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MODELS AND MEASURES FOR EFFICIENCY DOMINANCE IN DEA
Part I: Additive Models and MED Measures *

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Abstract The usual models in DEA (Data Envelopment Analysis) employ a postulate of continuity to obtain comparison points for the entities known as DMUs (Decision Making Units) whose input-output behavior is to be evaluated. In some applications, it may be desired to restrict attention to actual DMUs and hence to drop (or modify) the continuity assumptions in DEA. Using the concept of efficiency dominance, this is accomplished in the present paper in the form of mixed integer programming models which restrict the efficiency evaluations to comparisons with actually observed performances. Simple and easily interpreted scalar measures of efficiency are provided while retaining the ability to identify the sources and amounts of inefficiency in each DMU that is evaluated.

1. Introduction

This is one in a series of papers dealing with issues of efficiency dominance in Data Envelopment Analysis (DEA) beginning with Bowlin *et al.* [6] and continuing with Bardhan *et al.* [4]. We here move from the formulations in [6], the first paper in this series, to new extensions after anchoring our developments in more customary DEA models as follows.

The term "Data Envelopment Analysis" is derived from the left-hand (=primal) member of the following dual pair of linear programming problems:

$$
\begin{array}{llll}
\min\limits_{\lambda,s^+,s^-} & \theta - \varepsilon e^T s^+ - \varepsilon e^T s^- & & \max\limits_{\omega,\mu} \omega^T Y_o \\
\text{subject to:} & 0 = \theta X_o - \sum\limits_{j=1}^{n} X_j\lambda_j - s^+, & \text{subject to:} & 0 \geq \omega^T Y_j - \mu^T X_j, \\
& Y_o = \sum\limits_{j=1}^{n} Y_j\lambda_j - s^-, & & 1 = \mu^T X_o, \\
& 0 \leq s^+, s^-, \lambda_j, & & -\varepsilon e^T \geq -\omega^T, \quad -\varepsilon e^T \geq -\mu^T,
\end{array}
\tag{1}
$$

where X_j, Y_j, $j = 1, \cdots, n$, are vectors of dimension $m \times 1$ and $s \times 1$, respectively, which represent observed amounts of x_{ij}, $i = 1, \cdots, m$, (inputs) and y_{rj}, $r = 1, \cdots, s$, (outputs) for each of $j = 1, \cdots, n$ Decision Making Units (DMUs) regarded as entities responsible for converting inputs into outputs. The components are all assumed to be positive, that is, $x_{ij} > 0$, and $y_{sj} > 0$, for all i, s and j.[1] X_o and Y_o represent the vectors of inputs and outputs for DMU_o, the DMU being evaluated relative to the performance of the other DMUs. The data of DMU_o also appear on the right of the primal problem as one of the

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[1]These positivity conditions will be relaxed later in this paper. See [15] for a general treatment.

DMU_j to be used in the evaluation, so that there is no issue concerning the existence of solutions because $\theta = 1, \lambda_j = \lambda_o = 1$, and all other $\lambda_j = 0$ evidently satisfies all constraints. It also follows that $\min \theta = \theta^* \leq 1$. Evidently $\theta^* = 1$ represents one part of the condition for full (= 100 %) efficiency, since $\theta^* < 1$ means that some other combination of DMUs could have produced at least as much of all of the outputs recorded for DMU_o with smaller amounts of all inputs.

Here, the superscript T symbolizes "transpose", and e^T is the transpose of the column vector (of suitable length) with all elements equal to unity. s^+ and s^- are "slack vectors", and ω^T, μ^T and λ are all "structural vectors"[2] of variables for the primal and dual problems, respectively, as recorded under the "min"(= minimization) and "max" (=maximization) operators in the objectives.

We now define a "virtual output" and a "virtual input" via the expressions:

$$\omega^T Y_o = y_o, \qquad \mu^T X_o = x_o. \tag{2}$$

Then we interpret the problem on the right in (1) in terms of maximizing this virtual output, y_o, with a virtual input of $x_o = 1$, subject to the further condition that no virtual output can exceed its virtual input for any DMU_j, $j = 1, \cdots, n$. A necessary and sufficient condition for full (100%) efficiency for DMU_o can then be specified as

$$\max \omega^T Y_o = \omega^{*T} y_o = 1, \tag{3}$$

in which case we will have equality between the virtual output and virtual input values for DMU_o, —*viz.*,

$$y_o{}^* = x_o{}^* = 1. \tag{4}$$

The problem on the right in (1) is referred to as being in "production function form". Also called the "multiplier form", it bears a variety of other names as well. Because the problem on the left in (1) has a finite optimum, the problem on the right also has a finite optimum and

$$\min \theta - \varepsilon e^T s^+ - \varepsilon e^T s^- = \omega^{*T} Y_o \tag{5}$$

by the dual theorem of linear programming.

