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Modified α-ψ-contractive mappings with applications

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Abstract
The aim of this work is to modify the notions of α-admissible and α-ψ-contractive mappings and establish new fixed point theorems for such mappings in complete metric spaces. Presented theorems provide main results of Karapinar and Samet (Abstr. Appl. Anal. 2012:793486, 2012) and Samet *et al.* (Nonlinear Anal. 75:2154-2165, 2012) as direct corollaries. Moreover, some examples and applications to integral equations are given here to illustrate the usability of the obtained results.
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1 Introduction and preliminaries

Metric fixed point theory has many applications in functional analysis. The contractive conditions on underlying functions play an important role for finding solutions of metric fixed point problems. The Banach contraction principle is a remarkable result in metric fixed point theory. Over the years, it has been generalized in different directions by several mathematicians (see [1–25]). In 2012, Samet *et al.* [24] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces. Afterwards Karapinar and Samet [19] generalized these notions to obtain fixed point results. The aim of this paper is to modify further the notions of α-ψ-contractive and α-admissible mappings and establish fixed point theorems for such mappings in complete metric spaces. Our results are proper generalizations of the recent results in [19, 24]. Moreover, some examples and applications to integral equations are given here to illustrate the usability of the obtained results.

Denote with Ψ the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

The following lemma is obvious.

Lemma 1.1 *If* $\psi \in \Psi$, *then* $\psi(t) < t$ *for all* $t > 0$.

Definition 1.1 [24] Let T be a self-mapping on a metric space (X,d) and let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is an α-admissible mapping if

$$x,y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \geq 1.$$

Definition 1.2 [24] Let T be a self-mapping on a metric space (X,d). We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,+\infty)$ and $\psi \in \Psi$

such that

$$\alpha(x,y)d(Tx,Ty) \le \psi\big(d(x,y)\big)$$

for all $x, y \in X$.

For the examples of α-admissible and α-ψ-contractive mappings, see [19, 24] and the examples in the next section.

2 Main results

We first modify the concept of α-admissible mapping.

Definition 2.1 Let T be a self-mapping on a metric space (X,d) and let $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x,y) \ge \eta(x,y) \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge \eta(Tx,Ty).$$

Note that if we take $\eta(x,y) = 1$, then this definition reduces to Definition 1.1. Also, if we take $\alpha(x,y) = 1$, then we say that T is an η-subadmissible mapping.

Our first result is the following.

Theorem 2.1 *Let (X,d) be a complete metric space and let T be an α-admissible mapping with respect to η. Assume that*

$$x, y \in X, \quad \alpha(x,y) \ge \eta(x,y) \quad \Longrightarrow \quad d(Tx,Ty) \le \psi\big(M(x,y)\big), \tag{2.1}$$

where $\psi \in \Psi$ and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$;*
(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n,x) \ge \eta(x_n,x)$ for all $n \in \mathbb{N}\cup\{0\}$.*

Then T has a fixed point.

Proof Let $x_0 \in X$ be such that $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since T is a generalized α-admissible mapping with respect to η and $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$, we deduce that $\alpha(x_1,x_2) = \alpha(Tx_0,T^2x_0) \ge \eta(Tx_0,T^2x_0) = \eta(x_1,x_2)$. Continuing this process, we get $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$. Now, by (2.1) with $x = x_{n-1}$, $y = x_n$, we get

$$d(Tx_{n-1},Tx_n) \le \psi\big(M(x_{n-1},x_n)\big).$$

On the other hand,

$$\begin{aligned}M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2},\right.\\&\qquad\left.\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\right\}\\&= \max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2},\frac{d(x_{n-1},x_{n+1})}{2}\right\}\\&\leq \max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\}\\&\leq \max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\},\end{aligned}$$

which implies

$$d(x_n,x_{n+1}) \leq \psi\bigl(\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr).$$

Now, if $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ for some $n \in \mathbb{N}$, then

$$d(x_n,x_{n+1}) \leq \psi\bigl(\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr) = \psi\bigl(d(x_n,x_{n+1})\bigr) < d(x_n,x_{n+1}),$$

which is a contradiction. Hence, for all $n \in \mathbb{N}$, we have

$$d(x_n,x_{n+1}) \leq \psi\bigl(d(x_{n-1},x_n)\bigr).$$

By induction, we have

$$d(x_n,x_{n+1}) \leq \psi^n\bigl(d(x_0,x_1)\bigr).$$

Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n\geq N}\psi^n\bigl(d(x_n,x_{n+1})\bigr) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Let $m, n \in \mathbb{N}$ with $m > n \geq N$. Then, by the triangular inequality, we get

$$d(x_n,x_m) \leq \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \leq \sum_{n\geq N}\psi^n\bigl(d(x_n,x_{n+1})\bigr) < \epsilon.$$

