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Models for Supersymmetric Quantum Mechanics

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Abstract

An algebraic view of supersymmetric quantum mechanics is taken, emphasising the couplings between bosonic and fermionic modes in the supercharges. A class of model Hamiltonians is introduced wherein the fermionic (bosonic) operators are canonical and the bosonic (fermionic) ones satisfy a Lie algebra (superalgebra) whose representation theory permits the complete solution of the model in principle. The kinematical symmetry of such models is also described. The examples of one and two bosonic models, with SU(2) and SU(3) dynamical algebras respectively, are analysed in detail.

1. Introduction and Main Results

Supersymmetry provides a unified framework for the exact solutions of classical textbook nonrelativistic quantum mechanical models (Haymaker and Rau 1986; Fuchs 1986). In the simplest version the odd and even parts of the Hamiltonian correspond to 'partner' potentials for the system, whose form is related by supersymmetry and which produce strongly constrained spectra, phase shifts, etc.; in soluble cases the partners are typically reparametrisations of each other. More generally, one can consider conditions under which a wider class of Hamiltonians with a given number of bosonic and fermionic canonical coordinates and momenta possesses simple or extended supersymmetry (de Crombrugghe and Rittenberg 1983; Rittenberg and Yankielowicz 1985), allowing also for constraints (Flume 1985). Making reasonable assumptions about the couplings in the Hamiltonian and supercharges, there are only a limited number of solutions (up to $N=8$); the more extended models often require exotic values of magnetic moments, monopole charges and so on, conspiring with highly specific spatial potentials (D'Hoker and Vinet 1985).

In the present paper we point out a general class of $N=1$ supersymmetric Hamiltonians $H=\{Q,Q^\dagger\}$ which arise from supercharges involving couplings between fermionic and bosonic modes; one set of modes satisfies canonical commutation (or anticommutation) relations while the other set is assumed to satisfy certain Lie algebra (or superalgebra) relations. Primary amongst these is the requirement that they should mutually commute (or anticommute) in order to secure $Q^2=Q^{\dagger 2}=0$. Further conditions will dictate the type of kinematical and dynamical symmetry possessed by the model Hamiltonian.

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In the next section we treat the bosonic compact case of one canonical mode ($n=1$), which produces an SU(2) dynamical algebra.* When $n=2$ the bosonic compact case leads to an SU(3) dynamical algebra; this is considered in detail in the following section as an example of this class of model. Concluding remarks and generalisations are drawn in Section 4.

2. An SU(2) Example

As a simple illustration of the construction, consider the $n=1$ case consisting of one bosonic and one fermionic operator:

$$Q = Ea, \qquad Q^\dagger = Fa^\dagger,$$

where

$$\{a,a^\dagger\} = 1, \quad \{a,a\} = \{a^\dagger,a^\dagger\} = 0,$$

and F is the hermitian conjugate of E. The requirement $Q^2 = Q^{\dagger 2} = 0$ is now trivially satisfied and H is fixed as

$$H = EF - [E,F]N_F,$$

where $N_F = a^\dagger a$ is the number operator for fermionic modes.

If we now consider E and F as generators of a Lie algebra, then possible supersymmetric H are enumerated by giving the representation of the Lie algebra and the matrix elements of the various operators therein. The simplest possibility is that E and F generate the angular momentum SU(2):

$$[E,F] = 2J_0, \qquad [J_0,E] = E, \qquad [J_0,F] = -F.$$

Introducing now the orthonormal basis states $|j\ m\rangle$ with $m = -j, -j+1, \ldots, j-1, j$; $j = 0, 1/2, 1, \ldots$ and the well-known matrix elements

$$J_-|j\ m\rangle = \sqrt{(j+m)(j-m+1)}\ |j\ m-1\rangle,$$

we obtain the spectrum of bosonic ($n_F = 0$) and fermionic ($n_F = 1$) sectors,

$$E^{(0)}_{jm} = (j+m)(j-m+1), \qquad E^{(1)}_{jm} = (j-m)(j+m+1).$$

In each sector this corresponds to a singlet $E = 0$ state ($m = -j$ and $m = +j$, respectively) and a tower of doubly degenerate $E > 0$ states ranging up to $E_{max} = j(j+1)$ for $m = 0, 1$ and $m = 0, -1$, respectively.

