Modified Cohen–Lee Time–Frequency Distributions and Instantaneous Bandwidth of Multicomponent Signals

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***Abstract*—Cohen has introduced and extensively studied and developed the concept of the instantaneous bandwidth of a signal. Specifically, instantaneous bandwidth is interpreted as the spread in frequency about the instantaneous frequency, which is itself interpreted as the average frequency at each time. This view stems from a joint time–frequency distribution (TFD) analysis of the signal, where instantaneous frequency and instantaneous bandwidth are taken to be the first two conditional spectral moments, respectively, of the distribution. However, the traditional definition of instantaneous frequency, namely, as the derivative of the phase of the signal, is not consistent with this interpretation, and new definitions have therefore been recently proposed. In this paper, we show that similar problems arise with the Cohen–Lee instantaneous bandwidth of a signal and propose a new formulation for the instantaneous bandwidth that is consistent with its interpretation as the conditional standard deviation in frequency of a TFD. We give the kernel constraints for a distribution to yield this new result, which is a modification of the kernel proposed by Cohen and Lee. These new kernel constraints yield a modified Cohen–Lee TFD whose first two conditional moments are interpretable as the average frequency and bandwidth at each time, respectively.**

***Index Terms*—Conditional moments, instantaneous bandwidth, instantaneous frequency, time–frequency analysis.**

I. INTRODUCTION

THE instantaneous bandwidth of a signal is a concept that has been extensively developed by Cohen, particularly in the context of time–frequency distributions (TFDs), where it is taken to be the standard deviation in frequency at a given time [3]–[6]. This approach and view stem from the interpretation of instantaneous frequency as the mean frequency at a given time, which arises because many TFDs yield the instantaneous frequency (derivative of the phase) of a signal for their first conditional spectral moment [3]. As Cohen reasons, given the mean, it is natural to ask about the standard deviation, which gives rise to the concept of instantaneous bandwidth [3]–[6].

While this view and approach are appealing, difficulties arise in interpreting the traditional definition of instantaneous frequency (derivative of the phase) as the average frequency at a given time, particularly for so-called "multicomponent" signals [18]. Indeed, it is well known that the derivative of the phase of a signal often ranges beyond the spectral support of the signal [3], [18], yielding the paradox that the supposed average frequency at a given time is often outside the bandwidth of the signal. This difficulty in interpretation of the derivative of the phase has given rise to renewed efforts to find new definitions of instantaneous frequency that are amenable to clear physical interpretation [17]–[21].

As we show here, similar difficulties arise in the interpretation of the Cohen–Lee instantaneous bandwidth as the standard deviation in frequency at a given time. Namely, like the instantaneous frequency, the instantaneous bandwidth often ranges beyond the global bandwidth of the signal. Nevertheless, the interpretation itself is appealing, and we are therefore motivated to pinpoint the source of difficulties that arise in interpreting instantaneous bandwidth as the spread in frequency at a given time and to suggest a new candidate for instantaneous bandwidth that is consistent with this interpretation. Such is the purpose of this paper. Because our approach builds on recent results regarding instantaneous frequency, we summarize these results first and then present new results regarding instantaneous bandwidth. We give the kernel constraints for a Cohen-class TFD to yield this new candidate for instantaneous bandwidth and illustrate these results by computing TFDs with these conditional moments for synthetic and real signals. Because of the symmetry between the time-domain and frequency-domain formulations of Cohen-class TFDs [3], the results extend readily to conditional temporal moments, namely, the average time at a given frequency and the spread in time at a particular frequency (the "local duration").

II. BACKGROUND

A. Instantaneous Frequency and Bandwidth

For a signal $s(t)$, written in complex form in terms of its amplitude and phase[1]

$$s(t) = A(t)e^{j\varphi(t)} \tag{1}$$

the instantaneous frequency is defined as the derivative of the phase [9]

$$\omega_i(t) = \varphi'(t) \tag{2}$$

[1]We take the signal to be normalized to unit energy so that the temporal, spectral, and time–frequency densities integrate to one for consistency with distribution theory.

