external walls and roofs

The equation that governs the heat transfer through a homogeneous material can be derived through taking account of:

- (a) the rate of heat transfer across each boundary surface of an elemental control volume within the material that would arise corresponding to the temperature gradient that exists at the surface;
- (b) the rate of heat generation or removal by internal heat sources or sinks present within the control volume; and
- (c) the change in internal energy of the material in the control volume, which is reflected by the change in temperature of the material.

The resultant equation is:

$$\rho c \frac{\partial T}{\partial t} - \nabla \cdot k \nabla T - \dot{q} = 0 \quad (1.1)$$

where  $\rho$ ,  $c$  and  $k$  are respectively the density ( $\text{kg m}^{-3}$ ), specific heat ( $\text{kJ kg}^{-1} \text{K}^{-1}$ ) and thermal conductivity ( $\text{kW m}^{-1} \text{K}^{-1}$ ) of the material;  $T$  is the temperature (K) and  $\dot{q}$  the net rate of heat gain from the internal source or sink within the material ( $\text{kW m}^{-3}$ ); and  $t$  is time (s).

For applications to analysis of the heat transfer through walls and slabs in buildings, Equation 1.1 can be simplified by making the following assumptions:

- (a) only the heat transfer in the direction across the thickness of each wall or slab (assumed to be in the  $x$ -direction, Fig. 1.2) needs to be modelled whereas heat transfer in the other two directions (the  $y$ - and  $z$ -directions) can be ignored;
- (b) heat transfer in the material is isotropic;
- (c) properties of the material ( $\rho$ ,  $c$  and  $k$ ) are independent of temperature; and
- (d) no internal heat source or sink exists within the material ( $\dot{q} = 0$ ).

The governing equation can then be simplified to:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (1.2)$$

where  $\alpha = k/\rho c$  is the thermal diffusivity of the material. Equation 1.2 is a one-dimensional (in space) partial differential equation (PDE) governing the transient heat conduction in a homogeneous slab with no internal heat source or sink.

### **Boundary conditions**

Equation 1.2 needs to be solved in conjunction with the equations that define the boundary conditions that apply to the two surfaces of the fabric element. Typical boundary conditions that apply to building heat transfer problems include the following.

- (a) The temperature at either surface is known, i.e.:

$$T(x, t) = f(t) \quad (\text{for } x = 0, L) \quad (1.3)$$

where  $f(t)$  is a known function that relates the surface temperature to time, and  $L$  is the thickness of the fabric element (m). This includes the condition that the surface temperature is a steady temperature, i.e.  $f(t) = \text{constant}$ .

- (b) The surface is exposed to an ambient air at a known temperature and is exchanging radiant heat with the surroundings. In this case, the heat balance at the surface can be described by:

$$q(0, t) = -k \frac{\partial T(x, t)}{\partial x} = h_0 (T_{\text{Amb},0} - T(x, t)) + q_{r,\text{net},0} \quad (\text{for } x = 0) \quad (1.4a)$$

$$q(L, t) = -k \frac{\partial T(x, t)}{\partial x} = h_L (T(x, t) - T_{\text{Amb},L}) - q_{r,\text{net},L} \quad (\text{for } x = L) \quad (1.4b)$$

where  $h_0$  and  $h_L$  are the convective heat transfer coefficients ( $\text{kW m}^{-2} \text{K}^{-1}$ ),  $T_{\text{Amb},0}$  and  $T_{\text{Amb},L}$  are the ambient air temperatures (K) and  $q_{r,\text{net},0}$  and  $q_{r,\text{net},L}$  are the net radiant heat gains per unit area of the surfaces ( $\text{kW m}^{-2}$ ) at the surfaces at  $x = 0$  and  $x = L$  respectively.

Note that both the ambient air temperatures and the net radiant heat gains could be complex functions of time, such as the outdoor air temperature and the solar radiation absorbed by the external surface of an external wall or roof of a building.