Consequently, $\lim_{m,n,\to+\infty} d(x_n,x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now, if we suppose that T is continuous, then we have

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So, z is a fixed point of T. On the other hand, since

$$\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to z$ as $n \to \infty$, we get

$$\alpha(x_n, z) \geq \eta(x_n, z)$$

for all $n \in \mathbb{N} \cup \{0\}$. Then from (2.1) we have

$$d(x_{n+1}, Tz) \leq \psi\bigl(M(x_n, z)\bigr),$$

where

$$M(x_n, z) = \max\left\{d(x_n, z), \frac{d(x_n, x_{n+1}) + d(z, Tz)}{2}, \frac{d(x_n, Tz) + d(z, x_{n+1})}{2}\right\}.$$

Since $M(x_n, z) > 0$, then

$$d(x_{n+1}, Tz) \leq \psi\bigl(M(x_n, z)\bigr) < M(x_n, z).$$

By taking limit as $n \to \infty$ in the above inequality, we have

$$d(z, Tz) = \lim_{n\to\infty} d(x_{n+1}, Tz) \leq \lim_{n\to\infty} M(x_n, z) = \frac{d(z, Tz)}{2},$$

which implies $d(z, Tz) = 0$, *i.e.*, $z = Tz$. □

By taking $\eta(x, y) = 1$ in Theorem 2.1, we have the following result.

Corollary 2.1 *Let (X, d) be a complete metric space and let T be an α-admissible mapping. Assume that for $\psi \in \Psi$,*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi\bigl(M(x, y)\bigr).$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*;

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

By taking $\alpha(x, y) = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2 *Let (X, d) be a complete metric space and let T be an η-subadmissible mapping. Assume that for $\psi \in \Psi$,*

$$x, y \in X, \quad \eta(x, y) \leq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi\bigl(M(x, y)\bigr).$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\eta(x_0, Tx_0) \leq 1$*;

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Clearly, Corollary 2.1 implies the following results.

Corollary 2.3 (Theorem 2.1 and Theorem 2.2 of [24]) *Let (X,d) be a complete metric space and let T be an α-admissible mapping. Assume that for $\psi \in \Psi$,*

$$\alpha(x,y)d(Tx,Ty) \le \psi\big(d(x,y)\big)$$

holds for all $x,y \in X$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Corollary 2.4 (Theorem 2.3 and Theorem 2.4 of [19]) *Let (X,d) be a complete metric space and let T be an α-admissible mapping. Assume that for $\psi \in \Psi$,*

$$\alpha(x,y)d(Tx,Ty) \le \psi\big(M(x,y)\big) \quad \forall x,y \in X,$$

where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Example 2.1 Let $X = [0,\infty)$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{2(x+1)} & \text{if } x \in [0,1], \\ \ln x + |\sin x| & \text{if } x \in (1,\infty). \end{cases}$$

Define also $\alpha : X \times X \to [0,+\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 4 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \psi(t) = \frac{1}{2}t.$$

We prove that Corollary 2.1 can be applied to T. But Theorem 2.2 of [24] and Theorem 2.4 of [19] cannot be applied to T.

Clearly, (X,d) is a complete metric space. We show that T is an α-admissible mapping. Let $x,y \in X$, if $\alpha(x,y) \ge 1$, then $x,y \in [0,1]$. On the other hand, for all $x \in [0,1]$ we have $Tx \le 1$. It follows that $\alpha(Tx,Ty) \ge 1$. Hence, the assertion holds. In reason of the above arguments, $\alpha(0,T0) \ge 1$.

Now, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subset [0,1]$ and hence $x \in [0,1]$. This implies that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Let $\alpha(x,y) \geq 1$. Then $x, y \in [0,1]$. We get

$$\begin{aligned} d(Tx, Ty) &= Ty - Tx = \frac{y}{2(y+1)} - \frac{x}{2(x+1)} \\ &= \frac{y-x}{2(1+x)(1+y)} \\ &\leq \frac{y-x}{2} = \frac{1}{2} d(x,y) \leq \frac{1}{2} M(x,y) = \psi\big(M(x,y)\big). \end{aligned}$$

That is,

$$\alpha(x,y) \geq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi\big(M(x,y)\big).$$

All of the conditions of Corollary 2.1 hold. Hence, T has a fixed point. Let $x = 0$ and $y = 1$, then

$$\alpha(0,1) d(T0, T1) = 1 > 1/2 = \psi\big(d(0,1)\big).$$

That is, Theorem 2.2 of [24] cannot be applied to T.

Also, by a similar method, we can show that Theorem 2.4 of [19] cannot be applied to T.

By the following simple example, we show that our results improve the results of Samet *et al.* [24] and the results of Karapinar and Samet [19].