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and the instantaneous bandwidth, per Cohen and Lee [4]–[6], is given by the (absolute value of the) derivative of the log-amplitude

$$ib(t) = \left|\frac{d}{dt}\log A(t)\right| = \left|\frac{A'(t)}{A(t)}\right|. \tag{3}$$

The instantaneous bandwidth is related to the global bandwidth, given by

$$\sigma_\omega = \left(\int (\omega - \langle\omega\rangle)^2 |S(\omega)|^2\, d\omega\right)^{\frac{1}{2}} \tag{4}$$

where $S(\omega)$ is the Fourier transform of $s(t)$, and $\langle\omega\rangle = \int \omega |S(\omega)|^2\, d\omega$, according to [3]

$$\sigma_\omega^2 = \int ib^2(t) A^2(t)\, dt + \int (\langle\omega\rangle - \omega_i(t))^2 A^2(t)\, dt. \tag{5}$$

For the case of a "multicomponent signal"[2]

$$s(t) = A(t)e^{j\varphi(t)} = \sum_{k=1}^{N} a_k(t) e^{j\varphi_k(t)} \tag{6}$$

we have for the instantaneous frequency and bandwidth, per (2) and (3) (for the case $N = 2$)

$$A^2(t) = a_1^2(t) + a_2^2(t) + 2a_1(t)a_2(t)\cos\Delta\varphi_{12}(t) \tag{7}$$

$$\begin{aligned}\varphi'(t) = \frac{1}{A^2(t)} \big(&a_1^2(t)\varphi_1'(t) + a_2^2(t)\varphi_2'(t)\\ &+ (a_1'(t)a_2(t) - a_1(t)a_2'(t))\sin\Delta\varphi_{12}(t)\\ &+ a_1(t)a_2(t)(\varphi_1'(t) + \varphi_2'(t))\cos\Delta\varphi_{12}(t)\big)\end{aligned} \tag{8}$$

$$\begin{aligned}\left(\frac{A'(t)}{A(t)}\right)^2 = \frac{1}{A^2(t)} (&a_1(t)a_1'(t) + a_2(t)a_2'(t)\\ &+ (a_1(t)a_2'(t) + a_1'(t)a_2(t))\cos\Delta\varphi_{12}(t)\\ &- a_1(t)a_2(t)(\varphi_1'(t) - \varphi_2'(t))\sin\Delta\varphi_{12}(t))^2\end{aligned} \tag{9}$$

where $\Delta\varphi_{12}(t) = \varphi_1(t) - \varphi_2(t)$.[3]

Note the presence of oscillatory terms in the instantaneous frequency and bandwidth, which cause difficulties in interpretation of these quantities as conditional averages. Specifically, both quantities generally extend beyond the spectral range of the signal (except under special circumstances in the case of instantaneous frequency [19], [21]).

It is generally held that a more appropriate expression for the instantaneous frequency of a multicomponent signal is a "weighted average" of the instantaneous frequencies of the individual signal components [12]

$$\omega_i(t) = \frac{a_1^2(t)\varphi_1'(t) + a_2^2(t)\varphi_2'(t)}{a_1^2(t) + a_2^2(t)}. \tag{10}$$

It has recently been shown that the instantaneous frequency (derivative of the phase) *exactly* equals this weighted average when the components exhibit symmetry [19], [21]. For an odd number of components [$N = 2K + 1$ in (6)], the signal components have (even) symmetry when

$$\begin{aligned} a_{2K+1} &= a_1 & \varphi_{2K+1} - \varphi_{2K} &= \varphi_2 - \varphi_1\\ a_{2K} &= a_2 & \varphi_{2K} - \varphi_{2K-1} &= \varphi_3 - \varphi_2\\ &\vdots & &\vdots\\ a_{K+2} &= a_K & \varphi_{K+2} - \varphi_{K+1} &= \varphi_{K+1} - \varphi_K. \end{aligned} \tag{11}$$

See [19], [21] for details.

