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MODELS FOR CASH FLOW ESTIMATION IN CAPITAL BUDGETING

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BACKGROUND OF THIS PAPER

This paper is based on research in multi-period planning problems of a large paper products manufacturer. This manufacturer uses a large linear optimization model for production scheduling in the short run and wanted to extend its use over a number of time periods to assess the impact over time of possible changes in the configuration of fixed equipment in an initial period. The principal analytical tool used is that of recursive programming in which parametric programming tools are also applied.

MODELS FOR CASH FLOW ESTIMATION IN CAPITAL BUDGETING

1. Introduction

Mathematical models have been developed for capital budgeting decisions which require one to have reliable estimates of the cash flows that would be generated by alternative investment proposals (for example, Lorie and Savage[5]and Weingartner[8]. Unfortunately, the problem of estimating cash flows is oftentimes so complex that only informal estimates can be made, and in some cases even informal estimates appear to be impossible to develop. Moreover, in such decision models it is usually assumed that the investments under consideration are independent. In this paper we develop a model which, for a class of capital budgeting problems, will provide estimates of cash flows to be used in a capital budgeting decision model (we call this a model for data inputs). This model can be used to determine a state of dependence among investment alternatives and by means of it one can assess whether the investment alternatives can be regarded as independent or dependent. Furthermore, if the degree of dependence is such that an assumption of independence seems to be inappropriate, we show that the model for data inputs leads naturally to a capital budgeting decision model for an optimal selection of investments which does not require an assumption of independence.

The impetus for our model comes as a result of the complexity, referred to above, of obtaining cash flow estimates in an industrial setting where there are many machines and many products to be considered. Typically, extrapolations of the performances of old machines along with simple estimations for new machines, based on new machine capacity and experience, are the basis for capital budgeting analyses. Interdependencies among old and new machines in regard to resource usage and products produced are, at best, handled in an ad hoc and informal manner. In our model we offer a more formalized approach that takes account of interdependencies among old machines and newly proposed machines both on an inter-period and intra-period basis. A pragmatic approach rather than a theoretical one, it should be particularly appealing to industrial firms that now regularly use a formalized approach such as linear programming to deal with their short-run production allocation problems.

As we shall allude to later, we recognize that the real conceptual problem we are addressing ourselves to is a large dynamic stochastic problem that is calculationally intractable. We develop an approximative or heuristic approach to the problem using recursive linear programming. This model, in addition to its tractability, provides a firm with a means of using a well-established short-run optimization model, for which data already are available, as a basis for a long-run model to generate the necessary data inputs for a capital budgeting analysis.

II. A Model for Estimating Cash Flows

We begin with the Apex Corporation which manufactures paper products.[1] Optimal allocation of machine time to the production of various paper grades in the short-run is done by means of a linear programming model. Management wishes to continue to use linear programming and parameterization options

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[1]Apex is a fictitious name for a company some of whose investment problems the authors have worked on; the models presented in this paper are simplified versions of those developed in the course of this work.

(sensitivity analysis) for decision making in any short-run period. However, management also wishes to analyze investment alternatives, to determine the impact of various investments upon optimal production allocations and optimal profits over a sequence of specified time periods where optimality calculations for any period are to be made by means of linear programming. The problem then is the classical one of the relation between short-run and long-run optimization. To put it another way, we are concerned with a multi-period decision problem. Under some circumstances this problem could be treated by means of dynamic programming. However, in any short-run period the production allocation model must have the capability to handle over one thousand constraints having several thousand decision variables. This means that dynamic programming cannot be used since efficient solution proceedures for problems of this size are not available. We instead develop a multi-period decision model using recursive programming which is solvable, and which takes into account period by period (time dependent) linkages.[2] Furthermore, a difficult interpretational problem arises with respect to the modelling or

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[2]For a more complete description of recursive programming see Day [2]. Although in our exposition below sales forecasts for various periods are regarded as point estimates, management can change or sensitize sales forecasts in any period and replace the initial point estimates by other estimates if it wishes. Theoretically, if the sales forecasts were to be regarded as random variables with known probability distributions, the model for any period would become one of stochastic programming. Experience with these models has shown that they almost always lead to models which are non-linear in the decision vector X. Given the size of the model in this paper in each period, incorporating sales forecasts as random variables would mean that the model for any period would become intractable.

choice of recursion relationships in a model of this kind. These relationships can be assumed to be linear or non-linear, for example, or they might be made functions of the final stage variables, etc.[3]. Our modelling strategy can be simply stated: we link together a sequence of (short-run) linear programming models by recursion relationships which are also assumed to be linear. This permits one to take account of dynamic influences, it renders the multi-period model tractable and it also provides management with a linear programming decision model for use in any period.

