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Modification of Quasi-Particle Theory in Spherical Nuclei. I

——*Method*——

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A more useful number conserving basis which takes the place of the BCS quasi-particle basis is constructed. The nucleon operator is expressed in terms of two kinds of operators which represent $J=0$-coupled nucleon pairs and unpaired nucleons. The pairing states are treated by a linearization technique which makes use of the boson-like property of operators. The motion of unpaired nucleons is described by a modified quasi-particle. The method is applicable not only to the super-conducting phase but also to the normal one. It gives number conserving treatments of the pairing vibration and of the coupling with quadrupole phonon.

§ 1. Introduction

The BCS quasi-particle theory, which offers a useful basis to treat the residual interaction in nuclei, has been conventionally applied to the study of collective excitations. A serious defect of the theory is the fact that the basis states are not eigenstates of nucleon number. The problem has been extensively investigated since early works.[1] Recently, the importance of number conserving treatment has been recognized again in the study of multi-phonon states.[2),3] On the other hand, starting with the quasi-particle basis, it was indicated that the coupling between the pairing and surface vibrations plays an important role in some nuclei.[4),5] The accuracy of the BCS quasi-particle basis must be, however, investigated simultaneously when it is used as a starting point.

A number conserving basis was developed by the number fixed BCS (FBCS) formalism.[6)~10] The multi-phonon states are similar to the FBCS basis states. A characteristic of the formalism is that a distribution of $J=0$-coupled nucleon pairs among various orbits is determined by the variational principle with a number fixed trial wave function. The variational equation is, however, considerably complicated. The approach fails for forces which do not give a sufficient configuration mixing in the seniority zero space, because the trial wave function cannot describe such a non-superconducting situation.[10]

A theory proposed by Suzuki and Matsuyanagi[11),12] opened a hopeful prospect in the study of interplay between the pairing and other correlations. The pairing collective and intrinsic degrees of freedom are exactly separated in the "ideal boson-quasi-particle space". The nucleon operators are expressed by the pairing bosons and ideal quasi-particles in a form of the Holstein-Primakoff representation.

In the previous paper,[13] the authors separated the pairing collective and intrinsic degrees of feedom within the original nucleon space and developed a Dyson-like representation. A method suitable for describing proper eigenstates of nucleon and seniority numbers was obtained. The advantage of the Dyson-like representation in contrast with the Holstein-Primakoff one is the fact that it gives a finite and hermitian Hamiltonian, which is simply divided into three parts; the pairing collective, intrinsic and their coupling parts. The representation, however, leads to asymmetric matrices of some physical quantities like the two-nucleon transfer strength.

The purpose of this paper is to construct a more practicable number conserving basis which takes the place of the BCS or the FBCS version and is capable of systematic further development. For this purpose, making use of the advantage of the Dyson-like representation, we formulate a method in the Holstein-Primakoff-like representation without a perturbative expansion of operators. A special approximation, which makes it possible to contain the same eigenvalue equation determining the self-consistent pairing field in the Dyson-like representation, is adopted. It is essential to use the particle-hole representation so that the ground state may well represent not only the super-conducting phase but also the normal one.

The present treatment of the seniority zero space (in § 3) is analogous to the ones involving use of the equation of motion and linearization procedure.[14)~18)] The pairing modes are expressed in terms of the "quasi-bosons" which mean the $J=0$-coupled pairs of particle and hole. Once the description of the pairing collective subspace is finished, a "modified" quasi-particle, which represents the motion of unpaired nucleon, plays an important role instead of the BCS quasi-particle (in § 4). We have a basis suitable for describing configuration mixing in both the normal and super-conducting phases. We show a guide to treat the residual interaction on this basis (in § 6). The present number conserving treatment brings about a new knowledge regarding the coupling between pairing vibration and quadrupole phonon.

§ 2. Holstein-Primakoff-like representation and Dyson-like one of the pairing Hamiltonian

2.1. *Separation of the pairing collective and intrinsic degrees of freedom*

As was shown in the previous paper,[13] the nucleon operator $C_\alpha^\dagger$ is expressed in terms of two kinds of operators $S_{j\pm}$ and $(d_\alpha^\dagger, d_\alpha)$ (or $\mathscr{S}_{j\pm}$ and $d_\alpha^{(\pm)}$) which represent the motion of $J=0$-coupled nucleon pairs and that of unpaired nucleons respectively: The Holstein-Primakoff-like representation is given by

$$C_\alpha^\dagger = d_\alpha^\dagger\sqrt{1-\frac{S_{j+}S_{j-}}{2\hat{S}_j}}+\frac{S_{j+}}{\sqrt{2\hat{S}_j}}d_{\bar{\alpha}}\,, \tag{2·1}$$

where $S_{j-}^{\dagger}=S_{j+}$. The Dyson-like representation is given by

$$C_\alpha{}^{\dagger}=d_\alpha{}^{(+)}+\frac{\mathscr{S}_{j+}}{2\hat{S}_j}d_{\bar{\alpha}}{}^{(-)},$$

$$C_{\bar{\alpha}}=-\mathscr{S}_{j-}d_\alpha{}^{(+)}+\left(1-\frac{\mathscr{S}_{j+}\mathscr{S}_{j-}}{2\hat{S}_j}\right)d_{\bar{\alpha}}{}^{(-)}\,. \tag{2·2}$$

Here, for simplicity we denote $\alpha\equiv(j,m)$ and $C_{\bar{\alpha}}=(-)^{j-m}C_{j\bar{m}}$, omitting additional quantum numbers. The quasi-spin operators $\hat{S}_j$ and $\hat{S}_{j0}$ are written as

$$2\hat{S}_j=\Omega_j-\boldsymbol{n}_j\,,\qquad \hat{S}_j-\hat{S}_{j0}=\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j\,, \tag{2·3}$$

where

$$\boldsymbol{N}_j=\boldsymbol{S}_{j+}\boldsymbol{S}_{j-}=\mathscr{S}_{j+}\mathscr{S}_{j-}\,, \tag{2·4}$$

$$\boldsymbol{n}_j=\sum_m d_\alpha{}^{\dagger}d_\alpha=\sum_m d_\alpha{}^{(+)}d_\alpha{}^{(-)}. \tag{2·5}$$

The operator $\boldsymbol{N}_j$ represents the number of $J=0$-coupled nucleon pairs and $\boldsymbol{n}_j$ the number of unpaired nucleons, i.e., the seniority v_j of orbit j. We shall say that the states composed of $\boldsymbol{S}_{j+}$ span "pairing collective" subspace and the states composed of $d_{jm}^{\dagger}$ "intrinsic" subspace.

The detailed definitions and properties of $\boldsymbol{S}_{j\pm}$ and $(d_\alpha{}^{\dagger},d_\alpha)$ ($\mathscr{S}_{j\pm}$ and $d_\alpha{}^{(\pm)}$) are given in Ref. 13). We repeat only a few points. The two kinds of operators commute with each other

$$[\boldsymbol{S}_{j\pm},d_\alpha{}^{\dagger}]=[\boldsymbol{S}_{j\pm},d_\alpha]=0\,. \tag{2·6}$$

The commutation relations about $\boldsymbol{S}_{j+}$ and $\boldsymbol{S}_{j-}$ are written as

$$[\boldsymbol{S}_{j-},\boldsymbol{S}_{j'+}]=\delta_{jj'}\{\hat{\eta}_j+\boldsymbol{S}_{j+}\boldsymbol{S}_{j-}(\hat{\eta}_j-1)\},$$

$$[\boldsymbol{S}_{j-},\boldsymbol{S}_{j'-}]=[\boldsymbol{S}_{j+},\boldsymbol{S}_{j'+}]=0\,, \tag{2·7}$$

where the operator $\hat{\eta}_j$ is defined by

$$\hat{\eta}_j=\begin{cases}0 & \text{for}\quad |S_j,S_{j0}=S_j\rangle\,,\\ 1 & \text{for other states}\,.\end{cases} \tag{2·8}$$

