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Research Article

Modified Block Iterative Algorithm for Solving Convex Feasibility Problems in Banach Spaces

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The purpose of this paper is to use the modified block iterative method to propose an algorithm for solving the convex feasibility problems for an infinite family of quasi-ϕ-asymptotically nonexpansive mappings. Under suitable conditions some strong convergence theorems are established in uniformly smooth and strictly convex Banach spaces with *Kadec-Klee property*. The results presented in the paper improve and extend some recent results.

1. Introduction

The problem of finding a point in the intersection of closed and convex subsets $\{C_i\}_{i=1}^m$ of a Banach space is a frequently appearing problem in diverse areas of mathematics and physical sciences. This problem is commonly referred to as the *convex feasibility problem* (CFP). There is a considerable investigation on (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [1]. The advantage of a Hilbert space H is that the projection P_C onto a closed convex subset C of H is nonexpansive. So projection methods have dominated in the iterative approaches to (CFP) in Hilbert space. In 1993, Kitahara and Takahashi [2] deal with the convex feasibility problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banach space (see also, O'Hara et al. [3] and Chang et al. [4]). It is known that if C is a nonempty closed convex subset of a smooth, reflexive, and strictly convex Banach space E, then the *generalized projection* Π_C from E onto C is relatively nonexpansive. In 2005, Matsushita and Takahashi [5] reformulated the definition of the notion and obtained weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping. Recently, Qin et al. [6], Zhou and

Tan [7], Wattanawitoon and Kumam [8], Li and Su [9], and Takahashi and Zembayashi [10] extend the notion from relatively nonexpansive mappings or quasi-ϕ-nonexpansive mappings to quasi-ϕ-asymptotically nonexpansive mappings and also prove some weak and strong convergence theorems to approximate a common fixed point of finite or infinite family of quasi-ϕ-nonexpansive mappings or quasi-ϕ-asymptotically nonexpansive mappings.

It should be noted that the *block iterative algorithm* is a method which often used by many authors to solve the convex feasibility problem (see, e.g., Kikkawa and Takahashi [11], Aleyner and Reich [12]). Recently, some authors by using the block iterative scheme to establish strong convergence theorems for a finite family of relativity nonexpansive mappings in Hilbert space or finite-dimensional Banach space (see, e.g., Aleyner and Reich [12], Plubtieng and Ungchittrakool [13, 14]) or uniformly smooth and uniformly convex Banach spaces (see, e.g., Sahu et al. [15] and Ceng et al. [16–18]).

Motivated and inspired by these facts, the purpose of this paper is to use the modified block iterative method to propose an iterative algorithm for solving *the convex feasibility problems* for an infinite family of quasi-ϕ-asymptotically nonexpansive. Under suitable conditions some strong convergence theorems are established in a uniformly smooth and strictly convex Banach space with *Kadec-Klee property*. The results presented in the paper improve and extend the corresponding results in Aleyner and Reich [12], Plubtieng and Ungchittrakool [13, 14], and Chang et al. [19].

2. Preliminaries

Throughout this paper we assume that E is a real Banach space with the dual E^* and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \left\| f^* \right\|^2 \right\}, \quad x \in E. \tag{2.1}$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T, and use $\mathcal{R}$ and $\mathcal{R}^+$ to denote the set of all real numbers and the set of all nonnegative real numbers, respectively. We also denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and weak convergence of a sequence $\{x_n\}$, respectively.

A Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex* if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 \le 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \ge \epsilon$. E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\left\| x + ty \right\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. E is said to be *uniformly smooth* if the above limit exists uniformly in $x, y \in U$.

Remark 2.1. The following basic properties can be found in Cioranescu [20].

(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is hemicontinuous, that is, J^{-1} is norm-$weak^*$-continuous.

(iii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one, and onto.

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(v) Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Next we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E. In the sequel we always use $\phi : E \times E \to \mathcal{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.3}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{2.4}$$

Following Alber [21], the *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \inf_{y \in C} \phi(y,x), \quad \forall x \in E. \tag{2.5}$$

Lemma 2.2 (see [21]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) *$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y)$ for all $x \in C$ and $y \in E$;*

(b) *if $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \Longleftrightarrow \langle z - y, Jx - Jz\rangle \ge 0, \quad \forall y \in C, \tag{2.6}$$

(c) *for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$.*

Remark 2.3. If E is a real Hilbert space H, then $\phi(x,y) = \|x - y\|^2$ and Π_C is the metric projection P_C of H onto C.

Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, $T : C \to C$ a mapping, and $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all asymptotic fixed points of T by $\widetilde{F}(T)$.

Definition 2.4. (1) A mapping $T : C \to C$ is said to be *relatively nonexpansive* [5] if $F(T) \neq \emptyset$, $F(T) = \widetilde{F}(T)$, and

$$\phi(p, Tx) \le \phi(p,x), \quad \forall x \in C,\ p \in F(T). \tag{2.7}$$

(2) A mapping $T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Definition 2.5. (1) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \; p \in F(T). \tag{2.8}$$

(2) A mapping $T : C \to C$ is is said to be *quasi-ϕ-asymptotically nonexpansive* [7], if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1, \; x \in C, \; p \in F(T). \tag{2.9}$$

Remark 2.6. (1) From the definition, it is easy to know that each relatively nonexpansive mapping is closed.

(2) The class of quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-nonexpansive mappings as a subclass and the class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true.

Next, we give some examples which are closed and quasi-ϕ-asymptotically nonexpansive mappings.

Example 2.7 (see [7]). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ a maximal monotone mapping such that $A^{-1}0$ (the set of zero points of A) is nonempty. Then the mapping $J_r = (J + rA)^{-1} J$ is closed and quasi-ϕ-asymptotically nonexpansive from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.8. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset $C \subset E$. Then Π_C is relative nonexpansive, which in turn is a closed and quasi-ϕ-nonexpansive mapping, and so it is a closed and quasi-ϕ-asymptotically nonexpansive mapping.

Lemma 2.9 (see [13, 22]). *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{n=1}^{N} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$,*

$$\left\| \sum_{n=1}^{N} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{N} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.10}$$

Lemma 2.10. *Let E be a uniformly convex Banach space, $r > 0$ a positive number and $B_r(0)$ a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.11}$$

Proof. Since $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and $\lambda_i > 0$ for all $i \geq 1$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, we have

$$\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\| \leq \sum_{i=1}^{\infty} \lambda_i \|x_i\| \leq r. \tag{2.12}$$

Hence, for any given $\epsilon > 0$ and any given positive integers i, j with $i < j$, it follows from (2.12) that there exists a positive integer $N > j$ such that $\| \sum_{i=N+1}^{\infty} \lambda_i x_i \| \leq \epsilon$. Letting $\sigma_N = \sum_{i=1}^{N} \lambda_i$, by Lemma 2.9, we have

$$\begin{aligned}
\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\|^2 &= \left\| \sigma_N \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} + \sum_{i=N+1}^{\infty} \lambda_i x_i \right\|^2 \leq \left(\sigma_N \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\| + \left\| \sum_{i=N+1}^{\infty} \lambda_i x_i \right\| \right)^2 \\
&\leq \sigma_N^2 \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\|^2 + \epsilon^2 + 2\epsilon\sigma_N \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\| \\
&\leq \sigma_N^2 \sum_{i=1}^{N} \frac{\lambda_i}{\sigma_N} \|x_i\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|) + \epsilon \left(\epsilon + 2\sigma_N \left\| \sum_{i=1}^{N} \frac{\lambda_i x_i}{\sigma_N} \right\| \right) \\
&\leq \sum_{i=1}^{N} \lambda_i \|x_i\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|) + \epsilon \left(\epsilon + 2 \left\| \sum_{i=1}^{N} \lambda_i x_i \right\| \right) \\
&\leq \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|) + \epsilon \left(\epsilon + 2 \left\| \sum_{i=1}^{N} \lambda_i x_i \right\| \right).
\end{aligned} \tag{2.13}$$

Since $\epsilon > 0$ is arbitrary, the conclusion of Lemma 2.10 is proved. □

Lemma 2.11. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Then $F(T)$ is a closed convex subset of C.*

Proof. Letting $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ (as $n \to \infty$), we prove that $p \in F(T)$. In fact, from the definition of T, we have

$$\phi(p_n, Tp) \leq k_1 \phi(p_n, p) \longrightarrow 0 \quad (\text{as} \ \ n \longrightarrow \infty). \tag{2.14}$$

Therefore we have

$$\begin{aligned}
\lim_{n \to \infty} \phi(p_n, Tp) &= \lim_{n \to \infty} \left(\|p_n\|^2 - 2\langle p_n, JTp \rangle + \|Tp\| \right) \\
&= \|p\|^2 - 2\langle p, JTp \rangle + \|Tp\| = \phi(p, Tp) = 0,
\end{aligned} \tag{2.15}$$

that is, $p \in F(T)$.

