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Models of rotating magnetic stars

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Summary. Steady-state models are constructed of axially symmetric, uniformly rotating stars with poloidal magnetic fields and with self-consistent, thermally-driven circulation fields. Solutions are found for two types of magnetic field, having either basically quadrupolar or dipolar structure. In each case there must be flow across the field near the surface, and in the dipolar case, also near the equator deep in the star, so that the inclusion of a finite resistivity is essential. For both field parities solutions are found, over a limited range of parameters, in which deep in the star the solution is essentially the centrifugally-driven Eddington–Sweet circulation, with the magnetic field merely acting to keep the rotation uniform but otherwise dynamically unimportant. In the surface regions the magnetic field dominates and the velocities are very small. Other solutions are found which are modifications due to finite resistivity of zero circulation models. In all cases the predicted tendency of the circulation to concentrate the field in the deep interior of the star is confirmed. The results offer a possible explanation of why the rapidly rotating A stars usually do not possess *observable* magnetic fields; and also of recent observations showing an anticorrelation between the angular velocity and the effective field within the class of observably magnetic stars. The results are relevant to the controversy over whether the fields of the magnetic A stars are fossils, or maintained by contemporary dynamos.

1 Introduction

This paper is a continuation of a series of studies (Mestel 1965b, 1975; Moss 1973, 1974, 1975; Wright 1969; see also Davies 1968; Monaghan & Robson 1971; Monaghan 1973) which explores different aspects of the theory of rotating magnetic stars. From the start it must be emphasized that the discussion is not restricted to the class of stars that are *observably* magnetic, i.e. to those with an effective field of more than a few hundred Gauss. It is argued that many more stars may contain substantial internal flux, and in fact one of the principal aims of the series is to elucidate the relation between the total flux F_t and that fraction F_s which manages to emerge above the surface. We are also interested in whether the field of a strongly magnetic star is a slowly decaying 'fossil' (Cowling 1945), or whether it is being maintained by (contemporary) dynamo action. In Section 5 we discuss how the present work can shed light on this very important question.

An outstanding feature of the strongly magnetic stars is the periodic variation of the effective magnetic field, often with a change in sign, and in the same period as the variations in spectral features and luminosity. Plausible phenomenological models that account for this involve rotation of a quasi-rigid magnetic structure which is essentially non-symmetric about the rotation axis. The oblique rotator in its simplest form assumes a surface distribution of flux symmetric about an axis which is inclined to the rotation axis (e.g. Preston 1971). Alternatively, one can construct a model of a magnetic field with an axis more-or-less coincident with the rotation axis, but with an azimuthal-dependent structure. It is known (Spitzer 1958; Mestel & Takhar 1972; Mestel & Wright, in preparation) that there must be significant, *dynamically-driven* motions inside any magnetic star which lacks kinetic symmetry about the angular momentum vector. These include oblique rotators and all but the most symmetric of the aligned rotators. The motions are oscillatory, with a frequency given roughly by the angular velocity of the star, reduced by the ratio of total magnetic to gravitational energy. For even quite modest magnetic fluxes, this corresponds to a period much shorter than the Kelvin–Helmholtz time. These motions are inevitably subject to dissipation which, if efficient enough, will cause secular changes in the observable field over a stellar lifetime. Thus in an oblique rotator the angle between the angular momentum vector and the magnetic axis will tend to change, increasing or decreasing according as the star is dynamically prolate or oblate about the magnetic axis. This process could be important for our understanding of the observations as interpreted by the oblique rotator model (Preston 1971). Alternatively, the process of magnetic braking which we presume to be the cause of the abnormally slow rotations of the bulk of the magnetic stars (Havnes & Conti 1971; Mestel 1975) may also affect the obliquity (Mestel 1968; Mestel & Selley 1970). However, no such superficial effects could be observed if field-lines did not emerge from the surface. In this paper we study another secular effect – the consequences of *thermally-driven*, persistent circulation (analogous to the well-known Eddington–Sweet flow) on the magnetic field. The arguments for this phenomenon are as compelling for a non-axisymmetric as for a symmetric star; but as the analysis of both the time-dependent problem and the quasi-steady asymptotic state (studied in this paper) is much simpler than in non-axisymmetric cases (*cf.* Monaghan 1973), we shall restrict our studies to axial symmetry.

A basic condition that must clearly be satisfied by a magnetic star model is that of dynamical stability: there should be no spontaneous tendency for the field to convert its energy into kinetic energy of mass motion (with the possibility of its ultimate dissipation). Studies to date (Wright 1973; Markey & Tayler 1973, 1974) suggest strongly that stable fields must be topologically fairly complex, with toroidal flux linking the poloidal loops. (Dynamo-built fields automatically have this type of structure.) It is always much more difficult to prove stability against any possible set of displacements than to demonstrate instability by judicious selection of one such set. We shall provisionally assume that a non-trivial class of dynamically stable fields exists, at least in stellar zones that are themselves sub-adiabatic and therefore stable against spontaneous convective overturning. We also accept that in superadiabatic zones the turbulent convection will tend to expel even moderately strong fields (Spitzer 1957; Weiss 1966; Gough & Tayler 1966; Moss & Tayler 1969; Mestel 1970). Thus in a late-type star, with a violent sub-photospheric zone, any primeval field will be largely concentrated in the central regions and so would be unobservable (e.g. Moss 1975). By contrast in an early-type star the field would be expelled from the convective core, but should emerge from the surface (unless other effects intervene). In either case, provided the field in the radiative zone is dynamically stable, we may appeal to Cowling's classical discussion (1945) of decay of a large-scale field in a *stationary* medium. The very long timescales he finds are the basis of the fossil theory of stellar magnetism – applicable whether the field is a relic of the local galactic field, or was built up by dynamo action during an earlier phase of the star's life.

Several recent papers (Davies 1968; Wright 1969; Moss 1973, 1974; Monaghan & Robson 1971; Monaghan 1973) have studied the joint effect of rotation and magnetism on heat transport in a radiative zone. It is well known that a stellar domain subject to arbitrary, non-spherical perturbing forces cannot be in strict radiative equilibrium. In general there will be an excess or defect of heat flowing into each element; the consequent buoyancy (positive or negative) leads to large-scale circulation (Eddington 1929; Sweet 1950; Mestel 1965a). The circulation speed is slow over the bulk of the star, having a timescale at least as long as the Kelvin–Helmholtz time, so that only for centrifugal or magnetic forces that are at least 1 per cent of the gravitational force will there be a consequent serious distortion of the internal magnetic field. However, in the low-density surface regions the circulation speeds can be faster by many orders of magnitude (Baker & Kippenhahn 1959), so that *prima facie* these thermal effects cannot be ignored in early-type stars. In particular, if such a circulation persisted unmodified, it would tend to drag field-lines with it, so reducing the observable flux.

It should be emphasized that such an effect is by no means observationally undesirable. One of the puzzles which must be resolved – and which may be crucial for the fossil-versus-dynamo controversy (*cf.* Section 5) – is the fact that whereas normal A stars are rapid rotators, the rotations of the magnetic stars (and indeed of all the Ap and Am stars) are usually significantly lower. This is inferred both from statistical studies of line-widths, and more directly, from the periodicity as interpreted by the oblique rotator model. This may very well by the consequence of the large-scale Eddington–Sweet circulation in a uniformly rotating star (Mestel 1965b, 1967). In a rapid rotator with a weak magnetic field, the centrifugal forces will dominate over the poloidal magnetic forces even in the low-density photospheric regions; the magnetic field plays the vital role of keeping the rotation uniform against the advection of angular momentum by the circulation (Mestel 1961, 1965a; Roxburgh 1963), while the circulation concentrates the magnetic flux deep down.

In the earlier papers on thermal effects, the problem was inverted. Recognizing that circulation will act so as to prevent a star from appearing magnetic, the authors imposed the condition that the circulation be killed. The magnetic field was required to have a structure and strength sufficient to ensure that the disturbance to local radiative equilibrium due to the centrifugal forces of uniform rotation was cancelled by the corresponding effect of the magnetic forces. In the simplest case of a dipolar field, Davies (1968) and Wright (1969) showed that if the total magnetic flux F_t in the star is fixed, then the *observable* flux F_s steadily decreases as the rotation rate is supposed to go up. Equivalently, a star of fixed rotation and with its magnetic field determined by one of the Wright–Davies models will show relatively less F_s as F_t decays through Ohmic dissipation. In fact, F_s vanishes (for given Ω) when F_t reaches a critical, finite value; and as F_t approaches this value, a small change in F_t leads to a disproportionately large decrease in F_s. Similar effects appear when the field has a quadrupolar structure (Moss 1973). It is of interest to note that observers seem to be finding an *anti*-correlation within the class of magnetic stars between $V \sin i$ and the effective magnetic field (Landstreet *et al.* 1975; Wolff & Wolff, private communication) – though this may be due partly to the direct effect of more efficient magnetic braking.

In this paper we follow the argument a step further, and look for steady solutions which involve non-zero meridional circulation, again determined by energy conservation. We consider separately magnetic fields of even and odd parity, i.e. fields with basically quadrupolar structure (Sections 2 and 3) and dipolar structure (Section 4). In some (though not all) of the cases studied, our choice of parameters F_t and Ω is such that no solutions can be found in strict radiative equilibrium, so that meridional circulation is an essential ingredient of the solution.

A steady self-consistent model must include the effect of the circulation on the field. If the perfect conductivity approximation were adequate through the whole zone, so that the circulation velocity is everywhere parallel to the magnetic field, then the conservation of mass and magnetic flux jointly yield (e.g. Mestel 1961)

$$\frac{\rho v}{B} = \eta = \text{constant on field–streamlines.} \quad (1)$$

If v is given more-or-less by the first-order Eddington–Sweet theory, then it does not vary through the star anything like as much as ρ; if (1) were valid all the way from a mean density of $\simeq 1$ to a photospheric density of $\simeq 10^{-7}$, the surface radial component $(B_s)_r$ would be smaller than the field B_i deep down by this factor 10^{-7}, with the horizontal component $(B_s)_\theta$ larger by the factor $|v_\theta/v_r| \simeq |r\rho'/\rho| \simeq 10^2$. *Prima facie* we are finding even stronger central condensation than that predicted by the Wright–Davies (zero-circulation) solutions, which yield typically $(B_s)_r/B_i \simeq 5 \times 10^{-4}$.

