

Modelling the final state from binary black-hole coalescences

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Abstract. Over the last few years enormous progress has been made in the numerical description of the inspiral and merger of binary black holes. A particular effort has gone into the modelling of the physical properties of the final black hole, namely its spin and recoil velocity, as these quantities have direct impact in astrophysics, cosmology and, of course, general relativity. As numerical-relativity calculations still remain computationally very expensive and cannot be used to investigate the complete space of possible parameters, semi-analytic approaches have been developed and shown to reproduce with very high precision the numerical results. I here collect and review these efforts, pointing out the relative strengths and weaknesses, and discuss which directions are more promising to further improve them.

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1. Introduction

Despite the almost unnatural simplicity with which the problem can be formulated (black holes are after all the simplest macroscopical objects we know), the final evolution of a binary system of black holes is an impressively complex problem to solve. At the same time, this very simple process plays a fundamental role in astrophysics, in cosmology, in gravitational-wave astronomy, and of course in general relativity. Recent progress in numerical relativity initiated by the works in refs. [1, 2, 3], have made it now possible to compute the different stages of the evolution, starting from the inspiral at large separations, for which post-Newtonian (PN) calculations provide an accurate description, through the highly relativistic merger, and finally to the ringdown.

As long as the two black holes are not extremal and have masses which are not too different from each other (see however [4] for simulations with a mass-ratio of 1 : 10), no major technical obstacle now prevents the solution of this problem in full generality and with an overall error which can be brought down to less than 1% (see ref. [5, 6, 7, 8, 4] for some recent examples). Yet, obtaining such a solution still requires a formidable computational power sustained over several days. Even for the simplest set of initial data, namely those considering two black holes in quasi-circular orbits, the space of parameters is too vast to be explored entirely through numerical-relativity calculations. Furthermore, many studies of astrophysical interest, such as many-body simulations of galaxy mergers, or hierarchical models of black-hole formation, span a statistically large space of parameters and are only remotely interested in the evolution of the system during the last few tens of orbits and much interested in determining the properties of the final black hole when the system is still widely separated.

In order to accommodate these two distinct and contrasting needs, namely that of sampling the largest possible space of parameters and that of reducing the computational costs, a number of analytical or semi-analytic approaches have been developed over the last couple of years. In most of these approaches the inspiral and merger is considered as a process that takes, as input, two black holes of initial masses M_1 , M_2 and spin vectors \mathbf{S}_1 , \mathbf{S}_2 and produces, as output, a third black hole of mass M_{fin} , spin \mathbf{S}_{fin} and recoil velocity v_{kick} . Mathematically, therefore, one is searching for a mapping between the initial 7-dimensional space of parameters (*i.e.*, the one containing the six spin components $S_{1,2}^j$ and the mass ratio $q \equiv M_2/M_1$) to two distinct 3-dimensional spaces (*i.e.*, the ones containing the three components of the final spin and of the recoil velocity)[‡]. Clearly this is a degenerate mapping (two different initial configurations can lead to the same final one) and it would seem a formidable task to accomplish given the highly nonlinear features of the few last orbits. Yet surprisingly, or perhaps not so surprisingly given the underlying simplicity of the problem, all of these studies have shown that the final spin vector and the final recoil velocity vector, can be estimated to remarkably good accuracy if the initial parameters are known [11, 12, 13, 14, 15, 16, 17, 9, 10].

This paper is dedicated to review the different approaches developed so far to tackle this problem, pointing out the relative strengths and weaknesses, and finally comparing them against the numerical data. The manuscript is organized as follows: in Sect. 2 I review the different methods that have been suggested to model the final spin, distinguishing those that are purely analytic from those that employ, with different amounts, the results of numerical simulations. Sect. 3 is instead dedicated to the specific approach developed at the AEI to obtain a simple and robust algebraic expression for the final spin vector. In Sect. 3 I also

[‡] The space of final parameters could in principle be larger if the final mass of the black hole were to be modelled. Two attempts to do so have been recently proposed [9, 10] (see also the discussion in Sec. 2), but this quantity is not usually modelled.

exploit the advantages of such analytic formula to explore those configurations that may be astrophysically more interesting and then validate the accuracy of the formula against all of the available data and across alternative approaches. Finally, Sect. 4 is dedicated to the modelling of the recoil velocity vector and a summary is presented of the different contributions and of the debate still present on one them. Section 5 will present the conclusions and highlight the prospects of future work in the modelling of the final state.

As a final but important remark I note that all of the considerations made here apply to binary systems that inspiral from very large separations and hence through quasi-circular orbits. Such configurations are the ones more likely to occur astrophysically since any residual eccentricity is lost quickly by the gravitational-radiation reaction [18]. Furthermore, at least for nonspinning equal-mass black holes, recent work has shown that the final spin does not depend on the value of the eccentricity as long as the latter is not too large [19].

2. Modelling the final spin vector

A number of analytical approaches have been developed over the years to determine the final spin from a binary black hole coalescence [20, 21, 22, 23, 24]. A particularly active line of research is the one that has exploited the motion of test particles in black hole spacetimes. A first attempt in this direction was made by Hughes and Blandford [25], who assumed that the energy and angular momentum radiated in the merger and ringdown phase is much smaller than the one radiated during the inspiral. In this way they were able to express the conservation of energy and angular momentum when the two black holes reach the innermost stable circular orbit (ISCO) and hence compute the mass and spin of the final black hole[§]. Of course, the notion of the ISCO is well defined in the test-particle limit (*i.e.*, for $M_2 \rightarrow 0$) and the approach of Hughes and Blandford reproduces this limit by construction and provides reasonable results for a large set of parameters. However, it also leads to incorrect final spins (*i.e.*, $S_{\text{fin}}/M_{\text{fin}}^2 > 1$) for very rapidly rotating black holes and even at very small mass ratios. Inspired in part by this work, Buonanno, Lehner, and Kidder [12] have more recently proposed a variant in which the angular momentum of the final black hole is assumed to be the sum of the individual spins and of the orbital angular momentum of a test particle at the ISCO of a Kerr black hole with the *same spin* parameter as that of the final black hole, *i.e.*,

$$M^2 \mathbf{a}_{\text{fin}} = M_1 M_2 \mathbf{L}_{\text{tp}}(|\mathbf{a}_{\text{fin}}|, \theta_{\text{fin}}) + M_1^2 \mathbf{a}_1 + M_2^2 \mathbf{a}_2. \quad (1)$$

where $\mathbf{a}_{1,2} = \mathbf{S}_{1,2}/M_{1,2}^2$ are the two dimensionless spin vectors ($|\mathbf{a}_{1,2}| \in [0, 1]$), θ_{fin} is the angle of the final spin \mathbf{a}_{fin} with respect to the direction of the initial orbital angular momentum of the binary, and $\mathbf{L}_{\text{tp}}(|\mathbf{a}_{\text{fin}}|, \theta_{\text{fin}})$ is a vector whose direction is determined by assuming that the angle between \mathbf{L}_{tp} and the total spin $\mathbf{S}_1 + \mathbf{S}_2$ maintains constant during the inspiral, and whose magnitude is the dimensionless angular momentum at the ISCO for a test particle moving around a black hole with spin $|\mathbf{a}_{\text{fin}}|$ on an orbit with inclination θ_{fin} with respect to the equatorial plane.