This brings us to $\varepsilon > 0$, which appears in the objective of the problem on the left in (1) and in the constraints of the problem on the right. This ε is not a real number. It is, rather, a non-Archimedean infinitesimal defined so that no choice of slacks in the problem on the left can compensate for any increase this choice may cause in θ^*—the minimizing value of θ—so that θ^* and the optimal slacks define a two-component number in a manner analogous to the representations used for complex numbers.[3] For the problem on the right, the presence of $\varepsilon > 0$ means that all components of ω^T and μ^T are constrained to be positive. In short, all of the multipliers must assign "some" positive value to every component of Y_j and X_j for any DMU_j.

Returning to (5) and referring to the problem on the left in (1), we see that the necessary and sufficient condition for full (100%) efficiency of DMU_o requires *both*

$$\begin{array}{ll} \text{(i)} & \theta^* = 1, \\ \text{(ii)} & \text{All slacks are zero.} \end{array} \tag{6}$$

[2] I.e., these are vectors associated with the data from which the model is structures. For further discussion of this terminology see Chapter I in Charnes and Cooper [8].

[3] See [2] for further discussion and exploitation of this two-component property when it forms part of an optimal solution.

Now, returning to (4), we see the satisfaction of (6) means that no non-Archimedean components are present in the ω^{*T}, which is optimal for the problem on the right in (1). Slack values play an important role in what follows, so we pursue this topic a bit further. In fact, turning to the problem on the left in (1), we see that the non-Archimedean element requires the slacks to be maximized without worsening the optimal choice of $\min\theta = \theta^*$. In short, the choice of θ is given "preemptive priority" followed by maximization of the slack values. This insures that $\min\theta = \theta^*$ with non-zero slack will not be mistakenly identified as fulfilling the conditions for efficiency in (6) when alternate optima with non-zero slack are present. This follows because

$$\theta^* \geq \theta^* - \varepsilon e^T s^{+*} - \varepsilon e^T s^{-*}, \tag{7}$$

with strict inequality holding whenever any of the slacks are not zero. Conversely, if alternative solutions are available for $\min\theta = \theta^*$, and some of these have non-zero slack, then the expression on the left in (7) will not be optimal.

We can gain perspective for the developments that follow by noting that a use of (1) leads to evaluations of all of DMU$_o$'s inputs and outputs which can be represented as follows

$$\begin{aligned} x^*_{io} &= \theta^*_o x_{io} - s^{+*}_i \leq x_{io}, \qquad i = 1, \cdots, m, \\ y^*_{ro} &= y_{ro} + s^{-*}_r \geq y_{ro}, \qquad r = 1, \cdots, s \end{aligned} \tag{8}$$

with inefficiency present when any of these inequalities are strict. This use of (1) involves an assumption of continuity with the result that DMU$_o$ might be evaluated by reference to x^*_{io} and y^*_{ro} that do not correspond to actual observations in which case DMU$_o$ is evaluated by a hypothetical DMU$_j$ synthesized from (1) by a combination of actual DMUs.

In some circumstances it has proved desirable to restrict the choices so that evaluations are effected only by reference to actually observed behavior. See, Bitran and Valor-Sebastian [5] or Tulkens and Vanden Eeckaut [26] and [27], as well as Fried and Lovell [17] and [18] and Lovell and Pastor [21]. A natual way to accomplish this is to replace (8) with a search for an actually observed DMU$_j$ for which $X_j \leq X_o$ and $Y_j \geq Y_o$ with strict inequality holding for a least one input or one output. A DMU$_o$ is then said to be efficient if and only if *no* DMU$_j$ can be found which dominates the performance of DMU$_o$. This topic occupies the rest of this paper where we examine (a) model choices and (b) measures of efficiency which (i) reflect all sources of inefficiency and (ii) insure that only a "most dominant" DMU$_j$ is designated for the evaluations of each DMU$_o$.

