Electronic Journal of Probability

Vol. 4 (1999) Paper no. 17, pages 1–25.

Journal URL
http://www.math.washington.edu/~ejpecp
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol4/paper17.abs.html

MODERATE DEVIATIONS TYPE EVALUATION FOR INTEGRAL FUNCTIONALS OF DIFFUSION PROCESSES

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Abstract We establish a large deviations type evaluation for the family of integral functionals

$$\varepsilon^{-\kappa}\int_0^{T^\varepsilon}\Psi(X_s^\varepsilon)g(\xi_s^\varepsilon)ds,\ \varepsilon\searrow 0,$$

where Ψ and g are smooth functions, ξ_t^ε is a "fast" ergodic diffusion while X_t^ε is a "slow" diffusion type process, $\kappa\in(0,1/2)$. Under the assumption that g has zero barycenter with respect to the invariant distribution of the fast diffusion, we derive the main result from the moderate deviation principle for the family $(\varepsilon^{-\kappa}\int_0^t g(\xi_s^\varepsilon)ds)_{t\ge 0}$, $\varepsilon\searrow 0$ which has an independent interest as well. In addition, we give a preview for a vector case.

Keywords Large deviations, moderate deviations, diffusion

AMS 1991 subject classification: 60F10

Submitted to EJP on April 13, 1999. Final version accepted on September 28, 1999.

1 Introduction

In this paper, we consider a two scaled diffusion model with independent Wiener processes V_t and W_t:

$$\begin{aligned} d\xi_t^\varepsilon &= \frac{1}{\varepsilon} b(\xi_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}\sigma(\xi_t^\varepsilon)dV_t & (1)\\ dX_t^\varepsilon &= F(X_t^\varepsilon, \xi_t^\varepsilon)dt + G(X_t^\varepsilon, \xi_t^\varepsilon)dW_t. & (2)\end{aligned}$$

The fast component ξ_t^ε is assumed to be an ergodic Markov process while the slow component X_t^ε is a diffusion type process governed by the fast process $\xi^\varepsilon = (\xi^\varepsilon)_{t\geq 0}$ and independent of it Wiener process $W = (W_t)_{t\geq 0}$. Under appropriate conditions, a stochastic version of the Bogolubov averaging principle holds (see [12]), that is, the slow process is averaged with respect to the invariant density of the fast one, say $p(z)$. In other words, the X_t^ε process is approximated by a Markov diffusion process X_t with respect to some Wiener process $\overline{W}_t$:

$$dX_t = \overline{F}(X_t)dt + \overline{G}(X_t)d\overline{W}_t \tag{3}$$

with the averaged drift and diffusion parameters

$$\overline{F}(x) = \int_{\mathbb{R}} F(x,z)p(z)dz, \quad \overline{G}(x) = \Big(\int_{\mathbb{R}} G^2(x,z)p(z)dz\Big)^{1/2}.$$

Let us assume functions F and G are unknown and indicate here a statistical procedure for estimation of the averaged function $\overline{F}$ from the observation of the 'slow" process X_t^ε. The averaging principle suggests the following recipe: to proceed with the path of X_t^ε as if it is the path of X_t. For instance, it is well known the kernel estimate (with kernel K and bandwidth h) $\widetilde{F}(x) = \frac{\int_0^T K\left(\frac{X_t - x}{h}\right)dX_t}{\int_0^T K\left(\frac{X_t-x}{h}\right)dt}$ for $\overline{F}(x)$ via $X_t, 0 \leq t \leq T$ so that

$$\widetilde{F}^\varepsilon(x) = \frac{\int_0^T K\left(\frac{X_t^\varepsilon - x}{h}\right)dX_t^\varepsilon}{\int_0^T K\left(\frac{X_t^\varepsilon - x}{h}\right)dt}$$

is taken as estimate of $\overline{F}(x)$ via $X^\varepsilon, 0 \leq t \leq T$. An asymptotic analysis, as $\varepsilon \to 0$, for $\widetilde{F}^\varepsilon(x)$ leads to study of properties for integral functionals

$$\int_0^T F(x,\xi_t^\varepsilon)K\left(\frac{X_t^\varepsilon - x}{h}\right)ds \quad \text{and} \quad \int_0^T F^{(i)}(x,\xi_t^\varepsilon)K\left(\frac{X_t^\varepsilon - x}{h}\right)ds,\ i = 1,2,$$

where $F^{(i)}(x,y)$ is the i-th derivative of $F(x,y)$ in x. Namely, assuming that both T and h depend on ε: $T = T^\varepsilon$, $h = h^\varepsilon$ with $T^\varepsilon \nearrow \infty$, $h^\epsilon \searrow 0$, we need to show that for any of functions $F(x,y)$, $F^{(1)}(x,y)$, $F^{(2)}(x,y)$ specified as $H(x,y)$ the integral

$$\int_0^{T^\varepsilon}\left[H(x,\xi_t^\varepsilon) - \int H(x,y)\,p(y)\,dy\right]K\left(\frac{X_t^\varepsilon - x}{h^\varepsilon}\right)ds$$

goes to zero faster than ε^κ with some $\kappa > 0$ (see [18]). For fixed x, denote by $g(\xi_t) = H(x,\xi^\varepsilon_t) - \int H(x,y)\,p(y)\,dy$ and by $\Psi(X^\varepsilon_t) = K(\frac{X^\varepsilon_t - x}{h^\varepsilon})$. Then the desired property holds, if for instance with $T^\varepsilon\varepsilon \to 0$ we have

$$P\Big((T^\varepsilon\varepsilon)^{-\kappa}\sup_{t\le T^\varepsilon}\Big|\int_0^t \Psi(X^\varepsilon_s)g(\xi^\varepsilon_s)ds\Big| \ge z\Big) \lessapprox \exp\Big\{-\text{const.}\frac{z^2}{(T^\varepsilon\varepsilon)^{1-2\kappa}}\Big\}.$$

We avoid a straightforward verification of this asymptotic and prefer to consider first the special case, $\Psi \equiv 1$, which is of an independent interest and allows to clarify the main idea and to simplify the exposition. So, let

$$S^{\varepsilon,\kappa}_t = \frac{1}{\varepsilon^\kappa}\int_0^t g(\xi^\varepsilon_s)\,ds, \tag{4}$$

where $g = g(z)$ is an arbitrary function with zero barycenter with respect to the invariant density p. Our approach to the asymptotic analysis employs, so called, Poisson decomposition in the form used in [4] for proving the central limit theorem (CLT):

$$S^{\varepsilon,\kappa}_t = e^{\varepsilon,\kappa}_t + \varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t, \tag{5}$$

where $e^{\varepsilon,\kappa}_t$ is a negligible process and $\widetilde{S}^\varepsilon_t$ is a continuous martingale with the predictable quadratic variation $\langle S^{\varepsilon,\kappa}\rangle_t$ is "close" to a linear increasing function γt, so that the $S^{\varepsilon,\kappa}_t$ process is approximated in the distribution sense by Wiener process with the diffusion parameter γ. The Poisson decomposition allows to analyze the asymptotic behavior, in $\varepsilon \searrow 0$, of $S^{\varepsilon,\kappa}_t$ under $0 \le \kappa \le 1/2$. We exclude two extreme points $\kappa = 1/2$ and $\kappa = 0$ and emphasize only that for $\kappa = 1/2$ the family $(S^\varepsilon_t)_{t\ge 0}, \varepsilon \searrow 0$ obeys the CLT (see e.g [4]) while for $\kappa = 0$ a family of occupation measures of $(\xi^\varepsilon_t)_{t\ge0}, \varepsilon \searrow 0$, obeys the large deviation principle (LDP) (see [7] or e.g. [16]) and so, due to the contraction principle of Varadhan [25], the family $(S^{\varepsilon,\kappa}_t)_{t\ge0}, \varepsilon \searrow 0$, at least for bounded g, obeys the LDP as well. In contrast to both, the case $0 < \kappa < 1/2$ preserves the large deviation type property, which is the same as for a family of Wiener processes parametrized by diffusion parameter $\varepsilon^{1-2\kappa}\gamma$. In other words, the case $0 < \kappa < 1/2$ guarantees, so called, moderate deviation evaluation for the family $(S^{\varepsilon,\kappa}_t)_{t\ge0}, \varepsilon \searrow 0$. For establishing the moderate deviation principle (MDP), we use the conditions on the drift and diffusion parameters b and σ (see (A-1) in Section 2) proposed by Khasminskii, [13] and modified by Veretennikov, [26]. These conditions allow to verify that both $e^{\varepsilon,\kappa}_t$ and $\langle S^{\varepsilon,\kappa}\rangle_t - \gamma t$ are "exponentially negligible" processes with the rate of speed $\varepsilon^{1-2\kappa}$. Formally we could apply a recent Pukhalskii's result [23], which being adapted to the case considered is reformulated as: if $\langle\widetilde{S}^\varepsilon\rangle_t$ and γt are exponentially indistinguishable with the rate of speed $\varepsilon^{1-2\kappa}$, then the family $(\varepsilon^{1-2\kappa}\widetilde{S}^\varepsilon_t)_{t\ge0}, \varepsilon \searrow 0$ obeys the same type of LDP as $(\varepsilon^{1/2-\kappa}\sqrt{\gamma}\widetilde{W}_t)_{t\ge0}$ does, that is with the rate of speed $\varepsilon^{1-2\kappa}$ and the rate function of Freidlin-Wentzell's type (see [12]): for absolutely continuous function φ

$$J(\varphi) = \frac{1}{2\gamma}\int_0^\cdot \dot{\varphi}^2(t)\,dt.$$

Nevertheless, we give here another proof of the same implication, in which the role of the "fast" convergence of $\langle\widetilde{S}^\varepsilon\rangle_t$ to γt is discovered with more details and might be interested by itself.

For the integral functional $\frac{1}{\varepsilon^\kappa}\int_0^{T^\varepsilon}\Psi(X^\varepsilon_t)g(\xi^\varepsilon_t)dt$ we also apply the Poisson decomposition (5)

$$\frac{1}{\varepsilon^\kappa}\int_0^{T^\varepsilon}\Psi(X^\varepsilon_t)g(\xi^\varepsilon_t)dt = \int_0^{T^\varepsilon}\Psi(X^\varepsilon_t)de^{\varepsilon,\kappa}_t + \varepsilon^{1/2-\kappa}\int_0^{T^\varepsilon}\Psi(X^\varepsilon_t)d\widetilde{S}^\varepsilon_t$$

in which the first term is the Itô integral with respect to the semimartingale $e_t^{\varepsilon,\kappa}$ while the second one is the Itô integral with respect to the martingale $\widetilde{S}_t^{\varepsilon}$. As for "$\Psi \equiv 1$", the main contribution comes from the second term and many details of proof are borrowed from "$\Psi \equiv 1$".

Results on the MDP for processes with independent increments are well known from Borovkov, Mogulski [2], [3] and Chen [5], Ledoux [15]. For the depended case, the MDP estimations have attracted some attention as well. Some pertinent MDP results can be found in: Bayer and Freidlin [1] for models with averaging, Wu [27] for Markov processes, Dembo [8] for martingales with bounded jumps, Dembo and Zajic [9] for functional empirical processes, Dembo and Zeitouni [10] for iterates of expanding maps. The paper is organized as follows. In Section 2, we fix assumptions and formulate main results. Proofs of the main results are given in Section 3. Taking into account an interest to the vector case setting, in Section 4 we give a preview for the MDP with a vector fast process.