We first introduce the short-run linear programming model in the form of a simplified problem (a more complete statement of the mathematical model appears in Appendix A). Apex has three machines and produces eight grades of paper. Machine 1 can produce grades A, B, D, and E. Machine 2 produces grades F, G, and H, and Machine 3 can produces grades A, B, C, D, and E. All grades have upper limits in the form of maximum sales forecasts, and grades A, B, C, E, F, G have lower limits on production which come about because management specified that at least these amounts must be sold because of market penetration considerations. In addition management has specified some "force" inequalities: (1) not more than 75 percent of the total production of grade A can be made on Machine 1, (2) at least 20 percent of the production of grade A must be made on Machine 3, (3) not more than 80 percent of the production of grade D can be made on Machine 1, and (4) at least 15 percent of the production of grade D must be on Machine 3. Demand for grades A through E is forecasted to grow at a rate of 5 percent per period while a growth rate of 10 percent is forecasted for grades F, G, H.

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[3]For elaboration of this point, see Bellman[1], pp. 81-85.

Table 1 sets forth the production characteristics of the three machines and contains market data for the eight grades of paper for period 1. Profit contribution for each paper grade depends on the machine used since the machines are not of uniform efficiency. The data in Table 1, together with the force requirements specified by management, are the essential inputs into the short-run allocation (linear programming) model of Apex Corporation.

Table 1

Production Data: Machine Characteristics
(in hours)

Machine	Production Time Available	Time Required to Produce One Ton of Each Grade: A	B	C	D	E	F	G	H
1	6,000	.4128	.3586	--	.4752	.4492	--	--	--
2	6,000	--	--	--	--	--	.3132	.4266	.2654
3	6,000	.6914	.5688	.5140	.7028	.7148	--	--	--

Market Data for Paper Grades

Paper Grade	Upper Sales Forecast (in tons)	Lower Production Limit (in tons)	Profit Contribution Per Ton Machines		
			1	2	3
A	16,800	12,000	$141.33	--	$157.64
B	12,000	5,000	154.34	--	177.46
C	1,600	1,000	--	--	138.92
D	42,000	--	81.37	--	85.17
E	5,300	700	134.55	--	132.76
F	8,000	2,200	--	$123.63	--
G	5,930	3,930	--	76.40	--
H	20,000	--	--	183.03	--

This model can be represented mathematically if we let x_{ij} denote the amount of production of grade j on machine i where i = 1,2,3 and j can represent A, B, ..., G, H. The objective is to maximize total profit contribution (i.e., total sales revenue minus total variable costs): the Apex short-run allocation model appears on page 7 (a general formulation of this model appears in Appendix A). The objective function (1) and constraints (2) through (18) reflect the data contained in Table 1. Constraints (2), (3), (4) represent machine capacity and constraints (5) through (18) represent the demand forecasts and production lower limits for the eight paper grades. Constraints (19) through (22) reflect the four restrictions imposed by management. An optimal solution to this model is the basis for a short-run resource allocation by Apex.[4]

In order to develop our long-run model for analyzing investment alternatives we must establish how a sequence of short-run linear programming models provides a model for multi-stage decisions. We have been given a growth forecast of 5 percent for grades A through E and one of 10 percent for grades F, G, H. These increases form one link between the models for succeeding periods. A more important link is provided by optimal solution values of the dual problem of our short-run linear programming model. Specifically, Apex's management analyzes optimal dual values associated with the upper and

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[4]An optimal solution to the model given above is X_{1A}= 9,000, X_{3A}= 3,000, X_{1B}= 5,495, X_{3B}= 5,998, X_{3C}= 1,000, X_{1D}=0, X_{3D}=0, X_{1E}=700, X_{3E}=0, X_{2F}= 2,200, X_{2G}= 3,930, X_{2H}=13,694: optimal profit = $6,969,073.

(1) Maximize

$$141.33x_{1A}+157.64x_{3A}+154.34x_{1B}+177.46x_{3B}+138.92x_{3C}+81.37x_{1D}+85.17x_{3D}+134.55x_{1E}+132.76x_{3E}+123.63x_{2F}+76.40x_{2G}+183.03x_{2H}$$