In addition, even for signals without this symmetry, it has recently been shown that this weighted average instantaneous frequency (WAIF) can be obtained from the first conditional spectral moment of a spectrogram for an appropriately chosen (signal-dependent) window [15]. These results support the interpretation of the WAIF as the average frequency at each time. We build on these results and consider the interpretation of instantaneous bandwidth as a conditional spectral moment, namely, as the standard deviation in frequency at a given time. Such an interpretation requires that the result be positive and bounded by the global bandwidth of the signal. We give here a suitable candidate that meets these requirements and provide the (signal-dependent) kernel constraints for a TFD to yield this candidate for its second conditional spectral moment. We also give the kernel constraints for the TFD to yield the WAIF for its first conditional spectral moment (which is a generalization of the result previously obtained for the spectrogram [15]).

B. TFDs and Conditional Moments

From a TFD of the signal [2], [3]

$$\begin{aligned}P(t,\omega) = \frac{1}{4\pi^2}\iiint s^*\left(u - \frac{\tau}{2}\right) s\left(u + \frac{\tau}{2}\right)\\ \times \phi(\theta,\tau) e^{-j\theta t - j\omega\tau + j\theta u}\, du\, d\theta\, d\tau\end{aligned} \tag{12}$$

where $\phi(\theta,\tau)$ is a kernel that specifies the particular TFD, the conditional spectral moments are given by

$$\begin{aligned}\langle\omega^n\rangle_t = \frac{\int \omega^n P(t,\omega)\, d\omega}{\int P(t,\omega)\, d\omega} &= \frac{1}{P(t)}\int \omega^n P(t,\omega)\, d\omega\\ &= \int \omega^n P(\omega \,|\, t)\, d\omega\end{aligned} \tag{13}$$

where $P(t)$ is the time marginal of the TFD, and $P(\omega \,|\, t)$ is the time-conditional spectral density. The average frequency at each time is given by the first moment of $P(\omega \,|\, t)$, which is the conditional mean frequency, $\langle\omega\rangle_t$, and the spread in frequency at each time is obtained from the variance of $P(\omega \,|\, t)$,

$$\sigma_{\omega\,|\,t}^2 = \langle\omega^2\rangle_t - \langle\omega\rangle_t^2. \tag{14}$$

By substituting (12) into (13), we may express the conditional spectral moments in terms of the signal and kernel (see Appendix), as in (15), shown at the bottom of the next page, where $*$ denotes convolution, $s^{(n)}(t)$ denotes the nth derivative of $s(t)$, and

$$K_i(t) \triangleq \frac{1}{\sqrt{2\pi}}\int \left(\frac{\partial}{\partial\tau}\right)^i \phi(\theta,\tau)\bigg|_{\tau=0} e^{-j\theta t}\, d\theta. \tag{16}$$

[2]We consider a signal to be multicomponent when its TFD exhibits well-separated ridges in the time–frequency plane. One measure of this separation, given by Cohen and Lee [4], is that the instantaneous bandwidth of each component be less than the separation between the instantaneous frequencies of neighboring components: $(a_i'(t)/a_i(t))^2 < (\varphi_i'(t) - \varphi_k'(t))^2, i \neq k$. We take that to be the case here. Such signals may be categorized as "locally narrowband."

[3]The general N-component case is a straightforward but lengthy extension of the two-component case, which we consider here for simplicity and lucid presentation of the main ideas. See the Appendix for the more general case.