Because \mathbf{L}_{tp} depends implicitly on the final spin \mathbf{a}_{fin} , equation (1) cannot be solved analytically and needs to be computed numerically. Furthermore, attention must be paid that \mathbf{L}_{tp} refers to a prograde (retrograde) orbit if the final spin is aligned (anti-aligned).

Perhaps the most surprising aspect of this approach is that despite its simplicity, it is remarkably precise in a large portion of the space of parameters, with differences from the numerical-relativity results that can be as small as $\sim 1\%$ and of $\sim 10\%$ at most. As I will comment later on, I regard this result as an evidence that the dependence of the final spin on

[§] I note that although numerical simulations do not show evidence for the existence of an ISCO, the concept of an effective ISCO can nevertheless be useful for the construction of gravitational-wave templates [26, 27].

the initial conditions is particularly simple and that, as a consequence, the mapping between the initial and final state can be accomplished with rather simple expressions. After all, L_{tp} and $\mathbf{a}_{1,2}$ have a limited functional excursion and are they linearly related in equation (1).

An important ingredient in the above recipe is “mass conservation”, *i.e.*, the assumption that the mass of the final black hole is the same as the sum of the two initial masses. Because gravitational radiation carries away part of the mass of the system, such an assumption is trivially incorrect, at least from a formal standpoint. However, although the assumption is false, the resulting approximation is not too severe. Numerical simulations reveal in fact that the mass radiated in gravitational waves is $M_{\text{rad}}/M = 1 - M_{\text{fin}}/M \lesssim 5 - 7 \times 10^{-2}$, with $M \equiv M_1 + M_2$ being the total mass. Although this difference is of the order of the precision with which one wants to predict the final spin, a favourable series of cancellations reduces the impact of this otherwise unreasonable assumption.

In the spirit of improving the accord with the simulations and in particular to produce more reliable predictions in the case of the merger of maximally spinning black holes, Kesden [9] has recently proposed an improvement over the approach in ref. [12] which can be used for binaries with spins which are aligned or antialigned with the orbital angular momentum. Once again using the test-particle motion in a Kerr spacetime, Kesden has suggested that in addition to expression (1) being valid, the final mass M_{fin} for arbitrary mass ratios and initial spins can be computed as

$$M_{\text{fin}} = M - \mu[1 - E(a_{\text{fin}})], \quad (2)$$

where $\mu \equiv M_1 M_2 / M$ is the reduced mass and $E(a)$ is the energy per unit mass of a test particle on an equatorial orbit of a Kerr black hole with spin parameter a . This suggestion amounts to a minimal change in the calculation of the final spin, but provides a much more precise modelling of the final spin in the case of black holes with very large spins, *i.e.*, with $|\mathbf{a}_1| \simeq |\mathbf{a}_2| \gtrsim 0.99$.

The resulting picture is particularly simple for equal-spin black holes, *i.e.*, with $|\mathbf{a}_1| = |\mathbf{a}_2| = a$, (see Fig. 2 of ref. [9]). In this case the test-particle approach suggests that for $a \lesssim 0.9916$, the final spin a_{fin} is a function increasing monotonically with the symmetric mass ratio $\nu \equiv M_1 M_2 / (M_1 + M_2)^2 = q / (1 + q)^2$, so that equal-mass mergers (*i.e.*, with $\nu = 1/4$) are the most efficient ones in spinning up the final black hole. However, for values of $a > 0.9916$ the function $a_{\text{fin}}(\nu)$ is no longer monotonically increasing with ν but has a maximum at smaller and smaller values of ν , so that for these spins a suitable symmetric mass ratio $\nu \neq 1/4$ maximizes the final spin. This behaviour continues for increasingly large values of a and up to $a \sim 0.9987$, when $a_{\text{fin}}(\nu = 1/4) = a$; as a result, for $a \gtrsim 0.9987$ equal-mass mergers *spin down* black holes rather than spinning them up, *i.e.*, $a_{\text{fin}}(\nu = 1/4) < a$ for $a \gtrsim 0.9987$. Finally, when the initial black holes are maximally spinning, *i.e.*, $a = 1$, the maximum of $a_{\text{fin}}(\nu)$ is at the extreme mass-ratio limit (EMRL), *i.e.*, at $\nu = 0$, thus making $a_{\text{fin}}(\nu)$ monotonically decreasing with ν . As a direct consequence, maximally spinning black holes cannot be produced by the merger of two black holes, the only exception being offered by the EMRL. Rather, the merger of maximally spinning equal-mass black holes yields a black hole with $a_{\text{fin}} \simeq 0.9988$ ¶.

An alternative and distinct line of research for the calculation of the final spin has instead exploited numerical-relativity calculations as soon as the first successful evolutions were

|| Wald has shown in ref. [28] that the cosmic censorship hypothesis holds also in the neighbourhood of the EMRL, *i.e.*, that $a_{\text{fin}}(\nu) \leq 1$ for $\nu \simeq 0$.

¶ A different astrophysical route to the production of maximally spinning black holes is of course that of accretion and this was shown by Bardeen in ref. [29]. The arguments made there apply also here if one is considering a binary coalescence in the neighborhood of the EMRL, *i.e.*, for $\nu \simeq 0$.

possible [30, 31, 32]. While these first approaches essentially represented fits to the numerical data (which were initially scarce) they provided the first accurate description of the problem, albeit in a small region of the space of parameters. Subsequent work focussed instead on the derivation of analytic expressions which would model the numerical-relativity data but also exploit as much information as possible either from perturbative studies, or from the symmetries of the system when this is in the weak-field limit [11, 13, 14, 15, 16, 17]. In this sense, these approaches are not blind fits of the data, but, rather, use the numerical-relativity data to construct a physically consistent and mathematically accurate modelling of the final spin.