The operators $(\boldsymbol{S}_{j+},\boldsymbol{S}_{j-})$ satisfy the boson-type commutation relation except for the state $|S_j,S_{j0}=S_j\rangle$. In this sense we shall call the operators $(\boldsymbol{S}_{j+},\boldsymbol{S}_{j-})$ quasi-boson operators. The intrinsic operators $(d_\alpha{}^{\dagger},d_\alpha)$ satisfy the following relations:[11),13)]

$$\{d_\alpha,d_{\alpha'}^{\dagger}\}=\delta_{\alpha\alpha'}-\delta_{jj'}d_{\bar{\alpha}}{}^{\dagger}\frac{1}{2\hat{S}_j}d_{\bar{\alpha}'}\,,$$

$$\{d_\alpha{}^{\dagger},d_{\alpha'}^{\dagger}\}=\{d_\alpha,d_{\alpha'}\}=0\,. \tag{2·9}$$

As the relation $\boldsymbol{n}_j+\boldsymbol{N}_j\leqq\Omega_j$ is derived from Eq. (2·3), the square-root $\sqrt{\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j}$ in Eq. (2·1) never becomes imaginary. One can also show the relations such as

$$\begin{aligned} &\boldsymbol{S}_{j+}=\boldsymbol{S}_{j+}\hat{\eta}_j, && \boldsymbol{S}_{j-}=\hat{\eta}_j\boldsymbol{S}_{j-}\,, \\ &d_\alpha{}^\dagger=d_\alpha{}^\dagger\hat{\eta}_j, && d_\alpha=\hat{\eta}_j d_\alpha\,, \\ &[\hat{\eta}_j,\boldsymbol{N}_j]=0, && [\hat{\eta}_j,\boldsymbol{n}_j]=0 \end{aligned} \tag{2·10}$$

and $\sqrt{\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j}=\sqrt{\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j}\,\hat{\eta}_j$.

The operators $\mathscr{S}_{j\pm}$ and $d_\alpha{}^{(\pm)}$ in the Dyson-like representation satisfy the same relations as Eqs. (2·6) ~ (2·10). The Holstein-Primakoff-like and Dyson-like representations are connected with each other through the following transformation:

$$\begin{aligned} &\mathscr{S}_{j+}=\boldsymbol{O}_j\boldsymbol{S}_{j+}\boldsymbol{O}_j{}^{-1}=\boldsymbol{S}_{j+}\sqrt{\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j}\,, \\ &\mathscr{S}_{j-}=\boldsymbol{O}_j\boldsymbol{S}_{j-}\boldsymbol{O}_j{}^{-1}=\frac{1}{\sqrt{\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j}}\boldsymbol{S}_{j-}\,, \end{aligned} \tag{2·11}$$

$$\begin{aligned} &d_\alpha{}^{(+)}=\boldsymbol{O}_j d_\alpha{}^\dagger\boldsymbol{O}_j{}^{-1}=d_\alpha{}^\dagger\sqrt{\frac{\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j}{\Omega_j-\boldsymbol{n}_j}}\,, \\ &d_\alpha{}^{(-)}=\boldsymbol{O}_j d_\alpha\boldsymbol{O}_j{}^{-1}=\sqrt{\frac{\Omega_j-\boldsymbol{n}_j}{\Omega_j-\boldsymbol{n}_j-\boldsymbol{N}_j}}\,d_\alpha\,, \end{aligned} \tag{2·12}$$

where

$$\boldsymbol{O}_j=\sqrt{\frac{(2\hat{S}_j)!}{(\hat{S}_j-\hat{S}_{j0})!}},\quad \boldsymbol{O}_j{}^{-1}=\sqrt{\frac{(\hat{S}_j-\hat{S}_{j0})!}{(2\hat{S}_j)!}}\,. \tag{2·13}$$

It should be noticed that $\mathscr{S}_{j\pm}$ and $d_\alpha{}^{(\pm)}$ can be expressed in terms of $\boldsymbol{S}_{j\pm}$ and $(d_\alpha{}^\dagger, d_\alpha)$ and the converse is also true. We make use of the merit in the following treatment.

2.2. *The pairing Hamiltonian and operation formulae*

As the pairing correlation is superior to other long range correlations in spherical nuclei, it is useful to deal with the pairing force first. We then consider the usual pairing Hamiltonian

$$H=\sum_\alpha \epsilon_j C_\alpha{}^\dagger C_\alpha-G\sum_{jj'}\hat{S}_+(j)\hat{S}_-(j')\,, \tag{2·14}$$

where $\hat{S}_+(j)=\frac{1}{2}\sum_m C_\alpha{}^\dagger C_{\bar{\alpha}}{}^\dagger$ and $\hat{S}_-(j)=(\hat{S}_+(j))^\dagger$.

Let us divide single nucleon orbits into particle and hole orbits. In the limit of no perturbation, i.e., $G=0$ case, the orbits which are completely occupied by nucleons are the hole ones, and the other orbits are the particle ones. Hereafter we distinguish between the particle orbits and the hole ones by subscripts j and k, respectively. For the particle operators, we use the same notations as in the last subsection. We denote respectively the quasi-boson and intrinsic operators of the hole orbits by $\boldsymbol{T}_{k\pm}$ and $(d_{km}^\dagger, d_{km})$ in the Holstein-Primakoff-like representation, and denote them by $\mathcal{T}_{k\pm}$ and $(d_{km}^{(+)}, d_{km}^{(-)})$ in the Dyson-like representation. The number operators $\boldsymbol{N}_k$ of $J=0$-coupled hole pairs and $\boldsymbol{n}_k$ of unpaired holes are defined by the same equations as Eqs. (2·4) and (2·5). The relation between $(\mathcal{T}_{k\pm}, d_{km}^{(+)}, d_{km}^{(-)})$ and $(\boldsymbol{T}_{k\pm}, d_{km}^{\dagger}, d_{km})$ is given by

$$\mathcal{T}_{k+}=\boldsymbol{T}_{k+}\frac{1}{\sqrt{\Omega_k-\boldsymbol{n}_k-\boldsymbol{N}_k}},\quad \mathcal{T}_{k-}=\sqrt{\Omega_k-\boldsymbol{n}_k-\boldsymbol{N}_k}\,\boldsymbol{T}_{k-}\,,$$
$$d_{km}^{(+)}=d_{km}^{\dagger}\sqrt{\frac{\Omega_k-\boldsymbol{n}_k}{\Omega_k-\boldsymbol{n}_k-\boldsymbol{N}_k}},\quad d_{km}^{(-)}=\sqrt{\frac{\Omega_k-\boldsymbol{n}_k-\boldsymbol{N}_k}{\Omega_k-\boldsymbol{n}_k}}\,d_{km}\,. \tag{2·15}$$

It is an advantage of the Dyson-like representation that the pairing Hamiltonian is divided into three parts, the pairing collective, intrinsic and their coupling parts:

$$H=\text{const}+H_{\text{c}}+H_{\text{intr}}+H_{\text{coupl}}\,, \tag{2·16}$$

$$\begin{aligned}H_{\text{c}}=&\sum_m 2\epsilon_j\boldsymbol{N}_j-G\sum_{jj'}\mathcal{S}_{j'+}(\Omega_j-\mathcal{S}_{j+}\mathcal{S}_{j-})\mathcal{S}_{j-}\\&-\sum_k 2(\epsilon_k-G)\boldsymbol{N}_k-G\sum_{kk'}\mathcal{T}_{k+}(\Omega_k-\mathcal{T}_{k+}\mathcal{T}_{k-})\mathcal{T}_{k'-}\\&+G\sum_{jk}\{\mathcal{S}_{j+}\mathcal{T}_{k+}(\Omega_k-\mathcal{T}_{k+}\mathcal{T}_{k-})+(\Omega_j-\mathcal{S}_{j+}\mathcal{S}_{j-})\mathcal{S}_{j-}\mathcal{T}_{k-}\}\,,\end{aligned} \tag{2·17a}$$

$$H_{\text{intr}}=\sum_j\epsilon_j\boldsymbol{n}_j-\sum_k(\epsilon_k-G)\boldsymbol{n}_k\,, \tag{2·17b}$$

$$\begin{aligned}H_{\text{coupl}}=&G\sum_j\Big(\sum_{j'}\mathcal{S}_{j'+}\mathcal{S}_{j-}-\sum_k\mathcal{S}_{j-}\mathcal{T}_{k-}\Big)\boldsymbol{n}_j\\&+G\sum_k\Big(\sum_{k'}\mathcal{T}_{k+}\mathcal{T}_{k'-}-\sum_j\mathcal{S}_{j+}\mathcal{T}_{k+}\Big)\boldsymbol{n}_k\,.\end{aligned} \tag{2·17c}$$