Example 2.2 Let $X = [0, \infty)$ be endowed with the usual metric $d(x,y) = |x - y|$ for all $x, y \in X$ and let $T : X \to X$ be defined by $Tx = \frac{1}{4}x$. Also, define $\alpha : X^2 \to [0,\infty)$ by $\alpha(x,y) = 3$ and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{1}{2}t$.

Clearly, T is an α-admissible mapping. Also, $\alpha(x,y) = 3 \geq 1$ for all $x, y \in X$. Hence,

$$d(Tx, Ty) = \frac{1}{4}|x - y| \leq \frac{1}{2}|x - y| = \psi\big(d(x,y)\big) \leq \psi\big(M(x,y)\big).$$

Then the conditions of Corollary 2.1 hold and T has a fixed point. But if we choose $x = 4$ and $y = 8$, then

$$\alpha(4,8) d(T4, T8) = 3 > 2 = \psi\big(d(4,8)\big).$$

That is, Theorem 2.2 of [24] cannot be applied to T. Similarly, we can show that Theorem 2.4 of [19] cannot be applied to T. Further notice that the Banach contraction principle holds for this example.

Example 2.3 Let $X = [0, \infty)$ be endowed with the usual metric $d(x,y) = |x - y|$ for all $x, y \in X$ and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{4}x^2 & \text{if } x \in [0,1], \\ 2x^3 + 1 & \text{if } x \in (1, \infty). \end{cases}$$

Define also $\alpha, \eta : X \times X \to [0, +\infty)$ and $\psi : [0, \infty) \to [0, \infty)$ by

$$\eta(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 4 & \text{otherwise} \end{cases} \quad \text{and} \quad \psi(t) = \frac{1}{2}t.$$

We prove that Corollary 2.2 can be applied to T. But the Banach contraction principle cannot be applied to T.

Clearly, (X,d) is a complete metric space. We show that T is an η-subadmissible mapping. Let $x, y \in X$, if $\eta(x,y) \leq 1$, then $x, y \in [0,1]$. On the other hand, for all $x \in [0,1]$, we have $Tx \leq 1$. It follows that $\eta(Tx, Ty) \leq 1$. Also, $\eta(0, T0) \leq 1$.

Now, if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subset [0,1]$ and hence $x \in [0,1]$. This implies that $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

Let $\eta(x,y) \leq 1$. Then $x, y \in [0,1]$. We get

$$d(Tx, Ty) = \frac{1}{4}|x-y||x+y| \leq \frac{1}{2}|x-y| \leq \frac{1}{2}M(x,y) = \psi\big(M(x,y)\big).$$

That is,

$$\eta(x,y) \leq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi\big(M(x,y)\big).$$

Then the conditions of Corollary 2.2 hold. Hence, T has a fixed point. Let $x = 2$, $y = 3$ and $r \in [0,1)$. Then

$$d(T2, T3) = 38 > 1 > r = rd(2,3).$$

That is, the Banach contraction principle cannot be applied to T.

From our results, we can deduce the following corollaries.

Corollary 2.5 *Let (X,d) be a complete metric space and let T be an α-admissible mapping. Assume that*

$$\big(\alpha(x,y) + \ell\big)^{d(Tx,Ty)} \leq (1+\ell)^{\psi(d(x,y))} \tag{2.2}$$

holds for all $x, y \in X$, where $\psi \in \Psi$ and $\ell > 0$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Proof Let $\alpha(x,y) \geq 1$. Then by (2.2) we have

$$(1+\ell)^{d(Tx,Ty)} \leq \big(\alpha(x,y) + \ell\big)^{d(Tx,Ty)} \leq (1+\ell)^{\psi(d(x,y))}.$$

Then $d(Tx, Ty) \leq \psi(d(x,y))$. Hence, the conditions of Corollary 2.1 hold and f has a fixed point. □

Similarly, we have the following corollary.

Corollary 2.6 *Let (X,d) be a complete metric space and let T be an α-admissible mapping. Assume that*

$$\big(d(Tx,Ty)+\ell\big)^{\alpha(x,y)} \le \psi\big(d(x,y)\big)+\ell \tag{2.3}$$

hold for all $x,y\in X$, where $\psi\in\Psi$ and $\ell>0$. Also, suppose that the following assertions hold:

(i) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$;*
(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x$ as $n\to+\infty$, we have $\alpha(x_n,x)\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$.*

Then T has a fixed point.

Notice that the main theorem of Dutta and Choudhury [9] remains true if ϕ is lower semi-continuous instead of continuous (see, *e.g.*, [1, 8]).

We assume that

$$\Psi_1=\big\{\psi:[0,\infty)\to[0,\infty) \text{ such that } \psi \text{ is non-decreasing and continuous}\big\}$$

and

$$\Phi=\big\{\varphi:[0,\infty)\to[0,\infty) \text{ such that } \varphi \text{ is lower semicontinuous}\big\},$$

where $\psi(t)=\varphi(t)=0$ if and only if $t=0$.