The common ground shared by the approaches in this second class is given by the *assumption* that the final spin, when seen as the function $\mathbf{a}_{\text{fin}} = \mathbf{a}_{\text{fin}}(\mathbf{a}_1, \mathbf{a}_2, \nu)$, can be expressed as a Taylor expansion around $\mathbf{a}_1 = \mathbf{a}_2 = \nu = 0$. Given that $|\mathbf{a}_{1,2}| \leq 1$, this may seem as a mathematically reasonable assumption and the expectation that the series is convergent over the whole space of parameters as a legitimate one. However, this remains an assumption and more work is needed to make this mathematical assumption also a physically reasonable one. Sharing this assumption, then different routes are chosen to constrain the coefficients and these may invoke more mathematically-based considerations, as those proposed in ref. [13, 14, 10], or more physically-based considerations, as those proposed in refs. [11, 16, 17]. Both approaches often reach the same conclusions, but there is little doubt that the systematic approach proposed in refs. [13, 14] offers the advantage of being more easily expandable to capture higher-order effects.

Here, however, I will concentrate on reviewing the approach which, with a minimal number (4) of physically reasonable assumptions and with a minimal number (5) of free coefficients to be fixed from the numerical data, leads to a formula that can model generic initial spin configurations and mass ratios, thus covering all of the 7-dimensional space of parameters [11, 16, 17]. As I will show in Sect. 3, besides being simple and physically motivated, it is also remarkably accurate in reproducing the final spin of more than 150 simulations.

In essence, the approach developed in refs. [11, 16, 17] (hereafter simply the ‘‘AEI formula’’) amounts to considering the dimensionless spin vector of the final black hole as given by the sum of the two initial spins and of a ‘‘third’’ vector parallel to the initial orbital angular momentum when the binaries are widely separated. This ‘‘third’’ vector is an intrinsic ‘‘property’’ of the binary (it will be shown below that this is essentially the orbital angular momentum *not* radiated), thus depending on the initial spin vectors and on the black holes mass ratio, but not on the initial separation. The formula for the final spin then simply describes the properties of this vector (so far only of its length) in terms of the initial parameters of the binary and of a set of coefficients to be determined from a comparison with numerical simulations.

Let us now consider in more detail how to derive such a formula. As mentioned in the Introduction, four assumptions are needed in order to make the problem tractable analytically and these are listed in what follows.

(i) *The mass radiated to gravitational waves M_{rad} can be neglected i.e., $M_{\text{fin}} = M_1 + M_2$.*

As mentioned before, while this assumption is certainly not correct, its influence on the overall accuracy of the prediction is small, with the possible exception of binaries with very small mass ratios and with rapidly spinning black holes [9]. Work is in progress to relax this assumption and will be presented in a forthcoming paper [33].

(ii) *At a sufficiently large but finite initial separation the final spin vector \mathbf{S}_{fin} can be*

approximated as the sum of the two initial spin vectors and of a third vector $\tilde{\ell}$

$$\mathbf{S}_{\text{fin}} = \mathbf{S}_1 + \mathbf{S}_2 + \tilde{\ell},$$

When viewed as expressing the conservation of the total angular momentum, eq. (2) also defines the vector $\tilde{\ell}$ as the difference between the orbital angular momentum when the binary is widely separated \mathbf{L} , and the angular momentum radiated until the merger \mathbf{J}_{rad} , i.e., $\tilde{\ell} = \mathbf{L} - \mathbf{J}_{\text{rad}}$. Stated differently, the vector $\tilde{\ell}$ measures the orbital angular momentum that *cannot be radiated* and, assuming the “effective” ISCO as the radial separation at which the system essentially stops radiating, $\tilde{\ell}$ can then be thought of as the angular momentum of the binary at such ISCO.

(iii) *The vector $\tilde{\ell}$ is parallel to \mathbf{L} .*

This assumption (which is made also in ref. [12]) essentially enforces that the component of the final spin \mathbf{S}_{fin} in the orbital plane equals the one of the total initial spin $\mathbf{S}_1 + \mathbf{S}_2$ in that plane. This is correct when $\mathbf{S}_1 = -\mathbf{S}_2$ and $q = 1$ [this can be seen from the PN equations at 2.5 order], or by equatorial symmetry when the spins are aligned with \mathbf{L} , or when $\mathbf{S}_1 = \mathbf{S}_2 = 0$ (also these cases can be seen from the PN equations). However, for more general configurations one expects that ℓ will also have a component orthogonal to \mathbf{L} as a result, for instance, of spin-orbit or spin-spin couplings, which will produce in general a precession of ℓ . In practice, the component of ℓ orthogonal to \mathbf{L} will correspond to the angular momentum $\mathbf{J}_{\text{rad}}^\perp$ radiated in a plane orthogonal to \mathbf{L} , with a resulting error in the estimate of $|\ell|$ which is $\sim |\mathbf{J}_{\text{rad}}^\perp|^2 / |\tilde{\ell}|^2 \sim |\mathbf{J}_{\text{rad}}^\perp|^2 / (2\sqrt{3}M_1M_2)^2$. A direct consequence of assumption (iii) is a systematic overestimate of the final spin component along the direction of the orbital angular momentum $\mathbf{a}_{\text{fin}}^\parallel$ and, consequently, a systematic overestimate of $\lesssim 2\%$ in the cosine of the inclination angle θ_{fin} . In ref. [10] this assumption was relaxed and the modelling of $\mathbf{a}_{\text{fin}}^\parallel$ improved; however, a comparison with the available data will show that the error introduced by this assumption is not important for the estimate of the final spin magnitude $|\mathbf{a}_{\text{fin}}|$ (see Fig. 3 and the discussion in Sect. 4).

(iv) *When the initial spin vectors are equal and opposite ($\mathbf{S}_1 = -\mathbf{S}_2$) and the masses are equal ($q = 1$), the spin of the final black hole is the same as for nonspinning binaries.*

Stated differently, equal-mass binaries with equal and opposite-spins behave as nonspinning binaries, at least as far as the properties of the final black hole are concerned. This condition is met by the leading-order contributions to the spin-orbit and spin-spin point-particle Hamiltonians and by the spin-induced radiation flux [34, 22], but it is difficult to justify in full general relativity. Here it is introduced essentially to break a possible degeneracy in the formula (see below) and it reflects the expectation that if the spins are the same and opposite (independently on whether they are aligned/antialigned with the orbital angular momentum), their contributions to the final spin cancel (at least to the precision of interest here) for equal-mass binaries. Besides being physically reasonable, this expectation is met by all of the simulations performed to date, both for spins aligned with \mathbf{L} [11, 16] and orthogonal to \mathbf{L} [35, 15, 10]. A similar assumption is also made, although not explicitly, in ref. [12].