Subject to

(2) $$.4128x_{1A}+.3586x_{1B}+.4752x_{1D}+.4492x_{1E} \leq 6{,}000$$

(3) $$.3132x_{2F}+.4266x_{2G}+.2654x_{2H} \leq 6{,}000$$

(4) $$.6914x_{3A}+.5688_{3B}+.5140x_{3C}+.7028x_{3D}+.7148x_{3E} \leq 6{,}000$$

(5) $$x_{1A}+x_{3A} \leq 16{,}800$$

(6) $$x_{1A}+x_{3A} \geq 12{,}000$$

(7) $$x_{1B}+x_{3B} \leq 12{,}000$$

(8) $$x_{1B}+x_{3B} \geq 5{,}000$$

(9) $$x_{3C} \leq 1{,}600$$

(10) $$x_{3C} \geq 1{,}000$$

(11) $$x_{1D}+x_{3D} \leq 42{,}000$$

(12) $$x_{1E}+x_{3E} \leq 5{,}300$$

(13) $$x_{1E}+x_{3E} \geq 700$$

(14) $$x_{2F} \leq 8{,}000$$

(15) $$x_{2F} \geq 2{,}200$$

(16) $$x_{2G} \leq 5{,}930$$

(17) $$x_{2G} \geq 3{,}930$$

(18) $$x_{2H} \leq 20{,}000$$

(19) $$.25x_{1A}-.75x_{3A} \leq 0$$

(20) $$-.2x_{1A}+.8x_{3A} \geq 0$$

(21) $$.2x_{1D}-.8x_{3D} \leq 0$$

(22) $$-.15x_{1D}+.85x_{3D} \geq 0$$

(23) $$x_{ij} \geq 0 \quad (i = 1,2,3; \quad j = A,B,C,D,E,F,G,H).$$

lower production limits for the eight grades of paper. From linear programming theory and some well known interpretations of dual values we know that a paper grade with a positive dual associated with its upper limit or a negative dual associated with its lower limit provides management with opportunities for profitable strategy change regarding the upper and lower limits. For example, raising upper limits associated with positive duals or reducing lower limits associated with negative duals have the effect of increasing maximum profits. Thus, rather than simply using the forecasted growth rates alone to obtain the demand forecasts for each period, management can also use the dual values in one period to adjust demand forecasts for the next period.[5]

Rules for making adjustments can be flexible and in a given period it might be decided not to push a grade with a positive dual variable for any one of a number of reasons, including the influence of variables that are outside the model; on the other hand management could use a positive dual variable as a basis for review and examination of sales forecasts for subsequent periods. The recursive model we develop allows management to have a wide range of choices with respect to these problems and it can accept as inputs the results of management's decisions with respect to them. From our example problem we assume that Apex management has adopted the following rules which define how the dual values on the upper and lower production limits in one period influence the calculation of these limits for the eight grades of paper in the following period.

Let d_t = sales forecast in period t,
L_t = lower production limit in period t,
y_t = dual value for a grade of paper in period t.

[5]For a detailed analysis of the effects of short-run strategy changes guided by a consideration of dual values see Godfrey, Spivey and Stillwagon[4].

Then the recursive rule can be stated as follows,

A. Action regarding demand forecasts for period t:

1. If y_{t-1} for a grade is greater than zero, allow forecasted growth plus extra 20 percent of growth in order to "push" this grade, i.e., $d_t = d_{t-1} + (1.2)\ (.05)d_{t-1} = (1.06)d_{t-1}$ (if forecasted growth is .05);

2. If y_{t-1} for a grade is equal to zero, allow no increase, $d_t = d_{t-1}$.

B. Action regarding lower production limits for period t:

1. If y_{t-1} is equal to zero for some grade then $L_t = L_{t-1}$;

2. If y_{t-1} is less than zero for some grade, then reduce its lower limit by five percent, $L_t = (.95)L_{t-1}$.

Using these rules and the forecasted growth rates for the eight paper grades management can obtain allocation and profit estimates for as many periods in the future as it wishes; other parameters of the short-run model such as prices, costs and machine rates may also be varied over the time horizon of the analysis if current information and forecasts suggest that such variations are necessary. The full range of parametric options will be available in any period in the model we develop. Other linear recursive rules could also be used.

The linkages specified by our recursion rule above constitute the basis of our long-run model and provide the format for our analysis of investment alternatives.. We first use our long-run model to estimate future profit contributions assuming that no changes are made in our present equipment. Then through alterations in the machine constraints of our short-run allocation model and by means of the recursion relations we can examine the effects over time of alternate assumptions regarding acquisition of new machines or retirement of old machines and obtain corresponding estimates of future profit contributions. By comparing future profit contribution estimates with and without new machines on a period by period basis we can analyze the value of purchasing the new machines. In doing this we transform the estimated profit contributions into estimated cash flows which can then be used in a capital budgeting decision model (a detailed explanation of these procedures is given in the Appendices B and C).

For an example of how our long-run model can be used, suppose Apex is considering two new (but different) machines for purchase. The investment alternatives confronting Apex can be listed as follows:

(0) No new investments (base alternative).