Theorem 2.2 *Let (X,d) be a complete metric space and let T be an α-admissible mapping with respect to η. Assume that for $\psi\in\Psi_1$ and $\varphi\in\Phi$,*

$$\begin{aligned} &x,y\in X,\quad \alpha(x,Tx)\alpha(y,Ty)\ge\eta(x,Tx)\eta(y,Ty)\\ &\quad\Longrightarrow\quad \psi\big(d(Tx,Ty)\big)\le\psi\big(d(x,y)\big)-\varphi\big(d(x,y)\big). \end{aligned} \tag{2.4}$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge\eta(x_0,Tx_0)$;*
(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1})\ge\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x$ as $n\to+\infty$, we have $\alpha(x,Tx)\ge\eta(x,Tx)$ for all $n\in\mathbb{N}\cup\{0\}$.*

Then T has a fixed point.

Proof Let $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge\eta(x_0,Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n=T^nx_0=Tx_{n-1}$ for all $n\in\mathbb{N}$. If $x_{n+1}=x_n$ for some $n\in\mathbb{N}$, then $x=x_n$ is a fixed point for T and the result is proved. We suppose that $x_{n+1}\ne x_n$ for all $n\in\mathbb{N}$. Since T is an α-admissible mapping with respect to η and $\alpha(x_0,Tx_0)\ge\eta(x_0,Tx_0)$, we deduce that $\alpha(x_1,x_2)=\alpha(Tx_0,T^2x_0)\ge\eta(Tx_0,T^2x_0)=\eta(x_1,x_2)$. By continuing this process, we get $\alpha(x_n,x_{n+1})\ge\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}\cup\{0\}$. Clearly,

$$\alpha(x_{n-1},Tx_{n-1})\alpha(x_n,Tx_n)\ge\eta(x_{n-1},Tx_{n-1})\eta(x_n,Tx_n).$$

Now, by (2.4) with $x = x_{n-1}$, $y = x_n$, we have

$$\psi\left(d(Tx_{n-1}, Tx_n)\right) \leq \psi\left(d(x_{n-1}, x_n)\right) - \varphi\left(d(x_{n-1}, x_n)\right),$$

which implies

$$\psi\left(d(x_n, x_{n+1})\right) \leq \psi\left(d(x_{n-1}, x_n)\right) - \varphi\left(d(x_{n-1}, x_n)\right) \leq \psi\left(d(x_{n-1}, x_n)\right). \tag{2.5}$$

Since ψ is increasing, we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. That is, $\{d_n := d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. We shall show that $r = 0$. By taking the limit infimum as $n \to \infty$ in (2.5), we have

$$\psi(r) \leq \psi(r) - \varphi(r).$$

Hence $\phi(r) = 0$. That is, $r = 0$. Then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \quad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

Now, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \varepsilon + d(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Taking limit as $k \to +\infty$ in the above inequality and using (2.6), we get

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.7}$$

Since

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}),$$

then by taking the limit as $k \to +\infty$ in the above inequality, and by using (2.6) and (2.7), we deduce that

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{2.8}$$

On the other hand,

$$\alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)}) \geq \eta(x_{n(k)}, Tx_{n(k)})\eta(x_{m(k)}, Tx_{m(k)}).$$

Then, by (2.4) with $x = x_{n(k)}$ and $y = x_{m(k)}$, we get

$$\psi\big(d(x_{n(k)+1}, x_{m(k)+1})\big) \leq \psi\big(d(x_{n(k)}, x_{m(k)})\big) - \varphi\big(d(x_{n(k)}, x_{m(k)})\big).$$

By taking limit as $k \to \infty$ in the above inequality and applying (2.7) and (2.8), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon).$$

That is, $\varepsilon = 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \to z$. First we assume that T is continuous. Then we deduce

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So, z is a fixed point of T. On the other hand, since

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup 0$ and $x_n \to z$ as $n \to \infty$, so

$$\alpha(z, Tz) \geq \eta(z, Tz),$$

which implies

$$\alpha(x_n, x_{n+1})\alpha(z, Tz) \geq \eta(x_n, x_{n+1})\eta(z, Tz).$$

Now, by (2.4) we get

$$\psi\big(d(x_{n+1}, Tz)\big) = \psi\big(d(Tx_n, Tz)\big) \leq \psi\big(d(x_n, z)\big) - \varphi\big(d(x_n, z)\big).$$

Passing limit inf as $n \to \infty$ in the above inequality, we have

$$\psi\big(d(z, Tz)\big) = \lim_{n\to\infty} \psi\big(d(x_{n+1}, Tz)\big) = 0.$$

That is, $z = Tz$. □

By taking $\eta(x, y) = 1$ in Theorem 2.2, we deduce the following corollary.