Next we prove that $F(T)$ is convex. For any $p, q \in F(T)$, $t \in (0,1)$, putting $w = tp + (1-t)q$, we prove that $w \in F(T)$. Indeed, in view of the definition of $\phi(x, y)$ we have

$$\begin{aligned}
\phi(w, T^n w) &= \|w\|^2 - 2\langle w, JT^n w\rangle + \|T^n w\|^2 \\
&= \|w\|^2 - 2t\langle p, JT^n w\rangle - 2(1-t)\langle q, JT^n w\rangle + \|T^n w\|^2 \\
&= \|w\|^2 + t\phi(p, T^n w) + (1-t)\phi(q, T^n w) - t\|p\|^2 - (1-t)\|q\|^2 \\
&\le \|w\|^2 + tk_n\phi(p, w) + (1-t)k_n\phi(q, w) - t\|p\|^2 - (1-t)\|q\|^2 \\
&= (k_n - 1)\left(t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2\right).
\end{aligned} \tag{2.16}$$

Since $k_n \to 1$, we have $\phi(w, T^n w) \to 0$ (as $n \to \infty$). From (2.4) we have $\|T^n w\| \to \|w\|$. Consequently $\|JT^n w\| \to \|Jw\|$. This implies that $\{JT^n w\}$ is a bounded sequence. Since E is reflexive, E^* is also reflexive. So we can assume that

$$JT^n w \rightharpoonup f_0 \in E^*. \tag{2.17}$$

Again since E is reflexive, we have $J(E) = E^*$. Therefore there exists $x \in E$ such that $Jx = f_0$. By virtue of the weakly lower semicontinuity of norm $\|\cdot\|$, we have

$$\begin{aligned}
0 = \liminf_{n\to\infty} \phi(w, T^n w) &= \liminf_{n\to\infty}\left(\|w\|^2 - 2\langle w, J(T^n w)\rangle + \|T^n w\|^2\right) \\
&= \liminf_{n\to\infty}\left(\|w\|^2 - 2\langle w, J(T^n w)\rangle + \|J(T^n w)\|^2\right) \\
&\ge \|w\|^2 - 2\langle w, f_0\rangle + \|f_0\|^2 \\
&= \|w\|^2 - 2\langle w, Jx\rangle + \|Jx\|^2 \\
&= \|w\|^2 - 2\langle w, Jx\rangle + \|x\|^2 = \phi(w, x),
\end{aligned} \tag{2.18}$$

that is, $w = x$ which implies that $f_0 = Jw$. Thus from (2.17) we have $JT^n w \rightharpoonup Jw \in E^*$. Since $\|JT^n w\| \to \|Jw\|$ and E^* has the Kadec-Klee property, we have $JT^n w \to Jw$. Since E is uniformly smooth and strictly convex, by Remark 2.1(ii) it yields that $J^{-1} : E^* \to E$ is hemi-continuous. Therefore $T^n w \rightharpoonup w$. Again since $\|T^n w\| \to \|w\|$, by using the Kadec-Klee property of E, we have $T^n w \to w$. This implies that $TT^n w = T^{n+1} w \to w$. Since T is closed, we have $w = Tw$. This completes the proof of Lemma 2.11. □

3. Main Results

In this section, we will use the modified block iterative method to propose an iterative algorithm for solving the convex feasibility problem for an infinite family of quasi-ϕ-asymptotically nonexpansive mappings in uniformly smooth and strictly convex Banach spaces with the *Kadec-Klee property*.

Definition 3.1. (1) Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings. $\{S_i\}_{i=1}^{\infty}$ is said to be *a family of uniformly quasi -ϕ-asymptotically nonexpansive mappings*, if $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that for each $i \geq 1$

$$\phi(p, S_i^n x) \leq k_n \phi(p, x), \quad \forall p \in \bigcap_{n=1}^{\infty} F(S_n),\ x \in C,\ \forall n \geq 1. \tag{3.1}$$

(2) A mapping $S : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{3.2}$$

Theorem 3.2. *Let E be a uniformly smooth and strictly convex Banach space with Kleac-Klee property and C a nonempty closed convex subsets of E. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$. Suppose that for each $i \geq 1$, S_i is uniformly L_i-Lipschitz continuous and that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(S_i)$ is a nonempty and bounded subset in C. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
&x_0 \in C \text{ chosen arbitrary}, \quad C_0 = C,\\
&y_n = J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n\right),\\
&C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \xi_n\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.3}$$

where $\xi_n = \sup_{u \in \mathcal{F}} (k_n - 1)\phi(u, x_n)$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto the set C_{n+1} and for each $i \geq 0$, $\{\alpha_{n,i}\}$ is a sequence in $[0,1]$ satisfying the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all* $n \geq 0$;

(b) $\liminf_{n\to\infty} \alpha_{n,0} \cdot \alpha_{n,i} > 0$ *for all* $i \geq 1$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$.