(1) Purchase a new high-speed machine that can only produce grades A, B, C, D. Recall that these grades have only a 5 percent growth forecast but on the average they are more profitable than grades F, G, H which have a 10 percent growth forecast. The strategy of purchasing this machine would be to put future emphasis on expansion in those paper grades that are most profitable. The total cost is $25,200,000 with an assumed life of six years.

(2) Purchase a new machine, not as efficient as the one proposed in (1), but one that can produce grades A, B, D, E, F, G, H. The emphasis here, in contrast to proposal (1), would be to add capacity for almost all grades, particularly the three with the greatest growth potential (F, G, H). Total cost is $18,900,000 with an assumed life of six years.

(3) Purchase both of the machines described in (1) and (2) above. The total cost is assumed to be simply the sum of individual machine costs, $44,100,000.

Our example capital budgeting problem uses a six year horizon as the length of life of each of the two new machines proposed. This is unquestionably shorter than is characteristic of the life of capital equipment in this industry, where a machine life of thirty to forty years is not uncommon. However, in this paper the six year period is assumed to be the "long-run" specified by management. The length of time chosen is not a restrictive feature of the model since our analysis can be made for any finite sequence of time periods. The choice of six years is made merely for expository convenience.

We will use the data presented above for the Apex short-run model as the starting point for an example problem for our long-run model. By altering the machine constraints of the short-run model we can represent each of the investment alternatives and then by using our recursive rule and forecasted growth rates for the various grades of paper we can obtain future profit contribution estimates for each investment alternative. These estimates appear in Table 2. The difference between the estimated profit contributions for alternative 0 and those of each of the other alternatives provide estimates of additions to profit contributions resulting from choosing the given alternative instead of alternative 0. These differences appear in Table 3.

Table 2 Estimated Profit Contributions
(in thousands of dollars)

Period	Investment 0	1	2	3
1	$6,969	$12,882	$11,378	$17,516
2	7,044	13,185	11,640	18,674
3	7,117	13,500	11,817	19,750
4	7,191	13,828	11,991	20,525
5	7,264	14,171	12,175	21,233
6	7,301	14,498	12,327	21,641

Table 3 Estimated Additions to Profit Contribution
(in thousands of dollars)

Period	Investment 1	2	3
1	$5,913	$4,409	$10,547
2	6,141	4,596	11,630
3	6,383	4,700	12,633
4	6,637	4,800	13,334
5	6,907	4,911	13,969
6	7,197	5,026	14,340

One further operation on the data in Table 3 will be necessary to prepare the data for a capital budgeting analysis. We must convert the estimated additions to profit contribution into estimated additions to cash flow. A conversion procedure for doing this is given in Appendix C.[6] We assume a fifty percent income tax rate and use a sum-of-years digits depreciation policy assuming no salvage value for machines. The data in Table 4 are then

[6] It is also pointed out in Appendix C that what we have called profit contribution is simply cash flow before adjustments for depreciation and taxes.

in proper form to be used as input into a cash flow model for capital budgeting decision making problems.

Table 4 Estimated Additions to Cash Flow
(in thousands of dollars)

	Investment		
Period	1	2	3
1	$6,557	$4,905	$11,574
2	6,071	4,548	11,065
3	5,592	4,150	10,517
4	5,119	3,750	9,817
5	4,654	3,356	9,085
6	4,199	2,963	8,220

III. Assessing the State of Dependence Among Investment Alternatives

A fundamental question in the theory of capital budgeting concerns possible interactions among investment alternatives. Specifically, does the acceptance or rejection of any investment alternative affect the potential performance (that is, future benefits usually measured in terms of cash flow) of other investment alternatives in the set? If the answer to this question is "no" then we say we have an independent set of investment alternatives; if the answer is "yes" then the set is said to be dependent. In terms of cash flows a corresponding definition of an independent set of investments would be that the stream of future cash flows generated by any one of the investment alternatives in the set will not be affected by the selection or rejection of any of the remaining investments in the set. To illustrate this notationally, let us use the example capital budgeting problem of Section II above where the three investment alternatives were: (1) purchase machine 1, (2) purchase machine 2, and (3) purchase both machines 1 and 2. If we let $CF_1=(CF_{11},...,CF_{16})$, $CF_2=(CF_{21},...,CF_{26})$ and $CF_3=(CF_{31},...,CF_{36})$ denote sequences of estimated additions to cash flow for the three investment alternatives, respectively, then the set of investments 1 and 2 will be independent if $CF_3=(CF_{31},...,CF_{36}) = (CF_1+CF_2) = (CF_{11}+CF_{21},...,CF_{16}+CF_{26})$, i.e., $CF_{3t}=CF_{1t}+CF_{2t}$, $(t=1,...,6)$. On the other hand, if $(CF_1+CF_2) \neq CF_3$, i.e., if $(CF_1+CF_2) - CF_3 \neq 0$, we say the set containing investments 1 and 2 is dependent. To put it another way, the sequence $(CF_1+CF_2) - CF_3$ characterizes the state of dependence; if this is the zero sequence, then we say the set of investments is independent. Otherwise, the set is dependent. Our long-run model presented in Section II provides a way of obtaining CF_1, CF_2 and CF_3 so that the state of dependence between investments 1 and 2 may be estimated (it should be obvious from our illustration that a question about independence between investments 1 and 3, or 2 and 3 is irrelevant). The sequences CF_1, CF_2, and CF_3 for our example in Section II are shown in Table 4. Table 5 contains the data for a test of independence between investments 1 and 2 and is derived from Table 4.