Corollary 2.7 *Let (X, d) be a complete metric space and let T be an α-admissible mapping. Assume that for $\psi \in \Psi_1$ and $\varphi \in \Phi$,*

$$x, y \in X, \quad \alpha(x, Tx)\alpha(y, Ty) \geq 1 \quad \Longrightarrow \quad \psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big).$$

Also, suppose that the following assertions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

(ii) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $\alpha(x, Tx) \geq 1$.

Then T *has a fixed point.*

By taking $\alpha(x, y) = 1$ in Theorem 2.2, we deduce the following corollary.

Corollary 2.8 *Let* (X, d) *be a complete metric space and let* T *be an* η*-subadmissible mapping. Assume that for* $\psi \in \Psi_1$ *and* $\varphi \in \Phi$,

$$x, y \in X, \quad \eta(x, Tx)\eta(y, Ty) \leq 1 \quad \Longrightarrow \quad \psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big).$$

Also, suppose that the following assertions hold:

(i) *there exists* $x_0 \in X$ *such that* $\eta(x_0, Tx_0) \leq 1$;

(ii) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\eta(x_n, x_{n+1}) \leq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $\eta(x, Tx) \leq 1$.

Then T *has a fixed point.*

Example 2.4 Let $X = [0, \infty)$ be endowed with the usual metric

$$d(x, y) = \begin{cases} \max\{x, y\} & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in X$, and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x-x^2}{2} & \text{if } x \in [0, 1], \\ 2x & \text{if } x \in (1, \infty). \end{cases}$$

Define also $\alpha : X \times X \to [0, +\infty)$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ \frac{1}{2} & \text{otherwise}, \end{cases} \qquad \psi(t) = t \quad \text{and} \quad \varphi(t) = \frac{1}{2}t.$$

We prove that Corollary 2.7 can be applied to T, but the main theorem in [9] cannot be applied to T.

By a similar proof to that of Example 2.1, we show that T is an α-admissible mapping. Assume that $\alpha(x, Tx)\alpha(y, Ty) \geq 1$. Now, if $x \notin [0, 1]$, then $\alpha(x, Tx) = \frac{1}{2}$ and so $\alpha(x, Tx)\alpha(y, Ty) < 1$, which is contradiction. If $y \notin [0, 1]$. Similarly, $\alpha(x, Tx)\alpha(y, Ty) < 1$, which is contradiction. Hence, $\alpha(x, Tx)\alpha(y, Ty) \geq 1$ implies $x, y \in [0, 1]$. Therefore, we get

$$\psi\big(d(Tx, Ty)\big) = \max\left\{\frac{x - x^2}{2}, \frac{y - y^2}{2}\right\} \leq \frac{1}{2}\max\{x, y\} = \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big).$$

That is,

$$\alpha(x, Tx)\alpha(y, Ty) \geq 1 \quad \Longrightarrow \quad \psi\big(d(Tx, Ty)\big) \leq \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big).$$

The conditions of Corollary 2.7 are satisfied. Hence, T has a fixed point. Let $x = 2$ and $y = 3$, then

$$\psi\big(d(T2,T3)\big) = 6 > 1/2 = \psi\big(d(2,3)\big) - \varphi\big(d(2,3)\big).$$

That is, the main theorem in [9] cannot be applied to T.

Example 2.5 Let $X = [0,\infty)$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$, and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{5}(1-x^3) & \text{if } x \in [0,1], \\ |\sin(\frac{\pi}{2}x)| & \text{if } x \in (1,\infty). \end{cases}$$

Define also $\eta : X \times X \to [0,+\infty)$ and $\psi, \varphi : [0,\infty) \to [0,\infty)$ by

$$\eta(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 7 & \text{otherwise}, \end{cases} \qquad \psi(t) = t \quad \text{and} \quad \varphi(t) = \frac{2}{5}t.$$

We prove that Corollary 2.8 can be applied to T, but the main theorem in [9] cannot be applied to T.

By a similar proof to that of Example 2.3, we can show that T is an η-subadmissible mapping.

Assume that $\eta(x,Tx)\eta(y,Ty) \leq 1$. Now, if $x \notin [0,1]$, then $\eta(x,Tx)\eta(y,Ty) > 1$, which is a contradiction. Similarly, $y \notin [0,1]$ is a contradiction. Hence, $\eta(x,Tx)\eta(y,Ty) \leq 1$ implies $x,y \in [0,1]$. We get

$$\psi\big(d(Tx,Ty)\big) = \frac{1}{5}|x-y|\big|x^2+xy+y^2\big| \leq \frac{3}{5}|x-y| = \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big).$$

That is,

$$\eta(x,Tx)\eta(y,Ty) \leq 1 \quad \Longrightarrow \quad d(Tx,Ty) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big).$$

Then the conditions of Corollary 2.8 hold and T has a fixed point. Let $x = 2$, $y = 3$. Then $T2 = 0$ and $T3 = 1$, which implies

$$\psi\big(d(T2,T3)\big) = 1 > \frac{3}{5} = \psi\big(d(2,3)\big) - \varphi\big(d(2,3)\big).$$

That is, the main theorem in [9] cannot be applied to T.