In fact, strict field-freezing cannot hold everywhere. This is immediately clear when the field has odd parity, for the essentially quadrupolar circulation must flow across the field near the equator. More significantly, it is found (*cf.* Appendix) that whatever the field parity, a steady state requires explicit account of the finite resistivity which allows flow across the field in the low-density outer regions. Thus the factor 10^{-7} just quoted certainly overstates the case; however, detailed study confirms a strong central condensation of the field.

In Section 2 we begin by writing down approximations to the kinematic hydromagnetic equations appropriate to a quadrupolar circulation flowing in the presence of an even parity magnetic field. We then suppose that the field is sufficiently weak for centrifugal forces to dominate all the way to the photosphere, so that the circulation is essentially that due to the uniform rotation: we use this prescribed circulation field to find the magnetic field structure in the surface regions. In Sections 3 and 4 we suppose the magnetic field to be stronger, so that at least in the outer regions the circulation has to be determined by the joint centrifugal and magnetic disturbance to the thermal field. The significance of the results is discussed in Section 5.

2 Formulation of the general problem: centrifugally-dominated models

We confine attention to fields which are purely poloidal. As already noted, such fields are certainly dynamically unstable (Wright 1973; Markey & Tayler 1973, 1974) so that the present work must be regarded as a preliminary investigation; the analogous problem for a star with a mixed poloidal–toroidal field is discussed in a following paper (Moss 1976).

A field with dipolar structure, i.e. with $B_r \propto P_1(\cos\theta)$, exerts a poloidal force density **f** similar in angular structure to the centrifugal force of uniform rotation: there is a spherically symmetric radial component, which leads to a dilation of the star as a whole, and extra radial and transverse components proportional respectively to $P_2(\cos\theta)$, $dP_2/d\theta$. The consequent circulation pattern, due jointly to centrifugal and magnetic forces, would be quadrupolar (Sweet 1950), implying a flow of gas across the magnetic field near the equator. Such a model would not be consistent with a steady state, if Ohmic effects are ignored. The flow would in fact steadily distort the field until the $\mathbf{j}/\sigma$ term is comparable with the $\mathbf{v} \times \mathbf{B}/c$ term, yielding ultimately a field that would deviate markedly from a simple dipolar structure (though retaining the same parity properties with respect to the equator). Construction of such a field seemed, *a priori*, to present a difficult task, so initially we avoid it by assuming the field to have essentially quadrupolar parity properties. We make no claim that this is realistic; it is done purely for mathematical convenience. We return to the problem of the interaction of the even parity circulation with a basically dipolar magnetic field in Section 4.

The full equations for the even parity problem are given in Moss (1974), and the reader is referred there for any details omitted here. The present notation is the same as that used in that paper. Thus we write **B** in spherical polar components

$$\mathbf{B} = \bar{H}\left[a_2 P_2 + a_4 P_4 + \ldots, \frac{(a_2 x^2)'}{6x}\frac{dP_2}{d\theta} + \frac{(a_4 x^2)'}{20x}\frac{dP_4}{d\theta} + \ldots, 0\right] \tag{2}$$

(x being the non-dimensional radial distance r/R), so that the condition $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied. The inclusion of terms in $a_4, a_6 \ldots$ is inevitable, as the forces exerted by a P_2 field include terms in P_4 as well as P_0 and P_2. The problem grows rapidly in complication as more and more Legendre terms are included, and we shall in fact truncate after the P_4 terms.

As noted in the Introduction, if the medium were perfectly conducting, then in a steady state the meridian circulation **v** would be related (*cf.* equation (1)) to **B** by $\rho \mathbf{v} = \eta \mathbf{B}$, where η is forced to be a constant on field-streamlines by virtue of matter and flux conservation. Deep in a rapidly rotating star, with v of the order of the Eddington–Sweet velocity U_{ES}, the magnetic Reynolds number is high enough for this to be a good approximation. However, near the surface of the star, the conductivity becomes lower and also length-scales decrease, so it is not obvious that neglect of resistivity will be valid all the way to the surface. As already remarked it is found (*cf.* Appendix) that when the field is so large that the poloidal magnetic forces cannot be neglected near the photosphere, flow across the field is essential in a steady state. We anticipate this by adopting the perfect conductivity result (1) only as far as a radius r_1; further out we replace it by

$$\mathbf{v} \times \mathbf{B} = \frac{c^2}{4\pi\sigma} \nabla \times \mathbf{B} \tag{3}$$

where $\sigma \propto T^{3/2}$ as usual. The dominant terms in the velocity are written

$$\mathbf{v} = \left(V_2(r) P_2, \quad \frac{1}{6r}\left(\frac{\rho V_2 r^2}{\rho}\right)\frac{dP_2}{d\theta}, \quad 0\right), \tag{4}$$

ensuring that $\nabla \cdot (\rho \mathbf{v}) = 0$.

We now non-dimensionalize by writing $V_2 = v_2 V$, where the velocity V is conveniently chosen as

$$V = \frac{LR^2}{GM^2}\lambda_\Omega, \quad \left(\lambda_\Omega \equiv \frac{\Omega^2 R^2}{GM}\right) \tag{5}$$

a typical order-of-magnitude for the Eddington–Sweet velocity deep in the star (Sweet 1950). Then equation (3) yields

$$\tilde{v}_2\,[(a_2x^2)' + (a_4x^2)']/(42x) - (\rho\tilde{v}_2x^2)'[a_2/7 - 4a_4/21]/(6\rho x) = \frac{c^2}{4\pi\sigma}\frac{1}{VR}\frac{D_2(a_2)}{6x}, \tag{6}$$

$$\tilde{v}_2\,[3(a_2x^2)' + 17(a_4x^2)'/22]/(70x) - (\rho\tilde{v}_2x^2)'[3a_2/10 + a_4/22]/(7\rho x) = \frac{c^2}{4\pi\sigma}\frac{1}{VR}\frac{D_4(a_4)}{20x} \tag{7}$$

where

$$\left.\begin{aligned} D_2(a_2) &= x^2 a_2'' + 4x a_2' - 4a_2, \\ D_4(a_4) &= x^2 a_4'' + 4x a_4' - 18a_4. \end{aligned}\right\} \tag{8}$$

We are explicitly looking for steady states, in which Ohmic dissipation $\mathbf{j}^2/\sigma$ per unit volume is off-set by the work $-\mathbf{j}\times\mathbf{B}\cdot\mathbf{v}/c$ done on the field by the motion of the gas – the ultimate energy source being the nuclear sources of the star which maintain the star in a thermally steady state. But it is well-known that axisymmetric magnetic fields (and fields topologically similar) cannot be maintained against Ohmic decay: there is an inevitable decay of the field due to diffusion of closed field-lines into the O-type neutral points that such fields must have (Cowling 1934). The very long timescale of this decay – provided the medium remains at rest – is also well-known: stable magnetic fields can persist for longer than the star's lifetime, and we are interested only in effects that manifest themselves in times much less than this. Nevertheless, if we look for solutions of the *steady-state* hydromagnetic equations in the neighbourhood of a neutral point, then strictly accurate equations would yield velocities emerging from the neutral point and becoming singular there. If the neutral point is within the radius r_1, then the assumption $\sigma=\infty$ disposes of the problem. However, as noted in Moss (1974), even if finite σ is retained, approximate steady-state equations can often yield solutions near the neutral point, in spite of Cowling's theorem – a consequence of the truncation of the Legendre expansion. As we are mainly interested in the effect of the circulation on the field near the surface, while the neutral points are deep down, we are content to take advantage of this analytical accident here and in the following sections.

We now introduce explicitly the extra assumption that although the perfect conductivity approximation (1) must be replaced by (3) for $1>x>x_1$, the centrifugal forces are so much greater than the magnetic forces all the way to the photosphere that the Eddington–Sweet velocity remains an adequate approximation for $\tilde{v}_2$. This approximation reduces the problem to one in kinematic magnetohydrodynamics. We are required to solve equations (6) and (7), with $\tilde{v}_2(x)$ known, subject to boundary conditions of continuity of both components of $\mathbf{B}$ at x_1 and at the photosphere $x=1$. Beyond the photosphere the density drops so fast that the field becomes approximately curl-free: thus the boundary conditions at $x=1$ are

$$\left.\begin{aligned} x a_2' + 4a_2 &= 0, \\ x a_4' + 6a_4 &= 0. \end{aligned}\right\} \tag{9}$$

We specialize further by taking η in equation (1) to be constant over the whole domain $x<x_1$ rather than just constant on individual field-lines. At $x=x_1$, a_2 and so a_2' are known from (1) in terms of the Eddington–Sweet velocity, once a definite value of η is chosen. No condition on a_4 and a_4' is imposed at $x=x_1$; on the contrary, it is the solution for $x_1<x<1$ which will imply a P_4 correction to the field within x_1. Since equations (6) and (7) are all linear in $\mathbf{B}$, η scales out, and the only free parameter is λ_Ω. This is because we are requiring only that $\mathbf{B}$ be strong enough to keep Ω nearly uniform – a very weak condition over the bulk of the star; we are (for the moment) explicitly ignoring the contribution to $\mathbf{v}$ due to the magnetic disturbance to the thermal field.

For $\lambda_\Omega \gtrsim 10^{-3}$ – corresponding to a period of about 4 days for a typical A star – the solution for a_2 is fairly insensitive to the choice of x_1; we use a standard value of $x_1=0.8$. Further, we find $|a_4(x_1)| \ll |a_2(x_1)|$, so that the assumption of near parallelism of $\mathbf{v}$ and $\mathbf{B}$ within x_1 is a good approximation. Both a_2 and a_4 at $x=1$ are fairly insensitive to the choice of x_1. However, it turns out that $|a_4(1)| \gg |a_2(1)|$, so casting serious doubt on the accuracy of truncating at P_4. Typical values are given in Table 1. (Here, as elsewhere in this paper, zero

Table 1.