### 1.3 Analytical methods for solving the one-dimensional transient heat conduction equation

#### *Conventional time-domain analysis*

The one-dimensional transient heat conduction equation shown in Equation 1.2 can be solved in various ways; for instance, by using the conventional method of separation of variables to decompose the PDE into two ordinary differential equations (ODEs), followed by integration of the ODEs to yield a solution. The solution to Equation 1.2 for a wall, corresponding to the initial (at  $t=0$ ) and boundary (at  $x=0$  and  $x=L$ , for  $t>0$ ) conditions as given in Equations 1.5 to 1.7, is shown in Equation 1.8:

$$T(x,0) = 0 \quad (\text{for } 0 \leq x \leq L) \quad (1.5)$$

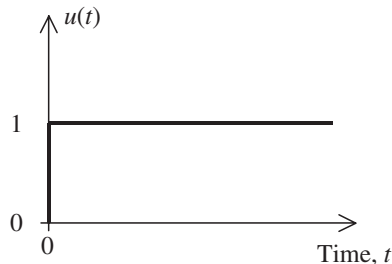
$$T(0,t) = u(t) = \begin{cases} 0 & (\text{for } t < 0) \\ 1 & (\text{for } t \geq 0) \end{cases} \quad (1.6)$$

$$T(L,t) = 0 \quad (\text{for } t > 0) \quad (1.7)$$

where  $u(t)$ , given in Equation 1.6, is a unit step function, as shown in Fig.1.5:

$$T(x,t) = 1 - \frac{x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp\left\{-\alpha \left(\frac{n\pi}{L}\right)^2 t\right\} \sin\left(\frac{n\pi}{L} x\right) \quad (1.8)$$

Since the conduction heat flux through a cross sectional plane within the wall,  $q(x,t)$ , is related to the temperature by Equation 1.9, the equation for the heat flux, in response to the unit temperature step excitation at  $x=0$ , can be found from Equation 1.8, as shown in Equation 1.10:



**Fig. 1.5** A unit step function.

$$q(x,t) = -k \frac{\partial T(x,t)}{\partial x} \quad (1.9)$$

$$q(x,t) = \frac{k}{L} + \frac{2k}{L} \sum_{n=1}^{\infty} \exp\left\{-\alpha\left(\frac{n\pi}{L}\right)^2 t\right\} \cos\left(\frac{n\pi}{L} x\right) \quad (1.10)$$

And, at  $x=L$ , the heat flux is:

$$q(L,t) = \frac{k}{L} + \frac{2k}{L} \sum_{n=1}^{\infty} (-1)^n \exp\left\{-\alpha\left(\frac{n\pi}{L}\right)^2 t\right\} \quad (1.11)$$

Although the above unit step response of the wall would hardly be directly applicable to solving actual problems, it is useful because its time derivative is the response of the wall to a unit impulse excitation, which is the basis for determining the response of the wall to an arbitrary excitation.

The unit impulse function,  $\delta(t)$ , as shown in Fig. 1.6, is defined as a function of time that has the following characteristics:

$$\delta(t) = \text{Lim}_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \quad \text{for } 0 \leq t \leq \Delta t \quad (1.12)$$

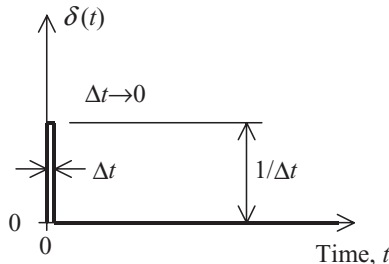
$$\delta(t) = 0 \quad \text{for } t < 0 \text{ and } t > \Delta t \quad (1.13)$$

The area under the unit impulse function is:

$$\int_{-\infty}^{+\infty} \delta(t) dt = \int_{-\infty}^{+\infty} \text{Lim}_{\Delta t \rightarrow 0} \frac{1}{\Delta t} dt = \text{Lim}_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_0^{\Delta t} dt = \text{Lim}_{\Delta t \rightarrow 0} \frac{\Delta t}{\Delta t} = 1 \quad (1.14)$$

For an impulse function  $p(t)$  that equals  $A/\Delta t$  from  $0 < t \leq \Delta t$ , with  $\Delta t$  tending to zero, the area under the curve of the impulse function is:

$$\int_{-\infty}^{+\infty} p(t) dt = \text{Lim}_{\Delta t \rightarrow 0} \frac{A}{\Delta t} \int_0^{\Delta t} dt = A \text{Lim}_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_0^{\Delta t} dt = A \int_{-\infty}^{+\infty} \delta(t) dt = A \quad (1.15)$$



**Fig. 1.6** A unit impulse function.

By comparing the above equation with Equation 1.14, it can be seen that the impulse function  $p(t)$  has an area under the curve  $A$  times as large as the unit impulse function. This area ( $A$  in this case) is called the strength of the impulse function, and it follows that the strength of a unit impulse function is one, which is the reason for calling it a unit impulse function.