The ground state of the pairing system under consideration is one of the state with the total seniority $v=\Sigma v_j=0$. Let us determine a self-consistent pairing field in this $v=0$ subspace (i.e., the pairing collective subspace), where $\boldsymbol{n}_j$ and $\boldsymbol{n}_k$ vanish, and then both H_{intr} and H_{coupl} do not contribute. We diagonalize the pairing collective Hamiltonian written in the Holstein-Primakoff-like representation:

$$\begin{aligned}\boldsymbol{H}_{\text{c}}=&\sum_j 2\epsilon_j\boldsymbol{N}_j-G\sum_{jj'}\boldsymbol{S}_{j+}R(\boldsymbol{N}_j)R(\boldsymbol{N}_{j'})\boldsymbol{S}_{j'-}\\&-\sum_k 2(\epsilon_k-G)\boldsymbol{N}_k-G\sum_{kk'}\boldsymbol{T}_{k+}R(\boldsymbol{N}_k)R(\boldsymbol{N}_{k'})\boldsymbol{T}_{k'-}\\&+G\sum_{jk}(\boldsymbol{S}_{j+}R(\boldsymbol{N}_j)\boldsymbol{T}_{k+}R(\boldsymbol{N}_k)+R(\boldsymbol{N}_j)\boldsymbol{S}_{j-}R(\boldsymbol{N}_k)\boldsymbol{T}_{k-})\,,\end{aligned} \tag{2·18}$$

where

$$R(\boldsymbol{N}_j)=\sqrt{\Omega_j-\boldsymbol{N}_j}\,,\quad R(\boldsymbol{N}_k)=\sqrt{\Omega_k-\boldsymbol{N}_k}\,. \tag{2·19}$$

With the use of Eqs. (2·7) and (2·10), we can obtain the following important formulae for the quasi-boson operators $\boldsymbol{S}_{j+}$:

$$[\boldsymbol{N}_j, \boldsymbol{S}_{j+}] = \boldsymbol{S}_{j+}, \tag{2·20}$$

$$[R(\boldsymbol{N}_j), \boldsymbol{S}_{j+}] = \boldsymbol{S}_{j+} d\boldsymbol{R}_j, \tag{2·21a}$$

$$d\boldsymbol{R}_j = R(\boldsymbol{N}_j + 1) - R(\boldsymbol{N}_j)$$
$$= -\frac{1}{2R(\boldsymbol{N}_j)} \left\{1 + \sum_{n=1} \frac{(2n-1)!!}{(n+1)!} (2R(\boldsymbol{N}_j)^2)^{-n}\right\}. \tag{2·21b}$$

These equations are formally rewritten as

$$[\boldsymbol{N}_j, \boldsymbol{S}_{j+}] = \frac{\partial \boldsymbol{N}_j}{\partial \boldsymbol{S}_{j-}}, \tag{2·22}$$

$$[R(\boldsymbol{N}_j), \boldsymbol{S}_{j+}] = \frac{\partial R(\boldsymbol{N}_j)}{\partial \boldsymbol{S}_{j-}}.^{*)} \tag{2·23}$$

In addition we have the relation

$$\frac{\partial \boldsymbol{H}_c}{\partial \boldsymbol{S}_{j-}} = [\boldsymbol{H}_c, \boldsymbol{S}_{j+}]. \tag{2·24}$$

The operators $\boldsymbol{T}_{k+}$ satisfy the same equations as Eqs. (2·20) $\sim$ (2·24).

§ 3. Self-consistent pairing field

3.1. *Linearized equation of motion*

As mentioned so far, we deal with the pairing collective Hamiltonian $\boldsymbol{H}_c$ to determine the self-consistent pairing field in the ground state with $v=0$. Let us introduce the particle-hole vacuum $|0\rangle$ with nucleon number A_0 which has no particles and no holes (i.e., $\boldsymbol{S}_{j-}|0\rangle = \boldsymbol{T}_{k-}|0\rangle = 0$) and consider the systems with even numbers of nucleons $A = A_0 + 2N$ and $A = A_0 + 2(N+1)$.**) We shall call these systems "N"-system and "$N+1$"-system and denote their ground states by $|\phi_0\rangle$ and $|\Psi_0(N+1)\rangle$, respectively. The unperturbed ground state of the "N"-system is written as

$$|\phi_0\rangle \to \frac{1}{\sqrt{N!}} (\boldsymbol{S}_{j_0+})^N |0\rangle \quad \text{when} \quad G \to 0, \tag{3·1}$$

where j_0 represents the lowest particle orbit and $0 \leqq N < \Omega_{j_0}$.

We aim at obtaining the one-quasi-boson mode from $|\phi_0\rangle$ to $|\Psi_0(N+1)\rangle$ by the linearized equation of motion in the following way:

*) If we replace $\partial R(\boldsymbol{N}_j)/\partial \boldsymbol{S}_{j-}$ by $\partial \boldsymbol{N}_j/\partial \boldsymbol{S}_{j-} \cdot \partial R(\boldsymbol{N}_j)/\partial \boldsymbol{N}_j$, approximating operators by c-numbers, it is reduced to $-\boldsymbol{S}_{j+} \cdot 1/2R(\boldsymbol{N}_j)$, which corresponds to pick up the first term in Eq. (2·21b).

**) In our definition of the particle and hole orbits, the particle-hole vacuum does not correspond to the closing of a major shell but the filling of each orbit i.e. $A_0 = \sum_{(\text{hole})} 2\Omega_j$.

$$X_0^\dagger=\sum_j \psi_j{}^0 \boldsymbol{S}_{j+}-\sum_k \varphi_k{}^0 \boldsymbol{T}_{k-}\,, \tag{3·2}$$

$$\langle \Psi_0(N+1)\,|\,[\boldsymbol{H}_c, X_0^\dagger]\,|\phi_0\rangle = E_0\langle \Psi_0(N+1)\,|X_0^\dagger|\phi_0\rangle\,. \tag{3·3}$$

With the aid of the formulae in § 2, the commutation relation in Eq. (3·3) is calculated as

$$\begin{aligned}
[\boldsymbol{H}_c, \boldsymbol{S}_{j+}] &= 2\epsilon_j \boldsymbol{S}_{j+} + G\boldsymbol{S}_{j+}\{\boldsymbol{S}_{j+}\boldsymbol{S}_{j-}\} \\
&\quad - G\sum_{j'\neq j} \boldsymbol{S}_{j+}\{d\boldsymbol{R}_j \boldsymbol{S}_{j'+}\boldsymbol{S}_{j-}R(\boldsymbol{N}_{j'}) + R(\boldsymbol{N}_{j'})\boldsymbol{S}_{j+}\boldsymbol{S}_{j'-}d\boldsymbol{R}_j\} \\
&\quad + G\sum_k \boldsymbol{S}_{j+}\{\boldsymbol{S}_{j+}\boldsymbol{T}_{k+}R(\boldsymbol{N}_k)\,d\boldsymbol{R}_j + d\boldsymbol{R}_j R(\boldsymbol{N}_k)\,\boldsymbol{T}_{k-}\boldsymbol{S}_{j-}\} \\
&\quad - G\sum_{j'} \boldsymbol{S}_{j'+}\{R(\boldsymbol{N}_{j'})\,R(\boldsymbol{N}_j)\} + G\sum_k \{R(\boldsymbol{N}_j)\,R(\boldsymbol{N}_k)\}\,\boldsymbol{T}_{k-}\,,
\end{aligned} \tag{3·4a}$$

$$\begin{aligned}
[\boldsymbol{H}_c, \boldsymbol{T}_{k-}] &= 2(\epsilon_k - G)\,\boldsymbol{T}_{k-} - G\{\boldsymbol{T}_{k+}\boldsymbol{T}_{k-}\}\,\boldsymbol{T}_{k-} \\
&\quad + G\sum_{k'\neq k} \{d\boldsymbol{R}_k \boldsymbol{T}_{k'+}\boldsymbol{T}_{k-}R(\boldsymbol{N}_{k'}) + R(\boldsymbol{N}_{k'})\,\boldsymbol{T}_{k+}\boldsymbol{T}_{k'-}d\boldsymbol{R}_k\}\,\boldsymbol{T}_{k-} \\
&\quad - G\sum_j \{\boldsymbol{S}_{j+}\boldsymbol{T}_{k+}R(\boldsymbol{N}_j)\,d\boldsymbol{R}_k + d\boldsymbol{R}_k R(\boldsymbol{N}_j)\,\boldsymbol{T}_{k-}\boldsymbol{S}_{j-}\}\,\boldsymbol{T}_{k-} \\
&\quad + G\sum_{k'} \{R(\boldsymbol{N}_{k'})\,R(\boldsymbol{N}_k)\}\,\boldsymbol{T}_{k'-} - G\sum_j \boldsymbol{S}_{j+}\{R(\boldsymbol{N}_j)\,R(\boldsymbol{N}_k)\}\,.
\end{aligned} \tag{3·4b}$$