Proof. We divide the proof of Theorem 3.2 into five steps.

Step 1. We first prove that $\mathcal{F}$ and C_n both are closed and convex subset of C for all $n \geq 0$.

In fact, It follows from Lemma 2.11 that $F(S_i)$, $i \geq 1$, is closed and convex. Therefore $\mathcal{F}$ is a closed and convex subset in C. Furthermore, it is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 1$. Since the inequality $\phi(v, y_n) \leq \phi(v, x_n) + \xi_n$ is equivalent to

$$2\langle v, Jx_n - Jy_n\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \xi_n, \tag{3.4}$$

therefore, we have

$$C_{n+1} = \left\{v \in C_n : 2\langle v, Jx_n - Jy_n\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \xi_n\right\}. \tag{3.5}$$

This implies that C_{n+1} is closed and convex. The desired conclusions are proved. These in turn show that $\Pi_{\mathcal{F}} x_0$ and $\Pi_{C_n} x_0$ are well defined.

Step 2. We prove that $\{x_n\}$ is a bounded sequence in C.

By the definition of C_n, we have $x_n = \Pi_{C_n} x_0$ for all $n \geq 0$. It follows from Lemma 2.2(a) that

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{C_n} x_0) \\ &\leq \phi(u, x_0), \quad \forall n \geq 0, \ u \in \mathcal{F}. \end{aligned} \tag{3.6}$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (2.4), $\{x_n\}$ is bounded. Denote

$$M = \sup_{n \geq 0} \{\|x_n\|\} < \infty. \tag{3.7}$$

Step 3. Next, we prove that $\mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i) \subset C_n$ for all $n \geq 0$.

It is obvious that $\mathcal{F} \subset C_0 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 0$. Since E is uniformly smooth, E^* is uniformly convex. For any given $u \in \mathcal{F} \subset C_n$ and for any positive integer $j > 0$, from Lemma 2.10 we have

$$\begin{aligned} \phi(u, y_n) &= \phi\left(u, J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n\right)\right) \\ &= \|u\|^2 - 2\alpha_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle u, JS_i^n x_n\rangle + \left\|\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n x_n\right\|^2 \\ &\leq \|u\|^2 - 2\alpha_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle u, JS_i^n x_n\rangle + \alpha_{n,0}\|Jx_n\|^2 \\ &\quad + \sum_{i=1}^{\infty} \alpha_{n,i}\|JS_i^n x_n\|^2 - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \qquad \text{(by Lemma 2.10)} \\ &= \|u\|^2 - 2\alpha_{n,0}\langle u, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle u, JS_i^n x_n\rangle + \alpha_{n,0}\|x_n\|^2 \\ &\quad + \sum_{i=1}^{\infty} \alpha_{n,i}\|S_i^n x_n\|^2 - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &= \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(u, S_i^n x_n) - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &\leq \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i} k_n \phi(u, x_n) - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &\leq k_n \phi(u, x_n) - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &\leq \phi(u, x_n) + \sup_{u \in \mathcal{F}} (k_n - 1)\phi(u, x_n) - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \\ &= \phi(u, x_n) + \xi_n - \alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \qquad \forall u \in \mathcal{F}. \end{aligned} \tag{3.8}$$

Hence $u \in C_{n+1}$ and so $\mathcal{F} \subset C_n$ for all $n \geq 0$. By the way, from the definition of $\{\xi_n\}$, (2.4), and (3.7), it is easy to see that

$$\xi_n = \sup_{u\in\mathcal{F}}(k_n - 1)\phi(u, x_n) \leq \sup_{u\in\mathcal{F}}(k_n - 1)(\|u\| + M)^2 \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.9}$$

Step 4. Now, we prove that $\{x_n\}$ converges strongly to some point $p \in \mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i)$.