λ_Ω	$a_2(0.80)$	$a_4(0.80)$	$a_2(0.99)$	$a_4(0.99)$
10^{-1}	0.098	$-1.36 10^{-4}$	1.8×10^{-5}	-4.5×10^{-5}
10^{-3}	0.098	$-1.44 10^{-2}$	5.7×10^{-4}	-4.8×10^{-3}
10^{-5}	0.098	-1.70	4.6×10^{-3}	-4.7×10^{-1}

order quantities are those appropriate to a Cowling model star with a simple Kramers opacity law $\kappa \propto \rho/T^{7/2}$.) The significant result is that $a_2(1)$ and $a_4(1)$ are much less than $a_2(x_1)$ unless λ_Ω is very small; although, because of finite resistivity, **v** and **B** cannot be taken as strictly parallel near the surface, the estimated flux that is able to emerge from the surface is still very small.

3 Models including magnetic forces

The models of the last section have essential limitations. For fields of interesting magnitude – 10^3 G or more – neglect of the poloidal magnetic forces is invalid before photospheric densities are reached. One cannot then use the Eddington–Sweet velocity field all the way to the surface, as the magnetic fields given by the previous work will also disturb the thermal field, along with the centrifugal forces. We must therefore use for these regions the hydrostatic equation (in standard notation)

$$-\nabla p - \rho\nabla\phi + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} + \rho\Omega^2\boldsymbol{\varpi} = 0, \tag{10}$$

where $\boldsymbol{\varpi}$ is the vectorial distance from the axis of symmetry; ϕ satisfies Poisson's equation

$$\nabla^2\phi = 4\pi G\rho, \tag{11}$$

and the usual equation of state $p = R\rho T/\mu$ relates T to p and ρ. If small perturbation analysis is adequate, p, ρ, ϕ and T will each consist of a spherically symmetric, zero-order part, and perturbations p_Ω, p_B etc., which contain both spherical and non-spherical parts. In a steady state the equation of heat conservation in a radiative envelope has the form

$$C_v\rho T\mathbf{v}\cdot\nabla[\ln(T/\rho^{\gamma-1})] = -\nabla\cdot\mathbf{F} \tag{12}$$

with

$$\mathbf{F} = -\frac{4ac}{3\kappa\rho}\,T^3\nabla T. \tag{13}$$

Thus $\nabla\cdot\mathbf{F}$ and so also the radial component v_r have non-spherical parts that depend on Ω^2 (the Eddington–Sweet term) and on $a_2, a_4, a_6 \ldots$ etc. In practice we truncate after the a_4 terms, and relation (4) is replaced by

$$V_r = V_2P_2 + V_4P_4,$$
$$V_\theta = \frac{1}{6r}\frac{(\rho r^2V_2)'}{\rho}\frac{dP_2}{d\theta} + \frac{1}{20r}\frac{(\rho r^2V_4)'}{\rho}\frac{dP_4}{d\theta}. \tag{14}$$

Note that the models of this section and Section 4 satisfy the truncated equations (10) and (11) everywhere, and (12) and (13) outside the convective core.

The final steady state condition to be satisfied is (3) (subject to our earlier remarks about O-type neutral points). The essential difference from Section 2 is that whereas there, **B** was determined from the kinematic constraints (6) and (7), with **v** independent of **B**, in the present problem **v** is a functional of **B** as well as Ω. It should be noted that even if the magnetic force per gram is locally less than the centrifugal acceleration, the effect on $\nabla \cdot \mathbf{F}$ and so on **v** can be greater in low-density regions. This is because uniform rotation is a special case, yielding a first-order contribution to $\nabla \cdot \mathbf{F}$ with a factor ρ, which cancels the same factor on the left of (12); whereas the magnetic forces (and non-uniform rotation) yield in general a velocity field $\propto 1/\rho$ (Baker & Kippenhahn 1959).

One of us (Moss 1974) has already constructed one class solutions of these equations – see that paper for details of the analysis. The magnetic fields found there are very similar to the (quadrupolar parity) zero circulation solutions (Moss 1973). The associated velocities are typically of order 10^{-7} cm s^{-1} over the bulk of the star – much below the typical Eddington–Sweet values if the rotation period is less than a few days. Near the surface the radial component falls off like $(1-x)^{2.75}$ (the particular value of the index coming from the choice of the simple Kramers opacity law). When combined with a conductivity appropriate to the bulk of the star, and with a length-scale $\simeq 2 \times 10^{10}$ cm, this velocity yields a magnetic Reynolds number of order unity. This indicates that in this first class of solutions the velocities are non-zero only by virtue of finite resistivity: they exist only so as to off-set the diffusion of the field. As already remarked, such solutions of the strictly steady equations can be found only because of the truncation procedure which enables one to bypass Cowling's theorem; in reality the total flux of the field must slowly decay. However, the field structure given by the zero circulation condition $\nabla \cdot \mathbf{F} = 0$ is quite distinct from any of the eigenmodes of Cowling's decay equation (1945). We can suppose the field at time $t = 0$ to be expanded as a series of these eigenmodes; then at later times, and in the absence of mass-motions, the modes with the shorter *e*-folding times will have decayed proportionately more than the largest-scale mode, resulting in a new field structure which will certainly not satisfy $\nabla \cdot \mathbf{F} = 0$. From Cowling's theorem we know that there do not exist motions which both satisfy mass conservation and keep the field strictly steady; but it is legitimate to ask if there exist motions – consistent with both mass and energy conservation – which keep the field almost steady over times short compared with the decay-time of the largest scale component. The velocity fields of Moss (1974) are approximately solutions of this form.

From the earlier work, we know that for a star of given rotation to be in radiative equilibrium, its total magnetic flux F_t must be above a minimum. For the reason we call the class of solutions in Moss (1974) the 'high flux' (HF) solutions. As noted above, they are essentially zero-circulation solutions modified by finite resistivity. They are found, however, for only a limited range of the two parameters λ_Ω and λ_H, where

$$\lambda_H = \frac{4\pi \bar{H}^2 R^4}{GM^2} \tag{15}$$

and $\bar{H}$ is the polar field-strength at the surface. Specifically, if $\lambda_\Omega \gtrsim 10^{-3}$ – fast rotators – these solutions exist for all values of λ_H; but if $\lambda_\Omega \lesssim 10^{-3}$, then such solutions were found only for $\lambda_H \gtrsim 3 \times 10^{-8}$. They have another limitation mentioned briefly in Moss (1974). Clearly the important parameter for the overall structure of a star is the total magnetic flux it contains, rather than the size of the surface field, although of course it is some part of the latter that we see. This suggests describing the models in terms of a parameter defined from the total flux rather than the surface flux. Thus, if as in Moss (1973) we define the total flux $F_t = \bar{H}R^2 f$, then

$$\lambda_H^* = \frac{4\pi F_t^2}{GM^2} = f^2 \lambda_H \tag{16}$$

is a measure of the total flux. If the HF models are plotted in a $(\lambda_\Omega, \lambda_H^*)$ diagram (see Fig. 1),

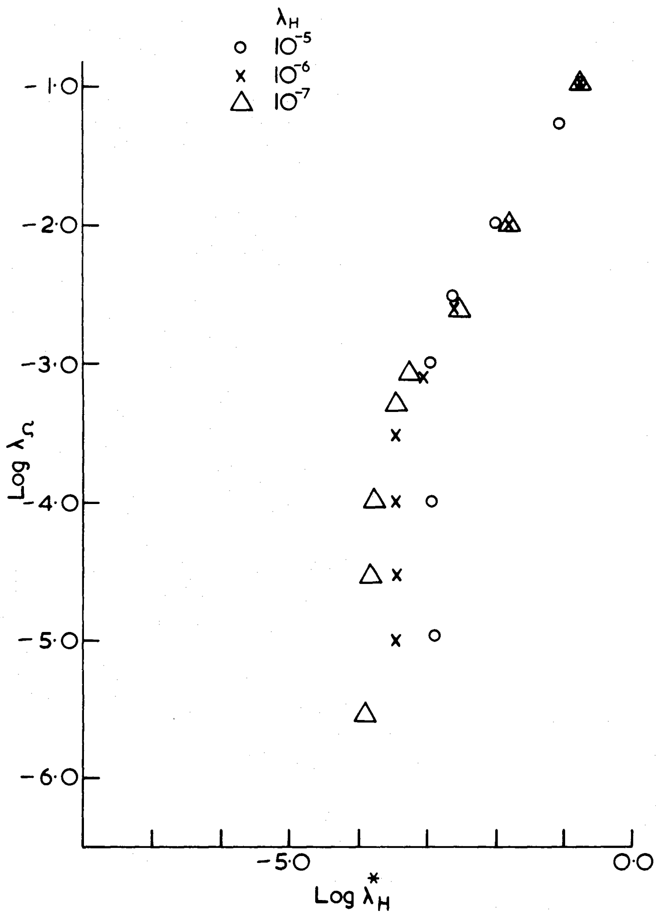

Figure 1. The position of the HF models in the λ_Ω, λ_H^* plane.

it can be seen that for $\lambda_\Omega \gtrsim 10^{-3}$, models can be found only for an almost uniquely defined value of the total flux at given λ_Ω, whatever the value of λ_H. For $\lambda_\Omega \lesssim 10\;^{3}$, λ_H^* does vary as a function of λ_H at fixed λ_Ω, but solutions can be found only for $\lambda_H \gtrsim 3 \times 10^{-8}$, and so there is a minimum allowed value of the total flux in these cases also. Thus the problem of what happens to stars which contain the 'wrong' flux for their angular velocity, which was encountered in the discussion of the zero-circulation models, is not overcome by the introduction of the circulation found in these HF models. In fact the minimum flux available at given λ_Ω is quite similar in both the zero circulation and HF models.