For the signal $s(t)=A(t)e^{j\varphi(t)}$, the first two moments are

$$\langle\omega\rangle_t=\frac{A^2(t)\varphi'(t)*K_0(t)-jA^2(t)*K_1(t)}{A^2(t)*K_0(t)} \tag{17}$$

and

$$\langle\omega^2\rangle_t=\frac{-A^2(t)*K_2(t)-2jA^2(t)\varphi'(t)*K_1(t)}{A^2(t)*K_0(t)}+\frac{\frac{1}{2}(A'^2(t)+2A^2(t)\varphi'^2(t)-A(t)A''(t))*K_0(t)}{A^2(t)*K_0(t)}. \tag{18}$$

Thus, for the Wigner distribution, with kernel $\phi(\theta,\tau)=1$, the conditional moments are

$$\langle\omega^n\rangle_t=\frac{\left(-\frac{j}{2}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l s^{*(l)}(t)s^{(n-l)}(t)}{|s(t)|^2}=\frac{(-j)^n\left(\frac{d}{du}\right)^n\left\{s\left(t+\frac{u}{2}\right)s^*\left(t-\frac{u}{2}\right)\right\}\Big|_{u=0}}{|s(t)|^2} \tag{19}$$

from which we obtain the well-known results for the first two moments of the Wigner distribution [3]

$$\langle\omega\rangle_t=\varphi'(t) \tag{20}$$

$$\langle\omega^2\rangle_t=\frac{1}{2}\left[\left(\frac{A'(t)}{A(t)}\right)^2-\frac{A''(t)}{A(t)}\right]+\varphi'^2(t) \tag{21}$$

$$\sigma^2_{\omega\,|\,t}=\frac{1}{2}\left[\left(\frac{A'(t)}{A(t)}\right)^2-\frac{A''(t)}{A(t)}\right]. \tag{22}$$

For the Rihaczek distribution [23] with kernel $\phi(\theta,\tau)=e^{j(\theta\tau)/2}$ [3], we obtain the moments in (23), shown at the bottom of the page, which can be written equivalently, and more compactly, as

$$\langle\omega^n\rangle_t=\left[\frac{\left(\frac{1}{j}\frac{d}{dt}\right)^n s(t)}{s(t)}\right]^*. \tag{24}$$

This latter expression was derived by Poletti [22] by direct calculation of the conditional moments of the Rihaczek distribution $P(t,\omega)=s(t)S^*(\omega)e^{-j\omega t}$. The first two moments are

$$\langle\omega\rangle_t=\varphi'(t)+j\frac{A'(t)}{A(t)} \tag{25}$$

$$\langle\omega^2\rangle_t=\varphi'^2(t)-\frac{A''(t)}{A(t)}+j\left(\varphi''(t)+2\frac{A'(t)}{A(t)}\varphi'(t)\right) \tag{26}$$

$$\sigma^2_{\omega\,|\,t}=\left(\frac{A'(t)}{A(t)}\right)^2-\frac{A''(t)}{A(t)}+j\varphi''(t). \tag{27}$$

Note that the conditional spectral variance of the Wigner distribution can be negative and that of the Rihaczek distribution is complex. Accordingly, for both distributions, there are difficulties in interpreting the conditional standard deviation in the usual way, namely, as the spread about the mean at each time.

It has been shown by Cohen and Lee [4] that with rather mild constraints on the kernel, a conditional variance that is always non-negative for all signals can be obtained. Specifically, for $\phi(\theta,0)=1$, $\partial\phi(\theta,\tau)/\partial\tau|_{\tau=0}=0$, and $\partial^2\phi(\theta,\tau)/\partial\tau^2|_{\tau=0}=\theta^2/4$, the conditional mean frequency and the conditional spectral variance are given by

$$\langle\omega\rangle_t=\varphi'(t) \tag{28}$$

$$\langle\omega^2\rangle_t=\left(\frac{A'(t)}{A(t)}\right)^2+\varphi'^2(t) \tag{29}$$

$$\sigma^2_{\omega\,|\,t}=\left(\frac{A'(t)}{A(t)}\right)^2. \tag{30}$$

Although, unlike the Wigner and most other TFDs, the second moment here is real and positive (as it should be), TFDs that yield these results are nevertheless generally not non-negative. Because the TFD contains negative values, it may produce first and second conditional spectral moments that range beyond the spectral support of the signal—such as the instantaneous frequency and instantaneous bandwidth above. For such a TFD of a multicomponent signal ($N=2$), the second conditional moment [see (29)] is evaluated as