3. Case Study of $n = 2$

Turning now to the $n = 2$ bosonic case we consider the Lie algebra generated by two commuting operators E^1, E^2 and their hermitian conjugates F_1, F_2. For example if $[F_1, E^2] \propto E^1$, then the Lie algebra is SU(3) or a non-compact real form thereof. In principle any rank 2 semi-simple Lie algebra is admissible, which then becomes the dynamical or spectrum-generating symmetry for the

* The non-compact SU(1,1) case is mentioned in the Conclusions.

supercharges

$$Q = E^1 a_1 + E^2 a_2, \qquad Q^\dagger = F_1 a^1 + F_2 a^2$$

and their associated supersymmetric Hamiltonian; a_1, a_2 and their conjugates a^1, a^2 stand for fermionic creation and annihilation operators.

In the general bosonic case we would have

$$Q = \sum_\alpha E_\alpha a_\alpha, \qquad Q^\dagger = \sum_\alpha E_{-\alpha} a_\alpha^\dagger,$$

where the E_α are a set of mutually commuting positive root vectors of the dynamical algebra $\mathcal{L}$ and $a_\alpha, a_\alpha^\dagger$ are the corresponding set of canonical fermionic operators. The representations of $\mathcal{L}$ together with the fermionic Fock space provide the physical Hilbert space of the model; H is in the enveloping algebra of $\mathcal{L}$. In addition, there normally is an identifiable kinematical symmetry $\mathcal{K} \subset \mathcal{L}$ which commutes with H and accounts for some of the degeneracies of the spectrum, apart from the usual supersymmetric doubling of states (Haymaker and Rau 1986). For models generalising $Q, Q^\dagger$ above, $\mathcal{L}$ is SU(n) and $\mathcal{K}$ is SU($n-1$); other choices of H lead to different $\mathcal{K}$'s.

It is our contention that, in contrast to the highly constrained nature of the supersymmetric systems alluded to above, the present type of model may well be realised in simple physical situations, given the simple *ansatz* taken and the minimal structural assumptions about the bosonic sector. There is thus the hope that models of this type may have a utility akin, say, to the use of group theory for two-level systems, namely SU(2) realised via the Pauli matrices (Dirac 1936). To this extent, the proposal is intermediate between the highly constrained models and models based on explicit supergroup chains, as applied to nuclear spectra (for a review of other applications see Kostelecky and Campbell 1985; Balantekin 1985; Sukumar 1985). We now turn to a detailed examination of the SU(3) case, for both matrix and oscillator realisations.

Fixed Irreducible Representation of SU(3)

For ease of notation we introduce the entire set of SU(3) generators in matrix form E^i_j, $1 \le i,j \le 3$,

$$[E^i_j, E^k_\ell] = \delta^k_j E^i_\ell - \delta^i_\ell E^k_j, \tag{1}$$

where $\sum_{i=1}^3 E^i_i = 0$ and $E^j_i = (E^i_j)^\dagger$ in unitary representations. We may choose E^1 and E^2 as any suitable pair of mutually commuting generators; for example,

$$E^1 \equiv E^1_2, \quad E^2 \equiv E^1_3, \quad \text{and put} \quad a \equiv a_1, \quad b \equiv a_2$$

so

$$Q \equiv aE^1 + bE^2 \qquad \text{and} \qquad H \equiv \{Q, Q^\dagger\}. \tag{2}$$

Upon using the SU(3) algebra (1), we get

$$H = E^2_1 E^1_2 + E^3_1 E^1_3 + (1 - N_a)(E^1_1 - E^2_2) + (1 - N_b)(E^1_1 - E^3_3) + a^\dagger b E^2_3 + b^\dagger a E^3_2, \tag{3}$$

where N_a and N_b are number operators for fermionic modes of type a and b, respectively.

In order to describe states in representations of SU(3) so as to choose a symmetry-adapted basis for H, we adopt the Gel'fand labelling scheme (Biedenharn and Louck 1981)

$$\left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle$$

where the rows are ordered by lexicality and successive rows are subject to 'betweenness' conditions,

$$p \geq p' \geq 0, \quad q \geq q' \geq 0, \quad r \geq 0,$$

$$p \geq q \geq p', \quad p' \geq q' \geq 0, \quad q \geq r \geq q',$$

for the integer labels p,p',q,q' and r; the pair $\{p,p'\}$ labels the highest weight of the representation corresponding to the partition of two parts of length p and p'. A generic state then has weight given by

$$E_1^1\left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle = [r-(p+p')/3]\left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle,$$

$$E_2^2\left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle = [q+q'-r-(p+p')/3]\left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle, \tag{4}$$

$$E_3^3\left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle = [2(p+p')/3-q-q']\left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle.$$