Acknowledgement The authors thank S. Pergamenshchikov and A. Veretennikov for helpful remarks and suggestions leading to significant improvement of the paper.

2 Assumptions. Formulation of main results

For a generic positive constant, notation 'ℓ' will be used hereafter. We fix the following assumptions. The initial conditions ξ_0 and X_0 for the Itô equations (1) and (2) respectively are deterministic and independent of ε.

(A-0) The function $g(x)$ is continuously differentiable and

$$|g(x)| \le \ell(1+|x|).$$

(A-1)
1. Functions b, σ are continuously differentiable (b once, σ twice);
2. $\sigma^2(x)$ is uniformly positive and bounded; its derivatives are bounded as well;
3. there exist constants $C > 1$ and $c > 0$ such that for $|x| > C$

$$\begin{aligned} &xb(x) \le -c|x|^2 \\ &b^2(x) + b'(x)\sigma^2(x) \ge (1/c)b^2(x). \end{aligned}$$

(A-2) Functions $F = F(x,z)$, $G = G(x,z)$ are bounded, continuous, and Lipschitz continuous in x uniformly in z.

(A-3) Function $\Psi = \Psi(x)$ is twice continuously differentiable and bounded jointly with its derivatives (the value $\Psi^* = \sup_x |\Psi(x)|$ is involved in the formulation of the main result).

(A-4) For some $d > 0$ and every $\varepsilon > 0$, $T^\varepsilon \ge d$ and for a chosen $\kappa \in (0, \frac{1}{2})$

$$\lim_{\varepsilon \to 0} T^\varepsilon \varepsilon^{1-2\kappa} = 0.$$

It is well known from [14], [24] that under (A-1) the process ξ^ε_t is ergodic with the uniquely defined invariant density $p(z) = \text{const.}\dfrac{\exp\Big\{2\int_0^z \frac{b(y)}{\sigma^2(y)}dy\Big\}}{\sigma^2(z)}$. Under (A-0), we have $\int_{\mathbb{R}} |g(z)|p(z)dz < \infty$ and in addition to (A-0) assume

$$\int_{\mathbb{R}} g(z)p(z)dz = 0. \tag{6}$$

Then, the function

$$v(z) = \frac{2}{\sigma^2(z)p(z)} \int_{-\infty}^{z} g(y)p(y)dy \tag{7}$$

is well defined and bounded. Set

$$\gamma = \int_{\mathbb{R}} v^2(z)\sigma^2(z)p(z)dz. \tag{8}$$

Our main result is formulated in the theorem below.

Theorem 1 *Assume (A-0)-(A-4), (6). Then for every $z > 0$ and $\kappa \in (0, 1/2)$*

$$\overline{\lim_{\varepsilon\to 0}}\, (\varepsilon T^\varepsilon)^{1-2\kappa} \log P\Big(\frac{1}{\Psi^*(T^\varepsilon\varepsilon)^\kappa} \sup_{t\le T^\varepsilon} \Big| \int_0^t \Psi(X^\varepsilon_s)g(\xi^\varepsilon_s)ds\Big| \ge z\Big) \le -\frac{z^2}{2\gamma}.$$

For $\Psi \equiv 1$, we give more refined evaluation. Let us recall the definition of the LDP in the space $\mathbb{C}$ of continuous functions on $[0, \infty)$ supplied by the local supremum topology: $r(X', X'') = \sum_{n\ge 1} 2^{-n}\Big(1 \wedge \sup_{t\le n} |X'_t - X''_t|\Big)$, $\forall\ X', X'' \in \mathbb{C}$. Following Varadhan, [25], the family $S^{\varepsilon,\kappa} = (S^{\varepsilon,\kappa}_t)_{t\ge 0}, \varepsilon \searrow 0$ (for fixed $\kappa \in (0, 1/2)$) is said to obey the LDP in the metric space $(\mathbb{C}, r)$ with the rate of speed $\varepsilon^{1-2\kappa}$ and the rate function $J_\kappa = J_\kappa(\varphi)$, $\varphi \in \mathbb{C}$, if

0. level sets of J_κ are compacts in $(\mathbb{C}, r)$;

1. for any open set G from $(\mathbb{C}, r)$

$$\underline{\lim}_{\varepsilon\to 0}\, \varepsilon^{1-2\kappa} \log P\Big(S^{\varepsilon,\kappa} \in G\Big) \ge -\inf_{\varphi\in G} J_\kappa(\varphi);$$

2. for any closed set F from $(\mathbb{C}, r)$

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon^{1-2\kappa} \log P\Big(S^{\varepsilon,\kappa} \in F\Big) \le -\inf_{\varphi\in F} J_\kappa(\varphi).$$

In the case $J_\kappa \equiv J$, $0 < \kappa < \frac{1}{2}$, the family $S^{\varepsilon,\kappa}$ is said to obey the MDP in $(\mathbb{C}, r)$ with the rate function J.

Theorem 2 *Assume (A-0.), (A-1), (6), and $0 < \kappa < 1/2$. Then, the family $S^{\varepsilon,\kappa}$ obeys the MDP in the metric space $(\mathbb{C}, r)$ with the rate of speed $\varepsilon^{1-2\kappa}$ and the rate function*

$$J(\varphi) = \begin{cases} \frac{1}{2\gamma}\int\limits_0^\infty \dot\varphi^2(t)\,dt, & d\varphi(t) = \dot\varphi(t)\,dt,\ \varphi(0) = 0 \\ \infty, & \textit{otherwise.} \end{cases} \tag{9}$$

3 Proofs

3.1 Exponential supermartingale

Hereafter, random processes are assumed to be defined on some stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ with the general conditions (see, e.g. [19], Ch. 1, §1).

Let M_t be a continuous local martingale with the predictable quadratic variation $\langle M\rangle_t$. It is well known that $\langle M\rangle_t$ is a continuous process and $(\lambda \in \mathbb{R})$

$$Z_t(\lambda) = \exp\Big(\lambda M_t - \frac{\lambda^2}{2}\langle M\rangle_t\Big)$$

is a continuous local martingale. Being positive, the Z_t process is a supermartingale (see e.g. Problem 1.4.4 in [18]) and therefore for every Markov time τ (on the set $\{\tau = \infty\}$, $Z_\tau = \lim_{t\to\infty} Z_t$)

$$EZ_\tau(\lambda) \leq 1. \tag{10}$$

We apply this property for the following useful

Lemma 1 *Let τ be a stopping time and $\mathfrak{A}$ be an event from $\mathcal{F}$.*

1. If there exists a positive constant α so that $M_\tau - \frac{1}{2}\langle M\rangle_\tau \geq \alpha$ on the set $\mathfrak{A}$, then

$$P(\mathfrak{A}) \leq e^{-\alpha}.$$

2. Let η and B be positive constants so that $M_\tau \geq \eta$, $\langle M\rangle_\tau \leq B$ on the set $\mathfrak{A}$. Then

$$P(\mathfrak{A}) \leq \exp\Big(-\frac{\eta^2}{2B}\Big).$$

3. If for fixed $T > 0$, $B > 0$ it holds $\langle M\rangle_T \leq B$, then

$$P(\sup_{t\leq T}|M_t| \geq \eta, \langle M\rangle_T \leq B) \leq 2\exp\Big(-\frac{\eta^2}{2B}\Big).$$

4. If for fixed $T > 0$, $B > 0$ it holds $\langle M\rangle_T \leq B$ on the set $\mathfrak{A}$, then

$$P(\sup_{t\leq T}|M_t| \geq \eta, \langle M\rangle_T \leq B, \mathfrak{A}) \leq 2\exp\Big(-\frac{\eta^2}{2B}\Big).$$

Proof: 1. By virtue of (10), $1 \geq EI_{\mathfrak{A}}Z_\tau(1) \geq P(\mathfrak{A})e^\alpha$ and the result holds.

2. Analogously, $1 \geq EI_{\mathfrak{A}}Z\tau(\lambda) \geq P(\mathfrak{A})\exp\Big(\lambda\eta - \frac{\lambda^2 B}{2}\Big) \geq P(\mathfrak{A})\exp\Big(\frac{\eta^2}{2B}\Big)$ and the assertion follows.

3. Introduce Markov times $\tau_\pm = \inf\{t : \pm M_t \geq \eta\}$, where $\inf\{\varnothing\} = \infty$, and two sets $\mathfrak{A}_\pm = \{\tau_\pm \leq T, \langle M\rangle_T \leq B\}$. Since by 2. $P(\mathfrak{A}_\pm) \leq \exp\Big(-\frac{\eta^2}{2B}\Big)$, it remains to note only that $\{\sup_{t\leq T}|M_t| \geq \eta\} \subseteq \mathfrak{A}_+ \cup \mathfrak{A}_-$.

4. The proof is the same as for 3. with $\mathfrak{A}_\pm = \{\tau_\pm \leq T, \langle M\rangle_T \leq B\}$ replaced by $\mathfrak{A}_\pm = \{\tau_\pm \leq T, \langle M\rangle_T \leq B, \mathfrak{A}\}$. $\square$

3.2 Proof of Theorem 2

3.2.1 Poisson decomposition for $S^{\varepsilon,\kappa}$

Let $S_t^{\varepsilon,\kappa}$ and $v(y)$ be defined in (4) and (7) respectivelly. Set $u(z) = \int_0^z v(y)dy, z \in R$. It is well known (see, e.g. [14]) that the invariant density $p(z)$ of the fast process satisfies the Fokker-Plank-Kolmogorov equation $0 = -(b(z)p(z))' + \frac{1}{2}(\sigma^2(z)p(z))''$ or, in the equivalent form,

$$\frac{1}{2}(\sigma^2(z)p(z))' = b(z)p(z). \tag{11}$$

Lemma 2 *Under (A-0), (A-1), and (6), the Poisson decomposition holds* $S_t^{\varepsilon,\kappa} = \varepsilon^{1/2-\kappa}\widetilde{S}_t^\varepsilon + e_t^{\varepsilon,\kappa}$ *with*

$$\begin{aligned} \widetilde{S}_t^\varepsilon &= -\int_0^t v(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)dV_s \\ e_t^{\varepsilon,\kappa} &= \varepsilon^{1-\kappa}[u(\xi_t^\varepsilon) - u(\xi_0)]. \end{aligned} \tag{12}$$

Moreover, for every $t > 0$ *and a suitable constant* ℓ, $|e_t^{\varepsilon,\kappa}| \le \ell\varepsilon^{1-\kappa}(1 + |\xi_t^\varepsilon|)$ *and so*

$$\sup_{s\le t}|e_s^{\varepsilon,\kappa}| \le \ell\varepsilon^{1-\kappa}(1 + \sup_{s\le t}|\xi_s^\varepsilon|), \ t > 0.$$

Proof: Let us consider the conjugate to (11) equation with g from (4):

$$\frac{1}{2}\sigma^2(z)v'(z) + b(z)v(z) = g(z).$$

It is clear the function v, defined in (7), is one of solutions of this equation. The Itô formula, applied to $u(\xi_t^\varepsilon)$, gives the required decomposition

$$u(\xi_t^\varepsilon) = u(\xi_0) + \varepsilon^{-1}\int_0^t g(\xi_s^\varepsilon)ds + \varepsilon^{-1/2}\int_0^t v(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)dV_s.$$

Under (6) $v(z)$ is bounded function and so, the last statement of the lemma holds. □

3.2.2 Exponential negligibility of $e^{\varepsilon,\kappa}$

We establish here the exponential negligibility

$$\lim_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|e_t^{\varepsilon,\kappa}| > \eta\Big) = -\infty. \tag{13}$$

(13) is derived from Lemma 2 and the following lemma below.