$$\langle\omega^2\rangle_t=\frac{1}{A^2(t)}\Big(a_1'^2(t)+a_2'^2(t)+a_1^2(t)\varphi_1'^2(t)+a_2^2(t)\varphi_2'^2(t)-2(a_1(t)a_2'(t)\varphi_1'(t)-a_1'(t)a_2(t)\varphi_2'(t))\sin\Delta\varphi_{12}(t)+2(a_1'(t)a_2'(t)+a_1(t)a_2(t)\varphi_1'(t)\varphi_2'(t))\times\cos\Delta\varphi_{12}(t)\Big). \tag{31}$$

Note the oscillatory terms, which tend to cause difficulty in interpretation of this quantity, as well as of $\sigma^2_{\omega\,|\,t}$.

TFDs that are non-negative and satisfy the marginals [7], [16] do not generally yield these results and, as such, cast further doubt on the interpretation of these definitions of instantaneous frequency and instantaneous bandwidth as the average frequency at a given time and the spread in frequency at a given time. One of our aims is to obtain an expression for the instantaneous bandwidth that can be interpreted as a true conditional spectral variance, analogous to the results recently obtained for instantaneous frequency (namely, the WAIF).

$$\langle\omega^n\rangle_t=\frac{(-j)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k\left\{\sum_{l=0}^{k}\binom{k}{l}(-1)^l s^{*(l)}(t)s^{(k-l)}(t)\right\}*K_{n-k}(t)}{|s(t)|^2*K_0(t)} \tag{15}$$

$$\langle\omega^n\rangle_t=\frac{\left(\frac{j}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{d}{dt}\right)^{n-k}\left\{\sum_{l=0}^{k}\binom{k}{l}(-1)^l s^{*(l)}(t)s^{(k-l)}(t)\right\}}{|s(t)|^2} \tag{23}$$

III. Results

In this section, we derive an expression for the instantaneous bandwidth of multicomponent signals from two different approaches. Examples are then provided to demonstrate the results.

A. First Approach: Eliminate Oscillatory Terms from Expression

As we remarked earlier, under certain conditions of signal symmetry, the oscillatory terms in the usual definition of instantaneous frequency drop out, resulting in the WAIF [see (10)] for such signals [19], [21]. In addition, even for signals without this symmetry, the first conditional spectral moment of a spectrogram can eliminate the oscillatory terms from the instantaneous frequency and yield the WAIF [15]. We are therefore naturally led to consider the result one obtains for the instantaneous bandwidth by simply dropping the oscillatory terms from the second moment [see (31)] as well. Doing so, we obtain

$$\langle\omega^2\rangle_t = \frac{a_1'^2(t)+a_2'^2(t)}{a_1^2(t)+a_2^2(t)} + \frac{a_1^2(t)\varphi_1'^2(t)+a_2^2(t)\varphi_2'^2(t)}{a_1^2(t)+a_2^2(t)}. \tag{32}$$

Taking this result together with the WAIF [see (10)], we obtain the conditional spectral variance

$$\sigma_{\omega\,|\,t}^2 = \frac{a_1'^2(t)+a_2'^2(t)}{a_1^2(t)+a_2^2(t)} + \frac{a_1^2(t)a_2^2(t)(\varphi_1'(t)-\varphi_2'(t))^2}{\left(a_1^2(t)+a_2^2(t)\right)^2}. \tag{33}$$

Note that this expression for $\sigma_{\omega\,|\,t}^2$ is non-negative for all signals, which is good. Further, it exhibits reasonable behavior as a measure of spread in frequency at each time: As the separation between the instantaneous frequencies increases, so too does this measure of "instantaneous bandwidth," and as the amplitude modulations become more pronounced (i.e., such that $a_1'(t)$ and $a_2'(t)$ are very different from zero), the instantaneous bandwidth again increases. Conversely, as the amplitude modulations decrease (i.e., $a_1(t)$ and $a_2(t)$ become constants), the instantaneous bandwidth also decreases.