The conduction heat flux through the wall in response to a unit temperature impulse applied to the side at  $x=0$  can be derived by differentiating Equation 1.10 with respect to time, which yields:

$$q(x,t) = -\frac{2\alpha k \pi^2}{L^3} \sum_{n=1}^{\infty} n^2 \exp\left\{-\alpha\left(\frac{n\pi}{L}\right)^2 t\right\} \cos\left(\frac{n\pi}{L} x\right) \quad (1.16)$$

And, at  $x=L$ , the conduction heat flux is:

$$q(L,t) = \frac{2\alpha k \pi^2}{L^3} \sum_{n=1}^{\infty} (-1)^n n^2 \exp\left\{-\alpha\left(\frac{n\pi}{L}\right)^2 t\right\} \quad (1.17)$$

Since the PDE that governs the conduction heat transfer through the wall is linear, the response of the wall to an impulse of strength different from unity will be the unit impulse response of the wall scaled by the impulse strength. For instance, if the heat flux through the wall at  $x=L$  is regarded as the ‘output’ of a system (the wall in this case), denoted as  $y(t)$ , in response to an ‘input’ (the temperature excitation at  $x=0$ ) to the system, and let  $y_0(t)$  be the system’s output in response to the unit impulse input  $\delta(t)$  (given in Equation 1.17), the output of the system to an impulse input of strength  $A$  will be:

$$y(t) = A y_0(t) \quad (1.18)$$

If the impulse is applied at a time greater than zero instead, say at  $t = \tau$ , the output will become:

$$y(t) = A y_0(t - \tau) \quad (1.19)$$

Assume that the wall is subject to an arbitrary time-varying temperature excitation at  $x=0$ , denoted by an excitation function  $f(t)$ ; the value of  $f(t)$  at  $t = \tau$ , is  $f(\tau)$ . The wall’s response to an impulse of strength that equals  $f(\tau)$ , applied to the wall at  $t = \tau$ , is:

$$y(t) = f(\tau) y_0(t - \tau) \quad (1.20)$$

Note that:

$$\int_{-\infty}^{+\infty} f(t) \delta(t - \tau) dt = f(\tau) \quad (1.21)$$

Equation 1.21 shows that the excitation applied to the wall at time  $\tau$ , given by the value  $f(\tau)$ , can be represented by an impulse of strength  $f(\tau)$  that takes place at  $t = \tau$ . The excitation, as described by the time function  $f(t)$ , can, therefore, be represented by a consecutive sequence of impulses, each with a strength that equals the value of the function at the time of its occur-

rence, as shown in Fig. 1.7. It follows that the response of the wall to this excitation function, at an arbitrary time  $t$ , can be determined by summing the responses of the wall to each of the sequence of impulses that took place from  $\tau=0$  to  $\tau=t$ .

This means that the response of the wall to a particular impulse applied to the wall at time  $\tau$ , as shown in Equation 1.20, will only be an incremental part of the overall response of the wall, denoted as  $\Delta y(t)$ , when the effect of an additional impulse, taking place between  $t = \tau$  and  $t = \tau + \Delta\tau$ , has been accounted for. Taking the limit that  $\Delta\tau$  tends to zero:

$$\lim_{\Delta\tau \rightarrow 0} \frac{\Delta y(t)}{\Delta\tau} = \frac{dy(t)}{d\tau} = f(\tau)y_0(t-\tau) \quad (1.22)$$

Therefore, the overall response of the wall observable at time  $t$  to all the impulses taking place from  $\tau=0$  to  $\tau=t$  can be determined by integrating Equation (1.22) over this lapsed time as follows:

$$y(t) = \int_0^t f(\tau)y_0(t-\tau)d\tau \quad (1.23)$$

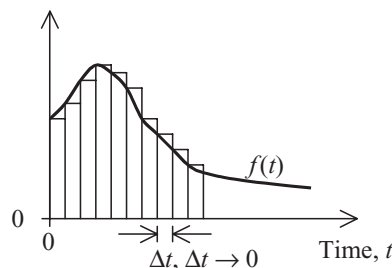
Note that since  $y_0(t)$  is the response to a unit impulse  $\delta(t)$ ,  $y_0(t)$  will have zero value for  $t < 0$ . Likewise,  $y_0(t-\tau)$  will have zero value for  $t < \tau$ . The upper limit of integration can, therefore, be extended to infinity without affecting the result. Hence, Equation 1.23 can be re-written as:

$$y(t) = \int_0^{\infty} f(\tau)y_0(t-\tau)d\tau \quad (1.24)$$

Equation 1.24 is called a convolution integral, which allows the output of a system to an arbitrary excitation to be found from the unit impulse response of the system.