2. A Model for Evaluating Efficiency Dominances

We formalize the model that forms the basis for our further developments as follows:

$$\begin{aligned} & \max \sum_{i=1}^{m} s_i^+ + \sum_{r=1}^{s} s_r^- \\ \text{subject to:} \quad x_{io} &= \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^+, \qquad i = 1, \cdots, m, \\ y_{ro} &= \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^-, \qquad r = 1, \cdots, s, \\ 1 &= \sum_{j=1}^{n} \lambda_j, \end{aligned} \tag{9}$$

where $\lambda_j \in \{0,1\}$—i.e., these variables are bivalent for all j—and $s_i^+, s_r^- \geq 0$ all i and r.

Evidently (9) is a mixed integer programming problem. It can be viewed as a bi-valency variant of the "additive model" for DEA as introduced in [10] or as a problem of goal-programming variety with only one-sided deviations allowed. See [7]. The resulting maximization problem involves a signed measure of distance in an l_1 (absolute value) metric. This is in contrast with (1) which can be interpreted so that (i) an l_2 (Euclidean) metric is supplied for the θ component[4] while (ii) the slack values associated with coefficients representing the non-Archimedean element $\varepsilon > 0$ are measured in an l_1 metric.[5] The non-Archimedean elements are not needed in (9) and hence the objective with its corresponding use of an l_1 metric may be interpreted as a model which guides the choices in terms of equally weighted deviations below the inputs and above the outputs of DMU$_o$. Using these equally weighted slacks, the problem stated in (9) is to determine a non-dominated vector (=DMU$_j$) which is maximally distant (in l_1 measure) from the observed values for the DMU$_o$ being evaluated. We again note there is no issue of existence since the choice $\lambda_j = \lambda_o = 1$ and all other variables zero satisfies the constraints.

The two conditions in (6) can now be replaced by the following single condition,

> DMU$_o$ is to be rated as fully efficient if and only if all slacks are zero in an optimal solution to (9). Moreover, each non-zero slack identifies an inefficiency amount in the corresponding input or output for DMU$_o$ measured in the same units as the corresponding x_{io} or y_{ro}. (10)

We will turn next to methods for measuring and interpreting the results secured from (9). Before doing so, however, we quote from Nemhauser [23, page 9] as follows:[6]

> "In the last five years, the capability has arrived to solve to optimality some MIPS [Mixed Integer Programming Systems] with thousands of binary variables on a workstation or personal computer."

The formulation in (9) should therefore prove computationally tractable over a wide range of potential applications and capable of being incorporated in DEA codes now available as described in [2].

3. MED—A Measure of Efficiency Dominance

Further discussions of relations between (9) and (1) are delayed until after we examine other measures which are also "units invariant" in the sense that their values do not depend on the units used to measure the different inputs and outputs. We also seek a measure of efficiency which achieves a maximum value of unity if and only if there is no DMU$_j$ which is dominatingly more efficient than the DMU being evaluated.

To obtain such a measure we note that

$$\begin{aligned} 0 \le s_i^{+*} &= x_{io} - x_{ik} \le x_{io}, \qquad i = 1, \cdots, m, \\ 0 \le s_r^{-*} &= y_{rk} - y_{ro} \le y_{rk}, \qquad r = 1, \cdots, s, \end{aligned} \tag{11}$$

where x_{ik} and y_{rk} are associated with $\lambda_k^* = 1$, and hence all other $\lambda_j^* = 0$, in the optimizing

[4] θ^* is commonly referred to as a "radial measure". Other metrics may be used, of course, but we use the Euclidean metric as a natural way of interpreting the term "radial measure".

[5] Also called the "city-block metric". See Appendix A in Charnes and Cooper [8] for a detailed development of these and other metrics.

[6] See also Nemhauser *et al.* [24].

solution to (9). But then, also,

$$0 \leq \frac{s_i^{+*}}{x_{io}} = \frac{x_{io} - x_{ik}}{x_{io}} \leq 1, \qquad i = 1, \cdots, m,$$
$$0 \leq \frac{s_r^{-*}}{y_{rk}} = \frac{y_{rk} - y_{ro}}{y_{rk}} \leq 1, \qquad r = 1, \cdots, s. \tag{12}$$

We interpret these as proportional measures of the inefficiency in each input and output with the first expression, measuring the proportion of input excess in x_{io}; while the second measures the proportion of output shortfall from y_{rk}. All of these proportions are units invariant.