But can we justify simply dropping the troublesome oscillatory terms from the second conditional moment expression? At least for the first conditional moment, the oscillatory terms were "naturally" eliminated under certain conditions to yield the WAIF, namely, for symmetric components and in the conditional mean of a spectrogram in many cases. Do the same conditions eliminate the oscillatory terms in the second conditional moment, as well? Unfortunately, no. It is easy to verify that the oscillatory terms do not cancel out by setting $a_1(t)=a_2(t)$ in (9); see [13] for a specific example. In addition, the second conditional moment of a spectrogram is typically distorted by the window [5]. This distortion, which cannot generally be eliminated, makes it virtually impossible for the conditional spectral variance of a spectrogram to exactly equal the expression in (33), even when the first conditional moment of the spectrogram does exactly equal the WAIF [see (10)].

However, the second conditional spectral moment and variance expressions above can be obtained from another rather different approach: a Gaussian mixture model of the time-varying spectrum of the multicomponent signal. This approach, which we present next, also yields the WAIF. We then give kernel constraints for a Cohen-class TFD to yield the proposed conditional moments.

B. Second Approach: Gaussian Mixture Model

Let $x(t)=\sum_{i=1}^N x_i(t)$ be a multicomponent random process where the $x_i(t)$ are independent (and unknown). The spectral density is thus of the form $P(\omega)=\sum_{i=1}^N P_i(\omega)$, where the $P_i(\omega)$ are the (unknown) spectral densities of the individual (unknown) components $x_i(t)$. Suppose we are given the mean frequency μ_i and bandwidth (spectral variance) σ_i^2 of the individual independent components, and their relative power $p_i > 0$, $\sum_{i=1}^N p_i = 1$. Given this information, a Gaussian mixture model [24] of the spectral density can be made and is given by

$$P_{\mathrm{gm}}(\omega) = \sum_{i=1}^N \frac{p_i}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\omega-\mu_i)^2}{2\sigma_i^2}}. \tag{34}$$

(As an aside, note that $1/(\sqrt{2\pi\sigma_i^2})e^{-((\omega-\mu_i)^2)/2\sigma_i^2}$ is the (normalized) maximum entropy spectral estimate for $x_i(t)$. The entropy of the Gaussian mixture is given by

$$\begin{aligned} H[P_{\mathrm{gm}}(\omega)] &= -\int P_{\mathrm{gm}}(\omega)\ln P_{\mathrm{gm}}(\omega)\,d\omega \\ &= -\int \sum_{i=1}^N \frac{p_i}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\omega-\mu_i)^2}{2\sigma_i^2}} \\ &\quad \times \ln\left(\sum_{j=1}^N \frac{p_j}{\sqrt{2\pi\sigma_j^2}} e^{-\frac{(\omega-\mu_j)^2}{2\sigma_j^2}}\right) d\omega \\ &\ge -\sum_{i=1}^N p_i \int \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\omega-\mu_i)^2}{2\sigma_i^2}} \\ &\quad \times \ln\left(\frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\omega-\mu_i)^2}{2\sigma_i^2}}\right) d\omega \\ &= \sum_{i=1}^N p_i H\left[\frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\omega-\mu_i)^2}{2\sigma_i^2}}\right] \end{aligned} \tag{35}$$

from which we observe that the entropy of the Gaussian mixture is lower bounded by the weighted sum of the entropies of the individual (maximum entropy) densities [14].[4] Thus, the Gaussian mixture may also be viewed as a "minimax" solution to a maximum entropy statement of the problem, given the individual means and variances.)