In order to linearize the right-hand sides of Eqs. (3·4), let us replace the pairs of quasi-boson operators in the brackets { } by their expectation values with respect to $|\phi_0\rangle$ as follows:

$$\begin{aligned}
&\boldsymbol{S}_{j+}\boldsymbol{S}_{j'-} \to \langle \boldsymbol{S}_{j+}\boldsymbol{S}_{j'-}\rangle, \qquad \boldsymbol{T}_{k+}\boldsymbol{T}_{k'-} \to \langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k'-}\rangle\,, \\
&\boldsymbol{S}_{j+}\boldsymbol{T}_{k+} \to \langle \boldsymbol{S}_{j+}\boldsymbol{T}_{k+}\rangle, \qquad \boldsymbol{S}_{j-}\boldsymbol{T}_{k-} \to \langle \boldsymbol{S}_{j-}\boldsymbol{T}_{k-}\rangle\,.
\end{aligned} \tag{3·5}$$

Then the operators $\boldsymbol{N}_j$ and $\boldsymbol{N}_k$ appearing in $R(\boldsymbol{N}_j)$, $R(\boldsymbol{N}_k)$, $d\boldsymbol{R}_j$ and $d\boldsymbol{R}_k$ are replaced by their expectation values

$$N_j \equiv \langle \boldsymbol{N}_j\rangle = \langle \boldsymbol{S}_{j+}\boldsymbol{S}_{j-}\rangle, \qquad N_k \equiv \langle \boldsymbol{N}_k\rangle = \langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k-}\rangle\,. \tag{3·6}$$

For example, $R(\boldsymbol{N}_j)$ and $d\boldsymbol{R}_j$ are replaced by

$$R(\boldsymbol{N}_j) \;\to\; R(N_j) = \sqrt{\Omega_j - N_j}\,, \tag{3·7a}$$

$$d\boldsymbol{R}_j \;\to\; \langle d\boldsymbol{R}_j\rangle = -\frac{1}{2R(N_j)}\left\{1+\frac{1}{4}R(N_j)^{-2}+\cdots\right\}. \tag{3·7b}$$

In this approximation, Eq. (3·3) leads to the following equation for the amplitudes $\psi_j{}^0$ and $\varphi_k{}^0$:

$$\boldsymbol{\Sigma}\begin{pmatrix} h_{jj'} & h_{jk'} \\ h_{kj'} & h_{kk'} \end{pmatrix}\begin{pmatrix} \psi^0_{j'} \\ \varphi^0_{k'} \end{pmatrix} = E_0\begin{pmatrix} \psi_j{}^0 \\ \varphi_k{}^0 \end{pmatrix}, \tag{3·8}$$

where

$$h_{jj'}=\delta_{jj'}\{2\epsilon_j-2G\langle d\boldsymbol{R}_j\rangle\sum_{j''\neq j}R(N_{j''})\langle \boldsymbol{S}_{j''+}\boldsymbol{S}_{j-}\rangle+G\langle \boldsymbol{S}_{j+}\boldsymbol{S}_{j-}\rangle$$

$$+2G\langle d\boldsymbol{R}_j\rangle\sum_k R(N_k)\langle \boldsymbol{T}_{k-}\boldsymbol{S}_{j-}\rangle\}-GR(N_j)R(N_{j'}), \tag{3·9a}$$

$$h_{kk'}=\delta_{kk'}\{2(\epsilon_k-G)+2G\langle d\boldsymbol{R}_k\rangle\sum_{k''\neq k}R(N_{k''})\langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k''-}\rangle-G\langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k-}\rangle$$

$$-2G\langle d\boldsymbol{R}_k\rangle\sum_j R(N_j)\langle \boldsymbol{S}_{j+}\boldsymbol{T}_{k+}\rangle\}+GR(N_k)R(N_{k'}), \tag{3·9b}$$

$$h_{jk}=-h_{kj}=GR(N_j)R(N_k). \tag{3·9c}$$

In Eq. (3·8), the symbol $\sum$ means that repeated indices (j' and k') are summed. We shall use the convention hereafter. Equation (3·8) is an eigenvalue equation, although the expectation values (3·5) are not yet fixed.

3.2. *Structure of the eigenvalue equation*

Equation (3·8) approximately determining $X_0^\dagger$ has additional solutions. Denoting the numbers of the particle orbits and hole ones by J and K respectively, we have $(J+K)$ sets of solutions. If we arrange the eigenvalues according to their magnitudes, the large J eigenvalues correspond to the pair additional modes $X_\mu^\dagger$ and the other K eigenvalues correspond to the pair removal modes Y_ν. The lowest $X_\mu^\dagger$, i.e., the J-th eigenmode is $X_0^\dagger$ between $|\phi_0\rangle$ and $|\Psi_0(N+1)\rangle$. The new modes $X_\mu^\dagger$ and Y_ν are written as

$$X_\mu^\dagger=\sum_j \psi_j{}^\mu \boldsymbol{S}_{j+}-\sum_k \varphi_k{}^\mu \boldsymbol{T}_{k-}, \qquad (X_{\mu=0}^\dagger\equiv X_0^\dagger)$$

$$Y_\nu=\sum_k \psi_k{}^\nu \boldsymbol{T}_{k-}-\sum_j \phi_j{}^\nu \boldsymbol{S}_{j+}. \tag{3·10}$$

The orthonormality relation can be expressed by

$$\sum\begin{pmatrix}\psi_j{}^\mu & -\varphi_k{}^\mu\\ \phi_j{}^\nu & -\psi_k{}^\nu\end{pmatrix}\begin{pmatrix}\psi_j{}^{\mu'} & \phi_j{}^{\nu'}\\ \varphi_k{}^{\mu'} & \psi_k{}^{\nu'}\end{pmatrix}=\begin{pmatrix}\delta_{\mu\mu'} & 0\\ 0 & -\delta_{\nu\nu'}\end{pmatrix}. \tag{3·11}$$

The completeness relation is given by the inverse transformation of Eq. (3·11) and we have

$$\boldsymbol{S}_{j+}=\sum_\mu \psi_j{}^\mu X_\mu^\dagger+\sum_\nu \phi_j{}^\nu Y_\nu,$$

$$\boldsymbol{T}_{k-}=\sum_\nu \psi_k{}^\nu Y_\nu+\sum_\mu \varphi_k{}^\mu X_\mu^\dagger. \tag{3·12}$$

In the approximation $\hat{\eta}_j|\phi_0\rangle\simeq\hat{\eta}_k|\phi_0\rangle\simeq|\phi_0\rangle$, or the boson approximation

$$[\boldsymbol{S}_{j-},\boldsymbol{S}_{j'+}]|\phi_0\rangle\simeq\delta_{jj'}|\phi_0\rangle, \qquad [\boldsymbol{T}_{k-},\boldsymbol{T}_{k'+}]|\phi_0\rangle\simeq\delta_{kk'}|\phi_0\rangle, \tag{3·13}$$

the new modes $(X_\mu^\dagger, Y_\nu^\dagger)$ satisfy similar relations

$$[X_\mu, X_{\mu'}^\dagger]|\phi_0\rangle\simeq\delta_{\mu\mu'}|\phi_0\rangle, \qquad [Y_\nu, Y_{\nu'}^\dagger]|\phi_0\rangle\simeq\delta_{\nu\nu'}|\phi_0\rangle. \tag{3·14}$$