Matrix elements of other E_j^i are then prescribed; for instance (Biedenharn and Louck 1981)

$$\left\langle\begin{matrix} p & p' & 0 \\ & q+1 \quad q' & \\ & r+1 & \end{matrix}\right| E_3^1 \left|\begin{matrix} p & p' & 0 \\ & q \quad q' & \\ & r & \end{matrix}\right\rangle = \sqrt{\frac{(p-q)(q-p'+1)(q+2)(r-q'+1)}{(q-q'+1)(q-q'+2)}}. \tag{5}$$

From (2) and (3) the Hamiltonian H acts on the tensor product $\mathcal{F} \otimes \mathcal{V}_{\{p,p'\}}$ of the 4 dimensional fermionic Fock space $\mathcal{F}$ with the bosonic carrier space $\mathcal{V}$ of the SU(3) representation with highest weight $\{p,p'\}$. Diagonalisation can be carried out explicitly using (5) and similar matrix elements; however the analysis is simplified by identifying which operators provide the kinematical or degeneracy symmetry of the problem by commuting with H.

From the form of H it is clear that E_1^1 and $N_F = N_a + N_b$ are two such operators. Moreover, Q itself can be regarded as a scalar under transformations of the 2,3 indices if there are also contragedient transformations of a,b. To this end, we introduce the V-spin generators (Lichtenberg 1978)

$$V_+ = E_3^2, \qquad V_- = E_2^3, \qquad V_3 = (E_2^2 - E_3^3)/2 \tag{6}$$

and their fermionic counterparts*

$$v_+ = -b^\dagger a, \qquad v_- = -a^\dagger b, \qquad v_3 = (b^\dagger b - a^\dagger a)/2, \tag{7}$$

form the total V-spin operators

$$\mathcal{V}_\pm = V_\pm + v_\pm, \qquad \mathcal{V}_3 = V_3 + v_3 \tag{8}$$

and confirm that

$$[\vec{\mathcal{V}}, Q] = [\vec{\mathcal{V}}, Q^\dagger] = 0, \tag{9}$$

where $\mathcal{V}_\pm \equiv \mathcal{V}_1 \pm i\mathcal{V}_2$ as usual.

In order to find the energy spectrum we note that E_1^1 is diagonal in the Gel'fand–Tseytlin basis, and that states with $n_F = 0$ or $n_F = 2$ have $v = 0$ and hence $\mathcal{V} = V$, whereas states with $n_F = 1$ have $v = \frac{1}{2}$ and need to be combined with states having $V = \mathcal{V} \pm \frac{1}{2}$. However these $v = \frac{1}{2}$ states are accessible from the $n_F = 0, 2$ states, with which they are degenerate since $[Q,H] = 0 = [Q^\dagger, H]$, by the action of Q or $Q^\dagger$; therefore it is sufficient to examine the spectrum of H in the former cases.

For definiteness consider the case $p' \equiv 0$. Then the representation $\{p, 0\}$ contains V-spin multiplets $0, \frac{1}{2}, 1, \ldots, p/2$ and the states

$$\left| \begin{matrix} p & 0 & 0 \\ & q \quad 0 & \\ & r & \end{matrix} \right\rangle$$

are diagonal in V and V_3 with eigenvalues

$$V = (p - r)/2, \qquad V_3 = q - (p + r)/2. \tag{10}$$

In this case E_1^1 and V give equivalent information since from (4) and (9), $E_1^1 = 2p/3 - 2V$. Thus we may label the states as

$$|n_F; n_a, n_b\rangle \otimes |p; V, V_3\rangle$$

with the understanding that v-spin is given by (7) and is to be coupled with V to total $\mathcal{V}$ for the energy eigenstates.