An issue may arise when $x_{io}, y_{rk} \leq 0$ for some i or r. However, Ali and Seiford showed in [1] that the additive model of DEA is "translation invariant".[7] That is, an arbitrary constant d_i can be added to all x_{ij} in row $i = 1, \cdots, m$, and, similarly, an arbitrary constant d_r can be added to all outputs y_{rj} in row $r = 1, \cdots, s$, of any additive model without affecting the optimum solutions. This property of translation invariance carries over into the integer programming formulation given in (9) since the Ali-Seiford proof derives from the fact that $\sum_{j=1}^{n} \lambda_j = 1$ will result in canceling the d_i and d_r from both sides of every constraint without affecting the optimal variable choices or values. This property is not affected by the bi-valency requirement. Hence, the presence of zero or negative values in some x_{ij} or y_{rj} need not be a concern since they can be eliminated by introducing new variables:

$$\hat{x}_{ij} = x_{ij} + d_i > 0, \qquad i = 1, \cdots, m,$$
$$\hat{y}_{rj} = y_{rj} + d_r > 0, \qquad r = 1, \cdots, s. \tag{13}$$

This property of translation invariance does not apply to the ratio measures given in (12). The following conventions may be employed, however, if access to these measures is desired: If $x_{io} = 0$, then $s_i^{+*} = 0$ and the proportional input inefficiency value for this observation is then set equal to zero. If $y_{rk} = 0$, then y_{ro} and $s_r^{-*} = 0$, so we can assign a value of unity to this measure and interpret this to mean that there was a failure to achieve *any* of the potential of output y_{rk}. This assumes, of course, that at least one $y_{rj} > 0$ in row r in order to provide evidence that some positive amount of this output was possible.[8]

The above formulas can be extended to provide inefficiency measures, component by component, when wanted, as in (12), or by other subdivisions, such as "input inefficiency" proportions and "output inefficiency" proportions, or both, as in the following overall measure,

$$0 \leq \frac{\sum_{i=1}^{m} s_i^{+*}/x_{io} + \sum_{r=1}^{s} s_r^{-*}/y_{rk}}{m+s} \leq 1, \tag{14}$$

which, as can be seen, is the simple average of these proportions. This is a measure of the *average inefficiency* proportion in all inputs and all outputs which we can label MID (=Measure of Inefficiency Dominance). When a measure of the *efficiency* attained by DMU$_o$ is wanted, we simply replace (14) by

$$0 \leq 1 - \frac{\sum_{i=1}^{m} s_i^{+*}/x_{io} + \sum_{r=1}^{s} s_r^{-*}/y_{rk}}{m+s} \leq 1, \tag{15}$$

[7] An alternate approach is described in Charnes *et al.* [14] which uses generalized inverses to deal with x_{io} or $y_{rk} = 0$.

[8] Generally one will have at least one $y_{rj} > 0$ unless all entries in row r are zero—in which case this row should be omitted from the analysis. See the discussion of "data domains" on pp. 201 ff. in [15].

which we will refer to as MED (=Measure of Efficiency Dominance) where MED= 1 only when all of the ratios in the numerator are zero.[9]

4. Additional Choices of Weights and Measures

The MED measure developed in the preceding section is to be used after a solution has been achieved to (9). But this is not the only possibility. One way to extend the range of applications for the formulation given in (9) is to assign relative (or even preemptive)[10] weights to the different s_i^+ and s_r^- and to incorporate them in the objective in order to reflect their importance. The objective in (9) may then be viewed as representing the special situation in which all inefficiencies are of equal importance and hence are equally weighted.

When unit costs and unit prices are available, one can retain the constraints of (9) and replace its objective with

$$\max \sum_{i=1}^{m} c_i s_i^+ + \sum_{r-1}^{s} p_r s_r^- = \text{M–MID} \tag{16}$$

where the c_i and p_r are the unit cost and unit prices associated with s_i^+ and s_r^-, respectively. Thus M-MID provides a Monetary Measure of Inefficiency Dominance which can be identified with (a) excess costs in the first term and (b) lost revenues in the second term of (16).