We thus obtain an orthogonal set of "one-body" modes in the self-consistent pairing field. The ground state is the lowest energy state composed of the new modes. By analogy with the unperturbed ground state (3·1), we approximate the ground state $|\phi_0\rangle$ of the "N"-system by

$$|\phi_0\rangle \simeq \frac{1}{\sqrt{N!}}(X_0^\dagger)^N|\tilde{0}\rangle, \tag{3·15}$$

where $|\tilde{0}\rangle$ is the new vacuum for $X_\mu^\dagger$ and $Y_\nu^\dagger$, which is defined by $X_\mu|\tilde{0}\rangle = Y_\nu|\tilde{0}\rangle = 0$. Then we have the relation

$$X_{\mu\neq 0}|\phi_0\rangle = 0, \qquad Y_\nu|\phi_0\rangle = 0. \tag{3·16}$$

After these preparations, we can evaluate the expectation values of the pairs of quasi-boson operators (3·5)

$$\begin{aligned}
&\langle \boldsymbol{S}_{j+}\boldsymbol{S}_{j'-}\rangle \simeq N\psi_j{}^0\psi_{j'}^0 + \sum_\nu \phi_j{}^\nu\phi_{j'}^\nu, \\
&\langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k'-}\rangle \simeq N\varphi_k{}^0\varphi_{k'}^0 + \sum_\mu \varphi_k{}^\mu\varphi_{k'}^\mu, \\
&\langle \boldsymbol{S}_{j+}\boldsymbol{T}_{k+}\rangle = \langle \boldsymbol{T}_{k-}\boldsymbol{S}_{j-}\rangle \\
&\qquad \simeq N\psi_j{}^0\varphi_k{}^0 + \sum_\nu \phi_j{}^\nu\psi_k{}^\nu = N\varphi_k{}^0\psi_j{}^0 + \sum_\mu \varphi_k{}^\mu\psi_j{}^\mu.
\end{aligned} \tag{3·17}$$

Since the matrix in Eq. (3·8) itself contains the amplitudes ψ_j, φ_k, ϕ_k and ϕ_j, the eigenvalue equation (3·8) must be solved self-consistently. The expectation value $\langle d\boldsymbol{R}_j\rangle$ ($\langle d\boldsymbol{R}_k\rangle$) in Eq. (3·7b) is expanded in the power series of $(2R(N_j)^2)^{-1}$ ($(2R(N_k)^2)^{-1}$). As the value N_j/Ω_j ($R(N_k)^2/\Omega_k$) corresponds to the occupation probability in a orbit $j(k)$, it is expected that $N_j/\Omega_j < 1/2$, $R(N_k)^2/\Omega_k > 1/2$ and $N_{j_0} \leqq \Omega_{j_0} - 1$ in our particle-hole treatment. Therefore the series in Eq. (3·7b) converges to a finite number. Even if we adopt the lowest order approximation:

$$\langle d\boldsymbol{R}_j\rangle \simeq -\frac{1}{2R(N_j)}, \tag{3·18}$$

the eigenvalue equation (3·8) is expected to be of use. Under this approximation, Eq. (3·8) is actually reduced to the equation obtained in the Dyson-like representation.[13),*)]

3.3. *Low-lying states in the seniority $v=0$ space*

The ground state of the "N"-system is approximately written as Eq. (3·15)

---

*) In Ref. 13), the second terms of the right-hand sides in Eqs. (3·17) were neglected. The seemingly asymmetric equation is symmetrized by the approximate relations:

$$\langle \Psi_0(N+1)|\mathcal{S}_{j+}|\phi_0\rangle \simeq \langle \Psi_0(N+1)|\boldsymbol{S}_{j+}|\phi_0\rangle R(N_j),$$
$$\langle \phi_0|\mathcal{S}_{j-}|\Psi_0(N+1)\rangle \simeq \langle \phi_0|\boldsymbol{S}_{j-}|\Psi_0(N+1)\rangle / R(N_j), \text{ etc.}$$

in our theory. In this self-consistent pairing field, the occupation probabilities $V_j^2(V_k^2)$ introduced in the BCS theory are given by

$$
\begin{aligned}
V_j &= \sqrt{\langle \boldsymbol{N}_j \rangle / \Omega_j} = \sqrt{N(\psi_j{}^0)^2 + \sum_\nu (\phi_j{}^\nu)^2} / \sqrt{\Omega_j}\,, \\
V_k &= R(N_k) / \sqrt{\Omega_k} = \sqrt{1 - \{N(\varphi_k{}^0)^2 + \sum_\mu (\varphi_k{}^\mu)^2\} / \Omega_k}\,.
\end{aligned}
\tag{3·19}
$$

Equation (3·11) and the completeness relation lead to the relation $\sum_{k\mu}(\varphi_k{}^\mu)^2 = \sum_{j\nu}(\phi_j{}^\nu)^2$. Combining this relation and Eq. (3·11), one can show the number conservation relation

$$
\langle \phi_0 | \sum_j \boldsymbol{N}_j - \sum_k \boldsymbol{N}_k | \phi_0 \rangle = N\,. \tag{3·20}
$$

We can define an orthogonal basis of the $v=0$ states in the "$N+1$"-system or "$N-1$"-system by using the new "one-body" modes obtained in the previous subsections as follows:

$$
\begin{aligned}
&X_\mu{}^\dagger|\phi_0\rangle,\ X_\mu{}^\dagger X_{\mu'}^\dagger Y_\nu{}^\dagger|\phi_0\rangle,\ \cdots \qquad \text{for the "}N+1\text{"-system}\,, \\
&Y_\nu{}^\dagger|\phi_0\rangle,\ Y_\nu{}^\dagger Y_{\nu'}^\dagger X_\mu{}^\dagger|\phi_0\rangle,\ \cdots \qquad \text{for the "}N-1\text{"-system}\,.
\end{aligned}
\tag{3·21}
$$

These states are not eigenstates of the pairing collective Hamiltonian $\boldsymbol{H}_c$ in general. The matrix elements of $\boldsymbol{H}_c$ with respect to this basis define a Hamiltonian matrix, and the more precise determination of low-lying states may proceed from an approximate diagonalization of this matrix. However, as shown in the two-level model calculation in Ref. 13), it will be a good approximation to describe the low-lying states of the "$N+1$" ("$N-1$")-system by $X_\mu{}^\dagger|\phi_0\rangle (Y_\nu{}^\dagger|\phi_0\rangle)$ and to evaluate the diagonal elements of $\boldsymbol{H}_c$. We adopt this first order perturbation theory. The energies $\boldsymbol{E}_\mu$ and $\boldsymbol{E}_\nu$ of $X_\mu{}^\dagger|\phi_0\rangle$ and $Y_\nu{}^\dagger|\phi_0\rangle$ measured from the ground state $|\phi_0\rangle$ are calculated by

$$
\begin{aligned}
\boldsymbol{E}_\mu &= \langle \phi_0 | X_\mu [\boldsymbol{H}_c, X_\mu{}^\dagger] | \phi_0 \rangle\,, \\
\boldsymbol{E}_\nu &= \langle \phi_0 | Y_\nu [\boldsymbol{H}_c, Y_\nu{}^\dagger] | \phi_0 \rangle\,.
\end{aligned}
\tag{3·22}
$$

For the ground state $|\Psi_0(N+1)\rangle \simeq (1/\sqrt{N+1}) X_0{}^\dagger|\phi_0\rangle$, the solution E_0 of Eq. (3·8) can be regarded as an approximate value of $\boldsymbol{E}_0$. If we define a chemical potential λ_N by the half of the energy difference between $|\phi_0\rangle$ and $|\Psi_0(N+1)\rangle$, we can write

$$
\lambda_N = E_0/2\,. \tag{3·23}
$$

In the case $N=0$, i.e., $|\phi_0\rangle = |\tilde{0}\rangle$, the lowest $Y_\nu{}^\dagger$ (we denote by $Y_0{}^\dagger$) is the mode between $|\phi_0\rangle$ and the ground state of the system with nucleon number A_0-2 ($|\Phi_0; A_0-2\rangle \simeq Y_0{}^\dagger|\tilde{0}\rangle$).