### **Laplace transformation**

A more widely adopted method for solving the PDE governing the conduction heat transfer through a slab of material is to apply Laplace transformation to the PDE to yield an ODE in the Laplace domain. The solution to the ODE can then be inverse-transformed to yield the



**Fig. 1.7** Representing an arbitrary function by a sequence of unit impulses.

solution in the time domain. One of the major advantages of using Laplace transformation to solve the PDE is that a far greater variety of situations can be dealt with and many important observations can be made in the analysis in the Laplace domain.

The Laplace transform of a function  $f(t)$ , denoted by  $L\{f(t)\}$ , which would yield a function in the Laplace domain, denoted as  $\phi(s)$ , is defined as:

$$L\{f(t)\} = \phi(s) = \int_0^{\infty} f(t) \exp(-st) dt \quad (1.25)$$

On the basis of this definition, it can be shown that:

$$L\{\partial f(t)/\partial t\} = s\phi(s) - f(0) \quad (1.26)$$

For avoiding the need to assign two different symbols to denote each function and its Laplace transform, the following nomenclature is defined for representing the Laplace transform of the temperature  $T(x,t)$  and heat flux  $q(x,t)$ :

$$L\{T(x,t)\} = T(x,s) \quad (1.27)$$

$$L\{q(x,t)\} = q(x,s) \quad (1.28)$$

Here, the substitution of the argument  $t$  by  $s$  in the functions means the Laplace transform of the functions rather than a simple substitution of the variable  $t$  by  $s$  in the function. The same convention will apply also to other time functions and their respective Laplace transforms. Note, however, that where zero appears as the time argument, the function remains the time domain function, for instance:

$$T(x,0) = T(x,t)|_{t=0} \quad (1.29)$$

and:

$$q(x,0) = q(x,t)|_{t=0} \quad (1.30)$$

Following the convention defined above, Equation 1.2 when transformed, becomes:

$$\frac{\partial^2 T(x,s)}{\partial x^2} = \frac{s}{\alpha} T(x,s) - \frac{1}{\alpha} T(x,0) \quad (1.31)$$

Note that Equation 1.31 is an ODE in  $x$ , which can be solved when the boundary conditions are defined.

Equation 1.9 can also be transformed into:

$$q(x,s) = -k \frac{\partial T(x,s)}{\partial x} \quad (1.32)$$

The initial and boundary conditions assumed in the preceding time domain analysis were:

$$T(x,0) = 0 \quad (1.33)$$

$$T(0,t) = f(0,t) \quad (1.34)$$

$$T(L,t) = 0 \quad (1.35)$$

Equations 1.34 and 1.35 are transformed to:

$$T(0,s) = f(0,s) \quad (1.36)$$

$$T(L,s) = 0 \quad (1.37)$$

Based on the initial condition as defined in Equation 1.33, Equation 1.31 then becomes:

$$\frac{\partial^2 T(x,s)}{\partial x^2} = \frac{s}{\alpha} T(x,s) \quad (1.38)$$

Equation 1.38 is an ODE in  $x$  that has the following solution:

$$T(x,s) = c_1 \exp(\sqrt{s/\alpha} \cdot x) + c_2 \exp(-\sqrt{s/\alpha} \cdot x) \quad (1.39)$$

where  $c_1$  and  $c_2$  are coefficients to be evaluated from the defined boundary conditions. Applying the boundary conditions as defined in Equations 1.36 and 1.37:

$$f(0,s) = c_1 + c_2 \quad (1.40)$$

$$0 = c_1 \exp(\sqrt{s/\alpha} \cdot L) + c_2 \exp(-\sqrt{s/\alpha} \cdot L) \quad (1.41)$$