Variations are possible so, for instance, the p_r may represent unit profits or other measures of like interest. However, in many public sector applications—and even in some private sector applications—there may be no easy accesss to such unit price and cost information. It may also be difficult and even impossible[11] to secure a collection of weights that can be readily agreed upon. One may then turn to a variety of devices such as obtaining a collection of solutions to (9) for review by potential users en route to selecting from one or more of these alternatives. See, e.g. [11] for the use of such an approach with "preemptive" as well as "absolute" and "relative" priorities used to deal with budgetary allocations for public health programs, dealing with contagious diseases, where neither market values nor easy access to relative weights for all such outputs were available for use in choosing between alternative program possibilities.

In some cases one may want to eliminate the problem of choosing a suitable unit of measure for each input and output in the objective of (9). For this purpose we can replace the objective in (9) with

$$\max \sum_{i=1}^{m} \frac{s_i^+}{x_{io}} + \sum_{r=1}^{s} \frac{s_r^-}{y_{ro}}. \tag{17}$$

Observing that the s_i^+ and x_{io} are in the same units of measure and that the same is true for the s_r^- and y_{ro}, we conclude that the units of measure cancel for each variable in the objective and this enables us to choose these units in whatever manner is convenient for treatment in the constraints.

[9]These measures may be applied in a suitably modified manner to other DEA models. See the Appendix to Banker and Cooper [3].

[10]See A. Charnes and W. W. Cooper [7] for uses of "preemptive", "absolute" and "relative priority" weights in goal programming.

[11]One may, for instance, think of the weights that might be assigned to "the increase in self esteem of a disadvantaged child" which represents one of the outputs in the large-scale social experiment associated with Program Follow Through discussed in [13].

The terms in the denominator of (17) can be interpreted as weights so that solutions with this objective will generally differ from those secured from (9) except when the weights $1/x_{io} = 1/y_{ro}$ for all i and r—in which case the objective in (17) becomes the same as the objective in (9). Although the numerators in (17) retain the property of translation invariance, this is not true for the denominators.[12] We can also have values of $s_r^{-*} \geq y_{ro}$ so that the value of the ratio in (17) may exceed unity even when averaged as was done for MED and MID. Recourse to the MID and MED measures defined by (14) and (15) are available when wanted, however, for use with solutions obtained from (17) and the same interpretations apply as before. Allowance must also be made for the possible presence of alternate optima. The differing values of s_i^{+*} and s_r^{-*} associated with such alternate optima may then yield different MED or MID values. Differences in MED or MID values associated with different optima may, of course, still be regarded as measures and used for effecting further choices of the programs with which they are associated.

Now we note that the objectives formulated in (16) and (17) may be used to obtain rankings of DMUs by reference to their inefficiencies.[13] When the units of measure for any input or output are changed in (16), a corresponding change in the associated weights must generally be made to preserve the value of M-MID. This is not true for (17), however, since each of the numerators and denominators in this expression is expressed in the same units.

This does not exhaust the possibilities. Very simple alterations in (17) can also produce a measure of total inefficiency which can be incorporated in the objective which cannot exceed unity in value. Such a measure (which we can refer to as SUMED) can be obtained by replacing the y_{ro} in (17) with values $y_{ro}^* = \max\{y_{rj} \mid j = 1, \cdots, n\}$ for $r = 1, \cdots, s$, in which case each s_r^{-*}/y_{ro}^* would represent the proportion of the maximal output recorded for any DMU. A still further extension would replace the x_{io} by $x_{io}^* = \min\{x_{ij} \mid j = 1, \cdots, n\}$ for each $j = 1, \cdots, m$—while using the Ali-Seiford translation theorem to avoid difficulties from the possible occurrence of values $x_{io}^* \leq 0$. See Bardhan *et al.* [4] for additional choices.