The energies of the excited states $X_{\mu\neq0}^\dagger|\phi_0\rangle$ and $Y_{\nu\neq0}^\dagger|\phi_0\rangle$ in the "$N+1$"- and "$N-1$"-system have correction terms in addition to the eigenvalues E_μ and E_ν of

Eq. (3·8). The correction terms are somewhat complicated. If we neglect the terms such as $d\boldsymbol{R}_j \cdot d\boldsymbol{R}_{j'}$, $[S_{j-}, d\boldsymbol{R}_j]$ etc. in the double commutators $[X_\mu, [\boldsymbol{H}_c, X_\mu^\dagger]]$ and $[Y_\nu, [\boldsymbol{H}_c, Y_\nu^\dagger]]$, we can get the same form of corrections as those obtained in Ref. 13). The approximated values of $\boldsymbol{E}_\mu$ and $\boldsymbol{E}_\nu$ are shown in the Appendix.

§ 4. A modified quasi-particle

As we have finished the treatment of the pairing collective Hamiltonian, we consider the intrinsic motion.

The single particle and single hole states in the odd nuclei with $A \pm 1$ ($A = A_0 + 2N$) nucleons are written as

$$d^\dagger_{jm}|\phi_0\rangle = d^{(+)}_{jm}\sqrt{\frac{\Omega_j}{\Omega_j - \boldsymbol{N}_j}}|\phi_0\rangle\,,$$
$$d^\dagger_{km}|\phi_0\rangle = d^{(+)}_{km}\sqrt{\frac{\Omega_k - \boldsymbol{N}_k}{\Omega_k}}|\phi_0\rangle\,. \tag{4·1}$$

We approximate the operators $\boldsymbol{N}_j$ and $\boldsymbol{N}_k$ in Eqs. (4·1) by their expectation values N_j and N_k with respect to $|\phi_0\rangle$. We can then regard the single particle (hole) states as $d^{(+)}_{jm}|\phi_0\rangle$ ($d^{(+)}_{km}|\phi_0\rangle$) except for the normalization constant. As $d^{(+)}_{jm}$ and $d^{(+)}_{km}$ commute with H_c, the energies of the single particle and single hole states are calculated by

$$\varepsilon_j d^{(+)}_{jm}|\phi_0\rangle = [H_{\text{intr}} + H_{\text{coupl}},\ d^{(+)}_{jm}]|\phi_0\rangle\,,$$
$$-\varepsilon_k d^{(+)}_{km}|\phi_0\rangle = [H_{\text{intr}} + H_{\text{coupl}},\ d^{(+)}_{km}]|\phi_0\rangle\,. \tag{4·2}$$

The energies ε_j and $(-\varepsilon_k)$ measured from $|\phi_0\rangle$ are given by

$$\varepsilon_j = \epsilon_j + G\Big(\sum_{j'} \frac{R(N_{j'})}{R(N_j)}\langle \boldsymbol{S}_{j'+}\boldsymbol{S}_{j-}\rangle - \sum_k \frac{R(N_k)}{R(N_j)}\langle \boldsymbol{S}_{j-}\boldsymbol{T}_{k-}\rangle\Big),$$
$$-\varepsilon_k = -(\epsilon_k - G) + G\Big(\sum_{k'} \frac{R(N_{k'})}{R(N_k)}\langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k'-}\rangle - \sum_j \frac{R(N_j)}{R(N_k)}\langle \boldsymbol{S}_{j+}\boldsymbol{T}_{k+}\rangle\Big). \tag{4·3}$$

Inside the space without $X^\dagger_{\mu\neq 0}$ and $Y^\dagger_{\nu\neq 0}$, one can replace the pairs of quasi-boson operators in $H_{\text{intr}} + H_{\text{coupl}}$ by their expectation values:

$$\boldsymbol{H}_{\text{intr}} = H_{\text{intr}} + H_{\text{coupl}} = \sum_j \varepsilon_j \boldsymbol{n}_j - \sum_k \varepsilon_k \boldsymbol{n}_k\,. \tag{4·4}$$

Equation (4·4) introduces a modified quasi-particle. The energy $\varepsilon_j - \lambda_N$ corresponds to the quasi-particle energy in the $A+1$ nucleus. The energy $\lambda_{N-1} - \varepsilon_k$ corresponds to the quasi-particle energy in the $A-1$ nucleus, where λ_{N-1} should be determined in the "$N-1$"-system. It should be noticed that $d_\alpha^\dagger$ and d_α ($d_\alpha^{(+)}$ and $d_\alpha^{(-)}$)

satisfy the commutation relations (2·9) and represent the motion of unpaired nucleons. In our treatment, the nucleon number is conserved and then there is no spurious states.

In the same manner, we can approximately obtain the energies of the $v=2$ states. For example, the energy $E_{v=2}(jk)$ of the state $(d_j^\dagger d_k^\dagger)_{JM}|\phi_0\rangle$ is given by $E_{v=2}(jk)=\varepsilon_j-\varepsilon_k$, and the energy $E_{v=2}(jj')$ of the state $(d_j^\dagger d_{j'}^\dagger)_{JM}|\Phi_0(N-1)\rangle$ is given by $E_{v=2}(jj')=(\varepsilon_j^{(N-1)}-\lambda_{N-1})+(\varepsilon_{j'}^{(N-1)}-\lambda_{N-1})$. Here $\varepsilon_j^{(N-1)}$ and λ_{N-1} should be determined for the ground state $|\Phi_0(N-1)\rangle$ in the "$N-1$"-system. If N dependence of these quantities is negligible, we can use ε_j and λ_N determined in the "N"-system like the BCS quasi-particle treatment.

§ 5. Contents of the approximation

It is easily shown that the basic equation (3·8) is reduced to the RPA one for the pairing vibration on the particle-hole basis, in the limit $\langle \boldsymbol{S}_{j+}\boldsymbol{S}_{j'-}\rangle =\langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k'-}\rangle=\langle \boldsymbol{S}_{j+}\boldsymbol{T}_{k+}\rangle=\langle \boldsymbol{T}_{k-}\boldsymbol{S}_{j-}\rangle=0$.

On the other hand, the BCS equation is obtained as follows: Let us approximate $\boldsymbol{S}_{j\pm}$ as

$$\bar{\boldsymbol{S}}_{j\pm}=\sqrt{\Omega_j}\,V_j\,, \tag{5·1}$$

where the particle-hole transformation is not introduced and the subscripts j imply all orbits. The variational equation,

$$\frac{\partial\overline{\boldsymbol{H}_c}}{\partial\overline{\boldsymbol{S}_{j-}}}-2\lambda\frac{\partial\overline{\Sigma\boldsymbol{N}_j}}{\partial\overline{\boldsymbol{S}_{j-}}}=0\,, \tag{5·2}$$

leads to the BCS equation

$$2(\epsilon_j-\lambda)\,U_jV_j=(U_j^2-V_j^2)\,G\sum_{j'}\Omega_{j'}U_{j'}V_{j'}\,, \tag{5·3}$$

where $U_j=\sqrt{1-V_j^2}$. The bar over operators represents replacing $\boldsymbol{S}_{j\pm}$ by Eq. (5·1). Replacing operators by c-numbers suggests us to approximate $\partial\overline{\boldsymbol{H}_c}/\partial\overline{\boldsymbol{S}_{j-}}\simeq\overline{\partial\boldsymbol{H}_c/\partial\boldsymbol{S}_{j-}}$ and $d\overline{\boldsymbol{R}_j}\simeq\partial R(\overline{\boldsymbol{N}_j})/\partial\overline{\boldsymbol{N}_j}=-1/2\bar{R}(\overline{\boldsymbol{N}_j})$. In this approximation, Eq. (5·2) combined with Eq. (2·24) is written as

$$\overline{[\boldsymbol{H}_c,\boldsymbol{S}_{j+}]}-2\lambda\overline{\boldsymbol{S}_{j+}}=0\,. \tag{5·4}$$

This equation is similar to our basic equation (3·3).