The solutions of Moss (1973, 1974) are clearly at the other extreme from those of Section 2; for so far from the magnetic field's being so weak that it merely responds passively to the 'inexorable' Eddington–Sweet circulation – determined by the rotation alone – the field is strong enough through the whole of the radiative envelope either to stop or greatly reduce the circulation. This suggests that there should be an intermediate class of solutions, with fields that are passive over the denser regions of the envelope, but which are hydrostatically and thermally significant over some domain below the photosphere. This class was indeed found, initially by following as a pseudo-time-dependent problem the evolution to a steady state of the stellar field from a configuration of hydrostatic but not thermal equilibrium. The most obvious property of these models is illustrated by Fig. 2, which shows a plot of λ_H^* against λ_H at fixed $\lambda_\Omega(=10^{-4})$. The models are seen to fall on to two sequences with a transition region around $\lambda_H = 4 \times 10^{-8}$. The ambiguity of solutions at $\lambda_H = 3 \times 10^{-8}$ may not be real. The models with $\lambda_H \lesssim 4 \times 10^{-8}$ are the low flux (LF) models, with λ_H^* at

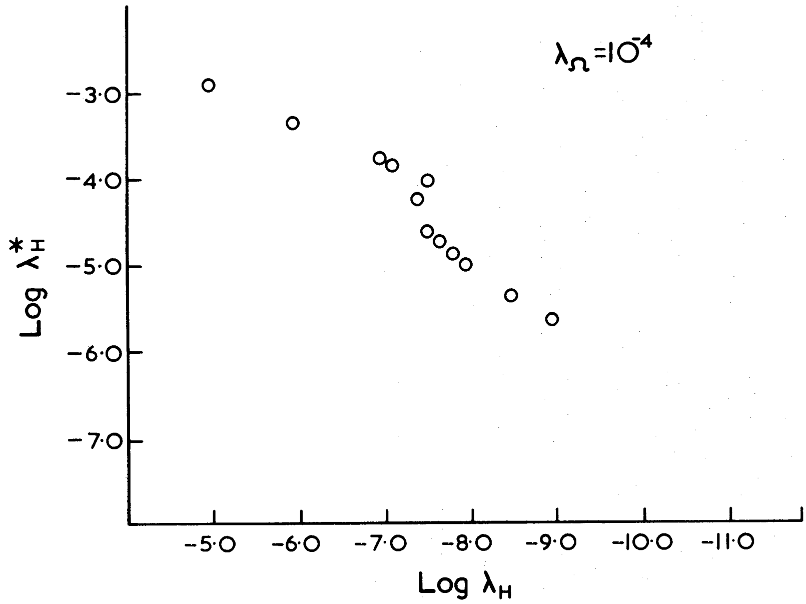

Figure 2. The variation of λ_H^* with λ_H at $\lambda_\Omega = 10^{-4}$ for models with even parity fields.

least an order of magnitude smaller at $\lambda_H = 10^{-9}$ than might have been expected from an extrapolation of the HF sequence. LF and HF models are not found in the same region of the $(\lambda_\Omega, \lambda_H)$ plane (with the exception of possible ambiguities at the transition region, as mentioned above). For $\lambda_\Omega \lesssim 10^{-3}$, λ_H^* is a monotonic function of λ_H, and models with internal fluxes as small as desired can be obtained by decreasing λ_H at fixed λ_Ω. However, models cannot be found falling to the left of the locus of Fig. 1 with $\lambda_\Omega \gtrsim 10^{-3}$, although HF models *are* found for all values of λ_H.

In LF solutions, with $\lambda_\Omega/\lambda_H \gtrsim 10^4$ through the bulk of the star the magnetic perturbation is much less than the centrifugal, and the circulation is basically the Eddington–Sweet circulation that would be found in the rotating star in the absence of a magnetic field. Near the surface the centrifugal perturbation (proportional to the zero-order density) decreases rapidly; eventually the magnetic terms dominate, and the solution for the velocity goes over to the type found in the HF solutions, with $v_r \propto (1-x)^{2.75}$. Fig. 3 shows how, at $\lambda_\Omega = 10^{-4}$, the circulation remains close to the Eddington–Sweet value out to larger and larger radii as λ_H (and λ_H^*) decreases. In these solutions the P_4 components of the velocity and magnetic fields are much smaller than the P_2 components, in contrast to some of the HF models, where they may be locally comparable. Typical magnetic field-lines are shown in Fig. 4(a). It should also be noted that it is only in the outer few per cent of the radius that the radial component of the velocity becomes much smaller than its value deep inside the star. However, in these low-density regions the value of $\nabla \times \mathbf{B}$ adjusts itself so that the total value of $\nabla \cdot \mathbf{F} \equiv (\nabla \cdot \mathbf{F})_\Omega + (\nabla \cdot \mathbf{F})_B$ falls off more rapidly than ρ, so yielding a net circulation speed that not only stays finite but vanishes as $\rho \to 0$. The horizontal component is larger by the factor $|r\rho'/\rho|$, behaving like $(1-x)^{1.75}$ and so also vanishing as $\rho \to 0$.

The viscous forces, which have been implicitly ignored through the whole discussion, would in fact begin to dominate at vanishingly small ρ and T, essentially because the mean free path becomes long. However, even with quite modest magnetic fields the viscous forces remain small out to photospheric densities, beyond which the whole theory must in any case be replaced by one involving non-local radiative transport (*cf.* Smith 1970).

In a perfectly conducting star condition (1) will hold with η a constant on field-lines. This can be expected to be a good approximation when the circulation is rapid enough for

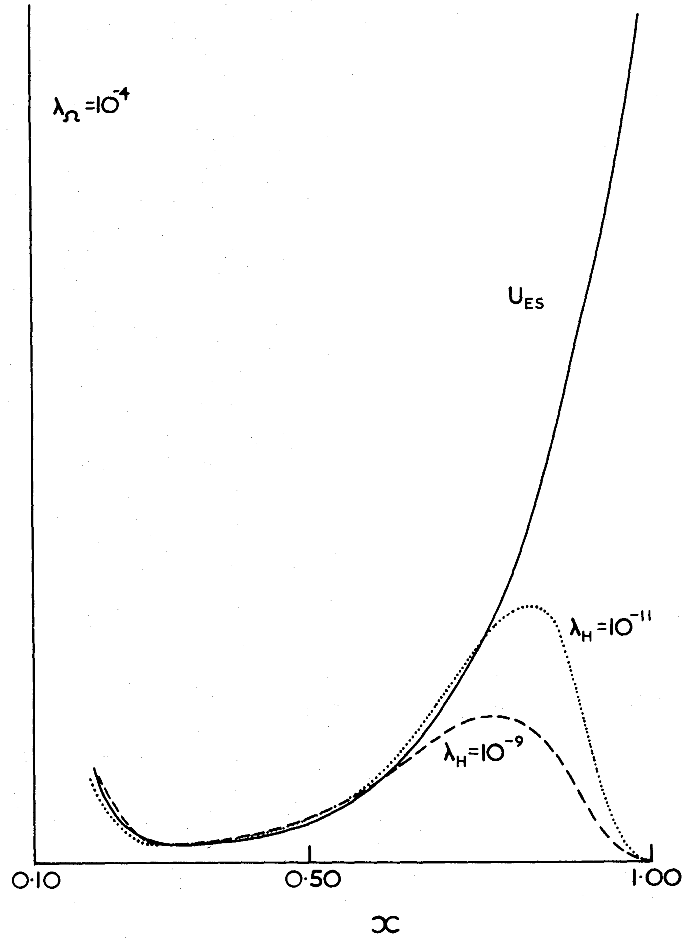

Figure 3. $\lambda_\Omega = 10^{-4}$, even parity fields. The curve labelled U_{ES} gives the radial component of the Eddington–Sweet velocity field along the axis of the star, the curves labelled with values of λ_H give the axial radial components of the computed circulations.

the magnetic Reynolds number to be large (which was not always the case in the HF solutions, where the velocity field often broke up into zones). The program does not impose any functional form for η, but in the simplest cases, where η is the *same* constant on all field-lines ψ = constant in a perfectly conducting star, condition (1) implies $\rho_0 V_2/a_2 = \eta$ = constant. The ratio $R_{\text{R}} = \rho_0 V_2/a_2$ is plotted against radius at $\lambda_\Omega = 9 \times 10^{-4}$, 10^{-4} and 10^{-6} in Fig. 5. It can be seen that R_{R} is quite accurately constant over much of the interior of the star at $\lambda_\Omega = 9 \times 10^{-4}$, but at $\lambda_\Omega = 10^{-4}$ and 10^{-6} the circulation (essentially Eddington–Sweet) is slower and so field freezing is less strictly adhered to. Again as in the HF case (Moss 1974), this suggests that the choice η = constant everywhere is realistic when discussing sufficiently rapid circulation in magnetic stars, although the possibility cannot be dismissed that other models with more complicated functional forms for $\eta(\psi)$ exist, whether or not they could be found by the methods used here. Field-freezing holds to greater radii in models with larger λ_Ω: the larger Eddington–Sweet velocity causes the point where $R_{\text{m}} = 1$, say, to move nearer the surface.