5. Radial Measures and Free Disposal

We now return to (1) and study its relations to the preceding developments in the following manner. First we adjoin to (1) and restricts the possible choices of DMUs to ones which dominate the DMU_o to be evaluated. It also provides access to modifications of (8) which we can relate to our MED and MID measures by replacing (11) with $\theta^* x_{io} = x_{ik} + s_i^{+*}$, so

$$0 \leq x_{io} - (\theta^* x_{io} - s_i^{+*}) = x_{io} - x_{ik} \leq x_{io}, \qquad i = 1, \cdots, m \tag{18}$$

and

$$y_{rk} - y_{ro} \leq y_{rk}, \qquad r = 1, \cdots, s. \tag{19}$$

This can designate a different DMU as "maximally dominating" the DMU_o being evaluated partly because of a difference in the metric employed. Allowing for this difference, these x_{ik} and y_{rk} may be employed in (11) with the same interpretations as before. It is to be noted that the slacks, as well as the results obtained from minimizing θ, are thus incorporated in

[12] An alternative treatment which utilizes generalized inverses is given in [14] Lovell and Pastor [22] have introduced a version of the additive model which preserves this property of translation invariance by replacing the x_{io} and y_{ro} in (17) by the standard deviations σ_i and σ_r associated with the corresponding x_{ij} or y_{rj}, $j = 1, \cdots, n$.

[13] Other criteria may also be used. See, for example Charnes, *et al.* [9] for a discussion of the use of use of total cost due to inefficiencies (including lost revenues) by the Public Utility Commission of Texas to rank the order in which electric cooperatives under their jurisdiction are submitted to efficiency audits.

a single measure. Insertion in (14) or (15) then produces a measure of average proportion of efficiency or inefficiency which, as before, eliminates the units in which the x_{io} and y_{ro} are measured.

Figure 1 below can help to clarify matters. The points $P_1, \cdots, P_5$ geometrically portray five DMUs, each of which produced a single unit of the same output with differing amounts of two inputs represented by their (x_1, x_2) coordinates. The solid line connecting P_1 and P_2 is the efficiency frontier that would be obtained from the ordinary DEA evaluations represented in (1).

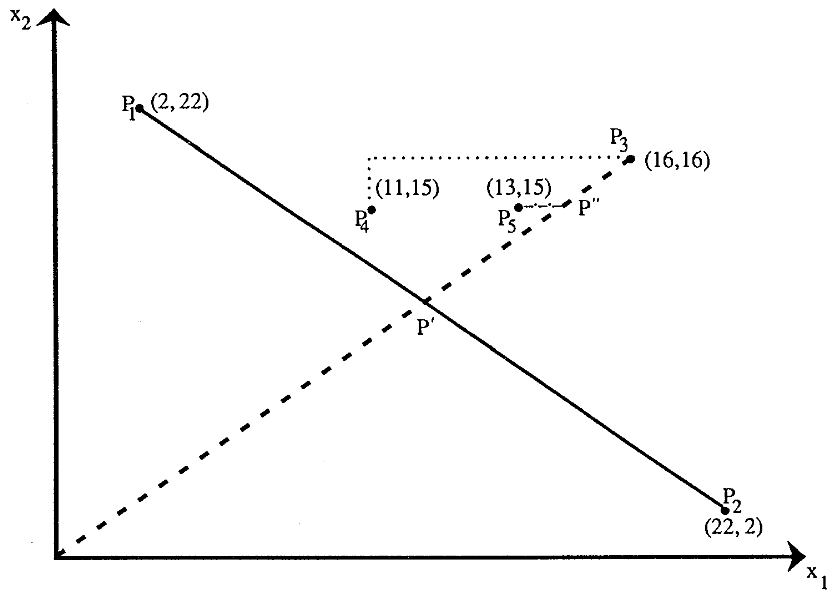

Figure 1: Dominance with Alternate Optima

We illustrate with the following use of (1) to evaluate P_3

$$\begin{array}{ll} \min \theta - \varepsilon s_1 - \varepsilon s_2 & \\ \text{subject to:} & 0 = 16\theta - 2\lambda_1 - 22\lambda_2 - 16\lambda_3 - 11\lambda_4 - 13\lambda_5 - s_1, \\ & 0 = 16\theta - 22\lambda_1 - 2\lambda_2 - 16\lambda_3 - 15\lambda_4 - 15\lambda_5 - s_2, \\ & 1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5, \\ & 0 \leq \lambda_1, \lambda_2, \cdots, \lambda_5, s_1, s_2, \end{array} \tag{20}$$

which has $\theta^* = 3/4$ with $\lambda_1^* = \lambda_2^* = 1/2$ and all other variables equal to zero as an optimal solution. Applying $\theta^* = 3/4$ to the coordinates of P_3 generates $P' = \lambda_1^* P_1 + \lambda^* P_2$ as a synthetic DMU which dominates P_3 in the indicated proportions. P_3 is therefore rated as being only 75% efficient in its performance because this combination of P_1 and P_2 supplies evidence that DMU_3 should have been able to produce its one unit of output with 25% less of each of its two inputs.