It is actually shown that Eq. (3·8) without the amplitudes $\psi_j^{\mu\neq0}$, φ_k^{μ}, ψ_k^{ν} and ϕ_j^{ν} is reduced to the same form as Eq. (5·3) under the approximation (3·18) and $\langle \boldsymbol{S}_{j+}\boldsymbol{S}_{j-}\rangle\simeq N(\psi_j^0)^2=\Omega_jV_j^2$. However, if N is larger than Ω_j of the lowest orbit, the boson approximations (3·13), (3·15) and (3·17) are not good. The introduction of the particle-hole basis guarantees these approximations and smoothly connects the correlated ground state (3·15) with the unperturbed one (3·1) in small G limit. This makes it possible to describe the normal phase as well as the superconducting phase. The denominators $R(N_j)$ and $R(N_k)$ appearing in Eq. (3·8) never become zero in our particle-hole treatment, whereas U_j in the BCS equation is zero for small G below a critical value G_c.

Within an extent where the BCS approximation is good, one may substitute the BCS equation (5·3) for Eq. (3·8) to obtain the occupation probabilities V_j of orbits j (including k). From Eqs. (4·3), (5·1) and (5·3), the excitation energies of the single nucleon states in the $A+1$ nucleus are approximated by

$$\varepsilon_j-\lambda_N \simeq \epsilon_j-\lambda+\frac{V_j}{U_j}\Delta=\sqrt{(\epsilon_j-\lambda)^2+\Delta^2}\,, \tag{5·5}$$

where $\Delta\equiv\sum_j G\Omega_j U_j V_j$. The Hamiltonian representing the intrinsic motion is written as

$$\boldsymbol{H}_{\rm intr}^{\rm BCS}=\sum_j\sqrt{(\epsilon_j-\lambda)^2+\Delta^2}\sum_m d_{jm}^\dagger d_{jm}\,, \tag{5·6}$$

where $\boldsymbol{n}_j=\sum_m d_{jm}^\dagger d_{jm}$ is still the seniority number operator and $(d_{jm}^\dagger, d_{jm})$ satisfy the commutation relations (2·9).

§ 6. Application to pairing plus quadrupole force model

The pairing plus quadrupole force model, which is capable of accounting approximately for low-lying states of spherical nuclei, has been conventionally treated by the BCS quasi-particle theory. The present theory offers an alternative basis to describe the model. The pairing Hamiltonian (2·16) is written as

$$\begin{aligned} H=&\sum_\mu \boldsymbol{E}_\mu X_\mu^\dagger X_\mu+\sum_\nu \boldsymbol{E}_\nu Y_\nu^\dagger Y_\nu\\ &+\sum_j \varepsilon_j\boldsymbol{n}_j-\sum_k\varepsilon_k\boldsymbol{n}_k+H_{\rm p}^{\rm res}. \end{aligned} \tag{6·1}$$

For the present, we neglect the pairing residual interaction $H_{\rm p}^{\rm res}$. The remaining problem is to take account of the quadrupole force

$$\begin{aligned} H_{QQ}&=-\frac{\chi}{2}\sum_M Q_{2M}^\dagger Q_{2M}\,,\\ Q_{2M}^\dagger&=-\sum_{jj'}q(jj')\sum_{mm'}\langle jmj'm'|2M\rangle C_{jm}^\dagger C_{\widetilde{j'm'}}\,. \end{aligned} \tag{6·2}$$

In our representation, the quadrupole operator $Q_{2M}^\dagger$ can be expressed in terms of $(d_{jm}^\dagger, d_{jm})$, $(d_{km}^\dagger, d_{km})$, $(X_\mu^\dagger, X_\mu)$ and $(Y_\nu^\dagger, Y_\nu)$. The expression of H_{QQ} which includes both the particle and hole operators is straightforward but lengthy.

For simplicity of the explanation, we study the case in which the hole orbits do not exist and N is smaller than Ω_j of the lowest orbit. In this case, our equation (3·8) is reduced to the BCS equation (5·3),*) if we adopt the approximation (3·18). Let us approximate $C^{\dagger}_{jm}$ in $Q^{\dagger}_{2M}$ as follows:

$$C^{\dagger}_{jm} \simeq d^{\dagger}_{jm}\boldsymbol{U}_j + \boldsymbol{V}_j{}^{\dagger} d_{\widetilde{jm}}\,,$$
$$\boldsymbol{U}_j = R(\boldsymbol{N}_j)/\sqrt{\Omega_j},\ \ \boldsymbol{V}_j{}^{\dagger} = \boldsymbol{S}_{j+}/\sqrt{\Omega_j}\,, \tag{6·3}$$

where $2\hat{S}_j = \Omega_j(1-\boldsymbol{n}_j/\Omega_j)$ is approximated by Ω_j. The neglect of $\boldsymbol{n}_j/\Omega_j$ may be good for the collective states with small seniority, because the occupation probability of $d^{\dagger}_{jm}$ in a orbit j is small for these states. We approximately write the quadrupole force H_{QQ} as

$$H_{QQ} \simeq \boldsymbol{H}_X + \boldsymbol{H}_Y + \boldsymbol{H}_V\,, \tag{6·4}$$
$$\boldsymbol{H}_X = -\chi \sum q(j_1j_2)\,q(j_3j_4) \sum_M (d^{\dagger}_{j_1}d^{\dagger}_{j_2})_{2M}(d_{j_4}d_{j_3})_{2M}\boldsymbol{V}^{\dagger}_{j_4}\boldsymbol{U}_{j_3}\boldsymbol{U}_{j_1}\boldsymbol{V}_{j_2}\,,$$
$$\boldsymbol{H}_Y = -\chi \sum q(j_1j_2)\,q(j_3j_4) \sum_M (d^{\dagger}_{j_1}d^{\dagger}_{j_2})_{2M}(d^{\dagger}_{j_3}d_{j_4})_{2M}\boldsymbol{U}_{j_1}\boldsymbol{V}_{j_2}(\boldsymbol{U}_{j_3}\boldsymbol{U}_{j_4} - \boldsymbol{V}^{\dagger}_{j_3}\boldsymbol{V}_{j_4}) + \text{h.c.},$$
$$\boldsymbol{H}_V = -\frac{\chi}{2} \sum q(j_1j_2)\,q(j_3j_4) \sum_M (d^{\dagger}_{j_1}d^{\dagger}_{j_2})_{2M}(-)^{2-M}(d^{\dagger}_{j_3}d^{\dagger}_{j_4})_{2\bar{M}}\boldsymbol{U}_{j_1}\boldsymbol{V}_{j_2}\boldsymbol{U}_{j_4}\boldsymbol{V}_{j_3} + \text{h.c.}$$

The three parts $\boldsymbol{H}_X$, $\boldsymbol{H}_Y$ and $\boldsymbol{H}_V$ transfer the seniority number 0, 2 and 4, respectively.

First, we consider the subspace which does not include the excited $v=0$ modes $X^{\dagger}_{\mu\neq 0}$. For example, the matrix element of $\boldsymbol{V}^{\dagger}_{j_4}\boldsymbol{U}_{j_3}\boldsymbol{U}_{j_1}\boldsymbol{V}_{j_2}$ in this subspace is calculated as

$$\langle\phi_0|\boldsymbol{V}^{\dagger}_{j_4}\boldsymbol{U}_{j_3}\boldsymbol{U}_{j_1}\boldsymbol{V}_{j_2}|\phi_0\rangle \simeq R(N_{j_1})R(N_{j_3})\sqrt{N_{j_2}N_{j_4}}/\prod_{i=1}^{4}\sqrt{\Omega_{j_i}}\,, \tag{6·5}$$

where we approximate as $(N/\Omega_j)(\psi_j{}^0)^2 \simeq ((N-1)/\Omega_j)\,(\psi_j{}^0)^2$. Equation (6·5) is rewritten as the familiar form $U_{j_1}V_{j_2}U_{j_3}V_{j_4}$ by using the expression $V_j = \sqrt{N_j/\Omega_j}$ and $U_j = R(N_j)/\sqrt{\Omega_j}$. This means that the operators $\boldsymbol{U}_j$ and $\boldsymbol{V}_j$ within the subspace under consideration can be replaced by their expectation values U_j and V_j in the approximation neglecting terms of the order of $1/\Omega_j$. The replacement in Eq. (6·4) directly leads to the same form as usual quasi-particle expression of H_{QQ}. This is a foundation of the quasi-particle treatment of the quadrupole phonon states. The simple replacement is, however, not correct in general cases including the pairing vibrational modes $X^{\dagger}_{\mu\neq 0}$.