One reason for constructing models of magnetic stars is to infer the interior magnetic field from observations of the surface field. Thus our models predict the ratio of the interior field to the largest-scale component of the surface field: since we have adopted the normalization $a_2(1) = 1$, by equations (2) and (16) this is essentially measured by the quantity f. Clearly, from the above remarks LF models have smaller f values at the same value of λ_Ω

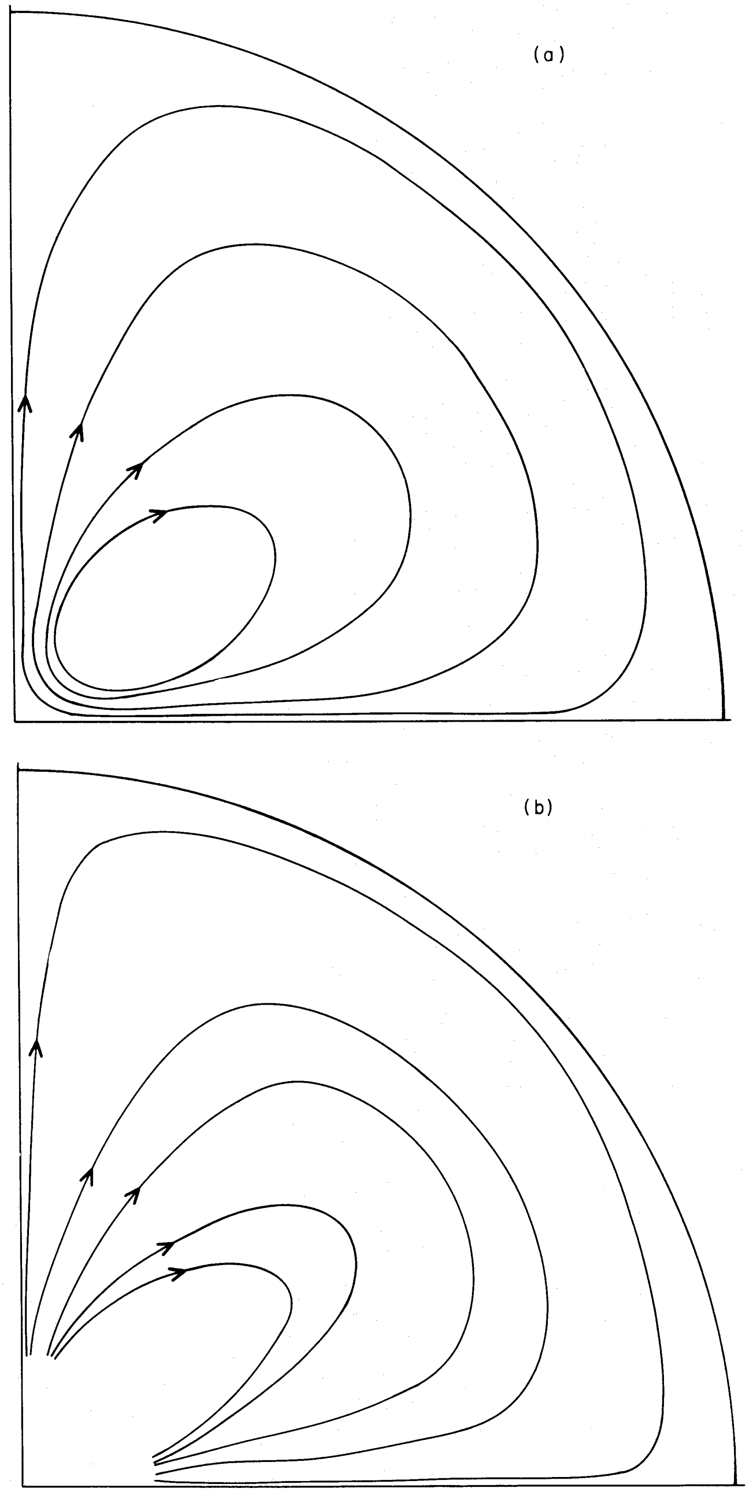

Figure 4. (a) P_2 magnetic field and (b) P_2 velocity field for even parity field, $\lambda_\Omega = 10^{-3}$, $\lambda_H = 10^{-12}$. No magnetic field lines are shown emerging from the surface in (a), but a very few do, mostly near the poles.

than the HF models (or the zero-circulation models). Equivalently, at the same value of λ_Ω/λ_H, but with λ_Ω varying, f is smaller by up to an order of magnitude in the LF models. This can be understood fairly simply. In the HF models the magnetic field is required to modify seriously the Eddington–Sweet circulation deep inside the star; thus at given λ_Ω, if the total flux decreases, then the remaining flux must become more concentrated inside the star, and

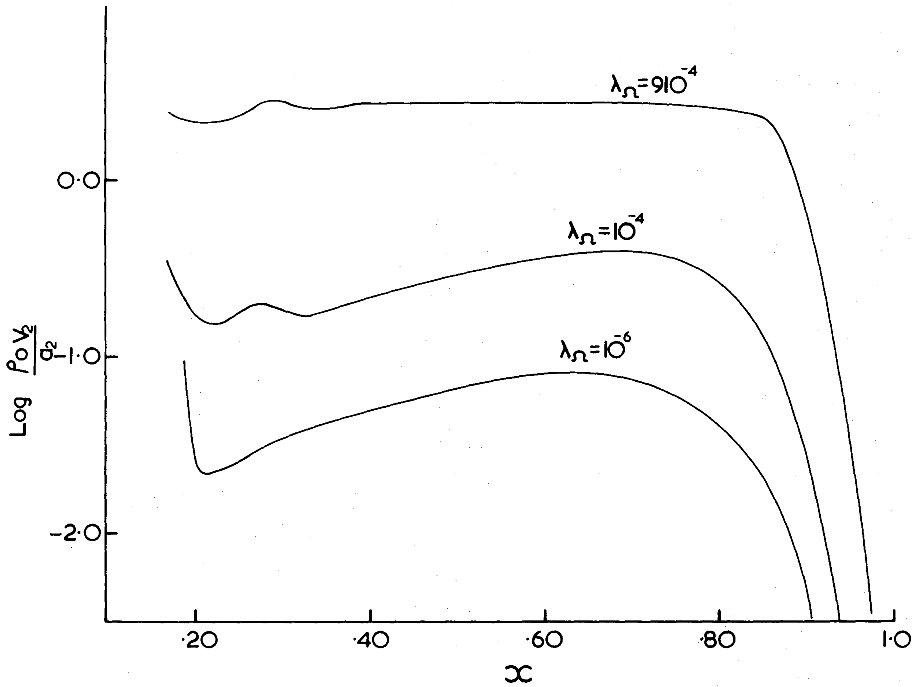

Figure 5. The ratio $\rho_0 V_2/a_2$ as a function of radius for three values of λ_Ω (even parity models).

so f increases. However, in the LF solutions the Eddington–Sweet circulation flows unimpeded deep in the star and so the magnetic field is only concentrated to the extent demanded by equation (1), more or less independent of the value of the total flux. However, equation (1) *does not* apply all the way to the surface – see, e.g. Fig. 5. Thus at given λ_Ω in a LF solution, f is determined essentially by where field freezing breaks down (as noted in the Introduction).

However, this does not throw any light on what happens in the region of the $(\lambda_\Omega, \lambda_H^*)$ plane with $\lambda_\Omega \gtrsim 10^{-3}$, to the left of the locus of Fig. 1. Preliminary results suggest that the presence of a toroidal field may at least ameliorate this problem, as well as helping to stabilize the poloidal field-lines against instabilities such as described by Wright (1973) and Markey & Tayler (1973); but it does not seem that it will be possible to find models with large λ_Ω and arbitrarily small values of λ_H^*. In the next section it is found that a similar problem arises with a magnetic field of odd parity.

4 Models with odd parity magnetic fields

The analysis of the observations of magnetic A stars according to the oblique rotator theory requires a strong dipolar component of magnetic field. The models of the preceding sections – with even-parity fields – were studied first because it was feared that the odd-parity cases would be intractable. If the circulation is fairly rapid – for example of the order of the Eddington–Sweet value in a reasonably rapidly rotating star – so that a ‘global’ magnetic Reynolds number is considerably greater than unity, then in order for the fluid to flow across the magnetic field lines deep in the star near the equator, severe local distortion of the field will be necessary in order to reduce sufficiently the local length scale of the flow, and hence the local magnetic Reynolds number. Such distorted field will not be adequately represented by an expansion in Legendre polynomials which is truncated after just two or three terms. However, there are two situations in which an analysis similar to that described above may lead to plausible models. The first possibility is when, as in Moss 1974, the solution is only a small perturbation to the zero circulation solution, and the resulting velocities are everywhere small. Alternatively the rotational perturbation may dominate the magnetic through the bulk of the star, with an essentially Eddington–Sweet circulation (as in Section 3), but if λ_Ω is so small that the global magnetic Reynolds number of this circulation remains low enough, fluid may be able to flow across the field lines deep in the star near the equator without unduly distorting the field. Such models are described in this section.

The magnetic field is now expanded in the form

$$B_r = \bar{H}(a_1 P_1(\cos\theta) + a_3 P_3(\cos\theta)), \tag{17}$$

otherwise the analysis proceeds as before, and equations similar to those in Moss (1974), are again obtained, the only differences occurring in the equations derived from the hydrostatic and mhd equations. Suitable boundary conditions on a_1 and a_3 can be found, and the solutions normalized to, e.g. $a_1(1) = 1$.

Starting from a solution to the circulation-free problem with a purely dipolar field, solutions were readily found for a range of values of the parameters λ_Ω and λ_H. However, unlike the situation discussed in Section 3 where the magnetic field had even parity, models were not found for all values of λ_Ω and λ_H. The region in which models were found is hatched in Fig. 6. Models were not sought for $\lambda_\Omega < 10^{-6}$ and $\lambda_H < 10^{-7}$, but there is no reason to expect that models cannot be found there, as $\lambda_\Omega/\lambda_H \to 0$ at fixed λ_H, say. Similarly models were not looked for to the right of the hatched region ($\bar{H} = 10^5$ G corresponds to $\lambda_H \simeq 310^{-5}$), but there is no reason to suppose that solutions to the right of this boundary in Fig. 5 cannot be found. For $\lambda_\Omega \gtrsim 10^{-4}$ solutions are found only for $\lambda_H \gtrsim 10^{-6}$, and are essentially modifications of the zero circulation solutions, analogous to the HF solutions previously discussed. For $\lambda_\Omega \lesssim 10^{-4}$ solutions seem to exist for arbitrarily small values of λ_H, and as λ_H is decreased at fixed λ_Ω the solutions for the circulation change over smoothly into an essentially Eddington–Sweet circulation in the bulk of the star. If the Eddington–Sweet circulation is