In fact, since $\{x_n\}$ is bounded in C and E is reflexive, we may assume that $x_n \rightharpoonup p$. Again since C_n is closed and convex for each $n \geq 1$, it is easy to see that $p \in C_n$ for each $n \geq 0$. Since $x_n = \Pi_{C_n} x_0$, from the definition of Π_{C_n}, we have

$$\phi(x_n, x_0) \leq \phi(p, x_0), \quad \forall n \geq 0. \tag{3.10}$$

Since

$$\begin{aligned}\liminf_{n\to\infty} \phi(x_n, x_0) &= \liminf_{n\to\infty}\left\{\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\right\} \\ &\geq \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 = \phi(p, x_0),\end{aligned} \tag{3.11}$$

we have

$$\phi(p, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(p, x_0). \tag{3.12}$$

This implies that $\lim_{n\to\infty}\phi(x_n, x_0) = \phi(p, x_0)$, that is, $\|x_n\| \to \|p\|$. In view of the Kadec-Klee property of E, we obtain that

$$\lim_{n\to\infty} x_n = p. \tag{3.13}$$

Now we prove that $p \in \bigcap_{i=1}^{\infty} F(S_i)$.In fact, by the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. Therefore by Lemma 2.2(a) we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \longrightarrow 0 \quad (\text{as } \ n \longrightarrow \infty).\end{aligned} \tag{3.14}$$

In view of $x_{n+1} \in C_{n+1}$ and note the construction of C_{n+1} we obtain that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \xi_n \longrightarrow 0 \quad (\text{as } \ n \longrightarrow \infty). \tag{3.15}$$

From (2.4) it yields $(\|x_{n+1}\| - \|y_n\|)^2 \to 0$. Since $\|x_{n+1}\| \to \|p\|$, we have

$$\|y_n\| \longrightarrow \|p\| \quad (\text{as } \ n \longrightarrow \infty), \tag{3.16}$$

Hence we have

$$\|Jy_n\| \longrightarrow \|Jp\| \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.17}$$

This implies that $\{Jy_n\}$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, we can assume that $Jy_n \rightharpoonup f_0 \in E^*$. In view of the reflexive of E, we see that $J(E) = E^*$. Hence there exists $x \in E$ such that $Jx = f_0$. Since

$$\begin{aligned} \phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2. \end{aligned} \tag{3.18}$$

Taking $\liminf_{n\to\infty}$ on the both sides of equality above and in view of the weak lower semicontinuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, f_0\rangle + \|f_0\|^2 = \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 = \phi(p, x), \end{aligned} \tag{3.19}$$

that is, $p = x$. This implies that $f_0 = Jp$, and so $Jy_n \rightharpoonup Jp$. It follows from (3.17) and the Kadec-Klee property of E^* that $Jy_n \to Jp$ (as $n \to \infty$). Note that $J^{-1} : E^* \to E$ is hemi-continuous, it yields that $y_n \rightharpoonup p$. It follows from (3.16) and the Kadec-Klee property of E that

$$\lim_{n\to\infty} y_n = p. \tag{3.20}$$

From (3.13) and (3.20) we have that

$$\|x_n - y_n\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.21}$$

Since J is uniformly continuous on any bounded subset of E, we have

$$\|Jx_n - Jy_n\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.22}$$

For any $j \geq 1$ and any $u \in \mathcal{F}$, it follows from (3.8), (3.13), and (3.20) that

$$\alpha_{n,0}\alpha_{n,j} g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \leq \phi(u, x_n) - \phi(u, y_n) + \xi_n \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.23}$$

In view of condition (b) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$, we see that

$$g\left(\left\|Jx_n - JS_j^n x_n\right\|\right) \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.24}$$

It follows from the property of g that

$$\left\|Jx_n - JS_j^n x_n\right\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.25}$$

Since $x_n \to p$ and J is uniformly continuous, it yieads $Jx_n \to Jp$. Hence from (3.25) we have

$$JS_j^n x_n \longrightarrow Jp \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.26}$$

Since $J^{-1} : E^* \to E$ is hemi-continuous, it follows that

$$S_j^n x_n \rightharpoonup p, \quad \text{for each} \quad j \geq 1. \tag{3.27}$$

On the other hand, for each $j \geq 1$ we have

$$\left|\left\|S_j^n x_n\right\| - \|p\|\right| = \left|\left\|J\left(S_j^n x_n\right)\right\| - \|Jp\|\right| \leq \left\|J\left(S_j^n x_n\right) - Jp\right\| \longrightarrow 0 \qquad (\text{as} \quad n \longrightarrow \infty). \tag{3.28}$$