The resulting character of energy levels is shown in Table 1. States of pure V-spin with $v = 0$ are designated I_0 or II_2 depending on n_F (which is

Table 1. Spectrum of energy levels in the SU(3) example

$\mathcal{N} = 0$		$\mathcal{N} = 1$	E
I_0	$\rightarrow$	$(Q^\dagger I)_1$	$2V(p - 2V + 1)$
$(QII)_1$	$\leftarrow$	II_2	$(2V + 2)(p - 2V)$
	Φ		0

* This assignment makes a, b transform contragrediently to E_2^1, E_3^1 under the total V-spin generators, so that the supercharges become scalars.

given as a subscript); these are degenerate with mixed $v=\frac{1}{2}$, $V=\mathcal{V}\pm\frac{1}{2}$ states designated $(Q^{\dagger}I)_1$ and $(QII)_2$. The last column contains the energy eigenvalue; the corresponding eigenvectors are

$$Q^{\dagger}|0,0\rangle\otimes|V,V_3\rangle=\sqrt{(V+V_3+1)(p-2V)}\;|1,0\rangle\otimes|V+\tfrac{1}{2},V_3+\tfrac{1}{2}\rangle$$

$$+\sqrt{(V-V_3+1)(p-2V)}\;|0,1\rangle\otimes|V+\tfrac{1}{2},V_3-\tfrac{1}{2}\rangle \qquad (11)$$

for type *I*, and

$$Q|1,1\rangle\otimes|V,V_3\rangle=\sqrt{(V+V_3)(p-2V+1)}\;|0,1\rangle\otimes|V-\tfrac{1}{2},V_3-\tfrac{1}{2}\rangle$$

$$+\sqrt{(V-V_3)(p-2V+1)}\;|1,0\rangle\otimes|V-\tfrac{1}{2},V_3+\tfrac{1}{2}\rangle \qquad (12)$$

for type *II*. Finally, the columns labelled $\mathcal{N}=0,\mathcal{N}=1$ refer to the eigenvalue of the operator $\mathcal{N}=[Q^{\dagger},Q]/H$ which plays the role of the fermion number for states in the positive energy spectrum (Stedman 1985) in applications where spin is not explicitly defined or where the spin-statistics connection is not physically relevant.*

The final category of states Φ has $E=0$ and indefinite $\mathcal{N}$ and includes states annihilated by both Q and $Q^{\dagger}$. From the form (2) of the supercharges, these include both the highest weight state of the SU(3) representation with $n_F=2, V=V_3=0$,

$$|2;1,1\rangle\otimes\left|\begin{matrix}p & 0 & 0\\ & p\;\;0 & \\ & p & \end{matrix}\right\rangle$$

and the lowest weight state of the SU(3) representation with $n_F=0, V=V_3=p/2$,

$$|0;0,0\rangle\otimes\left|\begin{matrix}p & 0 & 0\\ & 0\;\;0 & \\ & 0 & \end{matrix}\right\rangle$$

together with the V-spin multiplet in which it lies.§ Finally there are the mixed states of $\mathcal{V}=(p+1)/2$ which obviously cannot be degenerate with pure $V=(p+1)/2$ states; for example, it is readily checked from (5) that the maximal $\mathcal{V}_3$ state

$$|0,1\rangle\otimes\left|\begin{matrix}p & 0 & 0\\ & p\;\;0 & \\ & 0 & \end{matrix}\right\rangle$$

is indeed annihilated by both Q and $Q^{\dagger}$.

* Note that in this basis 'fermion number' $\mathcal{N}=0,1$ is not directly related to the evenness or oddness of N_F.

§ The V-spin shifting nature of Q and $Q^{\dagger}$ in (10) and (11) is due to the commutators $[E_1^1,Q]=Q$ and $[E_1^1,Q^{\dagger}]=-Q^{\dagger}$, plus the fact that for the present choice of representation $p'\equiv 0$, E_1^1 and V are equivalent.