Lemma 3 *Under (A-1), for every* $\eta > 0$ *and sufficiently large* L *,*

$$\lim_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\varepsilon^{1-\kappa}\sup_{t\le T}|\xi_t^\varepsilon| > \eta\Big) = -\infty,$$

$$\lim_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\frac{1}{T}\int_0^T |\xi_s^\varepsilon|^2 ds > L\Big) = -\infty.$$

Proof: We start with useful remarks:
1. the second statement of the lemma is valid provided that for some positive constant C and L large enough

$$\lim_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\frac{1}{T}\int_0^T I(|\xi^\varepsilon_s|>C)|\xi^\varepsilon_s|^2ds>L\Big)=-\infty \tag{14}$$

(hereafter we will use the constant C from the assumption (A-1.3));
2. the first statement of the lemma is valid, if for all L

$$\lim_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\varepsilon^{2(1-\kappa)}\sup_{t\le T}|\xi^\varepsilon_t|^2>\eta^2,\int_0^T|\xi^\varepsilon_s|^2ds\le LT\Big)=-\infty. \tag{15}$$

So, only (14) and (15) will be verified below. By the Itô formula we have

$$\varepsilon^{2(1-\kappa)}|\xi^\varepsilon_t|^2 \;=\; \varepsilon^{2(1-\kappa)}|\xi_0|^2+\varepsilon^{1-2\kappa}\int_0^t\Big(2\xi^\varepsilon_s b(\xi^\varepsilon_s)+\sigma^2(\xi^\varepsilon_s)\Big)ds+\varepsilon^{3/2-2\kappa}M^\varepsilon_t, \tag{16}$$

where $M^\varepsilon_t=\int_0^t 2\xi^\varepsilon_s\sigma(\xi^\varepsilon_s)dV_s$. Let us note that $2\xi^\varepsilon_t b(\xi^\varepsilon_t)+\sigma^2(\xi^\varepsilon_t)$ is bounded above. In fact, for $|\xi^\varepsilon_t|>C$ the value $2\xi^\varepsilon_t b(\xi^\varepsilon_t)$ is negative (see (A-1)) and, since σ^2 is bounded, for $|\xi^\varepsilon_t|>C$ we have $2\xi^\varepsilon_t b(\xi^\varepsilon_t)+\sigma^2(\xi^\varepsilon_t)\le\sigma^2(\xi^\varepsilon_t)\le$ const. For $|\xi^\varepsilon_t|\le C$ the value $2|\xi^\varepsilon_t b(\xi^\varepsilon_t)|+\sigma^2(\xi^\varepsilon_t)$ is bounded as well. For notation convenience, a positive constant r is chosen such that $2\xi^\varepsilon_t b(\xi^\varepsilon_t)+\sigma^2(\xi^\varepsilon_t)\le r$. Then, (16) implies

$$\varepsilon^{2(1-\kappa)}|\xi^\varepsilon_t|^2\le\varepsilon^{2(1-\kappa)}|\xi_0|^2+\varepsilon^{1-2\kappa}rT+\varepsilon^{3/2-2\kappa}M^\varepsilon_t.$$

Set ε_0 so small that $\eta^2>\varepsilon_0^{1-2\kappa}rT+\varepsilon_0^{2(1-\kappa)}|\xi_0|^2$ and note that for any $\varepsilon\le\varepsilon_0$

$$\begin{aligned}&\Big\{\varepsilon^{2(1-\kappa)}\sup_{t\le T}|\xi^\varepsilon_t|^2>\eta^2,\int_0^T|\xi^\varepsilon_s|^2ds\le LT\Big\}\\ \subseteq\;&\Big\{\sup_{t\le T}|M^\varepsilon_t|\ge\frac{\eta^2}{\varepsilon^{3/2-2\kappa}}-\frac{rT}{\varepsilon^{1/2}}-\varepsilon^{1/2}|\xi_0|^2,\;\int_0^T|\xi^\varepsilon_s|^2ds\le LT\Big\}.\end{aligned}$$

Denote by $\mathfrak{A}=\Big\{\int_0^T|\xi^\varepsilon_s|^2ds\le LT\Big\}$ and apply Lemma 1. Since on the set $\mathfrak{A}$ we have $\langle M^\varepsilon\rangle_T=\int_0^t 4|\xi^\varepsilon_s|^2\sigma^2(\xi^\varepsilon_s)ds$ $(\le\ell LT)$, by Lemma 1 the inequality holds

$$\begin{aligned}&P\Big(\varepsilon^{2(1-\kappa)}\sup_{t\le T}|\xi^\varepsilon_t|^2>\eta^2,\int_0^T|\xi^\varepsilon_s|^2ds\le LT\Big)\\ \le\;&2\exp\Big\{-\frac{\Big(\frac{\eta^2}{\varepsilon^{3/2-2\kappa}}-\frac{rT}{\varepsilon^{1/2}}-\varepsilon^{1/2}\xi_0^2\Big)^2}{2\ell LT}\Big\},\end{aligned}$$

i.e. (15) is valid.

Next, the function b obeys the following property (see (A-1)):

$$b(y) \le -cy, \ y > C \quad \text{and} \quad \ge -cy, \ y < -C$$

which provides the boundedness for the positive part of the function $\int_0^z b(y)dy$. Therefore for every fixed positive ν one can choose a positive constant c_ν such that the function $\psi(c_\nu, z) = c_\nu - \nu \int_0^z b(y)dy$ is nonnegative. Since the function b is smooth, the function $\psi(c_\nu, z)$ is twice continuously differentiable in z and the Itô formula, applied to $\psi(c_\nu, \xi_t^\varepsilon)$, gives

$$\psi(c_\nu, \xi_t^\varepsilon) = \psi(c_\nu, \xi_0) - \frac{1}{\varepsilon}\int_0^t \nu[b^2(\xi_s^\varepsilon) + \frac{1}{2}b'(\xi_s^\varepsilon)\sigma^2(\xi_s^\varepsilon)]ds - M_t^\varepsilon,$$

where $M_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\int_0^t \nu b(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)dV_s$ is the continuous martingale with the predictable quadratic variation $\langle M^\varepsilon\rangle_t = \frac{1}{\varepsilon}\int_0^t |\nu b(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)|^2 ds$. Let us estimate below the value $M_T^\varepsilon - \frac{1}{2}\langle M^\varepsilon\rangle_T$. Observe that

$$\begin{aligned}
& M_T^\varepsilon - \frac{1}{2}\langle M^\varepsilon\rangle_T \\
= \ & [\psi(c_\nu, \xi_T^\varepsilon) - \psi(c_\nu, \xi_0)] \\
& + \frac{1}{\varepsilon}\int_0^T \Big(\nu[b^2(\xi_s^\varepsilon) + \frac{1}{2}b'(\xi_s^\varepsilon)\sigma^2(\xi_s^\varepsilon)] - \frac{1}{2}|\nu b(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)|^2\Big)ds. \qquad (17)
\end{aligned}$$

An appropriate lower bound for the right side of (17) is constructed as follows. The nonnegative value $\psi(c_\nu, \xi_T^\varepsilon)$ is excluded from the right side of (17). Then, with C from assumption (A-1), we find

$$\begin{aligned}
& \int_0^T \Big[\nu\{b^2(\xi_s^\varepsilon) + \frac{1}{2}b'(\xi_s^\varepsilon)\sigma^2(\xi_s^\varepsilon)\} - \frac{1}{2}|\nu b(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)|^2\Big]ds \\
& \ge -\int_0^T \Big|\nu\{b^2(\xi_s^\varepsilon) + \frac{1}{2}b'(\xi_s^\varepsilon)\sigma^2(\xi_s^\varepsilon)\} - \frac{1}{2}|\nu b(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)|^2\Big| I(|\xi_s^\varepsilon| \le C)ds \\
& + \int_0^T \Big[\nu\{b^2(\xi_s^\varepsilon) + \frac{1}{2}b'(\xi_s^\varepsilon)\sigma^2(\xi_s^\varepsilon)\} - \frac{1}{2}|\nu b(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)|^2\Big] I(|\xi_s^\varepsilon| > C)ds. \qquad (18)
\end{aligned}$$

Due to assumption (A-1), there exists a positive constant $H(\nu, C)$, depending on ν and C, such that

$$\Big|\nu\{b^2(\xi_s^\varepsilon) + \frac{1}{2}b'(\xi_s^\varepsilon)\sigma^2(\xi_s^\varepsilon)\} - \frac{1}{2}|\nu b(\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)|^2\Big| I(|\xi_s^\varepsilon| \le C) \le H(\nu, C)$$

and thus, the first term in the right side of (17) is larger than $-H(\nu, C)T$. The second term in the right side of (17) is evaluated below by using (A-1): for $|x| > C$

$$\begin{aligned}
& b^2(x) + \frac{1}{2}b'(x)\sigma^2(x) \ge (1/c)b^2(x) \\
& b^2(x)\sigma^2(x) \le \ell b^2(x) \\
& b^2(x) \ge c^2x^2.
\end{aligned}$$

Hence, with $\nu = \nu^\circ = 1/c\ell$, it holds

$$\begin{aligned}
& \int_0^T \Big[\nu\{b^2(\xi^\varepsilon_s) + \frac{1}{2}b'(\xi^\varepsilon_s)\sigma^2(\xi^\varepsilon_s)\} - \frac{1}{2}|\nu b(\xi^\varepsilon_s)\sigma(\xi^\varepsilon_s)|^2\Big] I(|\xi^\varepsilon_s| > C)ds \\
\geq & \int_0^T \Big(b^2(\xi^\varepsilon_s)\Big[\frac{\nu^\circ}{c} - \ell\frac{(\nu^\circ)^2}{2}\Big]\Big) I(|\xi^\varepsilon_s| > C)ds \\
\geq & \int_0^T \frac{b^2(\xi^\varepsilon_s)}{2c^2\ell} I(|\xi_s|^\varepsilon) > C)ds \\
\geq & \int_0^T \frac{(\xi^\varepsilon_s)^2}{2\ell} I(|\xi^\varepsilon_s|) > C)ds
\end{aligned}$$

Therefore, on the set $\mathfrak{A} = \Big\{ \int_0^T I(|\xi^\varepsilon_s) > C)|\xi^\varepsilon_s|^2 ds > LT \Big\}$ we have

$$M^\varepsilon_T - \frac{1}{2}\langle M^\varepsilon\rangle_T \geq -\psi(c_{\nu^\circ}, \xi_0) - \frac{H(\nu^\circ, C)T}{\varepsilon} + \frac{LT}{2\ell\varepsilon}.$$

Since $H(\nu^\circ, C)$ is independent of L and ξ_0 is fixed, the value L is chosen so large to provide $\psi(c_{\nu^\circ}, \xi_0) + \frac{H(\nu^\circ,C)T}{\varepsilon} < \frac{LT}{2\ell\varepsilon}$. Then, with chosen L, by Lemma 1 we have

$$P(\mathfrak{A}) \leq \exp\Big(- \frac{LT}{2\ell\varepsilon} + \frac{H(\nu^\circ, C)T}{\varepsilon} + \psi(c_{\nu^\circ}, \xi_0)\Big) \tag{19}$$

and (14) follows. □

Remark 1 *We emphasize one estimate which is useful for verifying the statement of Theorem 1. For ε small and L large enough and any T*

$$\varepsilon^{1-2\kappa} \log P\Big(\varepsilon^{1-\kappa} \sup_{t\leq T} |\xi^\varepsilon_t| > \eta\Big) \leq -const.\Big\{\frac{(\eta^2 - T\varepsilon^{1-2\kappa})^2}{\varepsilon^{2(1-\kappa)}} \bigwedge \frac{T}{\varepsilon^{2\kappa}}\Big\} \tag{20}$$

3.2.3 Martingale $\widetilde{S}^\varepsilon_t$

The process $\widetilde{S}^\varepsilon_t = -\int_0^t v(\xi^\varepsilon_s)\sigma(\xi^\varepsilon_s)dV_s$, defined in Lemma 2, is the continuous martingale with the predictable quadratic variation

$$\langle \widetilde{S}^\varepsilon\rangle_t = \int_0^t v^2(\xi^\varepsilon_s)\sigma^2(\xi^\varepsilon_s)ds. \tag{21}$$

By assumption (A-1) the random process ξ^ε_t is ergodic in the following sense (see e.g. [14]): for every continuous and bounded function h and fixed t
$P - \lim\limits_{\varepsilon\to 0} \int_0^t h(\xi^\varepsilon_s)ds = \int_{\mathbb{R}} h(z)p(z)dz$. Hence, with γ defined in (8),

$$P - \lim_{\varepsilon\to 0}\langle \widetilde{S}^\varepsilon\rangle_t = \gamma t, t > 0.$$

The proof of Theorem 2 requires a stronger ergodic property.