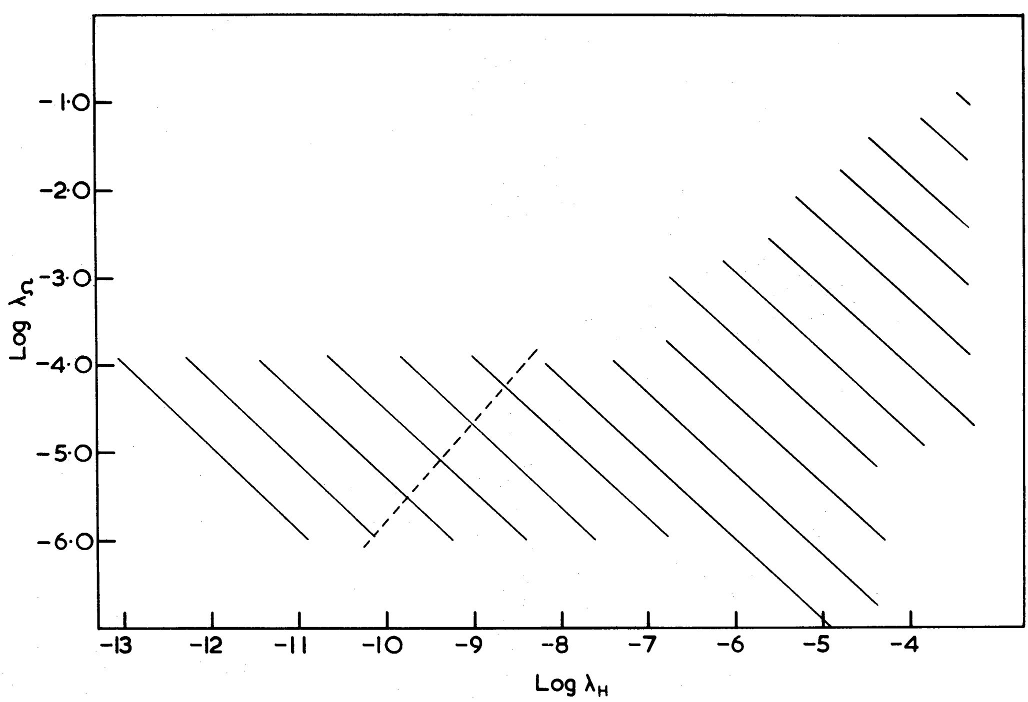

Figure 6. The region of the $(\lambda_\Omega, \lambda_H)$ plane where models with odd parity magnetic fields are found is shaded. To the left of the dashed line the models have essentially Eddington–Sweet circulations through much of their interiors – see text.

given by $V_r = U_{ES}P_2(\cos\theta)$, then these solutions typically first have $V_2 \sim U_{ES}$ between $x \simeq 0.40$ and $x \simeq 0.65$, and this region expands as λ_H decreases: see Fig. 7 where U_{ES}/V_2 is plotted at $\lambda_\Omega = 3 10^{-5}$ for four values of λ_H. In Fig. 6 models to the left of the dashed line have $V_2 \simeq U_{ES}$ through much of the interior of the star, and for some distance to the right there is a transitional region.

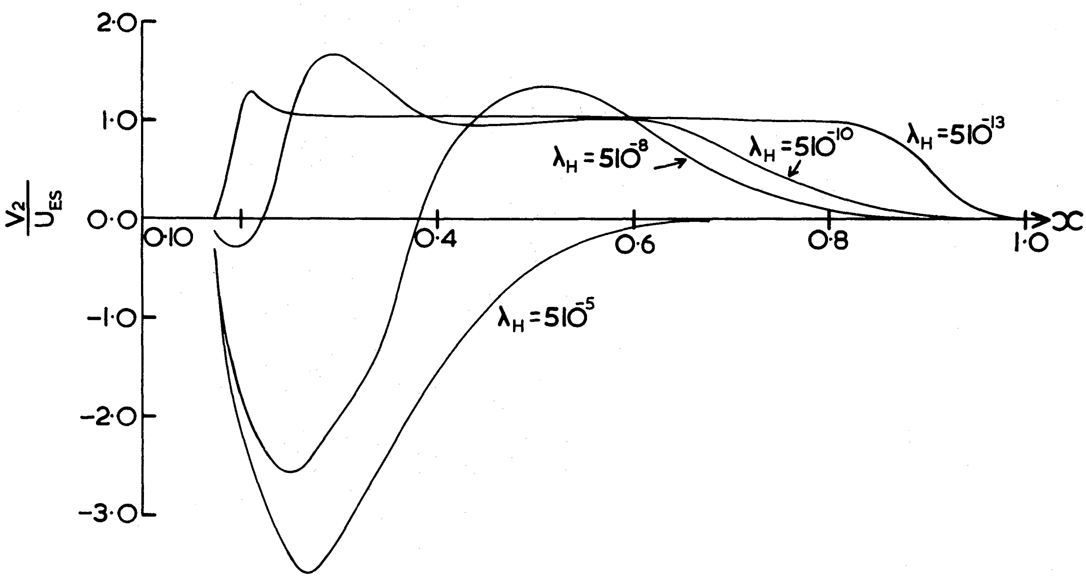

Figure 7. The ratio V_2/U_{ES} for several values of λ_H as a function of radius for $\lambda_\Omega = 3 10^{-5}$, odd parity fields.

For models with $\lambda_\Omega \lesssim 10^{-4}$, where solutions can be found for all values of λ_H, the total flux parameter λ_H^* – *cf.* definition (16) – decreases monotonically with λ_H, and models with arbitrarily small fluxes can be constructed. This behaviour is like that of the LF models of Section 3, but here there is no discontinuity such as was found previously between the LF and HF sequences. λ_H^* varies quite smoothly as λ_H is reduced at fixed λ_Ω. In the region $\lambda_\Omega \gtrsim 10^{-4}$ models can only be found for a limited range of values of λ_H^* – again similar to the results of Section 3 (but now the range of λ_H values is also restricted) – see Fig. 8. The ratio $|a_3/a_1|$ is less than one throughout all the models excepting those with $\lambda_\Omega \gtrsim 10^{-4}$ and $\lambda_H \lesssim 10^{-7}$ where it may locally be about 1.5 or so, and usually $|a_3/a_1| \ll 1$. The ratio $|V_4/V_2|$ is much less than unity in regions where the V_2 circulation is approximately Eddington–Sweet, but elsewhere locally we may have $|V_4| \sim |V_2|$. Although calculations with more terms in the angular expansions are needed to establish firmly the convergence of series such as (17), we feel – bearing in mind results of previous calculations (e.g. Moss 1973) – that the solutions for the leading terms in the expansions may be fairly reliable.

Field and streamlines from the leading terms in the expansions for one model (in which the circulation is largely Eddington–Sweet) are shown in Fig. 9. Flow across the field lines occurs both near the surface at all polar angles (as in Section 3), and near the equator throughout the star. There is no visible distortion of the field lines in this region, even if the P_3 component is included – the magnetic Reynolds number of the flow is quite small, and in any case the expansion (17) does not contain enough terms to permit this kind of resolution. This point is discussed further below.

Attempts were made to find solutions with small λ_H and $\lambda_\Omega \gtrsim 10^{-4}$, and the manner of their failure may throw some light on to the problem. At a fixed value of λ_H (say 10^{-9}) λ_Ω

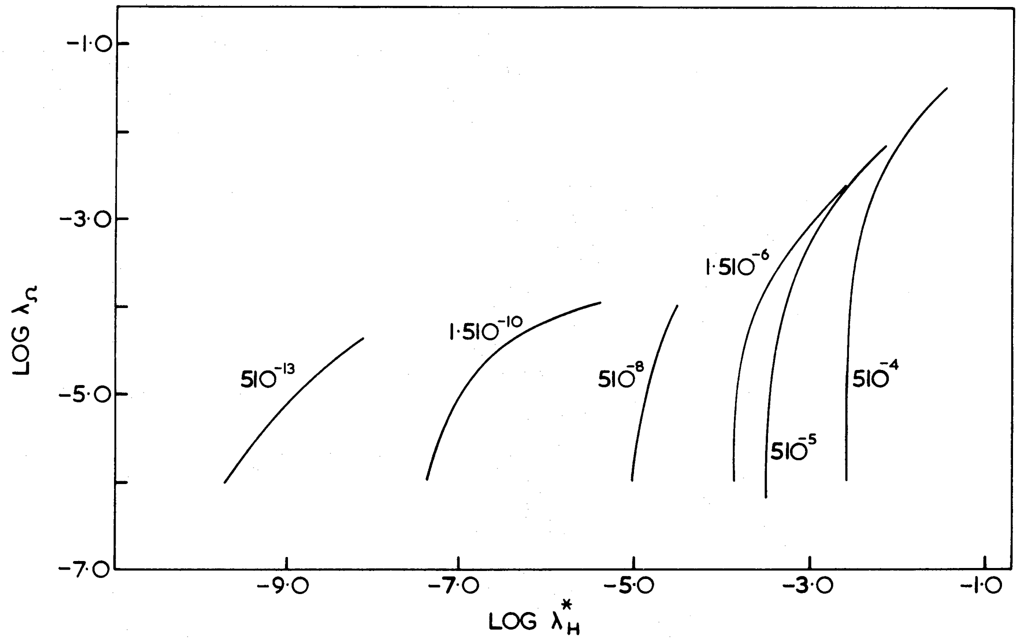

Figure 8. λ_Ω plotted against λ^*_H for the models with odd parity magnetic field. Curves are labelled with their values of λ_H.

was increased from 10^{-4} in small steps. Solutions were found up to $\lambda_\Omega = 1.5 \times 10^{-4}$, but the program failed to converge at $\lambda_\Omega = 2\,10^{-4}$. The solution at $\lambda_\Omega = 10^{-4}$ appeared to be quite well behaved, with $V_2 \simeq U_{\rm ES}$ through much of the interior, but at $\lambda_\Omega = 1.5 \times 10^{-4}$, although, formally, a solution to the finite difference equations had been found, V_2 and V_4 were very irregular in the vicinity of $x = 0.30$, and the solution could not be accepted as physically realistic. This suggests that the magnetic Reynolds number of the Eddington–Sweet circulation was becoming so large as λ_Ω was increased that the distortion to the magnetic field could no longer be adequately represented by the expansion (17). Moreover, the ratio $|a_3/a_1|$, which was less than unity everywhere with $\lambda_\Omega \lesssim 9\,10^{-5}$, increases to values slightly greater than unity through much of the star when $\lambda_\Omega \gtrsim 10^{-4}$. Again, with λ_Ω of the order of 10^{-2}, as λ_H is decreased the velocities increase towards the point where no solutions can be found. Of course it is not certain that solutions to the complete equations (rather than the truncated set used here) do exist with larger λ_Ω and small λ_H, but it seems plausible that with more resolution in this region such solutions could be found. Certainly this would be reassuring physically, since the result that models can only be found in a limited range of both $(\lambda_\Omega, \lambda_H)$ and $(\lambda_\Omega, \lambda^*_H)$ is rather strange. Either a calculation with further terms in the expansions, or a genuine two-dimensional treatment would be needed to resolve this point. Both are formidable undertakings.