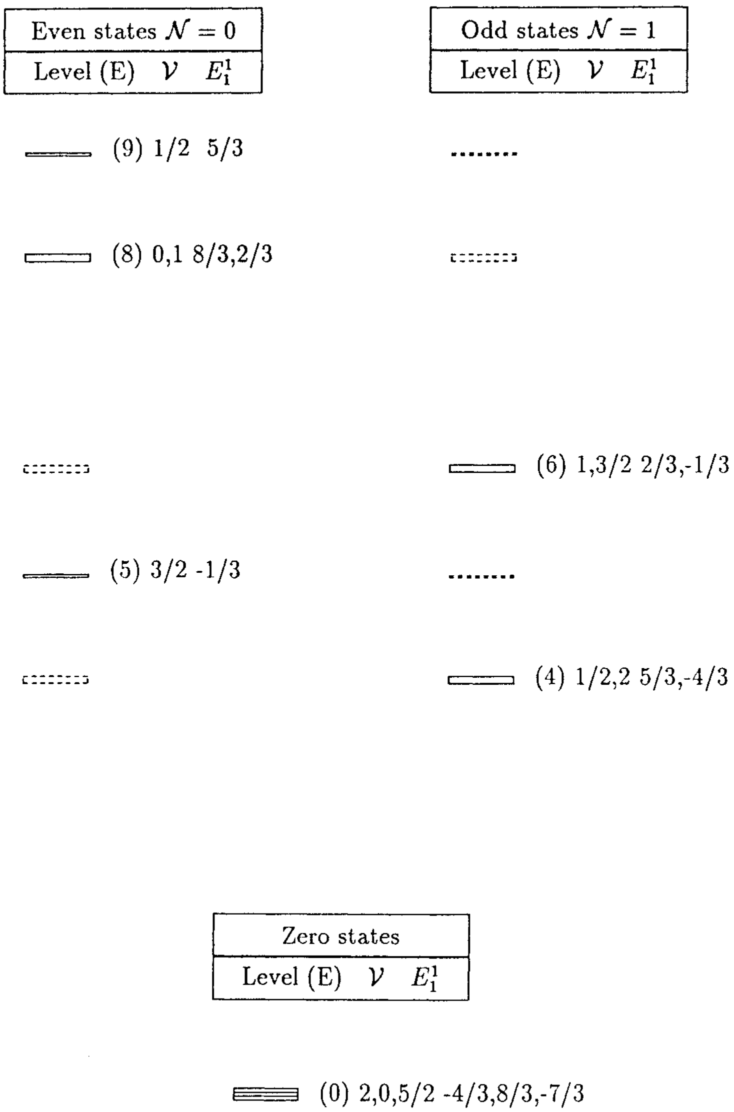

Fig. 1. Energy spectrum in the $n = 2$ case. For assignment of types $I_0, II_2, (Q^\dagger I)_1$ and $(QII)_1$ states, as well as Φ states, see Table 1.

This ends the analysis of the model for a fixed irreducible representation of the bosonic Lie algebra. The spectrum derived from Table 1 is given in Fig. 1 for the $p = 4$ case ($\underline{15}$-dimensional representation) for illustrative purposes.

Oscillator Realisations

We now turn to an investigation of the model for oscillator realisations in the $n = 2$ case. While realised in (possibly infinite-dimensional) Fock spaces, these can nevertheless be analysed in terms of the individual representations of the underlying dynamical algebras into which they decompose, as done previously. However it is of more interest to consider the form of the model Hamiltonian from the oscillator viewpoint as it is likely to be useful for applications in such a second-quantised formulation and can even be related to canonical phase space coordinates; moreover there exists a body of literature (Rittenberg and Yankielowicz 1985) on supersymmetric quantum mechanics with which the present *ansatz* may be compared.

Firstly, for the SU(3) case, we retain the fermions a, b but replace the operators E^i_j by the corresponding expressions in terms of bosonic creation/annihilation operators,

$$[r_i, r^j] = \delta^j_i, \quad r^i \equiv r_i^\dagger, \qquad E^i_j = r^i r_j - \delta^i_j N_B, \quad N_B = \sum_{i=1}^{3} r^i r_i. \tag{13}$$

For simplicity write $r_1 \equiv r$, $r_2 \equiv s$, $r_3 \equiv t$ and note that the combinations

$$\begin{aligned} 2T_1 &= E^1_2 + E^2_1, & 2T_2 &= i(E^2_1 - E^1_2), \\ 2T_4 &= E^1_3 + E^3_1, & 2T_5 &= i(E^3_1 - E^1_3), \\ 2T_6 &= E^3_2 + E^2_3, & 2T_7 &= i(E^3_2 - E^2_3), \\ 2T_3 &= E^1_1 - E^2_2, & 2T_8 &= -\sqrt{3}E^3_3, \end{aligned} \tag{14}$$

obey the standard SU(3) commutation rules

$$[T_a, T_b] = i f_{abc} T_c$$

with totally antisymmetric structure constants (Lichtenberg 1978).

As before, v-spin is generated by the fermion bilinears with Casimir operator

$$\vec{v}^2 = 3N_F(2 - N_F)/4, \tag{15}$$

while V-spin is generated in analogous way by s and t bilinears [or by T_6, T_7 and $(T_3 + T_8/\sqrt{3})/2$ if one prefers] with Casimir invariant

$$\vec{V}^2 = (N_B - r^\dagger r)(N_B - r^\dagger r + 2)/4 \tag{16}$$

from which we identify

$$p = n_B, \qquad q = n_r + n_s = n_B - n_t \qquad \text{and} \qquad r = n_r,$$

in comparison with the Gel'fand–Tseytlin labelling scheme for these totally symmetric representations; $V = (n_B - n_r)/2$ from (15).