Next, we consider the couplings between the pairing vibration and quadrupole phonon, the importance of which has been discussed recently.[5] We show a method how to calculate the couplings, taking up the following matrix element as an example:

*) In this case, the BCS equation has a non-trivial solution in any G.

$$\langle\phi_0|[(d_{j_1'}d_{j_2'})_2(d_{j_3'}d_{j_4'})_2]_{00}\boldsymbol{H}_V X^\dagger_{\mu\neq 0}X_0^\dagger|\phi_0\rangle$$

$$=-\frac{\chi}{2}\sum q(j_1j_2)q(j_3j_4)\sqrt{5}\langle 0[(d_{j_1'}d_{j_2'})_2(d_{j_3'}d_{j_4'})_2]_{00}[(d^\dagger_{j_1}d^\dagger_{j_2})_2(d^\dagger_{j_3}d^\dagger_{j_4})_2]_{00}|0\rangle$$

$$\times\langle\phi_0|[\boldsymbol{U}_{j_1}\boldsymbol{V}_{j_2}\boldsymbol{U}_{j_4}\boldsymbol{V}_{j_3}, X_\mu{}^\dagger]X_0{}^\dagger|\phi_0\rangle\,. \tag{6·6}$$

The part of commutation relation in Eq. (6·6) is calculated by using Eqs. (2·7) and (2·21). If we approximate $d\boldsymbol{R}_j$ by $-1/2R(N_j)$ and $(\boldsymbol{U}_j, \boldsymbol{V}_j)$ by (U_j, V_j), the result is as follows:

$$\langle\phi_0|[\boldsymbol{U}_{j_1}\boldsymbol{V}_{j_2}\boldsymbol{U}_{j_4}\boldsymbol{V}_{j_3}, \sum\psi_j{}^\mu\boldsymbol{S}_{j+}]X_0{}^\dagger|\phi_0\rangle$$

$$=\frac{\psi^\mu_{j_2}}{\sqrt{\Omega_{j_2}}}U_{j_1}V_{j_3}U_{j_4}+\frac{\psi^\mu_{j_3}}{\sqrt{\Omega_{j_3}}}U_{j_1}V_{j_2}U_{j_4}$$

$$-\frac{1}{2}\frac{\psi^\mu_{j_1}}{\sqrt{\Omega_{j_1}}U_{j_1}}V_{j_1}V_{j_2}V_{j_3}U_{j_4}-\frac{1}{2}\frac{\psi^\mu_{j_4}}{\sqrt{\Omega_{j_4}}U_{j_4}}U_{j_1}V_{j_2}V_{j_3}V_{j_4}\,. \tag{6·7}$$

We can also calculate other couplings in the same manner. Equation (6·7) contains the factors $1/(\sqrt{\Omega_j}U_j)$ which is derived from $d\boldsymbol{R}_j=R(\boldsymbol{N}_j+1)-R(\boldsymbol{N}_j)$. The factors never become extremely large numbers for small G in the case under consideration. In general case the factors do not diverge as mentioned in § 3, because we adopt the particle-hole basis.

The structure of the couplings is different from those obtained on the BCS quasi-particle basis in Refs. 5) and 12). The number non-conservation effect should be eliminated from the couplings. In order to avoid the mixing of nucleon number, the separation between the pairing collective and intrinsic degrees of freedom should be performed in the original nucleon space.

§ 7. Concluding remarks

We have developed a useful method to treat the pairing correlation, which takes the place of the BCS quasi-particle theory or the FBCS version. The principal aspects of the method are as follows: The distribution of the $J=0$-coupled nucleon pairs among various orbits is determined by a linearization technique, with respect to the number projected ground state. The method describes both the normal and super-conducting phases of the pairing system. It gives a number conserving treatment of the pairing vibrations. Truncated low-seniority states are given by the number conserving basis states which are suitable for dealing with configuration mixing caused by other residual interaction.

An advantage of the method is to be available for the normal and transitional regions in which the BCS approximation is not applicable. In a forthcoming paper, the applicability of the method will be examined in realistic nuclei. Another advantage is that the method excludes the number non-conservation effect in the couplings between pairing vibration and quadrupole phonon. This problem will be more extensively discussed in a subsequent paper.

Appendix

If we neglect the terms such as $d\boldsymbol{R}_j d\boldsymbol{R}_{j'}$, $[\boldsymbol{S}_{j-}, d\boldsymbol{R}_j]$, etc., in the double commutators $[X_\mu, [\boldsymbol{H}_c, X_\mu^\dagger]]$ and $[Y_\nu, [\boldsymbol{H}_c, Y_\nu^\dagger]]$ in Eqs. (3·22), the approximate energies $\boldsymbol{E}_\mu$ and $\boldsymbol{E}_\nu$ of $X^\dagger_{\mu\neq 0}|\phi_0\rangle$ and $Y^\dagger_{\nu\neq 0}|\phi_0\rangle$ are given by

$$\boldsymbol{E}_\mu = E_\mu + \sum(\psi_j{}^\mu - \varphi_k{}^\mu)\begin{pmatrix} C_{jj'} & C_{jk'} \\ C_{kj'} & C_{kk'} \end{pmatrix}\begin{pmatrix} \psi^\mu_{j'} \\ \varphi^\mu_{k'} \end{pmatrix},$$

$$\boldsymbol{E}_\nu = E_\nu + \sum(\phi_j{}^\nu - \psi_k{}^\nu)\begin{pmatrix} C_{jj'} & C_{jk'} \\ C_{kj'} & C_{kk'} \end{pmatrix}\begin{pmatrix} \phi^\nu_{j'} \\ \psi^\nu_{k'} \end{pmatrix},$$

where

$$C_{jj'} = \frac{G}{2}\left\{\frac{R(N_{j'})}{R(N_j)} N(\psi_j{}^0)^2 + \frac{R(N_j)}{R(N_{j'})} N(\psi^0_{j'})^2\right\}$$
$$+\delta_{jj'} G\left\{\sum_{j''} \frac{R(N_{j''})}{R(N_j)}\langle \boldsymbol{S}_{j''+}\boldsymbol{S}_{j-}\rangle - \sum_k \frac{R(N_k)}{R(N_j)}\langle \boldsymbol{T}_{k-}\boldsymbol{S}_{j-}\rangle\right\},$$

$$C_{kk'} = -\frac{G}{2}\left\{\frac{R(N_{k'})}{R(N_k)} N(\varphi_k{}^0)^2 + \frac{R(N_k)}{R(N_{k'})} N(\varphi^0_{k'})^2\right\}$$
$$-\delta_{kk'} G\left\{\sum_{k''} \frac{R(N_{k''})}{R(N_k)}\langle \boldsymbol{T}_{k+}\boldsymbol{T}_{k''-}\rangle - \sum_j \frac{R(N_j)}{R(N_k)}\langle \boldsymbol{S}_{j+}\boldsymbol{T}_{k+}\rangle\right\},$$

$$C_{jk} = -C_{kj} = \frac{G}{2}\left\{\frac{R(N_j)}{R(N_k)} N(\varphi_k{}^0)^2 + \frac{R(N_k)}{R(N_j)} N(\psi_j{}^0)^2\right\}.$$

Here $\langle \boldsymbol{S}_{j+} d\boldsymbol{R}_j \boldsymbol{S}_{j-}\rangle$ and $\langle \boldsymbol{T}_{k+} d\boldsymbol{R}_k \boldsymbol{T}_{k-}\rangle$ are approximated by $-N(\psi_j{}^0)^2/2R(N_j)$ and $-N(\varphi_k{}^0)^2/2R(N_k)$.

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