Using these assumptions it is now possible to derive the analytic expression for the final spin. Let us start by expressing the vector relation (2) as

$$\mathbf{a}_{\text{fin}} = \frac{1}{(1+q)^2} (\mathbf{a}_1 + \mathbf{a}_2 q^2 + \ell q), \quad (3)$$

where $\mathbf{a}_{\text{fin}} = \mathbf{S}_{\text{fin}}/M^2$ [cf. assumption (i)], $\ell \equiv \tilde{\ell}/(M_1M_2)$, $\mathbf{a}_{1,2} \equiv \mathbf{S}_{1,2}/M_{1,2}^2$, and its norm is then given by

$$|\mathbf{a}_{\text{fin}}| = \frac{1}{(1+q)^2} \left[|\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 q^4 + 2|\mathbf{a}_2||\mathbf{a}_1|q^2 \cos \alpha + 2(|\mathbf{a}_1| \cos \beta + |\mathbf{a}_2|q^2 \cos \gamma) |\ell|q + |\ell|^2 q^2 \right]^{1/2}, \quad (4)$$

where the three vector cosine α , β and γ are defined by

$$\cos \alpha \equiv \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{a}}_2, \quad \cos \beta \equiv \hat{\mathbf{a}}_1 \cdot \hat{\ell}, \quad \cos \gamma \equiv \hat{\mathbf{a}}_2 \cdot \hat{\ell}. \quad (5)$$

Because $\mathbf{a}_{1,2} \parallel \mathbf{S}_{1,2}$ and $\ell \parallel \mathbf{L}$, the angles α , β and γ are also those between the initial spin vectors and the initial orbital angular momentum. Indeed, it is to exploit this simplification that assumption (iii) was introduced and thus to replace $\hat{\mathbf{a}}_{1,2}$ with $\hat{\mathbf{S}}_{1,2}$ and $\hat{\ell}$ with $\hat{\mathbf{L}}$ in (5). Note that in this line of arguments I am making the underlying assumption that α , β and γ are well-defined and hence that the initial separation of the two black holes is sufficiently large that the changes in these angles due to precession is small. The error introduced in this way in the measure of $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are expected to be of the order of $\sim |\mathbf{J}_{\text{rad}}^\perp|/|\tilde{\ell}|$. All that is needed at this point is to “measure” $|\ell|$ by exploiting the results of numerical-relativity simulations and, in particular, those for binaries with spins parallel and aligned (*i.e.*, $\alpha = \beta = \gamma = 0$), for spins parallel and antialigned spins (*i.e.*, $\alpha = 0$, $\beta = \gamma = \pi$), and for antiparallel spins which are aligned or antialigned ($\alpha = \beta = \pi$, $\gamma = 0$ or $\alpha = \gamma = \pi$, $\beta = 0$).

As mentioned before, the matching with the numerical data is not unique but this degeneracy can be broken if assumption (iv) is adopted. Doing this and requiring in addition that $|\ell|$ depends linearly on $\cos \alpha$, $\cos \beta$ and $\cos \gamma$, one finally obtains that the modulo of the dimensionless, non-radiated angular momentum is given by (see ref. [17] for a more detailed derivation of this expression)

$$|\ell| = \frac{s_4}{(1+q^2)^2} (|\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 q^4 + 2|\mathbf{a}_1||\mathbf{a}_2|q^2 \cos \alpha) + \left(\frac{s_5 \nu + t_0 + 2}{1+q^2} \right) (|\mathbf{a}_1| \cos \beta + |\mathbf{a}_2|q^2 \cos \gamma) + 2\sqrt{3} + t_2 \nu + t_3 \nu^2. \quad (6)$$

The 5 coefficients s_4, s_5, t_0, t_2, t_3 appearing in (6) were determined by fitting a large number of binaries having equal-mass and unequal but aligned/antialigned spins [11] or unequal-mass but equal aligned/antialigned spins [16]. As a result of this fitting process, the coefficients have been determined to be

$$\begin{aligned} s_4 &\simeq -0.129, & s_5 &\simeq -0.384, \\ t_0 &\simeq -2.686, & t_2 &\simeq -3.454, \\ t_3 &\simeq 2.353. \end{aligned} \quad (7)$$

Expressions (4) and (6) fully and accurately determine the modulo of the final spin under generic spin configurations and mass-ratio, but they do not yet provide information about the different components of the final spin vector. Such information can be extracted from the angle between the final spin vector and the initial orbital angular momentum θ_{fin} , which can be easily calculated from $|\mathbf{a}_{\text{fin}}|$. Because of assumption (iii), the component of the final spin in the direction of \mathbf{L} is [*cf.* eq. (3)]

$$a_{\text{fin}}^\parallel \equiv \mathbf{a}_{\text{fin}} \cdot \hat{\ell} = \frac{|\mathbf{a}_1| \cos \beta + |\mathbf{a}_2|q^2 \cos \gamma + |\ell|q}{(1+q)^2}, \quad (8)$$

so that $\cos \theta_{\text{fin}} = a_{\text{fin}}^\parallel / |\mathbf{a}_{\text{fin}}|$, and the component orthogonal to the initial orbital angular momentum is $a_{\text{fin}}^\perp = |\mathbf{a}_{\text{fin}}| \sin \theta_{\text{fin}}$. Note that by construction our approach does not further decompose the perpendicular component of the spin a_{fin}^\perp .

Before moving on to a more detailed exploration in Sect. 4 of the physical implications of expressions (4) and (6), it is useful to remark that the approach reviewed here is intrinsically approximate. Although the comparison between the predictions of the formula and all of the available numerical data has revealed differences of a few percent at most, attention should be paid to the fact that the coefficients have been constrained with a rather small and statistically non-homogeneous set of configurations. Most importantly, however, the formula discussed above can be improved in a number of different ways which I list below: (a) by reducing the χ^2 of the fitting coefficients as new simulations are carried out; (b) by using fitting functions that are of higher-order than those used so far [*cf.* expressions (10) and (9)] in the spirit of the systematic spin expansion suggested in ref. [13, 14]; (c) by estimating $\mathbf{J}_{\text{rad}}^\perp$ through PN expressions or by measuring it via numerical simulations; (d) by introducing an independent modelling of the mass radiated in gravitational waves M_{rad} . Work is in progress to include some of these possible improvements and it will be presented in [33].

3. Exploring the AEI spin formula

In what follows I discuss in some detail the predictions of the AEI spin formula for some simple cases and highlight how to extract interesting physical considerations.