Table 5 Test for State of Dependence
Between Investments 1 and 2
(in thousands of dollars)

Period t	Investment 1 CF_{1t}	2 CF_{2t}	1 + 2 $CF_{1t}+CF_{2t}$	3 CF_{3t}	(1+2)-3 $(CF_{1t}+CF_{2t})-CF_{3t}$
1	\$6,557	\$4,905	\$11,462	\$11,574	\$ -112
2	6,071	4,548	10,619	11,065	-446
3	5,592	4,150	9,742	10,517	-775
4	5,119	3,750	8,869	9,817	-948
5	4,654	3,356	8,010	9,085	-1,075
6	4,199	2,963	7,162	8,220	-1,058

It can be seen that $CF_{1t}+CF_{2t} \neq CF_{3t}$ for all values of t (i.e., that $CF_1+CF_2-CF_3 \neq 0$), and therefore investment alternatives 1 and 2 are not (mutually) independent. It appears that a joint investment in alternatives 1 and 2 promises better returns than would be indicated by the sums of the individual cash flow estimates for alternatives 1 and 2 considered separately. The importance of the state of dependence shown in the last column of Table 5 is assessed as follows. We select a decision criterion and observe what the investment decision would be for two cases: first assuming independence and second, assuming dependence between investments in the two machines. If the investment decisions are different in these two cases, then we say that the state of dependence is important, otherwise we say that it is unimportant.

We select as the decision criterion that of Net Present Value. In using this criterion all estimated future cash inflows and outflows for each investment alternative are discounted at a given rate of interest to obtain a net present value calculation for each alternative. All alternatives with a positive net present value are then acceptable for undertaking.

For our example problem this means summing the results of our individual analyses of the two new machines (CF_1+CF_2) and calculating the net present value of this flow. This is contained in Table 6 along with a net present value calculation for the flow CF_3. Note that the Net Present Value criterion is applied to a joint investment in which independence is not assumed.

Table 6 Net Present Value Comparison
(in thousands of dollars)

	Investments	
	$CF_1 + CF_2$	CF_3
Initial Cost	$-44,100*	$-44,100*
Present Value of Estimated Cash Flows Discounted at 8%	43,934	47,131
Net Present Value	$-166	$3,031

*Recall that in the description of the example investment problem in Section II, it was assumed that the total cost of undertaking both investments 1 and 2 was simply the sum of the individual costs, $25,200 + $18,900 respectively.

We see in Table 6 that if Net Present Value is our criterion for acceptance or rejection and eight percent is our cost of capital, then using the individual cash flow estimates as data inputs ($CF_{1t}+CF_{2t}$) will result in a negative net present value (-$166). This would mean rejection of the joint purchase of machines 1 and 2. Yet by taking into consideration the dependence between investments 1 and 2 we see that a joint investment will produce a positive net present value ($3,031). Thus it can be said that the state of dependence between investments 1 and 2 is important since different assumptions about dependence lead to different investment decisions. The analysis indicates that the use of a capital budgeting decision model that requires an assumption of independence among investment alternatives would not be appropriate for our sample problem.

In this section we have shown how our long-run model can be used to characterize the state of dependence among investment alternatives. Any number of new machine proposals can be analyzed simultaneously by means of this model by making appropriate changes in the machine constraints of the short-run allocation model.

IV. A Dependent Investment Decision Model

Capital budgeting decision models that are based on a discounted cash flow approach usually include an assumption of independence among investment alternatives. Two classic models, Net Present Value (defined in Section III) and Rate of Return are examples of discounted cash flow models in which an independence assumption is conventionally made. In using the Rate of Return model a rate of return is found for each investment that will make the present value of estimated cash inflows equal to the present value of estimated cash outflows. All alternatives with estimated rates of return greater than or equal to some predetermined rate are regarded as acceptable. Once again, it is the convention in this model to assume that there are no interdependencies among those alternatives judged to be acceptable.