Lemma 4 *Assume (A-1). Then for every $T > 0$ and $\eta > 0$*

$$\lim_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\ge\eta\Big)=-\infty.$$

Proof: Set $\mathfrak{g}(x)=v^2(x)\sigma^2(x)-\gamma$ and note that

$$\langle\widetilde{S}^\varepsilon\rangle_t-\gamma_t=\int_0^t\mathfrak{g}(\xi^\varepsilon_s)ds:=\mathfrak{S}^\varepsilon_t. \tag{22}$$

Since the function $\mathfrak{g}$ is bounded, continuously differentiable, and $\int_{\mathbb{R}}\mathfrak{g}(z)p(z)dz=0$, the function $\mathfrak{v}(z)=\frac{2}{\sigma^2(z)p(z)}\int_{-\infty}^z\mathfrak{g}(y)p(y)dy$ (compare (7)) is continuously differentiable and bounded as well. Define also the function $\mathfrak{u}(z)=\int_0^z\mathfrak{v}(y)dy$. The same arguments, which have been applied for the proof of Lemma 2, provide the Poisson decomposition

$$\mathfrak{S}^\varepsilon_t=\varepsilon^{1/2}\widetilde{\mathfrak{S}}^\varepsilon_t+\mathfrak{e}^\varepsilon_t \tag{23}$$

with

$$\begin{aligned}\mathfrak{e}^\varepsilon_t&=\varepsilon[\mathfrak{u}(\xi^\varepsilon_s)-\mathfrak{u}(\xi_0)]\\ \widetilde{\mathfrak{S}}^\varepsilon_t&=-\int_0^t\mathfrak{v}(\xi^\varepsilon_s)\sigma(\xi^\varepsilon_s)dV_s.\end{aligned} \tag{24}$$

With $\varepsilon<1$ we have $\varepsilon<\varepsilon^{1-\kappa}$ and so similarly to the proof of (13) we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|\mathfrak{e}^\varepsilon_t|>\eta\Big)=-\infty.$$

Thus, it suffices to check that $\lim\limits_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sqrt{\varepsilon}\sup_{t\le T}|\widetilde{\mathfrak{S}}^\varepsilon_t|>\eta\Big)=-\infty$. The process $\widetilde{\mathfrak{S}}^\varepsilon_t$ is the continuous martingale and its predictable quadratic variation fulfills $\langle\sqrt{\varepsilon}\widetilde{\mathfrak{S}}^\varepsilon\rangle_T\le\varepsilon\ell T$. Then by Lemma 1

$$P\Big(\sqrt{\varepsilon}\sup_{t\le T}|\widetilde{\mathfrak{S}}^\varepsilon_t|>\eta\Big)\le 2\exp\Big(-\frac{\eta^2}{2\varepsilon\ell T}\Big)$$

and the required assertion follows. □

3.3 The MDP

We are now ready to complete the proof of Theorem 2. Due to Lemma 3 the families $(\varepsilon^{1/2-\kappa}S^{\varepsilon,\kappa}_t)_{t\ge 0}$ and $(\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t)_{t\ge 0}$ are exponentially indistinguishable with the rate of speed $\varepsilon^{1-2\kappa}$, that is if one of family obey the MDP with the rate of speed $\varepsilon^{1-2\kappa}$, then the another family possesses the same property. We will examine the MDP for the family of martingales $(\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t)_{t\ge 0}$. To this end, we apply the Dawson-Gärtner theorem (see e.g. [10] and [22]) which states that it suffices to check that the family $(\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t)_{t\le T}$ obeys the MDP (for every $T>0$) in the metric space $(\mathbb{C}_{[0,T]},r_T)$ (r_T is the uniform metric on $[0,T]$) with the rate of speed $\varepsilon^{1-2\kappa}$ and the rate function

$$J_T(\varphi)=\begin{cases}\frac{1}{2\gamma}\int\limits_0^T\dot\varphi^2(t)\,dt, & d\varphi(t)=\dot\varphi(t)\,dt,\ \varphi(0)=0\\ \infty, & \text{otherwise.}\end{cases} \tag{25}$$

For fixed T, for the verification of the above-mentioned MDP we use well known implication (see e.g. in [17] Theorem 1.3) formally reformulated here for the MDP case:

$$\left.\begin{array}{l}\text{Exponential tightness}\\ \text{Local MDP}\end{array}\right\} \Longrightarrow \text{MDP}. \tag{26}$$

Let us recall the definitions of *exponential tightness* and *local MDP*.

Following Deushel and Stroock [6] (see also Lynch and Sethuraman [20])), the family

$$(\varepsilon^{1/2-\kappa}\widetilde{S}^{\varepsilon}_t)_{0\le t\le T}, \varepsilon \searrow 0$$

is exponentially tight in $(\mathbb{C}_{[0,T]}, r_T)$ with the rate of speed $\varepsilon^{1-2\kappa}$, if there exists a sequence of compacts $(K_j)_{j\ge 1}$: $K_j \nearrow \mathbb{C}_{[0,T]}$ such that

$$\lim_{j}\limsup_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big((\varepsilon^{1/2-\kappa}\widetilde{S}^{\varepsilon}_t)_{0\le t\le T}\in \mathbb{C}_{[0,T]}\setminus K_j\Big) = -\infty \tag{27}$$

Effective sufficient conditions for (27) are known from Pukhalskii [22]

$$\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\varepsilon^{1/2-\kappa}\sup_{t\le T}|\widetilde{S}^{\varepsilon}_t| > L\Big) = -\infty \tag{28}$$

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log\sup P\Big(\varepsilon^{1/2-\kappa}\sup_{t\le\delta}|\widetilde{S}^{\varepsilon}_{\tau+t}-\widetilde{S}^{\varepsilon}_{\tau}| > \eta\Big) = -\infty,\ \eta > 0, \tag{29}$$

where "sup" is taken over all stopping times $\tau \le T$.

Following Freidlin and Wentzell [12], the family $(\varepsilon^{1/2-\kappa}\widetilde{S}^{\varepsilon}_t)_{0\le t\le T}, \varepsilon \searrow 0$ obeys the local MDP in $(\mathbb{C}_{[0,T]}, r_T)$ with the rate of speed $\varepsilon^{1-2\kappa}$ and the local rate function J_T, if for every $\varphi \in \mathbb{C}_{[0,T]}$

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^{\varepsilon}_t-\varphi_t|\le\delta\Big) \le -J_T(\varphi) \tag{30}$$

$$\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^{\varepsilon}_t-\varphi_t|\le\delta\Big) \ge -J_T(\varphi). \tag{31}$$

3.3.1 Verification of (28)

Introduce Markov times (with $\inf\{\varnothing\} = \infty$)

$$\sigma^{\pm}_{L,\varepsilon} = \inf\left\{t > 0 : \varepsilon^{1/2-\kappa}\widetilde{S}^{\varepsilon}_t\left\{\begin{array}{l} > +L\\ < -L\end{array}\right.\right\}$$

and note that (28) holds, if $\lim\limits_{L\to\infty}\limsup\limits_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sigma^{\pm}_{L,\varepsilon}\le T\Big) = -\infty$. Taking into account the statement of Lemma 4, is suffices to verify

$$\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sigma^{\pm}_{L,\varepsilon}\le T, \sup_{t\le T}|\langle\widetilde{S}^{\varepsilon}\rangle_t-\gamma t|\le\eta\Big) = -\infty,\ \eta > 0. \tag{32}$$

We consider separately cases "$\pm$". Since $\widetilde{S}^\varepsilon_t$ is the continuous martingale, the positive process

$$\mathfrak{z}_t(\lambda)=\exp\Big(\lambda\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\frac{\lambda^2}{2}\varepsilon^{1-2\kappa}\langle\widetilde{S}^\varepsilon\rangle_t\Big)$$

is the local martingale and the supermartingale as well, that is $E\mathfrak{z}_{\sigma^+_{L,\varepsilon}}\le 1$. Now, write $1\ge E\mathfrak{z}_{\sigma^+_{L,\varepsilon}}(\lambda)I\Big(\sigma^+_{L,\varepsilon}\le T,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta\Big)$. Under $\lambda>0$, the random variable $\mathfrak{z}_{\sigma^+_{L,\varepsilon}}(\lambda)$ is evaluated below on the set $\{\sigma^\pm_{L,\varepsilon}\le T,\sup\limits_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta\}$ as: $\mathfrak{z}_{\sigma^+_{L,\varepsilon}}(\lambda)\ge\exp\Big(\lambda L-\frac{\lambda^2}{2}\varepsilon^{1-2\kappa}\langle\widetilde{S}^\varepsilon\rangle_T\Big)\ge\exp\Big(\lambda L-\frac{\lambda^2}{2}\varepsilon^{1-2\kappa}(\gamma T+\eta)\Big)$ while the choice of $\lambda=\frac{L}{\varepsilon^{1-2\kappa}(\gamma T+\eta)}$ implies

$$\varepsilon^{1-2\kappa}\log P\Big(\sigma^+_{L,\varepsilon}\le T,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta\Big)\le-\frac{L^2}{\gamma T+\eta}\to-\infty,\ L\to\infty.$$

The proof for the case "$-$" is similar. $\square$

3.3.2 Verification of (29)

Due to the statement of Lemma 4, it suffices to show that, as long as $\eta'\to 0$,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log\sup P\Big(\varepsilon^{1/2-\kappa}\sup_{t\le\delta}|\widetilde{S}^\varepsilon_{\tau+t}-\widetilde{S}^\varepsilon_\tau|>\eta,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\Big)\to 0.$$

Let us first take $\tau=0$. Set $\sigma^\pm_{\eta,\varepsilon}=\inf\Big\{t>0:\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t\Big\{\begin{matrix}>+\eta\\<-\eta\end{matrix}\Big\}$ and use the obvious inclusion

$$\{\varepsilon^{1/2-\kappa}\sup_{t\le\delta}|\widetilde{S}^\varepsilon_t|>\eta,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\}\subseteq\{\sigma^\pm_{\eta,\varepsilon}\le\delta,\sup_{t\le T\wedge\delta}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\}.$$