This prompts the thought that perhaps a similar effect is responsible for the absence of LF models with $\lambda_\Omega \lesssim 10^{-3}$ in Section 3. Here the choice of an even parity magnetic field ensures that the problem of the fluid having to cross the field lines near the equator deep in the star vanishes. But as we have seen, there is necessarily a flow across the field lines near the surface as λ_H decreases at fixed λ_Ω; a similar effect is seen as λ_Ω is increased at fixed λ_H. It is again plausible that the distortion to the field caused by the Eddington–Sweet circulation becomes too great to be described adequately by the severely truncated expansions used in this paper. (It is, of course, possible that the distortions necessary at some stage become too large to exist in nature either, but it is plausible that the expansion breaks down first!) Again only more extensive calculations seem capable of resolving this point. From this point of view, it is consistent that the solutions with $V_2 \simeq U_{\rm ES}$ through much of the star would be

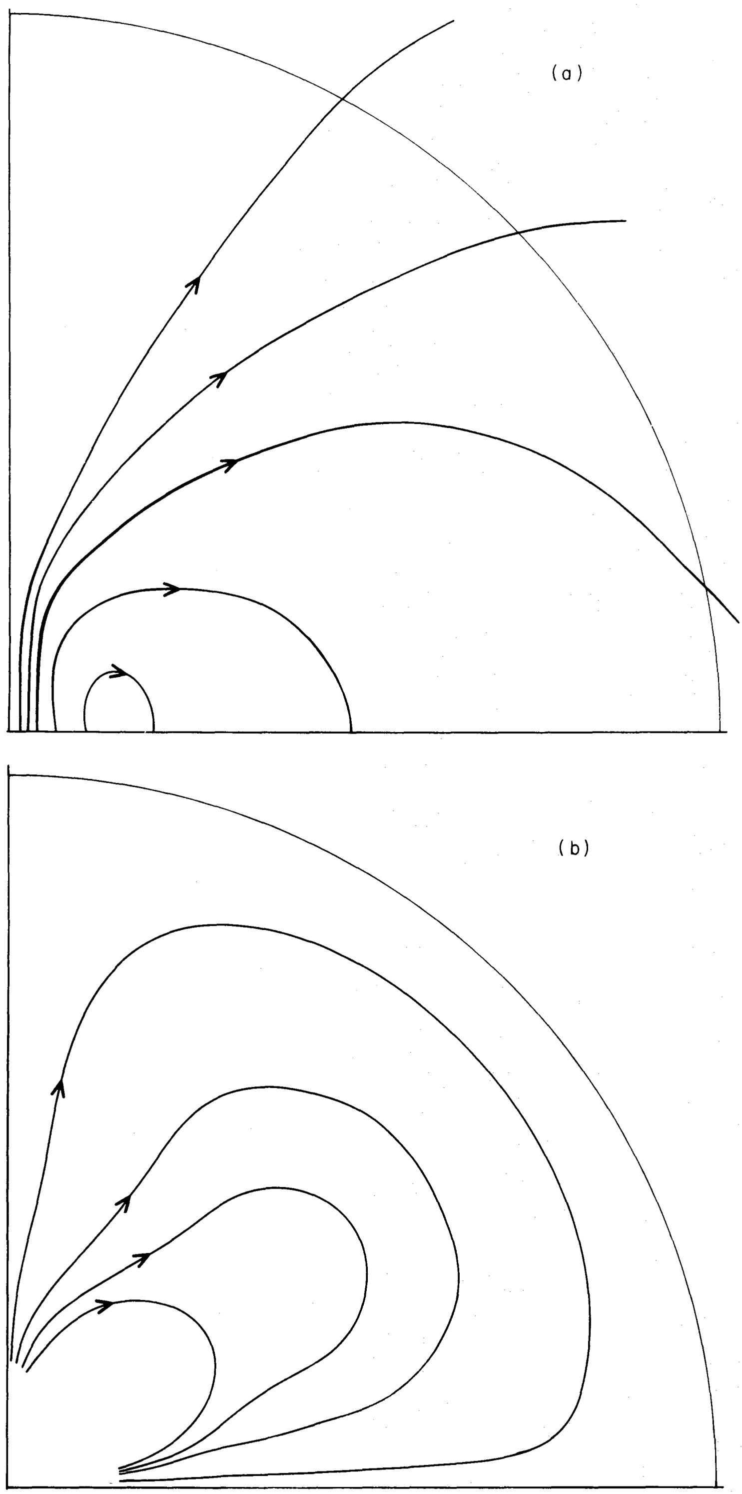

Figure 9. (a) P_1 magnetic field lines and (b) P_2 velocity stream lines for $\lambda_\Omega = 10^{-6}$, $\lambda_H = 5 10^{-13}$, odd parity magnetic field.

found for larger λ_Ω in the even parity case, as the condition due to flow across the field near the surface could be expected to be less restrictive than that due to flow across the field near the equator deep in the star in the odd parity models. However, the position is a little different for the even parity models as HF solutions do fill the region of the $(\lambda_\Omega, \lambda_H)$ plane where LF solutions cannot be found.

As is evident from the above discussion, the same general trends are seen in both the odd and even parity field models. This comparative insensitivity to field geometry is encouraging, as real magnetic stars, if describable by such models as these at all, would be expected to contain a mixture of field components. However, it should be kept in mind that many magnetic stars appear to have their magnetic and rotational axes nearly perpendicular and so the results of computations such as these should be interpreted with the appropriate degree of caution.

5 Discussion

In spite of the persisting incompleteness of the class of solutions (which as noted above may be due merely to the inadequacy of the expansions where the magnetic Reynolds number becomes large), and the absence of convergence proofs for the Legendre expansion, we regard the results of this paper as further support for the basic ideas outlined in the Introduction. First of all, the work is a worth-while addition to the theory of meridian circulation in stellar radiative zones. It has long been recognized that in order to maintain a uniformly rotating star in a steady state, one needs some process that will off-set the advection of angular momentum by the Eddington–Sweet circulation. An obvious candidate is a magnetic field, sufficiently strong for the Alfvén speed to be much faster than the circulation speed (Mestel 1961, 1965a; Roxburgh 1963). Over the bulk of the star the circulation speeds are so slow that this is quite consistent with the implicit neglect of the poloidal magnetic forces compared with the centrifugal forces: all that is required is a very slight twist in the magnetic field, so that the resulting toroidal magnetic force $(\mathbf{j}_p \times \mathbf{B}_p/c)$ can cancel the toroidal Coriolis force (seen by an observer rotating with the star). However, it has also been recognized that the circulation pattern will impose its structure on the field (allowing for the effect of finite resistivity), and that in the low-density surface regions we must include the poloidal magnetic forces. This paper and the earlier one (Moss 1974) go some way towards constructing fully self-consistent models of rotating magnetic stars with non-vanishing circulation fields.

We have noted that comparison of the normal A stars with the magnetic subset of the Ap stars requires an anti-correlation between rotation and observable field: not only is there no detectable field (nor spectral peculiarity) in most of the rapidly rotating A stars, but there is also evidence (Landstreet *et al.* 1975; Wolff & Wolff, private communication) that within the group of slowly rotating, magnetic Ap stars, the effective field is systematically larger for the slower rotators. This basic anti-correlation provided much of the original motivation for the work of this series, and the later results of Landstreet *et al.* and the Wolff's are *ex post facto* encouraging. However, we should note that their results are by their nature statistical. There remain observations that might be considered anomalous: e.g. the existence of stars that are similar in spectral type, effective field and Zeeman curve, yet which differ by a large factor in rotation rate (Cowling 1965). This can be adduced as evidence that the fields of these stars are not being maintained by contemporary dynamo action, but are fossils. The total magnetic flux in a star at the beginning of its main-sequence life is then not a function of its structure and its rotation, but is within wide limits a free parameter, dependent on the mode of formation of the star and on its pre-main sequence history. Likewise, as the star's rotation is braked, probably by magnetic interaction with interstellar gas clouds (Havnes & Conti 1971; Mestel 1975), there would not result an associated reduction in the total flux F_t, as with a dynamo-maintained field; instead, the present work predicts an increase in the observable flux F_s, as the star tries to approach a thermally steady state.

We can go further and argue that Cowling's classical formulation (1945) of the fossil theory should be inverted: rather than having to explain contemporary stellar magnetism by appealing to primeval flux present in the gas from which the star formed, we should rather direct our thoughts to explaining why stellar magnetic fields are not stronger, and not universal. Such a discussion would be out of place here. However, we emphasize that while the fossil theory certainly gives the extra degree of freedom that observation seems to demand, it in its turn poses its own problems. We have already referred to the requirement of dynamical stability; we should also mention that systems that are stable against adiabatic displacements may lose their stability if heat exchange is allowed. (The phenomenon of 'magnetic buoyancy' in a radiative zone is an 'Eddington–Sweet' effect: a toroidal flux loop will tend to rise at a rate determined by inflow of heat.) It is possible that to stabilize a magnetic field over times longer than a Kelvin–Helmholtz time we may need a negative gradient of mean molecular weight μ.