Weingartner [8] proposed a variation of the conventional Net Present Value Model by introducing budget constraints for the periods of analysis and structuring the problem as a linear programming model. In this model the investment alternatives were again assumed to be independent.

The primary reasons for making an independence assumption are that it is difficult to develop a model that allows for dependence among alternatives and even if dependence were accommodated for in a model, it still is very difficult to assess the state of dependence. Reiter, in [7] proposes a decision model that allows for pairwise dependence of alternative investments in a set. Yet when attempting to use Reiter's model it is still necessary to acquire a quantitative assessment of each pairwise dependency since his model assumes that this be known for each pair of investment alternatives.

In Section III above we described how our long-run model can be used to assess the state of dependence among alternative investments. In our example problem we had only two alternatives but it should be apparent that similar analyses could be made for three or more alternatives. We now suggest that our long-run model leads naturally to a decision model in which we need not worry about dependence among investment alternatives once it has been established by use of our state assessment. The model we propose utilizes the concept of a set of <u>mutually exclusive investments</u>. If a joint investment in two single investments is considered feasible then our analysis of the joint investment is made by simultaneously analyzing the effects of the joint investment rather than analyzing the single investments independently of each other and combining the separate analyses. Then a joint investment is judged on its own merits as a separate investment alternative. This is essentially what was done with our example problem in Section III when we used the Net Present Value criterion to analyze a joint investment in machines 1 and 2. In our example problem the investment alternatives were:

(0) Make no new investments,

(1) Purchase machine 1,

(2) Purchase machine 2,

(3) Purchase machines 1 and 2.

This set of four alternatives can be said to comprise a mutually exclusive set since the selection of any one of the alternatives precludes the acceptance of any other member of the set. We need not be concerned with interdependencies among alternatives; each alternative is judged on its own merits and selection is made on the basis of some criterion established by a decision maker.

Let us use the approach just described for our example problem. For illustrative purposes we choose both the net present value and rate of return models for investment selection. Unlike conventional usage of these models, however, we will not need to assume independence between new machine investments since a joint investment in the two machines is analyzed separately. Table 7 (on page 20) contains the data of our analyses. It appears that a decision based upon either criterion would be to select alternative 3 (purchase both new machines). A qualification should be given to the two models used in Table 7. It may be that because investment 3 costs so much more than either 1 or 2 that financial considerations may tend to reduce the attractiveness of investment 3. Obviously if the investment budget is fixed below $44,100,000 then investment alternative 3 cannot be chosen. But as long as the budget is sufficient and there are no alternative uses of the budget that promise better returns it appears that alternative 3 is the best.

This is the essence of our dependent investment decision model: to transform our original set of investment proposals into a set of mutually exclusive investment alternatives. We suggest that this concept can be generalized to any number of new machine proposals where one or more of the new machines can be purchased and where there may be investment interdependencies among the new machines. All possible combinations of new machine selections can be analyzed by our long-run model through proper modifications of matrix A_{11} and vectors b_1 and C^T. If we had three new machine proposals A, B, C, then our

Table 7 Mutually Exclusive Investment Models

Net Present Value Model
(in thousands of dollars)

	Investment Alternative			
	0	1	2	3
Initial Cost	0	\$ - 25,200	\$ - 18,900	\$ -44,100
Present Value of Estimated Cash Flows Discounted at 8 percent	0	25,291	18,643	47,131
Net Present Value	0	\$91	\$-257	\$3,031

Rate of Return Model

Investment Alternative			
0	1	2	3
0	.0813	.0751	.1037

alternatives would be as follows:

Alternative	Action
0	purchase no new machine
1	purchase A
2	purchase B
3	purchase C
4	purchase A, B
5	purchase A, C
6	purchase B, C
7	purchase A, B, C

Notice that there are $2^3 = 8$ mutually exclusive alternatives, i.e., 2^m where m is the number of new machines (investments) proposed. The number of mutually exclusive alternatives would be reduced if any alternative exceeded the total budget. Also, there would undoubtedly be other financial considerations since there will be differences in cost among the alternatives. These problems would have to be accounted for in the decision model used.[7]

V. Concluding Remarks

The long-run recursive analysis given above could be applied to the analysis of any investment problem for a firm when the firm's production allocation model within a time period is that of linear programming. Furthermore, the number of time periods considered need not be limited to that used above (which was selected for expository purposes). The problem of dependent investments which is not confronted directly in the usual approach to investment models was considered directly for any firm with the characteristics noted above. We have shown how the state of dependence of investment alternatives can be assessed and, if investments are dependent according to this assessment, we have proposed a decision model which makes use of the notion of mutually exclusive investments.