In 1984 Khan *et al.* [20] proved the following theorem.

Theorem 2.3 *Let (X,d) be a complete metric space and let T be a self-mapping on X. Assume that*

$$\psi\big(d(Tx,Ty)\big) \leq c\psi\big(d(x,y)\big) \quad \forall x,y \in X,$$

where $\psi \in \Psi_1$ and $0 < c < 1$. Then T has a unique fixed point.

Theorem 2.4 *Let (X,d) be a complete metric space and let T be a generalized α-admissible mapping with respect to η. Assume that*

$$x,y \in X, \quad \alpha(x,x)\alpha(y,y) \geq \eta(x,x)\eta(y,y) \quad \Longrightarrow \quad \psi\big(d(Tx,Ty)\big) \leq c\psi\big(d(x,y)\big), \tag{2.9}$$

where $\psi \in \Psi_1$ and $0 < c < 1$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,x_0) \geq \eta(x_0,x_0)$;*

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_n) \geq \eta(x_n,x_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x,x) \geq \eta(x,x)$.*

Then T has a fixed point.

Proof Let $x_0 \in X$ such that $\alpha(x_0,x_0) \geq \eta(x_0,x_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since T is a generalized α-admissible mapping with respect to η and $\alpha(x_0,x_0) \geq \eta(x_0,x_0)$, we deduce that $\alpha(x_1,x_1) = \alpha(Tx_0,Tx_0) \geq \eta(Tx_0,Tx_0) = \eta(x_1,x_1)$. By continuing this process, we get $\alpha(x_n,x_n) \geq \eta(x_n,x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly,

$$\alpha(x_{n-1},x_{n-1})\alpha(x_n,x_n) \geq \eta(x_{n-1},x_{n-1})\eta(x_n,x_n).$$

Now, by (2.9) with $x = x_{n-1}$, $y = x_n$, we have

$$\psi\big(d(x_n,x_{n+1})\big) = \psi\big(d(Tx_{n-1},Tx_n)\big) \leq c\psi\big(d(x_{n-1},x_n)\big) < \psi\big(d(x_{n-1},x_n)\big). \tag{2.10}$$

Since, ψ is increasing, we get

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n)$$

for all $n \in \mathbb{N}$. That is, $\{d_n := d(x_n,x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. We shall show that $r = 0$. By taking the limit as $n \to \infty$ in (2.10), we have

$$\psi(r) \leq c\psi(r),$$

which implies $\psi(r) = 0$, *i.e.*, $r = 0$. Then

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.11}$$

Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Proceeding as in the proof of Theorem 2.2, there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exist $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ such that

$$\lim_{k\to+\infty} d(x_{n(k)},x_{m(k)}) = \varepsilon \tag{2.12}$$

and

$$\lim_{k\to+\infty} d(x_{n(k)+1},x_{m(k)+1}) = \varepsilon. \tag{2.13}$$

Clearly,

$$\alpha(x_{n(k)}, x_{n(k)})\alpha(x_{m(k)}, x_{m(k)}) \geq \eta(x_{n(k)}, x_{n(k)})\eta(x_{m(k)}, x_{m(k)}).$$

Then, by (2.9) with $x = x_{n(k)}$ and $y = x_{m(k)}$, we get

$$\psi\big(d(x_{n(k)+1}, x_{m(k)+1})\big) = \psi\big(d(Tx_{n(k)}, Tx_{m(k)})\big) \leq c\psi\big(d(x_{n(k)}, x_{m(k)})\big).$$

Taking limit as $k \to \infty$ in the above inequality and applying (2.12) and (2.13), we get

$$\psi(\epsilon) \leq c\psi(\epsilon),$$

and so $\epsilon = 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \to z$. First, we assume that T is continuous. Then, we deduce

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So, z is a fixed point of T. On the other hand, since

$$\alpha(x_n, x_n) \geq \eta(x_n, x_n)$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to z$ as $n \to \infty$, we get

$$\alpha(z, z) \geq \eta(z, z),$$

which implies

$$\alpha(z, z)\alpha(x_n, x_n) \geq \eta(z, z)\eta(x_n, x_n).$$

Then by (2.12) we deduce

$$\psi\big(d(x_{n+1}, Tz)\big) = \psi\big(d(Tx_n, Tz)\big) \leq c\psi\big(d(x_n, z)\big).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\psi\big(d(z, Tz)\big) \leq \psi(0) = 0$$

and then $z = Tz$. □

Corollary 2.9 *Let (X, d) be a complete metric space and let T be an α-admissible mapping. Assume that*

$$x, y \in X, \quad \alpha(x, x)\alpha(y, y) \geq 1 \quad \Longrightarrow \quad \psi\big(d(Tx, Ty)\big) \leq c\psi\big(d(x, y)\big),$$

where $\psi \in \Psi_1$ and $0 < c < 1$. Also, suppose that the following assertions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, x_0) \geq 1$;
(ii) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_n) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $\alpha(x, x) \geq 1$.