This together with (3.27) shows that

$$S_j^n x_n \to p \qquad \text{for each } j \geq 1. \tag{3.29}$$

Furthermore, by the assumption that for each $j \geq 1$, S_j is uniformly L_i-Lipschitz continuous, hence we have

$$\begin{aligned}\left\|S_j^{n+1} x_n - S_j^n x_n\right\| &\leq \left\|S_j^{n+1} x_n - S_j^{n+1} x_{n+1}\right\| + \left\|S_j^{n+1} x_{n+1} - x_{n+1}\right\| + \|x_{n+1} - x_n\| + \left\|x_n - S_j^n x_n\right\| \\ &\leq (L_j + 1)\|x_{n+1} - x_n\| + \left\|S_j^{n+1} x_{n+1} - x_{n+1}\right\| + \left\|x_n - S_j^n x_n\right\|.\end{aligned} \tag{3.30}$$

This together with (3.13) and (3.29), yields $\|S_j^{n+1} x_n - S_j^n x_n\| \to 0$ (as $n \to \infty$). Hence from (3.29) we have $S_j^{n+1} x_n \to p$, that is, $S_j S^n x_n \to p$. In view of (3.29) and the closeness of S_j, it yields that $S_j p = p$, for all $j \geq 1$. This implies that $p \in \bigcap_{i=1}^{\infty} F(S_j)$.

Step 5. Finally we prove that $x_n \to p = \Pi_{\mathcal{F}} x_0$.

Let $w = \Pi_{\mathcal{F}} x_0$. Since $w \in \mathcal{F} \subset C_n$ and $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0. \tag{3.31}$$

This implies that

$$\phi(p, x_0) = \lim_{n \to \infty} \phi(x_n, x_0) \leq \phi(w, x_0). \tag{3.32}$$

In view of the definition of $\Pi_{\mathcal{F}} x_0$, from (3.32) we have $p = w$. Therefore, $x_n \to p = \Pi_{\mathcal{F}} x_0$. This completes the proof of Theorem 3.2. □

The following theorem can be obtained from Theorem 3.2 immediately.

Theorem 3.3. *Let E be a uniformly smooth and strictly convex Banach space with Kadec-Klee property , C a nonempty closed convex subset of E. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and quasi-ϕ-nonexpansive mappings. Suppose that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(S_i)$ is a nonempty subset in C. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
&x_0 \in C \text{ chosen arbitrary}, \quad C_0 = C, \\
&y_n = J^{-1}\left(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i x_n\right), \\
&C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n)\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0,
\end{aligned} \tag{3.33}$$

where $\{\alpha_{n,i}\}$, for each $i \ge 0$, is a sequence in $[0,1]$ satisfying the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *for all* $n \ge 0$;

(b) $\liminf_{n\to\infty} \alpha_{n,0} \cdot \alpha_{n,i} > 0$ *for all* $i \ge 1$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_0$.

Proof. Since $\{S_i\}_{i=1}^{\infty} : C \to C$ is an infinite family of closed quasi-ϕ-nonexpansive mappings, it is an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with sequence $\{k_n = 1\}$. Hence $\xi_n = \sup_{u \in \mathcal{F}} (k_n - 1)\phi(u, x_n) = 0$. Therefore the conditions appearing in Theorem 3.2: "$\mathcal{F}$ is a bounded subset in C" and "for each $i \ge 1$, S_i is uniformly L_i-Lipschitz continuous" are of no use here. In fact, by the same methods as given in the proofs of (3.13), (3.20) and (3.29), we can prove that $x_n \to p$, $y_n \to p$ and $S_j x_n \to p$ (as $n \to \infty$) for each $j \ge 1$. By virtue of the closeness of mapping S_j for each $j \ge 1$, it yields that $p \in F(S_j)$ for each $j \ge 1$, that is, $p \in \bigcap_{i=1}^{\infty} F(S_i)$. Therefore all conditions in Theorem 3.2 are satisfied. The conclusion of Theorem 3.3 is obtained from Theorem 3.2 immediately. □

Remark 3.4. Theorems 3.2 and 3.3 improve and extend the corresponding results in Aleyner and Reich [12], Plubtieng and Ungchittrakool [13, 14] and Chang et al. [19] in the following aspects.

(a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space with the Kadec-Klee property (note that each uniformly convex Banach space must have the Kadec-Klee property).

(b) For the mappings, we extend the mappings from nonexpansive mappings, relatively nonexpansive mappings or quasi-ϕ-nonexpansive mapping to an infinite family of quasi-ϕ-asymptotically mappings;

(c) For the algorithms, we propose a new modified block iterative algorithms which are different from ones given in [12–14, 19] and others.

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