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[7] It should be observed that 2^m is a large number even for fairly small values of the integer m. Our model is useful for examining the capital budgeting decision when there is a relatively small number of proposed investments, each of which may be large in total dollar amount. If $m \geq 8$, applicability of our model would then depend on the size of the short-run allocation model and the use of common sense in discarding some combinations. In the problem we are considering management is not attempting to determine which ten or twelve machines to buy but is considering the purchase of not more than two (very costly) machines.

Appendix A

Short-run Resource Allocation Model

The foundation for our data generating model is a sequence of short-run linear programming models. Therefore, it will be necessary to begin by describing our short-run model. This model is used by Apex Corporation to allocate its machines to the production of grades of paper for a period. Mathematically this model can be displayed as follows:

$$
\begin{aligned}
&\text{Maximize} \quad && C^T X \\
&\text{Subject to} \quad && A_{11} X \le b_1 \\
(1) \quad & && A_{21} X \le b_2 \\
& && A_{31} X \ge b_3 \\
& && A_{41} X \le b_4 \\
& && A_{51} X = b_5 \\
& && A_{61} X \ge b_6 \\
& && X \ge 0
\end{aligned}
$$

where X = the solution vector, the amount of each grade to make on each machine,

C^T = the profit contribution vector,

b_1 = vector of machine time available in the period,

b_2 = vector of sales forecasts for the period,

b_3 = vector of lower production limits for the grade (possibly not all grades),

$b_4, b_5, b_6 = 0$ and are the right hand sides of a number of balancing equations and inequalities that reflect management imposed relationships among some of the grades,

A_{11} = a matrix of machine rates for the time required to make each grade on the machine it can be made,

A_{21} = a matrix (ones and zeros) relating the grades to their sales forecasts for the period,

A_{31} = a matrix (ones and zeros) relating the grades to their lower production limits for the period,

A_{41}, A_{51}, A_{61} are matrices that reflect the required balancing relationships among the grades.

Appendix B

Long-run Recursive Model

Our long-run model consists of a sequence of short-run allocation models which are linked together by certain dependence relationships. The formulation for each of these short-run models is similar to (1). Differences among these models occur because each model represents a different time period; therefore the vectors, b_1 through b_6 and C^T, and matrices, A_{11} through A_{61}, contain forecasts and estimates pertaining to particular periods. If the time horizon of our analysis is k periods then we will have k short-run allocation models; each containing forecasts and estimates relevant to one of the k periods. These models serve as forecasts of the short-run allocation problem facing Apex in each of the next k periods and can be solved to obtain an estimated optimal allocation and profit contribution for each period.

If we would solve each one of the k models independently of the other k - 1 models this would imply that we assume that each short-run allocation plan and profit is not related to the others. If this assumption is valid then whatever happens to short-run allocation in any period depends only on marketing variables such as demand, prices and costs and production characteristics such as machine rates and machine time availability in the period.

For Apex Corporation and many other firms this is not the case. There are policy changes with respect to short-run profit contribution and allocation strategies that are often made. For Apex one of the more important of these strategies is to allow the dual values obtained in an optimal solution to one period's short-run model to influence the short-run model structure in the following period. For example, a grade of paper with a high dual value in one period's solution indicates that short-run profit contribution could be increased if more of that grade were produced and sold. The strategy for the next period could be for marketing personnel to give extra "push" to that grade with the high dual value in the preceding period. From the standpoint of model construction this would mean putting a higher forecast (than normal) for that grade into the short-run model for the next period. In this way next period's model would be different than if only the normal forecasts were put into it. Another strategy by Apex is with respect to lower production limits. When one of these has a negative dual value associated with it this means that more profitable use of production facilities could be made if this lower limit could be reduced still further. Apex allows these negative duals to influence its short-run allocation model formulation from one period to the next by decreasing the lower limits for grades with negative duals. These two types of influences (dual values of forecasts and production lower limits in one period on model construction in the following period) are policy decisions that cannot be forecasted and entered into the k short-run models all at once. We must know the optimal solution (in particular the dual optimal solution) in period t - 1 before our model for period t can be formulated. This is shown notationally in our long-run linear programming model below.

$$\text{Maximize } f_t[A_{11}, b_2^t(X_{t-1}, F_t), b_3^t(X_{t-1})] = C^T X_t$$

Subject to

(2)

$$A_{11}X_t \le b_1$$

$$A_{21}X_t \le b_2^t(X_{t-1}, F_t)$$

$$A_{31}X_t \ge b_3^t(X_{t-1})$$

$$A_{41}X_t \le b_4$$

$$A_{51}X_t = b_5$$

$$A_{61}X_t \ge b_6$$

$$X_t \ge 0$$

$$t = 1, \ldots, k$$

where

X_t = the solution (allocation) vector for period t,

$b_2^t(X_{t-1}, F_t)$ = the sales forecasts for period t which are a function of the solution vector of period t-1, X_{t-1}, and F_t the market demand forecasts for period t,

$b_3^t(X_{t-1})$ = production lower limits which are a function of the solution vector of period t-1.