The coefficients  $c_1$  and  $c_2$ , solved from the above equations, are:

$$c_1 = -\frac{\exp(-\sqrt{s/\alpha} \cdot L)}{\exp(\sqrt{s/\alpha} \cdot L) - \exp(-\sqrt{s/\alpha} \cdot L)} f(0,s) \quad (1.42)$$

$$c_2 = \frac{\exp(\sqrt{s/\alpha} \cdot L)}{\exp(\sqrt{s/\alpha} \cdot L) - \exp(-\sqrt{s/\alpha} \cdot L)} f(0,s) \quad (1.43)$$

Substituting  $c_1$  and  $c_2$  given in Equations 1.42 and 1.43 into Equation 1.39 yields:

$$T(x,s) = \frac{\exp(\sqrt{s/\alpha} \cdot (L-x)) - \exp(-\sqrt{s/\alpha} \cdot (L-x))}{\exp(\sqrt{s/\alpha} \cdot L) - \exp(-\sqrt{s/\alpha} \cdot L)} f(0,s)$$



The above can be expressed more succinctly as:

$$T(x,s) = \frac{\sinh(\sqrt{s/\alpha} \cdot (L-x))}{\sinh(\sqrt{s/\alpha} \cdot L)} f(0,s) \quad (1.44)$$

By substituting Equation 1.44 into Equation 1.32 the heat flux equation in the Laplace domain can be derived as:

$$q(x,s) = k\sqrt{\frac{s}{\alpha}} \left\{ \frac{\cosh(\sqrt{s/\alpha} \cdot (L-x))}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} f(0,s) \quad (1.45)$$

Note that if the temperature excitation at  $x=0, f(0,t)$ , is an unit impulse function, its Laplace transform,  $f(0,s)$ , equals 1. The right hand side of Equation 1.45 will then become the Laplace transform of the right hand side of Equation 1.16.

Through similar procedures, the temperature and heat flux equations corresponding to the initial and boundary conditions given in Equations 1.46 to 1.48 can be derived, which are given in Equations 1.49 and 1.50:

$$T(x,0) = 0 \quad (1.46)$$

$$T(0,t) = 0 \quad (1.47)$$

$$T(L,t) = f(L,t) \quad (1.48)$$

$$T(x,s) = \frac{\sinh(\sqrt{s/\alpha} \cdot x)}{\sinh(\sqrt{s/\alpha} \cdot L)} f(L,s) \quad (1.49)$$

$$q(x,s) = -k\sqrt{\frac{s}{\alpha}} \left\{ \frac{\cosh(\sqrt{s/\alpha} \cdot x)}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} f(L,s) \quad (1.50)$$

Let:

$$G_{q,0}(x,s) = k\sqrt{\frac{s}{\alpha}} \left\{ \frac{\cosh(\sqrt{s/\alpha} \cdot (L-x))}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} \quad (1.51)$$

and:

$$G_{q,L}(x,s) = -k\sqrt{\frac{s}{\alpha}} \left\{ \frac{\cosh(\sqrt{s/\alpha} \cdot x)}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} \quad (1.52)$$

It follows that:

$$G_{q,0}(0,s) = k\sqrt{\frac{s}{\alpha}} \left\{ \frac{\cosh(\sqrt{s/\alpha} \cdot L)}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} \quad (1.53)$$

$$G_{q,0}(L,s) = k\sqrt{\frac{s}{\alpha}} \left\{ \frac{1}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} \quad (1.54)$$

$$G_{q,L}(0,s) = -k\sqrt{\frac{s}{\alpha}} \left\{ \frac{1}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} \quad (1.55)$$

$$G_{q,L}(L,s) = -k\sqrt{\frac{s}{\alpha}} \left\{ \frac{\cosh(\sqrt{s/\alpha} \cdot L)}{\sinh(\sqrt{s/\alpha} \cdot L)} \right\} \quad (1.56)$$