The inequality $E\mathfrak{z}_{\sigma^\pm_{\eta,\varepsilon}}(\lambda)\le 1$ implies

$$1\ge E\mathfrak{z}_{\sigma^\pm_{\eta,\varepsilon}}(\lambda)I\Big(\sigma^\pm_{\eta,\varepsilon}\le\delta,\sup_{t\le T\wedge\delta}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\Big).$$

The lower bound $\mathfrak{z}_{\sigma^+_{\eta,\varepsilon}}(\lambda)\ge e^{\lambda\eta-\varepsilon^{1-2\kappa}\frac{\lambda^2}{2}(\eta'+\gamma(T\wedge\delta))}$ is valid for any positive λ on the set $\{\sigma^+_{\eta,\varepsilon}\le\delta,\sup\limits_{t\le T\wedge\delta}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\}$ while the choice of $\lambda=\frac{\eta}{\varepsilon^{1-2\kappa}(\eta'+\gamma(T\wedge\delta))}$ implies

$$\varepsilon^{1-2\kappa}\log P\Big(\sigma^+_{\eta,\varepsilon}\le\delta,\sup_{t\le T\wedge\delta}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\Big)\le-\frac{\eta^2}{2\gamma(\eta'+T\wedge\delta)}.$$

The same upper bound holds with $\sigma^-_{\eta,\varepsilon}$ and so that

$$\varepsilon^{1-2\kappa}\log P\Big(\sigma^\pm_{\eta,\varepsilon}\le\delta,\sup_{t\le T\wedge\delta}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\Big)\le-\log 2\frac{\eta^2}{2\gamma(\eta'+T\wedge\delta)}.$$

Furthermore, for $0 < \tau \le T$, we have the same estimate:

$$\begin{aligned}&\varepsilon^{1-2\kappa}\log\sup P\Big(\varepsilon^{1/2-\kappa}\sup_{t\le\delta}|\widetilde{S}^\varepsilon_{\tau+t}-\widetilde{S}^\varepsilon_\tau|>\eta,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\Big)\\ \le\;& -\log 2\frac{\eta^2}{2\gamma(\eta'+(T+\delta)\wedge\delta)}.\end{aligned} \tag{33}$$

In fact, one can consider a new martingale $\widetilde{S}^{\circ,\varepsilon}_{\tau+t}=\widetilde{S}^\varepsilon_{\tau+t}-\widetilde{S}^\varepsilon_\tau$ with respect to the filtration $(\mathcal{F}_{\tau+t})_{t\ge 0}$ with $\langle\widetilde{S}^{\circ,\varepsilon}\rangle_t=\langle\widetilde{S}^\varepsilon\rangle_{\tau+t}-\langle\widetilde{S}^\varepsilon\rangle_\tau$, and apply the same arguments.
The right side of (33) tends to $-\infty$, as $\delta\to 0,\eta'\to 0$, and (29) holds. □

3.3.3 Verification of (30)

By virtue of Lemma 4, we check only

$$\begin{aligned}&\limsup_{\eta'\to 0}\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\varphi_t|\le\delta,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\Big)\\ \le\;& -J_T(\varphi).\end{aligned} \tag{34}$$

For $\varphi_0\ne 0$ the left side of (34) is equal to $-\infty$. Hence, for $\varphi_0\ne 0$ the desired upper bound holds.

Let $\varphi_0=0$. Introduce the continuous local martingale $M^\varepsilon_t=\frac{1}{\varepsilon^{1/2-\kappa}}\int\limits_0^t\lambda(s)d\widetilde{S}^\varepsilon_s$ with $\langle M^\varepsilon\rangle_t=\frac{1}{\varepsilon^{1-2\kappa}}\int\limits_0^t\lambda^2(s)d\langle\widetilde{S}^\varepsilon\rangle_s$, where $\lambda(t)$ is piece wise constant right continuous function. The process $\mathfrak{z}_t(\lambda)=e^{M^\varepsilon_t-\frac12\langle M^\varepsilon\rangle_t}$ is the positive local martingale with $E\mathfrak{z}_T(\lambda)\le 1$. This inequality implies

$$1\ge E\mathfrak{z}_T(\lambda)I\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\varphi_t|\le\delta,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\Big).$$

Now, we find a lower bound for the random value $\mathfrak{z}_T(\lambda)$ on the set

$$\mathfrak{U}_{\varepsilon,\delta,\eta'}=\{\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\varphi_t|\le\delta,\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\eta'\}.$$

Since $\lambda(t)$ is right continuous having limit to the left function of bounded variation on $[0,T]$, the Itô formula for $\lambda(t)\widetilde{S}^\varepsilon_t$ is valid:

$$\lambda(T)\widetilde{S}^\varepsilon_T=\int_0^T\lambda(s)d\widetilde{S}^\varepsilon_s+\int_0^T\widetilde{S}^\varepsilon_s d\lambda(s).$$

Applying it, we find

$$\begin{aligned}M^\varepsilon_T &= \frac{1}{\varepsilon^{1-2\kappa}}\Big[\varepsilon^{1/2-\kappa}\Big(\lambda(T)\widetilde{S}^\varepsilon_T-\int_0^T\widetilde{S}^\varepsilon_s d\lambda(s)\Big)\Big]\\ &= \frac{1}{\varepsilon^{1-2\kappa}}\Big[\lambda(T)\varphi_T-\int_0^T\varphi_s d\lambda(s)\Big]+\frac{1}{\varepsilon^{1-2\kappa}}\lambda(T)\{\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_T-\varphi_T\}\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\varepsilon^{1-2\kappa}}\int_0^T \{\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_s-\varphi_s\}d\lambda(s) \\
\langle M^\varepsilon\rangle_T \ &= \ \frac{1}{\varepsilon^{1-2\kappa}}\Big(\lambda^2(T)\langle\widetilde{S}^\varepsilon\rangle_T-\int_0^T\langle\widetilde{S}^\varepsilon\rangle_s d\lambda^2(s)\Big) \\
&= \ \frac{1}{\varepsilon^{1-2\kappa}}\Bigg(\frac{1}{2}\int_0^T\lambda^2(s)\gamma ds+\lambda^2(T)\{\langle\widetilde{S}^\varepsilon\rangle_T-\gamma T\} \\
& -\int_0^T\{\langle\widetilde{S}^\varepsilon\rangle_s-\gamma s\}d\lambda^2(s)\Bigg).
\end{aligned}
$$

Hence, there is a constant ℓ, depending on T and λ, so that on the set $\mathfrak{U}_{\varepsilon,\delta,\eta'}$,

$$
\begin{aligned}
M^\varepsilon_T \ &\geq \ \frac{1}{\varepsilon^{1-2\kappa}}\Big[\lambda(T)\varphi_T-\int_0^T\varphi_s d\lambda(s)\Big]-\frac{\ell\delta}{\varepsilon^{1-2\kappa}} \\
\langle M^\varepsilon\rangle_T \ &\leq \ \frac{1}{\varepsilon^{1-2\kappa}}\Big[\frac{1}{2}\int_0^T\lambda^2(s)\gamma ds+\ell\eta'\Big],
\end{aligned}
$$

that is on $\mathfrak{U}_{\varepsilon,\delta,\eta'}$ the following nonrandom lower bound takes place

$$
\log\mathfrak{z}_T(\lambda)\geq\frac{1}{\varepsilon^{1-2\kappa}}\Big(\lambda(T)\varphi_T-\int_0^T\varphi_s d\lambda(s)-\frac{1}{2}\int_0^T\lambda^2(s)\gamma ds-\ell\delta-\ell\eta'\Big)
$$

and thereby

$$
\begin{aligned}
& \varepsilon^{1-2\kappa}\log P\Big(\sup_{t\leq T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\varphi_t|\leq\delta,\sup_{t\leq T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\leq\eta'\Big) \\
\leq & \ -\lambda(T)\varphi_T-\int_0^T\varphi_s d\lambda(s)-\frac{1}{2}\int_0^T\lambda^2(s)\gamma ds-\ell\delta-\ell\eta' \\
\to & \ -\lambda(T)\varphi_T-\int_0^T\varphi_s d\lambda(s)-\frac{1}{2}\int_0^T\lambda^2(s)\gamma ds, \ \ \delta,\eta'\to 0. \qquad (35)
\end{aligned}
$$

If φ_t is not absolutely continuous function, one can choose a sequence of piece wise constant and right continuous functions $\lambda^n(t)$'s so that the right side of (35) tends to $-\infty$ along with $n\to\infty$. If φ_t is absolutely continuous function the right side of (35) is transformed into $U(\lambda,\varphi)=-\int_0^T(\lambda(s)\dot\varphi_s-\frac{1}{2}\lambda^2(s)\gamma)ds$ and thus, $J_T(\varphi)=\sup\limits_\lambda(-U(\lambda,\varphi))$. □

3.3.4 Verification of (31)

We use the obvious lower bound

$$
P\Big(\sup_{t\leq T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\varphi_t|\leq\delta\Big)\geq P\Big(\sup_{t\leq T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\varphi_t|\leq\delta,\sup_{t\leq T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\leq\eta'\Big)
$$

and prove

$$
\begin{aligned}
& \liminf_{\eta'\to0}\liminf_{\delta\to0}\liminf_{\varepsilon\to0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\leq T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t-\varphi_t|\leq\delta,\sup_{t\leq T}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\leq\eta'\Big) \\
\geq & \ -J_T(\phi). \qquad (36)
\end{aligned}
$$

It is clear that the verification of (36) is required only for absolutely continuous functions φ_t with $\varphi_0 = 0$ and $J_T(\varphi) < \infty$. Moreover, this class of functions can be reduced to twice continuously differentiable functions $(\varphi_t)_{t\le T}$ with $\varphi_0 = 0$. In fact, if $\varphi_0 = 0$ and $J_T(\varphi) < \infty$ but φ_t is absolutely continuous only and even $\dot\varphi_t$ is unbounded, then one can choose a sequence $\varphi^n, n \ge 1$ of twice continuously differentiable functions with $\varphi^n_0 \equiv 0$ such that $\lim\limits_{n\to\infty} r_T(\varphi, \varphi^n) = 0$, $\lim\limits_{n\to\infty} J_T(\varphi^n) = J_T(\varphi)$. If for every n we have

$$\begin{aligned}&\liminf_{\eta'\to 0}\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t - \varphi^n_t| \le \delta, \sup_{t\le T}|\langle \widetilde{S}^\varepsilon\rangle_t - \gamma t| \le \eta'\Big)\\ \ge\ & -J_T(\varphi^n), \end{aligned} \tag{37}$$

then, choosing n° so that for $n \ge n^\circ$ it holds $r_T(\varphi, \varphi^n) \le \frac{\delta}{2}$, by virtue of the triangular inequality $r_T(\varepsilon^{1/2-\kappa}\widetilde{S}, \varphi) \le r_T(\varepsilon^{1/2-\kappa}\widetilde{S}, \varphi^n) + r_T(\varphi, \varphi^n)$ we get

$$\begin{aligned}&P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t - \varphi_t| \le \delta, \sup_{t\le T}|\langle \widetilde{S}^\varepsilon\rangle_t - \gamma t| \le \eta'\Big)\\ \ge\ & P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t - \varphi^n_t| \le \frac{\delta}{2}, \sup_{t\le T}|\langle \widetilde{S}^\varepsilon\rangle_t - \gamma t| \le \eta'\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\liminf_{\eta'\to 0}\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^{1-2\kappa}\log P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t - \varphi_t| \le \delta, \sup_{t\le T}|\langle \widetilde{S}^\varepsilon\rangle_t - \gamma t| \le \eta'\Big)\\ \ge\ & -J_T(\varphi^n) \to -J_T(\varphi),\ n\to\infty.\end{aligned}$$