Then T *has a fixed point.*

Corollary 2.10 *Let* (X, d) *be a complete metric space and let* T *be a generalized* α-*admissible mapping with respect to* η. *Assume that*

$$x, y \in X, \quad \eta(x,x)\eta(y,y) \leq 1 \quad \Longrightarrow \quad \psi\big(d(Tx, Ty)\big) \leq c\psi\big(d(x,y)\big),$$

where $\psi \in \Psi_1$ *and* $0 < c < 1$. *Also, suppose that the following assertions hold*:
(i) *there exists* $x_0 \in X$ *such that* $\eta(x_0, x_0) \leq 1$;
(ii) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\eta(x_n, x_n) \leq 1$ *for all* $n \in \mathbb{N} \cup 0$ *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $\eta(x, x) \leq 1$.

Then T *has a fixed point.*

Example 2.6 Let $X = [0, \infty)$ be endowed with the usual metric

$$d(x,y) = \begin{cases} \max\{x, y\} & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in X$, and let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{x^3 - x^5}{8} & \text{if } x \in [0,1], \\ 2x^2 + |(x-2)(x-3)| & \text{if } x \in (1, \infty). \end{cases}$$

Define also $\alpha : X \times X \to [0, +\infty)$ and $\psi, \varphi : [0, \infty) \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad \psi(t) = t^2.$$

We prove that Corollary 2.9 can be applied to T. But Theorem 2.3 cannot be applied to T.

By a similar proof to that of Example 2.1, we show that T is an α-admissible mapping. Assume that $\alpha(x,x)\alpha(y,y) \geq 1$. Now, if $x \notin [0,1]$, then $\alpha(x,x) = \frac{1}{2}$ and so $\alpha(x,x)\alpha(y,y) < 1$, which is contradiction. If $y \notin [0,1]$. Similarly, $\alpha(x,x)\alpha(y,y) < 1$, which is contradiction. Hence, $\alpha(x,x)\alpha(y,y) \geq 1$ implies $x, y \in [0,1]$. Therefore, we get

$$\psi\big(d(Tx, Ty)\big) = \left(\max\left\{\frac{x^3 - x^5}{8}, \frac{y^3 - y^5}{8}\right\}\right)^2 \leq \frac{1}{16}\big(\max\{x, y\}\big)^2 = \frac{1}{16}\psi\big(d(x,y)\big).$$

That is,

$$\alpha(x,x)\alpha(y,y) \geq 1 \quad \Longrightarrow \quad \psi\big(d(Tx, Ty)\big) \leq \frac{1}{16}\psi\big(d(x,y)\big).$$

Then the conditions of Corollary 2.9 hold. Hence, T has a fixed point. Let $x = 2$ and $y = 3$, then $T2 = 8$ and $T3 = 18$, and hence

$$\psi\big(d(T2, T3)\big) = 100 > \frac{1}{16} = \frac{1}{16}\psi\big(d(2,3)\big).$$

That is, Theorem 2.3 cannot be applied to T.

3 Application to the existence of solutions of integral equations

Integral equations like (3.1) were studied in many papers (see [2, 11] and references therein). In this section, we look for a nonnegative solution to (3.1) in $X = C([0,T],\mathbb{R})$. Let $X = C([0,T],\mathbb{R})$ be the set of real continuous functions defined on $[0,T]$ and let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x,y) = \|x - y\|_\infty$$

for all $x, y \in X$. Then (X, d) is a complete metric space.

Consider the integral equation

$$x(t) = p(t) + \int_0^T S(t,s) f\big(s, x(s)\big)\, ds, \tag{3.1}$$

and let $F : X \to X$ defined by

$$F(x)(t) = p(t) + \int_0^T S(t,s) f\big(s, x(s)\big)\, ds. \tag{3.2}$$

We assume that

(A) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B) $p : [0,T] \to \mathbb{R}$ is continuous;

(C) $S : [0,T] \times \mathbb{R} \to [0,+\infty)$ is continuous;