With reference to Equations 1.45, 1.50 and 1.53 to 1.56, the equations for the heat fluxes at  $x = 0$  and at  $x = L$ , when the wall is subject to excitations  $f(0,s)$  at  $x = 0$  and  $f(L,s)$  at  $x = L$ , can be written as:

$$q(0,s) = G_{q,0}(0,s)f(0,s) + G_{q,L}(0,s)f(L,s) \quad (1.57)$$

$$q(L,s) = G_{q,0}(L,s)f(0,s) + G_{q,L}(L,s)f(L,s) \quad (1.58)$$

which can be expressed in the following matrix form:

$$\begin{Bmatrix} q(0,s) \\ q(L,s) \end{Bmatrix} = \begin{bmatrix} G_{q,0}(0,s) & G_{q,L}(0,s) \\ G_{q,0}(L,s) & G_{q,L}(L,s) \end{bmatrix} \begin{Bmatrix} f(0,s) \\ f(L,s) \end{Bmatrix} \quad (1.59)$$

Equation 1.59 allows the heat fluxes at the two sides of the wall (the ‘output’) in response to the temperature excitations at the two wall surfaces (the ‘input’) to be determined. The coefficient matrix is, therefore, the transfer function of the wall (the ‘system’) relating the input to the output of the system.

Through inverting the coefficient matrix in the equation, the following equation can be obtained to relate the temperature response of the wall that can be observed at the surfaces at the two sides to heat inputs at the two surfaces:

$$\begin{Bmatrix} T(0,s) \\ T(L,s) \end{Bmatrix} = \begin{bmatrix} G_{q,0}(0,s) & G_{q,L}(0,s) \\ G_{q,0}(L,s) & G_{q,L}(L,s) \end{bmatrix}^{-1} \begin{Bmatrix} q(0,s) \\ q(L,s) \end{Bmatrix} \quad (1.60)$$

Note that  $f(0,s)$  and  $f(L,s)$  have been replaced by  $T(0,s)$  and  $T(L,s)$  respectively in the above equation, to reflect more appropriately that the temperatures are the ‘responses’ rather than the ‘forcing functions’. In the following,  $T(0,s)$  and  $T(L,s)$  are used to denote the Laplace transforms of the temperature functions for the two surfaces of the wall, irrespective of whether they represent forcing functions or responses to heat input.

Other useful forms of the equation can be derived by algebraic manipulations with Equations 1.57 and 1.58, for instance:

$$\begin{Bmatrix} T(0,s) \\ q(0,s) \end{Bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{Bmatrix} T(L,s) \\ q(L,s) \end{Bmatrix} \quad (1.61)$$

where:

$$A(s) = -\frac{G_{q,L}(L,s)}{G_{q,0}(L,s)}$$

$$B(s) = \frac{1}{G_{q,0}(L,s)}$$

$$C(s) = G_{q,L}(0,s) - \frac{G_{q,L}(L,s)G_{q,0}(0,s)}{G_{q,0}(L,s)}$$

$$D(s) = \frac{G_{q,0}(0,s)}{G_{q,0}(L,s)}$$

Substituting Equations 1.53 to 1.56 into the above:

$$A(s) = \cosh(\sqrt{s/\alpha} \cdot L) \quad (1.62)$$

$$B(s) = \frac{\sinh(\sqrt{s/\alpha} \cdot L)}{k\sqrt{s/\alpha}} \quad (1.63)$$

$$C(s) = k\sqrt{s/\alpha} \sinh(\sqrt{s/\alpha} \cdot L) \quad (1.64)$$

$$D(s) = \cosh(\sqrt{s/\alpha} \cdot L) \quad (1.65)$$

Note that the determinant of the coefficient matrix in Equation 1.61 ( $= A(s)D(s) - B(s)C(s)$ ) equals one. The equation can also be inverted into:

$$\begin{Bmatrix} T(L,s) \\ q(L,s) \end{Bmatrix} = \begin{bmatrix} D(s) & -B(s) \\ -C(s) & A(s) \end{bmatrix} \begin{Bmatrix} T(0,s) \\ q(0,s) \end{Bmatrix} \quad (1.66)$$

Equations 1.61 and 1.66 are particularly useful, as each of them relays the excitations at one side of the wall to the response of the wall at the other side. Therefore, when a composite wall involves  $N$  layers of different materials, the response of the wall can be modelled by recognising that the output of one layer becomes the input to the next layer. Hence, the transfer function matrices of the layers can be multiplied in sequence to provide an overall transfer function:

$$\begin{aligned} \begin{Bmatrix} T(0,s) \\ q(0,s) \end{Bmatrix} &= \begin{bmatrix} A_1(s) & B_1(s) \\ C_1(s) & D_1(s) \end{bmatrix} \begin{bmatrix} A_2(s) & B_2(s) \\ C_2(s) & D_2(s) \end{bmatrix} \cdots \\ &\quad \cdots \begin{bmatrix} A_{N-1}(s) & B_{N-1}(s) \\ C_{N-1}(s) & D_{N-1}(s) \end{bmatrix} \begin{bmatrix} A_N(s) & B_N(s) \\ C_N(s) & D_N(s) \end{bmatrix} \begin{Bmatrix} T(L,s) \\ q(L,s) \end{Bmatrix} \\ &= \begin{bmatrix} A_M(s) & B_M(s) \\ C_M(s) & D_M(s) \end{bmatrix} \begin{Bmatrix} T(L,s) \\ q(L,s) \end{Bmatrix} \end{aligned} \quad (1.67)$$

where:

$$L = \sum_{i=1}^N l_i \quad (1.68)$$

and  $l_i$  is the thickness of the  $i^{\text{th}}$  layer.

Note also that in the overall matrix in Equation 1.67,  $A_M(s)$  will not be equal to  $D_M(s)$  as in the single layer equations, but its determinant will still be equal to one.

So far, the heat transfer equations for the wall have been derived based solely on the temperature and heat flux at the surfaces. The convective and radiant heat exchange at the surfaces would need to be properly treated for applications to buildings.

Recall Equations 1.4a and 1.4b, with the convective heat transfer coefficients at the outdoor (assumed to be at  $x = 0$ ) and the indoor (at  $x = L$ ) sides denoted respectively as  $h_o$  and  $h_i$ , the time varying ambient air temperatures at the two sides as  $T_o(t)$  and  $T_i(t)$ , and the time varying net radiant heat gains at the two side as  $q_{r,o}(t)$  and  $q_{r,i}(t)$ :

$$q(0,t) = h_o(T_o(t) - T(0,t)) + q_{r,o}(t) \quad (1.69a)$$

$$q(L,t) = h_i(T(L,t) - T_i(t)) - q_{r,i}(t) \quad (1.69b)$$

When the two surfaces are each treated as one layer of a composite wall, the heat flow into the wall (at the surface at  $x = 0$ ) or out of the wall (at the surface at  $x = L$ ) (in the positive  $x$ -direction), which are given respectively by the right hand side of Equations 1.69a and 1.69b, should now be regarded as the heat flow into and out of the respective layers. Denoting these heat flows as  $q_o(t)$  and  $q_i(t)$ :

$$q(0,t) = q_o(t) \quad (1.70a)$$

$$q(L,t) = q_i(t) \quad (1.70b)$$

The above equations reflect the property of the ‘surface layers’ that each layer has zero heat capacity ( $\rho c = 0$ ), and thus any heat that flows in must flow out at the same rate.

It follows that:

$$q_o(t) = h_o(T_o(t) - T(0,t)) + q_{r,o}(t) \quad (1.71a)$$

$$q_i(t) = h_i(T(L,t) - T_i(t)) - q_{r,i}(t) \quad (1.71b)$$

Solving for the wall surface temperatures:

$$T(0,t) = T_o(t) + \frac{1}{h_o} q_{r,o}(t) - \frac{1}{h_o} q_o(t) \quad (1.72a)$$

$$T(L,t) = T_i(t) + \frac{1}{h_i} q_{r,i}(t) + \frac{1}{h_i} q_i(t) \quad (1.72b)$$

The indoor/outdoor temperature and the net absorbed radiant heat terms in the above equations can be combined and denoted by the environmental temperature  $T_{e,i}$  and the sol-air temperature  $T_{e,o}$ , allowing Equation 1.72 to be simplified to:

$$T(0,t) = T_{e,o}(t) - \frac{1}{h_o} q_o(t) \quad (1.73a)$$

$$T(L,t) = T_{e,i}(t) + \frac{1}{h_i} q_i(t) \quad (1.73b)$$

The Laplace transforms of Equations 1.70 and 1.73 are:

$$T(0,s) = T_{e,o}(s) - \frac{1}{h_o} q_o(s) \quad (1.74)$$

$$q(0,s) = q_o(s) \quad (1.75)$$

$$T(L,s) = T_{e,i}(s) + \frac{1}{h_i} q_i(s) \quad (1.76)$$

$$q(L,s) = q_i(s) \quad (1.77)$$

For the surface layer at  $x=L$ , the heat flow out of the wall and the surface temperature of the wall can be related to the heat flow out of the surface layer and the indoor environmental temperature as follows:

$$\begin{Bmatrix} T(L,s) \\ q(L,s) \end{Bmatrix} = \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} T_{e,i}(s) \\ q_i(s) \end{Bmatrix} \quad (1.78)$$