3.1. Unequal mass, Aligned/Antialigned Equal spins

If the black holes have *unequal mass* but spins that are *equal, parallel* and *aligned/antialigned* with the orbital angular momentum, *i.e.*, $|\mathbf{a}_1| = |\mathbf{a}_2| = a$, $\alpha = 0$; $\beta = \gamma = 0, \pi$, the prediction for the final spin is given by the simple expression [16]

$$a_{\text{fin}}(a, \nu) = a \cos \beta (1 + s_4 a \nu + s_5 \nu^2 + t_0 \nu) + 2\sqrt{3}\nu + t_2 \nu^2 + t_3 \nu^3. \quad (9)$$

where $\cos \beta = \pm 1$ for aligned/antialigned spins. Note that since the coefficients in (9) are determined by fits to the numerical data and the latter is scarcely represented at very large spins, the predictions of expression (9) for nearly maximal black holes are essentially extrapolations and are therefore accurate to a few percent at most. As an example, when $a = 1$, the formula (9) is a non-monotonic function with maximum $a_{\text{fin}} \simeq 1.029$ for $\nu \simeq 0.093$; this clearly is an artifact of the extrapolation.

The global behaviour of the final spin for unequal-mass and aligned/antialigned equal-spin binaries is summarized in Fig. 1, which shows the functional dependence of expression (9) on the symmetric mass ratio and on the initial spins. Squares refer to numerical-relativity values as reported in refs. [36, 37, 38, 16, 15, 39, 4], while circles to the EMRL constraints. A number of interesting considerations can now be made:

(a) Using (9) it is possible to estimate that the minimum and maximum final spins for an equal-mass binary are $a_{\text{fin}} = 0.3502 \pm 0.03$ and $a_{\text{fin}} = 0.9590 \pm 0.03$, respectively. While the value for the maximum spin is most likely underestimated (indeed Kesden predicts for this configuration $a_{\text{fin}} = 0.9987$ [9]) the minimum value is expected to be much more accurate than the estimate in ref. [9], which tends to underestimate the final spin for $a \lesssim -0.3$ (*cf.* Fig. 5 of ref. [9]).

(b) Using (9) it is straightforward to determine the conditions under which the merger will lead to a final *Schwarzschild* black hole. In practice this amounts to requiring $a_{\text{fin}}(a, \nu) = 0$ and this curve is shown as a (red) dashed line in Fig. 1. Several numerical simulations have been carried out to validate this condition [16, 39] and all of them have shown to produce black holes with $a_{\text{fin}} \lesssim 0.01$ (*cf.* squares in Fig. 1 with $\nu \simeq 0.16$). Overall, the behaviour captured by expression (9) shows that in order to produce a nonspinning black hole it is necessary to

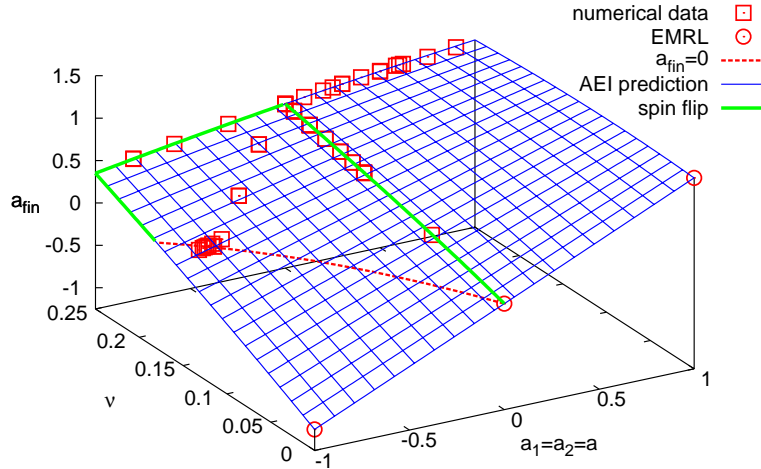


Figure 1. Global dependence of the final spin on the symmetric mass ratio and on the initial spins as predicted by expression (9) for equal-mass, aligned/antialigned equal-spin binaries. Squares refer to numerical-relativity values as reported in refs. [36, 37, 38, 16, 15, 39, 4], while circles to the EMRL constraints. Indicated with a (red) dashed line is the locus of points leading to a Schwarzschild black hole (*i.e.*, $a_{\text{fin}} = 0$), while (green) solid lines mark the region leading to a “spin-flip” (*i.e.*, $a_{\text{fin}} < 0$).

have unequal-masses (the largest possible mass ratio is $\nu \simeq 0.18$) and spins antialigned with the orbital angular momentum to cancel the contribution of the orbital angular momentum to the total one.

(*c*) Using (9) it is also straightforward to determine the conditions under which the merger will lead to a “*spin-flip*”, namely when the newly formed black hole will spin in the direction opposite to that of the two initial black holes. Mathematically this is equivalent to determine the region in the plane (a, ν) such that $a_{\text{fin}}(a, \nu) < a$ and it is shown in Fig. 1 as limited by the (green) solid lines and by the (red) dashed line. Overall, it is clear that a spin-flip can take place only for very large mass ratios if the black holes are initially rapidly spinning and that small mass ratios will lead to a spin-flip only for binaries with very small spins.

(*d*) Finally, using (9) it is also possible to determine the conditions under which the merger will lead to a final black hole with the *same* spin as the initial ones. This amounts to requiring that $a_{\text{fin}}(a, \nu) - a = 0$ and only a very small portion of the (a, ν) plane does satisfy this condition (*cf.* Fig. 5 of ref. [16]). For equal-mass binaries, for instance, the critical value is $a_{\text{crit}} \gtrsim 0.946$ and no spin-down is possible for $\nu \lesssim 0.192$. Because of the minuteness of the region for which $a_{\text{fin}} < a$, black holes from aligned-spins binaries are typically *spun-up* by mergers. As it will be shown also in the following Section, this statement is true also for other configurations and is probably true in general.

3.2. Equal-mass, Aligned/Antialigned Unequal spins

Equally interesting is to consider the prediction for the final spin in the case in which the initial black holes have *equal mass* but *unequal* spins that are either *parallel* or *antiparallel* to the orbital angular momentum, *i.e.*, for $q = 1$ and $\alpha = 0, \pi; \beta = 0, \pi; \gamma = 0, \pi$. Setting $2|a_1| \cos \beta = a_1 + a_2$ in expression (4) we obtain the simple expression for the final spin in

these cases [11]

$$a_{\text{fin}}(a_1, a_2) = p_0 + p_1(a_1 + a_2) + p_2(a_1 + a_2)^2, \quad (10)$$

where the coefficients p_0, p_1 and p_2 are given by

$$\begin{aligned} p_0 &= \frac{\sqrt{3}}{2} + \frac{t_2}{16} + \frac{t_3}{64} \simeq 0.6869, & p_1 &= \frac{1}{2} + \frac{s_5}{32} + \frac{t_0}{8} \simeq 0.1522, \\ p_2 &= \frac{s_4}{16} \simeq -0.0081. \end{aligned} \quad (11)$$

Note that the coefficients p_0, p_1, p_2 and s_4, s_5, t_0, t_2, t_3 were obtained through independent fits of two distinct data sets. The fact they satisfy the conditions (11) within the expected error-bars is an important consistency check.