(D) there exist $\psi \in \Psi$ and $\theta : X \times X \to \mathbb{R}$ such that if $\theta(x,y) \geq 0$ for $x, y \in X$, then for every $s \in [0,T]$ we have

$$\begin{aligned} 0 &\leq f\big(s, x(s)\big) - f\big(s, y(s)\big) \\ &\leq \psi\bigg(\max\bigg\{\big|x(s) - y(s)\big|, \frac{1}{2}\big[\big|x(s) - F\big(x(s)\big)\big| + \big|y(s) - F\big(y(s)\big)\big|\big], \\ &\quad \frac{1}{2}\big[\big|x(s) - F\big(y(s)\big)\big| + \big|y(s) - F\big(x(s)\big)\big|\big]\bigg\}\bigg); \end{aligned}$$

(F) there exists $x_0 \in X$ such that $\theta(x_0, F(x_0)) \geq 0$;

(G) if $\theta(x,y) \geq 0$, $x, y \in X$, then $\theta(Fx, Fy) \geq 0$;

(H) if $\{x_n\}$ is a sequence in X such that $\theta(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\theta(x_n, x) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$;

(J) $\int_0^T S(t,s)\, ds \leq 1$ for all $t \in [0,T]$ and $s \in \mathbb{R}$.

Theorem 3.1 *Under assumptions* (A)-(J), *the integral equation* (3.1) *has a solution in* $X = C([0,T],\mathbb{R})$.

Proof Consider the mapping $F : X \to X$ defined by (3.2).

By the condition (D), we deduce

$$\begin{aligned}
&\bigl|F(x)(t) - F(y)(t)\bigr| \\
&\quad = \biggl|\int_0^T S(t,s)\bigl[f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr]\,ds\biggr| \\
&\quad \le \int_0^T S(t,s)\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&\quad \le \int_0^T S(t,s)\biggl[\psi\biggl(\max\biggl\{\bigl|x(s)-y(s)\bigr|, \frac{1}{2}\bigl[\bigl|x(s)-F\bigl(x(s)\bigr)\bigr| + \bigl|y(s)-F\bigl(y(s)\bigr)\bigr|\bigr], \\
&\quad\quad \frac{1}{2}\bigl[\bigl|x(s)-F\bigl(y(s)\bigr)\bigr| + \bigl|y(s)-F\bigl(x(s)\bigr)\bigr|\bigr]\biggr\}\biggr)\biggr]\,ds \\
&\quad \le \int_0^T S(t,s)\biggl[\psi\biggl(\max\biggl\{\bigl\|x(s)-y(s)\bigr\|, \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(x(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(y(s)\bigr)\bigr\|\bigr], \\
&\quad\quad \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(y(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(x(s)\bigr)\bigr\|\bigr]\biggr\}\biggr)\biggr]\,ds \\
&\quad = \biggl(\int_0^T S(t,s)\,ds\biggr)\psi\biggl(\max\biggl\{\bigl\|x(s)-y(s)\bigr\|, \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(x(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(y(s)\bigr)\bigr\|\bigr], \\
&\quad\quad \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(y(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(x(s)\bigr)\bigr\|\bigr]\biggr\}\biggr) \\
&\quad \le \psi\biggl(\max\biggl\{\bigl\|x(s)-y(s)\bigr\|, \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(x(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(y(s)\bigr)\bigr\|\bigr], \\
&\quad\quad \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(y(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(x(s)\bigr)\bigr\|\bigr]\biggr\}\biggr).
\end{aligned}$$

Then

$$\begin{aligned}
\|Fx - Fy\|_\infty \le \psi\biggl(\max\biggl\{&\bigl\|x(s)-y(s)\bigr\|, \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(x(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(y(s)\bigr)\bigr\|\bigr], \\
&\frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(y(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(x(s)\bigr)\bigr\|\bigr]\biggr\}\biggr).
\end{aligned}$$

Now, define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \theta(x,y) \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $\alpha(x,y) \ge 1$ implies

$$\begin{aligned}
\|Fx - Fy\|_\infty \le \psi\biggl(\max\biggl\{&\bigl\|x(s)-y(s)\bigr\|, \frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(x(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(y(s)\bigr)\bigr\|\bigr], \\
&\frac{1}{2}\bigl[\bigl\|x(s)-F\bigl(y(s)\bigr)\bigr\| + \bigl\|y(s)-F\bigl(x(s)\bigr)\bigr\|\bigr]\biggr\}\biggr), \\
\|Fx - Fy\|_\infty \le \psi\biggl(\max\biggl\{&\|x-y\|_\infty, \frac{1}{2}\bigl[\bigl\|x-F(x)\bigr\|_\infty + \bigl\|y-F(y)\bigr\|_\infty\bigr], \\
&\frac{1}{2}\bigl[\bigl\|x-F(y)\bigr\|_\infty + \bigl\|y-F(x)\bigr\|_\infty\bigr]\biggr\}\biggr).
\end{aligned}$$

All of the hypotheses of Corollary 2.1 are satisfied, and hence the mapping F has a fixed point that is a solution in $X = C([0,T],\mathbb{R})$ of the integral equation (3.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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