For the surface at  $x=0$ , Equations 1.74 and 1.75 are re-arranged as:

$$T_{e,o}(s) = T(0,s) + \frac{1}{h_o} q(0,s) \quad (1.79)$$

$$q_o(s) = q(0,s) \quad (1.80)$$

The corresponding matrix form of the two equations is:

$$\begin{Bmatrix} T_{e,o}(s) \\ q_o(s) \end{Bmatrix} = \begin{bmatrix} 1 & 1/h_o \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} T(0,s) \\ q(0,s) \end{Bmatrix} \quad (1.81)$$

Equations (1.78) and (1.81) can now be substituted into Equation 1.67 to yield:

$$\begin{Bmatrix} T_{e,o}(s) \\ q_o(s) \end{Bmatrix} = \begin{bmatrix} 1 & 1/h_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_M(s) & B_M(s) \\ C_M(s) & D_M(s) \end{bmatrix} \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} T_{e,i}(s) \\ q_i(s) \end{Bmatrix} \quad (1.82)$$

which can be simplified to:

$$\begin{Bmatrix} T_{e,o}(s) \\ q_o(s) \end{Bmatrix} = \begin{bmatrix} A_O(s) & B_O(s) \\ C_O(s) & D_O(s) \end{bmatrix} \begin{Bmatrix} T_{e,i}(s) \\ q_i(s) \end{Bmatrix} \quad (1.83)$$

Equation 1.83 can be re-arranged (taking note that  $A_O(s)D_O(s) - B_O(s)C_O(s) = 1$ ), to yield the following, which would be more directly applicable for the prediction of the heat transfer across a wall when the wall is subject to variations in the indoor and outdoor temperatures and radiant heat gains:

$$\begin{Bmatrix} q_o(s) \\ q_i(s) \end{Bmatrix} = \begin{bmatrix} \frac{D_O(s)}{B_O(s)} & -\frac{1}{B_O(s)} \\ \frac{1}{B_O(s)} & -\frac{A_O(s)}{B_O(s)} \end{bmatrix} \begin{Bmatrix} T_{e,o}(s) \\ T_{e,i}(s) \end{Bmatrix} \quad (1.84)$$

The Laplace transformation analysis reviewed above is a common basis for several modelling methods that are in common use nowadays. These include the Response Factor Method, Transfer Function Method and the Admittance Method. A review of the theoretical basis of each of these methods is given in the following sections.

### ***Response Factor Method***

The Response Factor Method was first proposed by Stephenson and Mitalas (1967). The elements in the matrix in Equation 1.59, which relates the heat fluxes at the two surfaces of a homogeneous slab of material to the temperature excitations at the two surfaces (denoted as surfaces 0 and 1, and the temperature and heat flux at these surfaces denoted by subscripts 0 and 1 respectively), can be expressed in terms of the transfer functions  $A(s)$ ,  $B(s)$  and  $D(s)$  shown in Equations 1.62 to 1.65 as follows:

$$\begin{Bmatrix} q_0(s) \\ q_1(s) \end{Bmatrix} = \begin{bmatrix} \frac{D(s)}{B(s)} & -\frac{1}{B(s)} \\ \frac{1}{B(s)} & -\frac{A(s)}{B(s)} \end{bmatrix} \begin{Bmatrix} T_0(s) \\ T_1(s) \end{Bmatrix} \quad (1.85)$$

It can be seen from Equation 1.85 that the Laplace transforms of the heat flux at the two surfaces of the layer of material due to the temperature excitation  $T_0(s)$  alone are given by:

$$q_0(s) = \frac{D(s)}{B(s)} T_0(s) \quad (1.86)$$

$$q_1(s) = \frac{1}{B(s)} T_0(s) \quad (1.87)$$

For clarity,  $q_0(s)$  and  $q_1(s)$  in Equations 1.86 and 1.87 are denoted as  $q_{0,0}(s)$  and  $q_{1,0}(s)$  respectively, to denote that they are the heat fluxes due to excitation  $T_0(s)$ .