When seen as a power series of the initial spins, expression (10) suggests an interesting physical interpretation. Its zeroth-order term, p_0 , can be associated with the (dimensionless) orbital angular momentum not radiated in gravitational waves and thus amounting to $\sim 70\%$ of the final spin at most. Interestingly, the value for p_0 is in very good agreement with what is possibly the most accurate measurement of the final spin from this configuration and that has been estimated to be $a_{\text{fin}} = 0.68646 \pm 0.00004$ [40]. Similarly, the first-order term in (10), p_1 , can be seen as the contributions from the initial spins and from the spin-orbit coupling, amounting to $\sim 30\%$ of the final spin at most. Finally, the second-order term, p_2 , can be seen as accounting for the spin-spin coupling, with a contribution to the final spin which is of $\sim 4\%$ at most.

Another interesting consideration is possible for equal-mass binaries having spins that are equal and antiparallel, *i.e.*, $q = 1$, $\mathbf{a}_1 = -\mathbf{a}_2$. In this case, expressions (4) and (6) reduce to

$$|\mathbf{a}_{\text{fin}}| = \frac{|\ell|}{4} = \frac{\sqrt{3}}{2} + \frac{t_2}{16} + \frac{t_3}{64}. \quad (12)$$

Because for equal-mass black holes which are either nonspinning or have equal and opposite spins, the vector ℓ does not depend on the initial spins, expression (12) states that $|\ell|M_{\text{fin}}^2/4 = |\ell|M^2/4 = |\ell|M_1M_2$ is, for such systems, the orbital angular momentum at the effective ISCO. We can take this a step further and conjecture that $|\ell|M_1M_2$ is the series expansion of the dimensionless orbital angular momentum at the ISCO also for *unequal-mass* binaries which are either nonspinning or with equal and opposite spins. The zeroth-order term of this series (namely, the term $2\sqrt{3}M_1M_2$) is exactly the one predicted from the EMRL.

3.3. Generic (misaligned) binaries: unequal mass, unequal spins

When the binaries are generic, namely when the initial spins are oriented in generic directions and the two masses are different, the spin formula (4)–(6) does not reduce to a simple expression and the analysis of the physical implications becomes more complex.

Much more challenging is also the numerical solution in these cases, partly because they are computationally more expensive (no symmetries can be exploited to reduce the computational domain), and partly because the evolutions start at a finite separation which does not account for the earlier evolution of the orbital angular momentum vector and of the spins (both of which precess). In addition, because the final spin is oriented in directions which are in principle arbitrarily far from the main coordinate lines, the calculation of the inclination angle from the properties of the final apparent horizon is often non-trivial and suitable definitions need to be introduced (see, *e.g.*, [36]).

Notwithstanding this, accurate numerical simulations have been reported by several groups and Fig. 2 reports data for 75 of such binaries. Using a “dummy index” to order the different

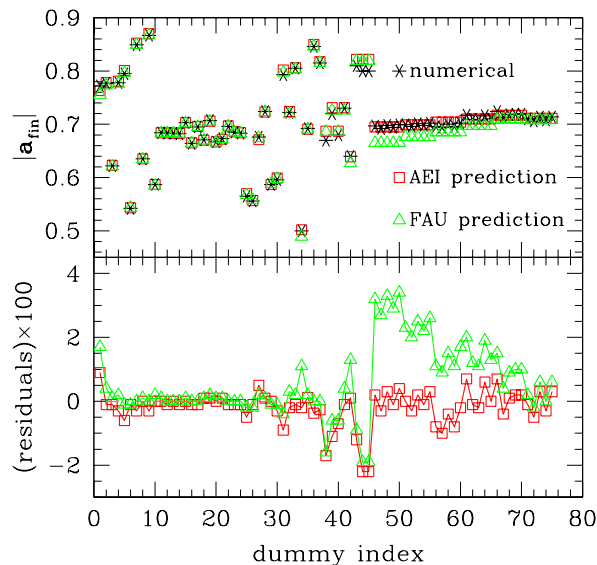


Figure 2. *Top panel:* Comparison between the final spin values $|\mathbf{a}_{\text{fin}}|$ obtained via numerical simulations and via the analytic formulas of AEI [17] (red squares) and of FAU [10] (green triangles). The numerical data has been published in ref. [10, 31, 41, 17, 42, 7] (see main text for a more detailed description). *Bottom panel:* Residuals $|\mathbf{a}_{\text{fin}}| - |\mathbf{a}_{\text{pred}}|$ for the AEI formulas (red squares) and for the FAU formula (green triangles).

binaries, here is where the different results were presented: the data from 1 to 31 was presented in ref. [10], the data from 32 to 34 in ref. [31, 41], the data from 35 to 37 in ref. [17], the data from 38 to 45 in ref. [42], and finally the data from 46 to 75 was presented in ref. [7]. The top panel of Fig. 2, in particular, reports a comparison between the final spin values $|\mathbf{a}_{\text{fin}}|$ obtained via numerical simulations and via the analytic formulas of AEI [17] (red squares) and of FAU [10] (green triangles). As mentioned before, the latter has been proposed as a suitable application of the systematic expansion suggested in [13] when this is truncated at first order.

Because derived from a generic expansion and not from some assumptions (as for the AEI formula), the the FAU formula offers the important improvement of going beyond the restrictions imposed by our assumption (iii) and, in particular, models the changes of the final spin orthogonal to the orbital angular momentum. I recall, in fact, that as a consequence of assumption (iii), the final spin component in the orbital plane, a_{fin}^{\perp} does not change in the AEI formula and it is simply that of the initial total spin $\mathbf{S}_1 + \mathbf{S}_2$. Tichy and Marronetti [10], instead, can model this change both via a contraction (of about 70%) and a rotation (which is rather small, with a rotation angle which is less than 9 degrees). This leads to a better estimate of the inclination angle⁺ but not necessarily of the final spin modulo.

This is shown in the bottom panel of Fig. 2, where I report the residuals $|\mathbf{a}_{\text{fin}}| - |\mathbf{a}_{\text{pred}}|$ for the AEI formulas (red squares) and for the FAU formula (green triangles). While the two approaches are overall comparable, the FAU one has larger residuals in general. This may be combined result of having used a small set of simulations to constrain the coefficients (the FAU formula has in fact used only the data from index 1 to index 31 and for these points

⁺ To fix the ideas and using as a reference the data published in ref. [7], the FAU formula has an average error on $\cos \theta_{\text{fin}}$ of $\sim 0.1\%$, while the AEI formula has an average error of $\sim 1.7\%$.