All vectors and matrices have the same interpretations as given in (1) but now X, b_2 and b_3 are indexed by t to indicate the time period.

Appendix C

Using Long-run Recursive Model to Obtain Cash Flow Estimates

Undertaking a cash flow approach to capital budgeting problems requires that we obtain estimates of future additions to cash flow for our investment alternatives. To do so we must first have an estimate of the firm's cash flow performance over the next k periods if <u>no</u> new investments are made. These estimates can then be subtracted from cash flow estimates given that new investments are made and these subtractions will produce estimated additions to cash flow for each new investment alternative. The firm's estimated performance assuming no new investments will be called the "base" sequence and will consist of k maximum short-run profit contributions obtained by successively solving the k short-run models described in (2) above in Appendix B. The key components of these k models which serve to indicate that no new investments have been made are the matrix A_{11} and the vector b_1. Recall that A_{11} is a matrix of machine rates with each row representing a different machine and the elements of b_1 are the time available on each machine for a period. No changes to A_{11} and b_1 over the k periods mean that no new investments or disinvestments are made and characterize the k short-run models of our base sequence. Investments in new machines and elimination of old machines can then be represented by changes in A_{11} and b_1. We can denote various investment alternatives by modifying (2) as follows:

$$\text{Maximize } f_{st}\left[A_{11}^{s}, b_2^t(X_{s,t-1}, F_t), b_3^t(X_{s,t-1})\right] = C_s^T X_{st}$$

$$\text{subject to} \quad A_{11}^{s} X_{st} \le b_1^s$$

$$A_{21} X_{st} \le b_2^t\,(X_{s,t-1}, F_t)$$

$$A_{31} X_{st} \ge b_3^t(X_{s,t-1})$$

$$A_{41} X_{st} \le 0$$

$$A_{51}X_{st} = 0$$

$$A_{61}X_{st} \geq 0$$

$$X_{st} \geq 0$$

$$t = 1,\ldots,k$$

$$s = 0,1,\ldots,p$$

The only change from (2) is the index s which has been added to X_t, A_{11}, and C^T. C^T will vary among the different investment alternatives because different machine-grade combinations will be present and therefore some changes in anticipated profit contributions will occur. No changes in equipment (base sequence) is denoted when s = 0 while s = 1 denotes the first new investment alternative, s = 2 denotes the second, etc. If $C_0^TX_{0t}^*$ represents optimal profit contribution in period t for our model with no changes in equipment, then the base sequence referred to above

is $(C_0^TX_{01}^*, C_0^TX_{02}^*, \ldots, C_0^TX_{0k}^*)$,

or $(C_0^TX_{0t}^*)$ $t = 1,\ldots,k$.

The following calculation,

$P_s = (C_s^TX_{st}^* - C_0^TX_{0t}^*)$, $t = 1,\ldots,k$, gives an estimate of additions to profit contribution of the s^{th} investment alternative in each of the k periods. The next step is to convert the estimated additions to profit contribution into additions to cash flow. We will make the assumption that all cash transactions resulting from the new investment (except the actual outlay for the purchase of the asset) are accounted for in the addition to profit figures P_s. Furthermore, we will assume that the only additional fixed cost that results from a new investment is depreciation. Thus what we have called profit contribution is actually cash flow before adjustments for depreciation and taxes. To make these adjustments we let

$$a_{st} = C_s^T X_{st}^* - C_0^T X_{0t}^* ,$$

and then

$$P_s = (a_{st}),\ s = 1,\ldots,\ p;\ t = 1,\ldots,k .$$

Assuming a 50 percent income tax rate then

$$CF_{st} = (a_{st} - .5a_{st} + .5Q_{st})$$

or

$$CF_{st} = (.5(a_{st} + Q_{st}))$$ where CF_{st} = the additional cash flow provided by the s^{th} alternative investment in period t, and Q_{st} equals the depreciation of the s^{th} alternative investment in period t. These cash flow estimates can then be used in a cash flow model for making investment decisions.

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