Thus, let φ_t be twice continuously differentiable function with $\varphi_0 = 0$. Set $\lambda^\circ(t) = \frac{\dot\varphi_t}{\gamma}$ and define the martingale $M^\varepsilon_t = \frac{1}{\varepsilon^{1/2-\kappa}}\int\limits_0^t \lambda^\circ(s)d\widetilde{S}^\varepsilon_s$ with

$$\langle M^\varepsilon\rangle_T = \frac{1}{\varepsilon^{1-2\kappa}}\int_0^T \frac{\dot\varphi^2_t}{\gamma^2}d\langle\widetilde{S}^\varepsilon\rangle_t = \frac{1}{\varepsilon^{1-2\kappa}}\int_0^T \frac{\dot\varphi^2_t}{\gamma^2}v^2(\xi^\varepsilon_t)\sigma^2(\xi^\varepsilon_t)dt \le \text{const.}(\varepsilon)$$

and so, $\mathfrak{z}_t(\lambda^\circ) = e^{M^\varepsilon_t - \frac12\langle M^\varepsilon\rangle_t}$ is the martingale, $E\mathfrak{z}_T(\lambda^\circ) = 1$. We use this equality to introduce new probability measure P°: $dP^\circ = \mathfrak{z}_T(\lambda^\circ)dP$. Since $\mathfrak{z}_T(\lambda^\circ) > 0$, P-a.s., not only $P^\circ \ll P$ but also $P \ll P^\circ$ with $dP = \mathfrak{z}^{-1}_T(\lambda^\circ)dP^\circ$. Write

$$\begin{aligned}&P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t - \varphi_t| \le \delta, \sup_{t\le T}|\langle \widetilde{S}^\varepsilon\rangle_t - \gamma t| \le \eta'\Big)\\ =\ & \int\limits_{\big\{\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t - \varphi_t|\le\delta\big\}\bigcap\big\{\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t - \gamma t|\le\eta'\big\}} \mathfrak{z}^{-1}_T(\lambda^\circ)dP^\circ.\end{aligned}$$

On the set $\{\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t - \varphi_t| \le \delta\}\bigcap\{\sup_{t\le T}|\langle\widetilde{S}^\varepsilon\rangle_t - \gamma t| \le \eta'\}$ the random variable $\mathfrak{z}^{-1}_T(\lambda^\circ)$ possesses nonrandom lower bound (ℓ is a generic constant):

$$\mathfrak{z}^{-1}_T(\lambda^\circ) \ge \exp\Big(-\frac{1}{\varepsilon^{1-2\kappa}}\Big\{J_T(\varphi) + \ell\delta + \ell\eta'\Big\}\Big).$$

Therefore

$$\begin{aligned}
&\frac{1}{\varepsilon^{1-2\kappa}}\log P\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde S^\varepsilon_t-\varphi_t|\le\delta,\sup_{t\le T}|\langle\widetilde S^\varepsilon\rangle_t-\gamma t|\le\eta'\Big)\\
\ge\ &-J_T(\varphi)-\ell(\delta+\eta')\\
&+\frac{1}{\varepsilon^{1-2\kappa}}\log P^\circ\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde S^\varepsilon_t-\varphi_t|\le\delta,\sup_{t\le T}|\langle\widetilde S^\varepsilon\rangle_t-\gamma t|\le\eta'\Big).
\end{aligned}$$

To finish the proof, it remains to check that for every $\delta>0,\eta'>0$

$$\lim_{\varepsilon\to0}P^\circ\Big(\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde S^\varepsilon_t-\varphi_t|>\delta\Big)=0. \tag{38}$$

$$\lim_{\varepsilon\to0}P^\circ\Big(\sup_{t\le T}|\langle\widetilde S^\varepsilon\rangle_t-\gamma t|>\eta'\Big)=0 \tag{39}$$

It is well known (see e.g. Theorem 4.5.2 in [19]) that the random process $(\widetilde S^\varepsilon_t)_{t\le T}$, being P-continuous martingale, is transformed to P°-continuous semimartingale with the decomposition $\widetilde S^\varepsilon_t=A^\varepsilon_t+N^\varepsilon_t$, where $(N^\varepsilon_t)_{t\le T}$ is the continuous local martingale having $\langle\widetilde S^\varepsilon\rangle_t$ as the quadratic variation and the drift A^ε_t is defined via the mutual variation $\langle\mathfrak{z}(\lambda^\circ),\widetilde S^\varepsilon\rangle_t$ of martingales $\mathfrak{z}(\lambda^\circ)_t$ and $\widetilde S^\varepsilon_t$ as:

$$A^\varepsilon_t=\int_0^t\mathfrak{z}_s^{-1}(\lambda^\circ)d\langle\mathfrak{z}(\lambda^\circ),\widetilde S^\varepsilon\rangle_s.$$

Hence $A^\varepsilon_t=\frac{1}{\varepsilon^{1/2-\kappa}}\int\limits_0^t\frac{\dot\varphi_s}{\gamma}d\langle\widetilde S^\varepsilon\rangle_s$ and we find that

$$\begin{aligned}
\varepsilon^{1/2-\kappa}\widetilde S^\varepsilon_t-\varphi_t&=\Big(\int_0^t\frac{\dot\varphi_s}{\gamma}d\langle\widetilde S^\varepsilon\rangle_s-\varphi_t\Big)+\varepsilon^{1/2-\kappa}N^\varepsilon_t\\
&=\frac{1}{\gamma}\int_0^t\dot\varphi_s d(\langle\widetilde S^\varepsilon\rangle_s-\gamma s)+\varepsilon^{1/2-\kappa}N^\varepsilon_t\\
&=\frac{1}{\gamma}\dot\varphi_T(\langle\widetilde S^\varepsilon\rangle_t-\gamma t)-\int_0^t\ddot\varphi_s(\langle\widetilde S^\varepsilon\rangle_s-\gamma s)ds+\varepsilon^{1/2-\kappa}N^\varepsilon_t.
\end{aligned}$$

Therefore, with a suitable constant ℓ, we obtain

$$\sup_{t\le T}|\varepsilon^{1/2-\kappa}\widetilde S^\varepsilon_t-\varphi_t|\le\ell\sup_{t\le T}|\langle\widetilde S^\varepsilon\rangle_t-\gamma t|+\varepsilon^{1/2-\kappa}\sup_{t\le T}|N^\varepsilon_t|.$$

As was mentioned above $\langle N^\varepsilon\rangle_T$ coincides with $\langle\widetilde S^\varepsilon\rangle_T$ ($P-$ and P°-a.s.) and so, it is bounded P°-a.s. Now, by the Doob inequality (E° is the expectation with respect to P°)

$$P^\circ(\varepsilon^{1/2-\kappa}\sup_{t\le T}|N^\varepsilon_t|>\eta'')\le\frac{\varepsilon^{1-2\kappa}}{(\eta'')^2}E^\circ\langle N^\varepsilon\rangle_T\to0,\ \varepsilon\to0,\ \forall\ \eta''>0.$$

The latter property allows to conclude that (38) holds provided that (39) is valid.

Thus, only (39) remains to prove.

Let us recall that $\langle \widetilde{S}^\varepsilon\rangle_t - \gamma t$ obeys the Poisson decomposition (see (23) and (24))

$$\langle \widetilde{S}^\varepsilon\rangle_t - \gamma t = \varepsilon^{1/2}\widetilde{\mathfrak{S}}^\varepsilon_t + \mathfrak{e}^\varepsilon_t \tag{40}$$

with $\mathfrak{e}^\varepsilon_t = \varepsilon[\mathfrak{u}(\xi^\varepsilon_s) - \mathfrak{u}(\xi_0)]$ and $\widetilde{\mathfrak{S}}^\varepsilon_t = -\int_0^t \mathfrak{v}(\xi^\varepsilon_s)\sigma(\xi^\varepsilon_s)dV_s$. The martingale $\widetilde{\mathfrak{S}}^\varepsilon_t$ obeys P°-decomposition $\widetilde{\mathfrak{S}}^\varepsilon_t = \mathfrak{A}^\varepsilon_t + \mathfrak{N}^\varepsilon_t$ with

$$\mathfrak{A}^\varepsilon_t = \int_0^t \mathfrak{z}_s^{-1}(\lambda^\circ)d\langle \mathfrak{z}(\lambda^\circ), \widetilde{\mathfrak{S}}^\varepsilon\rangle_t$$

and continuous local martingale $\mathfrak{N}^\varepsilon_t$ with $\langle \mathfrak{N}^\varepsilon\rangle_t \equiv \langle \mathfrak{S}^\varepsilon\rangle_t$ (P- and P°-a.s.). As was mentioned in Subsection 3.2.3, $\langle \mathfrak{S}^\varepsilon\rangle_T$ is bounded. Then, by the Doob inequality

$$P^\circ\Big(\varepsilon^{1/2}\sup_{t\le T}|\mathfrak{N}^\varepsilon_t| > \eta''\Big) \le \frac{\varepsilon}{(\eta'')^2}E^\circ\langle \mathfrak{N}^\varepsilon\rangle_T \to 0, \varepsilon\to 0, \ \forall\ \eta''>0. \tag{41}$$

Show now that

$$\lim_{\varepsilon\to 0} P^\circ\Big(\varepsilon^{1/2}\sup_{t\le T}|\mathfrak{A}^\varepsilon_t| > \eta''\Big) = 0, \ \forall\ \eta''>0. \tag{42}$$

To this end, we find an upper bound for $\sup\limits_{t\le T}|\mathfrak{A}^\varepsilon_t|$. Since $d\mathfrak{z}_t(\lambda^\circ) = \mathfrak{z}_t(\lambda^\circ)\lambda^\circ(t)d\widetilde{S}^\varepsilon_t$, we get $\mathfrak{A}^\varepsilon_t = \int_0^t \frac{\dot{\varphi}_s}{\gamma}d\langle \widetilde{\mathfrak{S}}^\varepsilon, \widetilde{S}^\varepsilon\rangle_s$, where the mutual variation $\langle \widetilde{\mathfrak{S}}^\varepsilon, \widetilde{S}^\varepsilon\rangle_t$ of P-martingales $\widetilde{\mathfrak{S}}^\varepsilon_t$ and $\widetilde{S}^\varepsilon_t$ is given, due to (12) and (24), by the formula

$$\langle \widetilde{\mathfrak{S}}^\varepsilon, \widetilde{S}^\varepsilon\rangle_t = \int_0^t v(\xi^\varepsilon_s)\mathfrak{v}(\xi^\varepsilon_s)\sigma^2(\xi^\varepsilon_s)ds.$$

The boundedness of v, $\mathfrak{v}$, and σ^2 implies $\sup\limits_{t\le T}|\mathfrak{A}^\varepsilon_t| \le$ const. and so (42) holds.