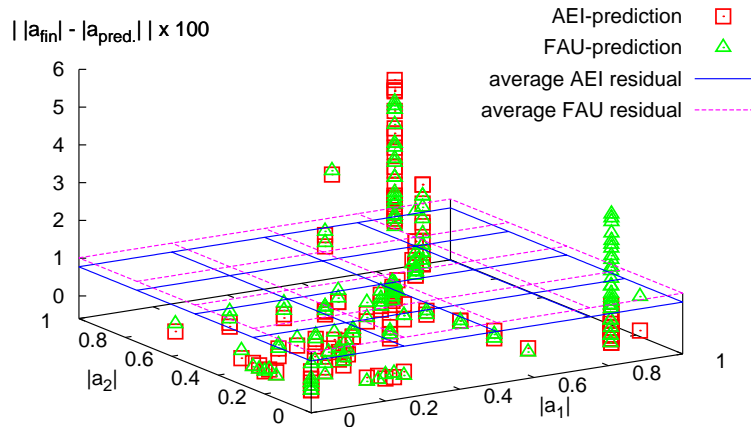


Figure 3. Absolute residuals between the numerically computed value for the final spin and the one predicted by either the AEI formula (red squares) or the FAU formula (green triangles), *i.e.*, $||\mathbf{a}_{\text{fin}}| - |\mathbf{a}_{\text{pred}}||$. The numerical data is the one published in refs [36, 37, 38, 11, 16, 15, 39, 4, 10, 31, 41, 17, 42, 7, 40, 4] and is shown simply in terms of the initial spin moduli $|\mathbf{a}_{1,2}|$.

the residuals are clearly smaller) and of truncating the series expansion at first order (the AEI formula has exploited a third-order expansion in the spins and mass ratio).

This conclusion applies also when comparing the two formulas against all of the available numerical data, that is all of the data reported for the 153 simulations presented in refs. [36, 37, 38, 11, 16, 15, 39, 4, 10, 31, 41, 17, 42, 7, 40, 4]. This large bulk of data is shown in a very synthetic form in Fig. 3, where I report the absolute residuals between the numerically computed value for the final spin and the one predicted by either the AEI formula (red squares) or the FAU formula (green triangles), *i.e.*, $||\mathbf{a}_{\text{fin}}| - |\mathbf{a}_{\text{pred}}||$. Also indicated with constant planes are the arithmetic average of the residuals for the AEI formula (*i.e.*, 0.775×10^{-2}) and the arithmetic average of the residuals for the FAU formula (*i.e.*, 1.011×10^{-2}).

A few remarks are worth making. Firstly, the two formulas yield very similar results and while the AEI one is better on average, it also has the largest residual and all for the data from ref. [43]. Secondly, configurations with very high spins are those that produce the largest residuals and this is partly due to the difficulty of having accurate calculations in those regimes and, in part, to the fact that the coefficients in AEI's formula were determined from simulations with moderate spins. Finally, and most importantly, the data refers to simulations from different codes, different numerical setups and hence different truncation errors. It is therefore remarkable that the scatter is so small, with the largest majority of residuals being below 1%. Stated differently, Fig. 3 can be interpreted as an absolute variance of the precision of present numerical-relativity calculations.

4. Modelling the final recoil vector

As mentioned in the Introduction, the final spin is not the only quantity which is possible and useful to model from binary black-hole coalescences. Rather, the gravitational recoil

is a common feature of binary black-hole coalescences and with multiple consequences. I recall that a binary with unequal masses or unequal spins radiates gravitational energy asymmetrically and the linear momentum lost at any given time is not compensated by the loss after one period. As a result, during the inspiral the linear momentum vector of the asymmetric binary changes continuously with time increasing and spiraling outwards up until the two black holes merge and the final black hole conserves the linear momentum at the time of the merger. This recoil velocity, or simply “kick”, was predicted well before numerical relativity simulations were able to prove its existence and measure its magnitude [44, 45]. Analytic estimates of the kick velocities have also been available for some time [46, 47, 48, 49, 50], but because the largest part of the system’s acceleration is generated in the final parts of the last orbit, when the motion is relativistic and the dynamics highly nonlinear, fully relativistic calculations are necessary in order to determine the kick accurately.

Indeed, modelling the recoil velocity has been one of the most exciting results of recent numerical-relativity calculations, and these have had a deep impact not only in general relativity but also on astrophysics and cosmology. Discussing in detail such implications would be very interesting but impossible with the space restrictions of this contribution. Hereafter, therefore, I will only briefly review the main results obtained over the last couple of years and provide a snapshot of the present understanding of the recoil velocity from binary black-hole coalescences.

Computations of the recoil velocity were made soon after the first successful binary black hole evolutions and the systems which were studied first were unequal-mass systems with moderate mass ratios [51, 52]. These initial studies were quite limited in the mass ratios they were able to consider and were further extended in ref. [53], where a large number of models with mass ratios between 0.25 and 1.0 was explored. A reasonably consistent picture has emerged from these simulations and it is one in which the recoil velocities generated by mass asymmetries are rather accurately bounded above by a maximum velocity of $\simeq 175$ km/s, which is attained for $\nu \simeq 0.195$, or $q \simeq 0.36$. Although very small ratios (*i.e.*, $q \lesssim 0.01$) still represent the frontier of what is presently possible to compute with standard finite-difference codes, the recent work of ref. [4] with a mass ratio $q = 0.1$ suggests that no major surprise should be expected in the functional dependence of the recoil velocity for $q < 0.1$.

As mentioned above, a recoil velocity is generated also when the asymmetry is in the spins rather than in the masses. A number of groups have therefore looked into the kick produced by equal-mass binaries having, however, unequal spins [43, 54, 55, 41, 5, 56]. In this context, particular attention has been paid to binary black hole systems with equal-mass and spins aligned with the orbital one, since systems of this type may represent a preferred end-state of the binary evolution. PN studies in vacuum have shown, in fact, that the gravitational spin-orbit coupling has a tendency to align the spins when they are initially close to the orbital one [57]. Furthermore, if the binary evolves in a disc, as expected for supermassive black holes, the matter can exert a torque tending to align the spins [58]. Overall, the results of several groups now agree that the recoil velocities generated from the merger of equal-mass, aligned/antialigned unequal-spins are rather accurately bounded above by a maximum velocity of $\simeq 450$ km/s, which is attained for $|\mathbf{a}_1| = -|\mathbf{a}_2| = 1$ (*cf.* Fig. 2 of ref. [11]). Such velocities are large enough to eject the merged black hole from the center of a dwarf elliptical or spheroidal galaxy, and which have escape velocities $\lesssim 300$ km/s [59].