The Hilbert space is just the direct sum of irreducible representations of the type $\{p, 0\}$ discussed previously, with all values $p = 0, 1, 2, \ldots$, and supercharges

$$Q = a r^\dagger s + b r^\dagger t, \qquad Q^\dagger = a^\dagger s^\dagger r + b^\dagger t^\dagger r, \tag{17}$$

leading to the Hamiltonian

$$H = N_r(2 + N_B - N_r - N_F) + N_a N_s + N_b N_t + v_+ t^\dagger s + v_- s^\dagger t. \tag{18}$$

This H is at most quartic in the creation and annihilation operators, although the supercharges were already trilinear. This shows in a simplistic toy model that the *ansatz* of de Crombrugghe and Rittenberg (1983) is certainly not general enough, because supercharges linear in both the fermionic and the bosonic creation and annihilation operators apparently define only an interesting subclass of quartic SUSY Hamiltonians. Another hint on necessary extensions of this subclass has been given by Baake *et al.* (1985).

As mentioned in the Introduction, there is a second natural class of model supersymmetric Hamiltonians with the same general features as those discussed so far; namely the case where canonical bosons are coupled in Q and $Q^\dagger$ to fermionic generators in a Lie superalgebra. In the $n = 2$ case one would have, for instance,

$$Q = r.\alpha^\dagger b + s.\alpha^\dagger c, \qquad Q^\dagger = r^\dagger.b^\dagger \alpha + s^\dagger.c^\dagger \alpha, \tag{19}$$

where the bosonic r,s operators are coupled to the odd generators of SU(2/1) formed from bilinears in the boson α and the fermions b,c. However, as can be seen by comparison with (17), this $n = 2$ case has the same structure as in the previous oscillator representation, given that the α,r,s boson bilinears generate SU(3) while the fermion b,c bilinears generate SU(2). In simple $n > 2$ generalisations of (2) the same situation prevails, although for higher n different possible types of realisation of the (super)algebra and different choices of (anti)commuting generators coupled in Q and $Q^\dagger$ require each case to be examined separately.

4. Conclusions

An algebraic perspective of supersymmetric quantum mechanics has been taken in which the structure of the supercharges has been analysed in terms of couplings between bosonic and fermionic modes. The $n = 1$ and $n = 2$ cases have been treated for the compact SU(2) and SU(3) algebras for fixed irreducible representations as well as in more general Fock spaces arising from oscillator descriptions.

The *ansatz* adopted in the present work is intermediate between the highly constrained supersymmetric potential models (Rittenberg and Yankielowicz 1985; Sukumar 1985) and models of dynamical superalgebras (Kostelecky and Campbell 1985). From the algebraic viewpoint it can be observed that the embeddings of supersymmetric quantum mechanics are in fact ubiquitous in appropriate superalgebras,* and it is contended that models with this richness may be rather interesting.

* A generalisation of the present *ansatz* is the coupling of arbitrary anticommuting generators of a Lie superalgebra to commuting generators of a Lie algebra. The simplest possibility is $Q = E_\alpha$ where α is an odd root and 2α is not a root, but in this case the boson/fermion coupling is not explicit. See Balantekin (1985) for such instances as OSp(1/2) and OSp(2/2).

One case of physical interest is the so-called Jaynes–Cummings model (Chaichian *et al.* 1990). Up to a factor, the Hamiltonian here is the operator

$$S = b^{\dagger} J_{-} + b J_{+},$$

which can be identified in the obvious way with the sum $S = Q + Q^{\dagger}$ of the two supercharges, each of the form (2) and leading to the non-compact dynamical algebra SU(1,1). Since $H = \{Q, Q^{\dagger}\} = S^2$, the analysis of H carries over to $S = \sqrt{H}$.

Finally we may point out that the present model bears some relation to the BRST construction (Van Holten 1990). There, however, one couples fermionic ghost fields with *every* root vector of the bosonic Lie algebra; Q contains additional trilinear ghost terms such that the Hamiltonian is proportional to the Casimir invariant and hence remains constant on any fixed irreducible representation.

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