When (1) is employed, the points P_3, P_4 and P_5 will all be similarly dominated by points that can be synthesized from P_1 and P_2. However, when the bi-valency condition is adjoined to (1), the situation is altered because the frontier connecting P_1 and P_2 can no longer be used to generate points to effect such evalutaions. P_1 and P_2 will continue to be undominated and hence be rated as efficient using (6). P_3 is not dominated by either P_1 or P_2 but it is dominated by P_4 and P_5, and a use of (9) will designate P_4 as "most dominant" with slack values of $s_1 = 5, s_2 = 1$ representing the distance shown by the dotted line going from P_3 to P_4 in Figure 1. P_4 will be undominated and receive a MED score of unity but P_5 is dominated by P_4 and hence will receive a MED score less than unity when (9) is employed.

This brings us to the treatment of non-zero slack possibilities. These slacks can differ in their amounts according to the metrics and models employed. This topic of slacks has been treated in other ways as well. For instance, Färe, Grosskopf and Lovell [16] make frequent use of concepts that they refer to as "strong" and "weak disposal" which represent refinements of the concept of "free disposal" as introduced by Koopmans in [19] and [20] for use in his "activity analysis" treatment of "slacks".[14] On these assumptions, we can use $\min \theta = \theta^*$ as a measure of efficiency and ignore the slacks in the second of the two conditions in (6) because (a) they each are associated with a free good or, more precisely, their availability in the form of input excesses or output shortfalls is of no value in improving the solution of (20) and (b) there is no cost associated with their disposal so that the coefficient $\varepsilon > 0$ in (20) is replaced by zero.

This last property drastically alters the objective in (20) so that we make its implications clear by replacing our definition of dominance with the following definition of efficiency:

> *Efficiency*: A DMU_o is fully (100%) efficient if and only if there is no other DMU which is strictly better than DMU_o in at least one input or output and is not worse than it in any other input or output.

Now we adjoin the bivalency condition $\lambda_j \in \{0, 1\}$ to (20) and replace $\varepsilon > 0$ in the objective by zero and obtain the solution $\min \theta = 15/16$. From this efficiency rating we have $\theta^*(16, 16) = (15, 15)$. Hence P_3 should have been able to reduce each of its two inputs by 1 unit—a reduction which would bring it into conformance with P''.

The evidence to justify this reduction is supplied by the performances of either P_4 or P_5 because these represent alternate optimum solutions to the thus modified (20). Slack being of no interest, either may be chosen. But then one faces a quandary. For example, P_5 dominates P'' because of the two units of slack in x_1 (indicated by the horizontal broken line) so the repair of P_3's performance is not completed by θ^*. Indeed, further adjustment would be needed even if P_3 were brought into coincidence with P_5 because the latter is dominated by P_4.

We conclude that assumptions like free disposal (including strong and weak disposal) require suitable safeguards if they are to be employed. Restoring $\varepsilon > 0$ to its place in (20), however, insures that P_4 will be designated for evaluating P_3's performance by virtue of the maximization of the slacks that must then be undertaken to complete the solution of (20). Concomitantly we can say that these requirements for optimization are consistent with the above definition of efficiency because the choice is determined by reference to which DMU is "most dominant"—and this will be DMU_o itself if and only if it is 100% efficient.

6. Conclusion

In this paper we have presented a variant of the "additive model" of DEA and associated it with a variety of objectives and measures to use in dealing with "efficiency dominance". We have also used these additive models to highlight shortcomings involved in the use of other approaches which are associated assumption of "free disposal", etc.

[14]Koopmans associated these variables with "disposal activities"—which was replaced by "slack variables ", a term which was developed when it was found that management had trouble in identifying disposal activities. See Charnes, Cooper and Mellon [12]. See also discussion in [4].

Further discussion of these points may be found in Bardhan *et al.* [4] which uses the models and measures developed in this paper to compare and analyze still other approaches to efficiency dominance as developed in work by H. Tulkens, C. A. K. Lovell and their associates.

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