To finish the proof of (39), it remains to check

$$\lim_{\varepsilon\to 0} P^\circ\Big(\sup_{t\le T}|\mathfrak{e}^\varepsilon_t| > \eta\Big) = 0. \tag{43}$$

¿From the definition of $\mathfrak{e}^\varepsilon_t$ it follows the existence of positive constants L_1, L_2 so that $\sup\limits_{t\le T}|\mathfrak{e}^\varepsilon_t| \le \varepsilon(L_1 + L_2\sup\limits_{t\le T}|\xi^\varepsilon_t|)$. Therefore, we prove below

$$\lim_{\varepsilon\to 0} P^\circ\Big(\sup_{t\le T}\varepsilon|\xi^\varepsilon_t| > \eta''\Big) = 0, \ \forall\ \eta''>0. \tag{44}$$

The verification of (44) uses the semimartingale decomposition of ξ^ε_t with respect to P°. Set $\Phi^\varepsilon_t = \frac{1}{\sqrt{\varepsilon}}\int\limits_0^t \sigma(\xi^\varepsilon_s)dV^\varepsilon_s$. The P-martingale Φ^ε_t is the semimartingale with respect to P° with the decomposition: $\Phi^\varepsilon_t = \mathfrak{L}^\varepsilon_t + \mathfrak{M}^\varepsilon_t$, where $\mathfrak{M}^\varepsilon_t$ is continuous local martingale with $\langle \mathfrak{M}^\varepsilon\rangle_t = \langle \Phi^\varepsilon\rangle_t$ (P- and P°-a.s.) and $\mathfrak{L}_t = \int\limits_0^t \mathfrak{z}^{-1}_s(\lambda^\circ) d\langle \mathfrak{z}(\lambda^\circ), U^\varepsilon\rangle_s$. Consequently,

$$\mathfrak{L}_t = \varepsilon^\kappa \int_0^t \frac{\dot\varphi_s}{\gamma} v(\xi^\varepsilon_s)\sigma^2(\xi^\varepsilon_s) ds$$

$$\mathfrak{M}^\varepsilon_t = \frac{1}{\sqrt{\varepsilon}} \int\limits_0^t \sigma(\xi^\varepsilon_s) dV^{\varepsilon,\circ}_s,$$

where $V^{\varepsilon,\circ}_t$ is P°-Wiener process.

Hence

$$P^\circ : \quad d\xi^\varepsilon_t = \frac{1}{\varepsilon}\Big(b(\xi^\varepsilon_t) + \varepsilon^{1+\kappa}\frac{\dot\varphi_t}{\gamma} v(\xi^\varepsilon_t)\sigma^2(\xi^\varepsilon_t)\Big)dt + \frac{1}{\sqrt{\varepsilon}}\sigma(\xi^\varepsilon_t) dV^{\varepsilon,\circ}_t$$
$$\xi_0 \text{ "is deterministic".}$$

By the Itô formula we find

$$\begin{aligned}\varepsilon^2(\xi^\varepsilon_t)^2 &= \varepsilon^2(\xi_0)^2 + 2\varepsilon\int_0^t \xi^\varepsilon_s(b(\xi^\varepsilon_s) + \varepsilon^{1+\kappa}\frac{\dot\varphi_t}{\gamma} v(\xi^\varepsilon_t)\sigma^2(\xi^\varepsilon_t) + \frac{1}{2}\sigma^2(\xi^\varepsilon_t))ds \\ &\quad + 2\varepsilon^{3/2}\int_0^t \xi^\varepsilon_s \sigma(\xi^\varepsilon_s) dV^{\varepsilon,\circ}_s.\end{aligned}$$

Due to (A-1), $\sup\limits_{t\le T}\Big(\int\limits_0^t \xi^\varepsilon_s(b(\xi^\varepsilon_s) + \varepsilon^{1+\kappa}\frac{\dot\varphi_t}{\gamma} v(\xi^\varepsilon_t)\sigma^2(\xi^\varepsilon_t) + \frac{1}{2}\sigma^2(\xi^\varepsilon_t))ds\Big)$ is bounded by a positive constant independent of ε. The Itô integral $\int\limits_0^t \xi^\varepsilon_s\sigma(\xi^\varepsilon_s)dV^{\varepsilon,\circ}_s$ is continuous local P°-martingale. Let $(\tau_n)_{n\ge 1}$ be its localizing sequence. Then

$$E^\circ \varepsilon^2(\xi^\varepsilon_{t\wedge\tau_n})^2 \le \varepsilon^2(\xi_0)^2 + \varepsilon \text{ const.}, \ n \ge 1$$

and by the Fatou lemma $E^\circ\varepsilon^2(\xi^\varepsilon_t)^2 \le \varepsilon^2(\xi^\varepsilon_0)^2 + \varepsilon$ const. Consequently, with $\sigma^2 \le \ell$,

$$\varepsilon^{9/4}\int_0^T E^\circ(\xi^\varepsilon_s)^2\sigma^2(\xi^\varepsilon_s)ds \le \ell T(\varepsilon^{1/4}(\xi_0)^2 + \varepsilon^{5/4} \text{ const.}).$$

Then, by the Doob inequality

$$P^\circ\Big(\sup_{t\le T}\varepsilon^{3/2}\Big|\int_0^t \xi^\varepsilon_s\sigma(\xi^\varepsilon_s)dV^\varepsilon_s\Big| > \eta''\Big) \le \frac{4\ell T(\varepsilon^{1/4}(\xi_0)^2 + \varepsilon^{5/4} \text{ const.})}{(\eta'')^2} \to 0, \ \varepsilon \to 0.$$

Consequently (44) holds. □

3.4 Proof of Theorem 1

As for $\Psi \equiv 1$, the proof uses the Poisson decomposition from Lemma 2: $\frac{1}{\varepsilon^\kappa}\int_0^t g(\xi^\varepsilon_s)\,ds = e^{\varepsilon,\kappa}_t + \varepsilon^{1/2-\kappa}\widetilde{S}^\varepsilon_t$, which for $U_t = \frac{1}{\varepsilon^\kappa}\int_0^t \Psi(X^\varepsilon_s)g(\xi^\varepsilon_s)ds$ implies "Poisson type" decomposition

$$U_t = \int_0^t \Psi(X^\varepsilon_s)\,de^{\varepsilon,\kappa}_s + \varepsilon^{1/2-\kappa}\int_0^t \Psi(X^\varepsilon_s)\,d\widetilde{S}^\varepsilon_s, \tag{45}$$

where the first term in this decomposition is the Itô integral with respect to the semimartingale $e_t^{\varepsilon,\kappa}$. Since the function Ψ is twice continuously differentiable, we decompose also $\int_0^t \Psi(X_s^\varepsilon)\, de_s^{\varepsilon,\kappa}$ applying the Itô formula to $\Psi(X_t^\varepsilon)e_t^{\varepsilon,\kappa}$:

$$\begin{aligned}\int_0^t \Psi(X_s^\varepsilon) de_s^{\varepsilon,\kappa} &= \Psi(X_t^\varepsilon)e_t^{\varepsilon,\kappa} \\ &\quad - \int_0^t e_s^{\varepsilon,\kappa}[\dot{\Psi}^\varepsilon(X_s^\varepsilon)F(X_s^\varepsilon,\xi_s^\varepsilon) + \frac{1}{2}\ddot{\Psi}^\varepsilon(X_s^\varepsilon)G^2(X_s^\varepsilon,\xi_s^\varepsilon)]\, ds \\ &\quad - \int_0^t e_s^{\varepsilon,\kappa}\dot{\Psi}^\varepsilon(X_s^\varepsilon)G(X_s^\varepsilon,\xi_s^\varepsilon)\, dW_s \\ &:= U_t^{(1)} + U_t^{(2)} + U_t^{(3)}.\end{aligned}$$

Now, with $U_t^{(4)} = \varepsilon^{1/2-\kappa}\int_0^t \Psi(X_s^\varepsilon)\, d\widetilde{S}_s^\varepsilon$, we arrive at the final decomposition $U_t = U_t^{(1)} + U_t^{(2)} + U_t^{(3)} + U_t^{(4)}$ and show that $U^{(4)}$ delivers the main contribution in the required estimate announced in Theorem 1:

$$\overline{\lim_{\varepsilon\to 0}}\, (\varepsilon T^\varepsilon)^{1-2\kappa} \log P\Big(\sup_{t\le T^\varepsilon} |U_t^{(4)}| \ge \Psi^*(T^\varepsilon)^\kappa z\Big) \le -\frac{z^2}{2\gamma},\ \forall z > 0 \tag{46}$$

while the others $U_t^{(i)},\ i = 1,2,3$ are exponentially negligible:

$$\lim_{\varepsilon\to 0}\, (\varepsilon T^\varepsilon)^{1-2\kappa} \log P\Big(\sup_{t\le T^\varepsilon} |U_t^{(i)}| > \delta\Psi^*(T^\varepsilon)^\kappa\Big) = -\infty,\ \forall\delta > 0,\ i = 1,2,3. \tag{47}$$

Lemma 5 *Under the assumptions of Theorem 1, (47) holds.*

Proof: (i=1): Since by (A-4) $T^\varepsilon \ge d > 0$, it suffices to verify only that

$$\lim_{\varepsilon\to 0}\, (\varepsilon T^\varepsilon)^{1-2\kappa} \log P\Big(\sup_{t\le T^\varepsilon} |U_t^{(1)}| > \delta\Big) = -\infty,\ \forall\ \delta > 0.$$

Recall that $\sup_{t\le T^\varepsilon} |e_t^{\varepsilon,\kappa}| \le \ell\varepsilon^{1-\kappa}(1 + \sup_{t\le T^\varepsilon} |\xi_t^\varepsilon|)$ (see Lemma 2). Then, by virtue of the upper bound $\sup_{t\le T^\varepsilon} |U_t^{(1)}| \le \Psi^* \sup_{t\le T} |e_t^{\varepsilon,\kappa}|$, it suffices to check only that

$$\lim_{\varepsilon\to 0}\, (T^\varepsilon\varepsilon)^{1-2\kappa} \log P\Big(\varepsilon^{1-\kappa} \sup_{t\le T^\varepsilon} |\xi_t^\varepsilon| > \delta\Big) = -\infty.$$

By (A-4) $\lim\limits_{\varepsilon\to 0} T^\varepsilon\varepsilon^{1-2\kappa} = 0$, so that applying (20) we find, with a suitable constant r,

$$(T^\varepsilon\varepsilon)^{1-2\kappa} \log P\Big(\varepsilon^{1-\kappa} \sup_{t\le T^\varepsilon} |\xi_t^\varepsilon| > \delta\Big) \le -rd^{1-2\kappa}\Big\{\frac{(\delta^2 - d\varepsilon^{1-2\kappa})^2}{\varepsilon^{2(1-\kappa)}} \bigwedge \frac{d}{\varepsilon^{2\kappa}}\Big\} \to -\infty$$

as $\varepsilon \to 0$.

(i=2): Write

$$\sup_{t\le T^\varepsilon}|U^{(2)}_t|\le\int_0^{T^\varepsilon}|e^{\varepsilon,\kappa}_t|\Big(\Big|\dot\Psi^\varepsilon(X^\varepsilon_s)F(X^\varepsilon_s,\xi^\varepsilon_s)+\frac{1}{2}\ddot\Psi^\varepsilon(X^\varepsilon_s)G^2(X^\varepsilon_s,\xi^\varepsilon_s)\Big|\Big)ds.$$

Due to (A-2) and (A-4) there exists a positive constant ℓ so that $\sup_{t\le T^\varepsilon}|U^{(2)}_t|\le\ell\sup\limits_{t\le t}|e^{\varepsilon,\kappa}_t|$ and the proof is completed as for (i=1).