The scepticism that was once felt about the possibility of stellar dynamo action has largely evaporated, and indeed there is a plethora of dynamo models in the literature, including some oblique rotators (Rädler 1975). One should therefore look again at the possibility that the fields of the Ap stars are dynamo-maintained, in spite of the *prima facie* difficulties in understanding the observations. Let us suppose that the motions in the convective core of an early-type star are such as to build up and maintain a *steady* rather than an oscillatory magnetic field. One's intuition is that in a more rapidly rotating star, dynamo action would be scaled up: differential rotation and Coriolis forces would be correspondingly larger, yielding a correspondingly larger magnetic flux. However, this refers to the total flux F_t of the poloidal field generated. The same arguments for meridian circulation acting so as to concentrate the field deep down – either directly, as in the models of this paper, or indirectly via the requirement that the circulation be killed – apply as much to dynamo-built fields as to fossil fields. What we are interested in is $\bar{H}$ (or λ_H) as a function of Ω, but with the extra constraint that F_t (or λ_H^*) is now not a given constant but is a function of Ω, say $F_t \propto \Omega^n$. Then

$$\bar{H}^2 \propto \lambda_H = \lambda_H^*/f^2 \propto \Omega^{2n}/f^2. \tag{18}$$

The present theory and its generalization (Moss 1976) to mixed poloidal–toroidal fields provides an estimate of f as a function of Ω and H: if we write $f \propto \Omega^a/\bar{H}^b$, then

$$\bar{H} \propto \Omega^{(n-a)/(1-b)}. \tag{19}$$

The above models show that in general $a,b < 1$; further at large λ_Ω/λ_H, f behaves asymptotically like $(\lambda_\Omega/\lambda_H)^{1/2}$, so that $a,b \to 1$ from below. When mixed poloidal–toroidal fields (to be expected if a dynamo is operating) are present the situation is more complicated, as the quantity of toroidal flux present in the calculations analogous to those of Section 3 can be varied to some degree at given λ_Ω and λ_H and so a unique relation $f = f(\Omega, \bar{H})$ is not obtained from the models (the quantity of toroidal flux present would have to be prescribed by the dynamo theory). However, restriction to models with dynamically stable poloidal field structure may tighten the relation. If a relation such as (19) is valid, then we can see that the crucial question is the value of the index n. If $n < 1$, then (at least asymptotically) the increase of F_t with Ω is insufficient to off-set the tendency of centrifugal forces to concentrate the field deep down, and the anti-correlation of F_s with Ω would be qualitatively explicable. But if $n \geqslant 1$, the more rapidly rotating stars should show systematically stronger surface fields, contrary to observation. One should not then conclude that dynamo action is not occurring deep down in A stars (and indeed in others with convective cores), but rather that the field we see at the surface is not part of this field, but is presumably a fossil, of total flux that is unrelated physically to the star's rotation.

Clearly, we need to know much more about stellar dynamo action, especially about the non-linear dynamical effects, e.g. magnetic interference with the rotation field – which presumably fix the strength of the total flux F_t generated by a particular dynamo. It may turn out that F_t is not a single-valued function of Ω. Theory must explain also why of the slowly rotating A stars, only some of the Ap stars and apparently none of the Am stars show strong surface fields; and also why there is apparently a class of slowly rotating stars near type A0 which exhibit neither strong surface fields nor any spectral peculiarities (Deutsch 1967). It is just this bewildering complexity of the observational data which leads us to postulate that the total magnetic flux possessed by a star of given structure is an extra parameter; and this may turn out to be the case. However, it should be noted that most of the theoretical work (including that of this paper) has been devoted to steady or quasi-steady models (though there have been some time-dependent studies, e.g. Mahasweran 1975; Milsom 1976). Detailed comparison with observation may therefore be misleading at this stage (especially as our understanding of the Ap and Am phenomena is still controversial, in spite of the advances of the last few years (Michaud 1970; Havnes 1974; Wolff & Wolff 1975). For example, a magnetic Ap star with a rapid rotation (an exception to the general rule) may be in the process of slowly concentrating its flux deep in the star, so that perhaps in time – unless the star is meanwhile braked – the surface flux will cease to be observable, and the spectral peculiarities will disappear. Again, consider a normal A star which begins its life as a rapidly rotating member of a moderately close binary system, but which steadily loses angular momentum of spin as tidal action forces it into synchronism with the orbital motion. A magnetic field – whether fossil or dynamo-generated – with most of its flux trapped inside the star would now be able to diffuse slowly outwards, so that the star may transform itself into a magnetic star. To sum up: for convincing comparison with observation, one needs a better theoretical understanding of (a) the origin of the Ap and Am phenomena; (b) the dependence of dynamo generation on rotation and stellar structure; and (c) the basic problem of this paper – the relation between observable and internal magnetic flux, as a function of time as well as of the stellar parameters.

A final word on the role of circulation in the observably magnetic stars. Suppose that the centrifugal forces initially dominate the magnetic sufficiently for the Eddington–Sweet circulation to flow from the rotational poles, so concentrating flux towards the equator. If the field is initially symmetric about the rotation axis, it remains so – an aligned rotator stays aligned. But if the magnetic axis is initially inclined by a sizeable angle to the rotation axis, then the concentration of flux towards the equator will make the star look more and more like a perpendicular rotator. One is left wondering whether it is this kinematic effect that is most important for interpretation of the observations, rather than the dynamical processes (referred to in the Introduction) which involve precession of the instantaneous axis of rotation through the star.

Acknowledgments

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Appendix

Proof for the purely dipolar problem that solutions with zero surface field do not exist

In the notation of Wright (1969) the equations of the problem, writing

$\mathbf{B} = \mathbf{H}(b/x^2 \cos\theta, b'/x \sin\theta, 0)$,

are

$$\rho_0 x\left(\frac{d\chi_{12}}{dx} + \frac{dp_{12}}{dx}\right) + \rho_0\tau\left(\frac{p_{12}}{N+1} - t_{12}\right) + \frac{2}{3}\frac{b'}{x}\left(\frac{2b}{x^2} - b''\right) = 0, \qquad \text{(A1)}$$

$$\rho_0(x_{12}+p_{12})+\frac{2}{3}\frac{b}{x^2}\left(\frac{2b}{x^2}-b''\right)=0, \tag{A2}$$

$$\frac{d^2\chi_{12}}{dx^2}+\frac{2}{x}\frac{d\chi_{12}}{dx}-\frac{6}{x^2}\chi_{12}+(t_{12}-p_{12})\frac{\rho_0}{t_0}=0, \tag{A3}$$

$$\frac{xdt_{12}}{dx}+l_{12}+(ep_{12}-st_{12})\frac{\tau}{N+1}=0, \tag{A4}$$

$$\frac{xdl_{12}}{dx}+l_{12}\left(1-\frac{s\tau}{N+1}+e\tau\right)+6\tau_{12}=0, \tag{A5}$$

where

$\tau=-\dfrac{d\log p_0}{d\log x}$, and the opacity is now given by $\kappa=\kappa_0\rho^{e-1}T^{-s+e+3}$.

Equations (A4) and (A5) are replaced by equations suitable for a convective core in $x<x_c$, e.g.

$$\frac{d}{dx}\left(\frac{p_{12}}{t_{12}}\right)=0, \tag{A4$'$}$$

$$\frac{dl_{12}}{dx}=0. \tag{A5$'$}$$

These are for the $\mathbf{v}=\mathbf{0}$ case, and the equations form a closed set – there is no truncation anywhere of expansions in Legendre polynomials. The boundary conditions can be taken in the form: at $x=0$,

$$x\frac{db}{dx}-2b=0, \tag{A6}$$

$$x\frac{d\chi_{12}}{dx}-2\chi_{12}=0, \tag{A7}$$

$$p_{12}=2.5t_{12}; \tag{A8}$$

and at $x=1$:

$$\frac{xdb}{dx}+b=0, \tag{A9}$$

$$\frac{xd\chi_{12}}{dx}+3\chi_{12}=\frac{5}{3}\frac{\lambda_\Omega}{\lambda_H}x^2, \tag{A10}$$

$$ep_{12}=st_{12}. \tag{A11}$$

The normalization

$$b(1)=1 \tag{A12}$$

can be imposed.

These are seven boundary conditions on equations (A1)–(A5) which are three first order and two second order o.d.e.'s.

Expand the unknowns in series near the surface ($x = 1$). Define $y = 1 - x$. The zero order model has the properties $\rho_0 \sim y^{s/e-1}, p_0 \sim y^{s/e}, t_0 \sim y, \tau \sim 1/y$. Expand $b = 1/x + y^{\beta}(b_0 + b_1 y + \ldots)$, $\beta > 0, b_0 \neq 0$, which satisfies (A9) and (A10); and expand $\chi_{12}, b_{12}, l_{12}, t_{12}$ in the form

$$\chi_{12} = \chi_0 + \chi_1 y + \ldots, \text{ etc.}$$

Substituting in equations (A1)–(A5), (A9)–(A11), and equating coefficients of powers of y gives linear algebraic equations for the constant coefficients introduced above, and it is found that the solution near $y = 0$ is determined in terms of three arbitrary quantities, say b_0, χ_0, l_0, at any given λ_Ω/λ_H. So if we think about solving the equations by integrating inwards from $y = 0 (x = 1)$ we have three free parameters with which to start the integrations, and three boundary conditions, (A6), (A7), (A8) to satisfy at $x = 0$. Thus the problem is well posed, and it is possible to seek solutions for any value of λ_Ω/λ_H.

Now try imposing the condition $b(1) = 0$. The normalization $b(1) = 1$, (A12), obviously cannot now be used, but we can impose the condition

$$b/x^2 = 1 \tag{A13}$$

at $x = 0$, for example. λ_Ω/λ_H can now be considered a free parameter as we are seeking *any* solution with $b(1) = 0$. Now put

$$b = y^{\beta}(b_0 + b_1 y + \ldots), \quad b_0 \neq 0, \quad \beta > 0,$$

near $y = 0$, and expand χ_{12} etc. as before. Substitute in the equations (A1)–(A5), (A9)–(A11), equate coefficients of powers of y, and after some manipulation it is found that the solution near $y = 0$ is given in terms of three free parameters, *including* λ_Ω/λ_H. However, there are now four boundary conditions to satisfy – including (A13) – at $x = 0$, and so a solution cannot be found.

A similar but more cumbersome argument can be found for the zero circulation case with even parity field, where the radial component of field is expanded as

$$B_r = \mathrm{H}(a_2(x) P_2(\cos\theta) + a_4(x) P_4(\cos\theta)), \quad \text{as in Moss (1973).}$$

Further, including the meridional circulation as in Section 3 does not alter the essentials of the argument. The circulation only appears as an additional term in the two equations equivalent to equation (A4), where it does not affect the lowest terms in the series expansion near $y = 0$, and also in the equations derived from the MHD equation. The same comments apply to solutions with odd parity fields and circulation, as in Section 4.