Since the recoil is the result of an asymmetry in the binary system, it does not come as a surprise that the largest recoils are those produced in the most asymmetric systems, namely in those binaries where the spins are antiparallel and lie in the orbital plane. These are indeed the configurations leading to the “superkicks”, *i.e.*, extremely large kicks which have been computed to be of the order of 2000 – 2500 km/s and then extrapolated to yield velocities as

large as 4000 km/s for the maximally spinning case [41, 60, 61, 35, 62]. Such velocities are so large to be able to eject the merged black hole from the center of even massive elliptical galaxies (whose escape velocities are $\lesssim 2000$ km/s [59]) and may represent a problem if this process does indeed take place frequently during the merger of massive galaxies.

To clarify the picture of what contributes to what in the final recoil, it is useful to adopt the phenomenological splitting first proposed in ref. [41], and which distinguishes the various contributions to the recoil velocity as coming from two different sources of asymmetry in the system, namely the one relative to *mass-asymmetry* v_M and the one relative to the *spin-asymmetry* v_S . The latter is then further decomposed in the component contained in the orbital plane (which is produced by the component of the spins parallel to the orbital angular momentum) and in the component parallel to the orbital angular momentum (which is produced by the component of the spins orthogonal to the orbital angular momentum). Selecting a Cartesian coordinate system in which the unit vector in the z -direction \mathbf{e}_3 is aligned with the orbital angular momentum \mathbf{L} , the above decomposition leads to a generic expression

$$\mathbf{v}_{\text{kick}} = v_M \mathbf{e}_1 + v_S^\perp (\cos \xi \mathbf{e}_1 + \sin \xi \mathbf{e}_2) + v_S^\parallel \mathbf{e}_3. \quad (13)$$

It should be noted that expression (13) is essentially heuristic although the different contributions can have approximate expressions as derived from PN considerations. Over the last couple of years a considerable effort has been invested in modelling the different contributions which can now be expressed as follows. The mass-asymmetry contribution is given by [46, 53]

$$v_M = A\nu^2(1 + B\nu)\sqrt{1 - 4\nu}, \quad (14)$$

the ‘‘in-plane’’, spin-asymmetry is given by [43, 54, 54, 41, 5, 56, 11]

$$v_S^\perp = \frac{\nu^2}{(1+q)} \left| C \left(a_1^\parallel - qa_2^\parallel \right) + D \left[\left(a_1^\parallel \right)^2 - q^2 \left(a_2^\parallel \right)^2 \right] \right|, \quad (15)$$

while the ‘‘off-plane’’, spin-asymmetry is given by [60, 61, 35, 63, 64]

$$v_S^\parallel = E \frac{\nu^2}{(1+q)} \left| a_1^\perp - qa_2^\perp \right| \cos(\Theta - \Theta_0). \quad (16)$$

Here, $A \simeq 1.2 \times 10^4$ km/s, $B \simeq -0.93$, $C \simeq 7040$ km/s, $D \simeq 1460$ km/s, and $E \simeq 6.0 \times 10^4$ km/s. Note that the angle Θ is defined ‘‘a-posteriori’’ as the angle between the in-plane component of the vector $\mathbf{\Delta} = (M_1 + M_2)(\mathbf{S}_1/M_1 - \mathbf{S}_2/M_2)$ and the direction along which the two black holes approach each other at the merger, while Θ_0 is a simple constant offset.

A number of considerations are in order. Firstly, it should be noted that $v_S^\perp \propto a_i^\parallel$ and viceversa $v_S^\parallel \propto a_i^\perp$, thus suggesting that the unbalanced gravitational-wave emission (*i.e.*, the one responsible for the changes in of linear momentum vector) is always orthogonal to the black holes spins. Secondly, the in-plane spin-asymmetry component given by expression (16) shows that the largest contributions come from binaries with anti-parallel initial spins, *i.e.*, $a_1^\parallel = -a_2^\parallel$ and that, in this case, the quadratic component is identically zero. Thirdly, the mass dependence of the off-plane spin-asymmetry component given by expression (16) is presently under debate and indeed the work carried out in ref. [65] suggests instead a dependence of the type

$$v_S^\parallel = E \frac{\nu^3}{(1+q)} \left(qa_2^\perp \cos(\phi_2 - \Phi_2) - a_1^\perp \cos(\phi_1 - \Phi_1) \right). \quad (17)$$

where $\Phi_{1,2}$ are constant offsets and $\phi_{1,2}$ are the angles between $a_{1,2}^\perp$ and some reference direction in the equatorial plane. The most conspicuous and important difference between expressions (16) and (17) is in the dependence on the symmetric mass ratio (*i.e.*, ν^2 vs ν^3) and this is not a minor difference since it has important astrophysical consequences (see the discussion in ref. [65]). It is presently unclear which of the two expressions will find further confirmation and work is in progress [66] to resolve this debate through an independent set of simulations and a careful modelling of the integration constant. The latter, I recall, accounts for the nonzero initial linear momentum the spacetime has at the time the simulation is started. An accurate determination of the integration constant has played an important role in the determination of the second-order correction in the contribution for v_s^\perp (see discussion in ref. [5]).

5. Conclusions

The determination of the final spin and recoil velocity from the from the knowledge of the initial properties of the initial black holes is of great importance in several fields. In astrophysics, it provides information on the properties of isolated stellar-mass black holes produced at the end of the evolution of a binary system of massive stars. In cosmology, it can be used to model the distribution of masses and spins of the supermassive black holes produced through the merger of galaxies [67, 68] and whether massive galaxies should be expected to host a massive black hole at their centre (but see also refs. [69, 70]). In addition, in gravitational-wave astronomy, the a-priori knowledge of the final spin can help the detection of the ringdown [26].

I have reviewed those semi-analytic approaches to model the final state from binary black-hole coalescences and that have been derived to bypass the still very expensive numerical-relativity calculations and obtain a complete and (reasonably) accurate description of the full space of parameters. Although following different routes and approximations, the several approaches proposed so far have provided a rather accurate description of the final state from binary coalescences. For the first time after decades, these approaches and the numerical-relativity calculations behind them, have placed us in the position of *knowing* the properties of the final black hole. However, our *understanding* of these properties has not progressed equally rapidly. It is now possible to compute the second and third-order dependences on the initial spins and mass ratios but many of these dependences are still unclear. Because knowledge without understanding is often sterile, the works reported here should serve as stimulus to other studies where the numerical results represent the “background” to be used by perturbative or PN studies to go from knowledge to understanding. The recent works in refs. [23, 37, 39, 6, 71, 72] are encouraging and useful steps in this direction.

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