(i=3): The $U^{(3)}_t$ process is the continuous local martingale with the predictable quadratic variation $\langle U^{(3)}\rangle_t=\int_0^t\Big(e^{\varepsilon,\kappa}_s\dot\Psi^\varepsilon(X^\varepsilon_s)G(X^\varepsilon_s,\xi^\varepsilon_s)\Big)^2ds$. By virtue of assumptions (A-2) and (A-4), $\langle U^{(3)}\rangle_{T^\varepsilon}\le\ell\sup\limits_{t\le T}(e^{\varepsilon,\kappa}_t)^2$. The same arguments, as were used for the proof for (i=1), yield

$$(T^\varepsilon\varepsilon)^{1-2\kappa}\log P\Big(\langle U^{(3)}\rangle^{1/2}_{T^\varepsilon}>\delta\Big)\le-\text{const.}d^{1-2\kappa}\Big\{\frac{(\delta^2-d\varepsilon^{1-2\kappa})^2}{\varepsilon^{2(1-\kappa)}}\bigwedge\frac{d}{\varepsilon^{2\kappa}}\Big\}.$$

Taking now $\delta=\varepsilon^{\frac{1}{2}(1-\kappa)}$, we arrive at the upper bound: with a suitable constant ℓ

$$(T^\varepsilon\varepsilon)^{1-2\kappa}\log P\Big(\langle U^{(3)}\rangle^{1/2}_{T^\varepsilon}>\varepsilon^{\frac{1}{2}(1-\kappa)}\Big)\le-\ell d^{1-2\kappa}\Big\{\frac{(\varepsilon^{1-\kappa}-d\varepsilon^{1-2\kappa})^2}{\varepsilon^{2(1-\kappa)}}\bigwedge\frac{d}{\varepsilon^{2\kappa}}\Big\}$$

which tends to $-\infty$ as $\varepsilon\to0$. Hence, it remains to prove only that for every $\delta>0$

$$(T^\varepsilon\varepsilon)^{1-2\kappa}\log P\Big(\sup_{t\le T^\varepsilon}|U^{(3)}_t|>\delta,\langle U^{(3)}\rangle^{1/2}_{T^\varepsilon}\le\varepsilon^{\frac{1}{2}(1-\kappa)}\Big)\to-\infty,\ \varepsilon\to0.$$

To this end, we apply Lemma 1:

$$P\Big(\sup_{t\le T^\varepsilon}|U^{(3)}_t|>\delta,\langle U^{(3)}\rangle^{1/2}_{T^\varepsilon}\le\varepsilon^{\frac{1}{2}(1-\kappa)}\Big)\le2\exp\Big(-\frac{\delta^2}{2\varepsilon^{1-\kappa}}\Big)$$

and so that

$$(T^\varepsilon\varepsilon)^{1-2\kappa}\log\Big\{2\exp\Big(-\frac{\delta^2}{2\varepsilon^{1-\kappa}}\Big)\Big\}\le(T^\varepsilon\varepsilon)^{1-2\kappa}\log2-d^{1-2\kappa}\frac{\delta^2}{2\varepsilon^\kappa}\to-\infty,$$

as $\varepsilon\to0$. □

The proof of (46) uses the following auxiliary statement which slightly extends the result of Lemma 4.

Lemma 6 *Under assumptions of Theorem 1*

$$\lim_{\varepsilon\to0}(T^\varepsilon\varepsilon)^{1-2\kappa}\log P\Big(\sup_{t\le T^\varepsilon}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|>\delta\Big)=-\infty,\ \forall\delta>0.$$

Proof: We apply the Poisson decomposition $\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t=\varepsilon^{1/2}\widetilde{\mathfrak{S}}^\varepsilon_t+\mathfrak{e}^\varepsilon_t$ used in the proof of Lemma 4, and take into account that $\sup\limits_{t\le T}|\mathfrak{e}^\varepsilon_t|\le\ell\varepsilon(1+\sup\limits_{t\le T}|\xi^\varepsilon_t|)$ and $\widetilde{\mathfrak{S}}^\varepsilon_t$ is the continuous martingale with $\langle\widetilde{\mathfrak{S}}^\varepsilon\rangle_{T^\varepsilon}\le\ell T^\varepsilon$. Since $\varepsilon^{1-\kappa}>\varepsilon$ for small ε, the arguments, used in the proof of the statement (i=1) from Lemma 5, provide $\lim_{\varepsilon\to 0}(T^\varepsilon\varepsilon)^{1-2\kappa}\log P\Big(\varepsilon\sup_{t\le T^\varepsilon}|\xi^\varepsilon_t|>\delta\Big)=-\infty$. Next, since $\langle\widetilde{\mathfrak{S}}^\varepsilon\rangle_{T^\varepsilon}\le \ell T^\varepsilon$, P-a.s. by Lemma 1 we have $P\Big(\varepsilon^{1/2}\sup_{t\le T^\varepsilon}|\widetilde{\mathfrak{S}}^\varepsilon_t|>\delta\Big)\le 2\exp\Big(-\frac{\delta^2}{2\varepsilon\ell T^\varepsilon}\Big)$, so that

$$(T^\varepsilon\varepsilon)^{1-2\kappa}\log P\Big(\varepsilon^{1/2}\sup_{t\le T^\varepsilon}|\widetilde{\mathfrak{S}}^\varepsilon_t|>\delta\Big)\le -\frac{\delta^2}{2\ell(T^\varepsilon\varepsilon)^{2\kappa}}\le -\frac{\delta^2}{2\ell(d\varepsilon)^{2\kappa}}\to-\infty,\ \varepsilon\to 0$$

and the required assertion holds. $\square$

To complete the proof of Theorem 1, it remains to check the validity of (46). The $U^{(4)}_t$ process is the continuous martingale with the predictable quadratic variation

$$\langle U^{(4)}\rangle_t=\varepsilon^{1-2\kappa}\int_0^t(\Psi(X^\varepsilon_s))^2d\langle\widetilde{S}^\varepsilon\rangle_s$$

which is evaluated above as:

$$\langle U^{(4)}\rangle_{T^\varepsilon}\le\varepsilon^{1-2\kappa}(\Psi^*)^2\langle\widetilde{S}^\varepsilon\rangle_{T^\varepsilon}\le\varepsilon^{1-2\kappa}(\Psi^*)^2\gamma T^\varepsilon+\varepsilon^{1-2\kappa}(\Psi^*)^2\sup_{t\le T^\varepsilon}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|.$$

Let us note $\langle U^{(4)}\rangle_{T^\varepsilon}\le\varepsilon^{1-2\kappa}(\Psi^*)^2(\gamma T^\varepsilon+\delta)$ on the set $\mathfrak{A}=\{\sup_{t\le T^\varepsilon}|\langle\widetilde{S}^\varepsilon\rangle_t-\gamma t|\le\delta\}$. Then, by Lemma 1

$$P\Big(\sup_{t\le T^\varepsilon}|U^{(4)}_t|\ge\Psi^*(T^\varepsilon)^\kappa z,\mathfrak{A}\Big)\le\exp\Big(-\frac{z^2(T^\varepsilon)^{2\kappa}}{2\varepsilon^{1-2\kappa}[\gamma T^\varepsilon+\delta]}\Big),\ z>0. \tag{48}$$

Therefore $(T^\varepsilon\varepsilon)^{1-2\kappa}\log P\Big(\sup_{t\le T^\varepsilon}|U^{(4)}_t|\ge\Psi^*(T^\varepsilon)^\kappa z,\mathfrak{A}\Big)\le-\frac{z^2}{2[\gamma+\delta/T^\varepsilon]}$. Since also

$$P\Big(\sup_{t\le T^\varepsilon}|U^{(4)}_t|\ge\Psi^*(T^\varepsilon)^\kappa z\Big)\le 2\Big\{P\Big(\sup_{t\le T^\varepsilon}|U^{(4)}_t|\ge\Psi^*(T^\varepsilon)^\kappa z,\mathfrak{A}\Big)\bigvee P(\Omega\setminus\mathfrak{A}\Big\}$$

and by Lemma 6 $\lim_{\varepsilon\to0}(T^\varepsilon\varepsilon)^{1-2\kappa}\log P(\Omega\setminus\mathfrak{A})=-\infty$, it remains to recall only that $T^\varepsilon\ge d$ (see (A-4)), so that $-\frac{z^2}{2[\gamma+\delta/T^\varepsilon]}\le-\frac{z^2}{2[\gamma+\delta/d]}\to-\frac{z^2}{2\gamma},\ \delta\to0$. $\square$

4 Appendix. Vector case

It this Section, we formulate (without proof) the result in MDP evaluation for the vector diffusion process ξ_t with respect to vector Wiener process V_t with the unit diffusion matrix (both ξ_t and V_t are valued in $\mathbb{R}^d$):

$$d\xi^\varepsilon_t=\frac{1}{\varepsilon}b(\xi^\varepsilon_t)dt+\frac{1}{\sqrt{\varepsilon}}\sigma(\xi^\varepsilon_t)dV_t$$

The assumptions, under which the MDP for the family $S^{\varepsilon,\kappa}_t=\frac{1}{\varepsilon^\kappa}\int_0^t g(\xi^\varepsilon_s)\,ds$ holds are more restrictive. All elements of vector- and matrix- functions b and σ are Lipschitz continuous functions. We use two conditions from Pardoux and Veretennikov, [21].

(A_σ): $a = \sigma\sigma^*$ is nonsingular matrix and there exist positive constants λ_-, λ_+ such that for any $x \in \mathbb{R}^d \setminus \{0\}$ $(|x| = \sqrt{x^* x})$

$$0 < \lambda_- \le \Big(\sigma\sigma^*(x)\frac{x}{|x|}, \frac{x}{|x|}\Big) \le \lambda_+$$

(A_b): (recurrence condition) there exist positive constants C, r and $\alpha > -1$ such that for $|x| > C$

$$\Big(b(x), \frac{x}{|x|}\Big) \le -r|x|^\alpha.$$

It is shown in [21] that under (A_σ) and (A_b) ξ_t is an ergodic process with the unique invariant measure, μ.

We assume the function g is continuous, bounded, and

$$\int_{\mathbb{R}^d} g(x)\mu(dx) = 0.$$

As in the scalar case, we exploit the Poisson decomposition applying the result from [21] on the Poisson equation

$$\mathcal{L}u = g,$$

where $\mathcal{L} = \frac{1}{2}\sum a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum b_i(x)\partial_{x_i}$ is the diffusion operator. It is shown in [21] that the Poisson equation obeys a bounded solution the gradient of which $\nabla u = (\frac{\partial u(x)}{\partial x_1}, \dots, \frac{\partial u_i(z)}{\partial x_d})$ has bounded components . Introduce

$$\gamma = \int_{\mathbb{R}^d} \nabla u(x)\sigma(x)\sigma^*(x)\nabla^*(x))\mu(dx) \tag{49}$$

and note that since the matrix $\sigma\sigma*$ is nonsingular $\gamma > 0$ for any u with $\nabla u \not\equiv 0$.

Theorem 3 *Under the setting of this Section and $1/2 < \kappa < 1$, the family $(S_t^{\varepsilon,\kappa})_{t\ge 0}$, $\varepsilon \searrow 0$ obeys the MDP in the metric space $(\mathbb{C}, r)$ with the rate of speed $\varepsilon^{1-2\kappa}$ and the rate function (9) with γ from (49).*

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