The first- and second-order spectral moments of $P_{\mathrm{gm}}(\omega)$ are readily calculated and given by

$$\langle\omega\rangle_{\mathrm{gm}} = \int \omega P_{\mathrm{gm}}(\omega)\,d\omega = \sum_{i=1}^N p_i\mu_i \tag{36}$$

$$\langle\omega^2\rangle_{\mathrm{gm}} = \int \omega^2 P_{\mathrm{gm}}(\omega)\,d\omega = \sum_{i=1}^N p_i\left(\sigma_i^2+\mu_i^2\right) \tag{37}$$

[4]The inequality in (35) follows from the non-negativity of the cross-entropy between any two densities $P_a(\omega)$ and $P_b(\omega)$, i.e., $\int P_a(\omega) \ln\left(P_a(\omega)/P_b(\omega)\right) d\omega \ge 0$.

and the spectral variance is therefore

$$\sigma_{\rm gm}^2 = \langle\omega^2\rangle_{\rm gm} - \langle\omega\rangle_{\rm gm}^2 = \sum_{i=1}^{N} p_i\sigma_i^2 + \sum_{i=1}^{N} p_i(1-p_i)\mu_i^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{N} p_i p_j \mu_i\mu_j. \quad (38)$$

We extend this model to the time-varying case by letting the individual spectral means, variances, and relative powers depend on time; accordingly, the spectral density also depends on time. From the previous considerations, given the time-dependent mean frequency $\mu_i(t)$, spectral variance $\sigma_i^2(t)$, and relative power $p_i(t)$ of each component, the time-dependent spectral density is

$$P_{\rm gm}(\omega\,|\,t) = \sum_{i=1}^{N} \frac{p_i(t)}{\sqrt{2\pi\sigma_i^2(t)}} e^{-\frac{(\omega-\mu_i(t))^2}{2\sigma_i^2(t)}}. \quad (39)$$

The first and second (conditional) spectral moments are exactly as in (36)–(38), except that now they depend on time. For $N=2$, we have

$$\langle\omega\rangle_t = p_1(t)\mu_1(t) + p_2(t)\mu_2(t) \quad (40)$$

$$\langle\omega^2\rangle_t = p_1(t)\left(\sigma_1^2(t)+\mu_1^2(t)\right) + p_2(t)\left(\sigma_2^2(t)+\mu_2^2(t)\right) \quad (41)$$

$$\sigma_{\omega\,|\,t}^2 = p_1(t)\sigma_1^2(t) + p_2(t)\sigma_2^2(t) + p_1(t)p_2(t) \times (\mu_1(t)-\mu_2(t))^2 \quad (42)$$

[where (42) follows from the fact that $p_1(t)+p_2(t)=1$].

If we take for the time-dependent mean frequency, spectral variance, and relative power of each component the following quantities:

$$p_1(t) \triangleq \frac{a_1^2(t)}{a_1^2(t)+a_2^2(t)} \quad p_2(t) \triangleq \frac{a_2^2(t)}{a_1^2(t)+a_2^2(t)} \quad (43)$$

$$\mu_1(t) \triangleq \varphi_1'(t) \quad \mu_2(t) \triangleq \varphi_2'(t) \quad (44)$$

$$\sigma_1^2(t) \triangleq \left(\frac{a_1'(t)}{a_1(t)}\right)^2 \quad \sigma_2^2(t) \triangleq \left(\frac{a_2'(t)}{a_2(t)}\right)^2 \quad (45)$$

then on substituting (43)–(45) into (40)–(42), we obtain

$$\langle\omega\rangle_t = \frac{a_1^2(t)\varphi_1'(t) + a_2^2(t)\varphi_2'(t)}{a_1^2(t)+a_2^2(t)} \quad (46)$$

$$\langle\omega^2\rangle_t = \frac{a_1'^2(t)+a_2'^2(t)}{a_1^2(t)+a_2^2(t)} + \frac{a_1^2(t)\varphi_1'^2(t)+a_2^2(t)\varphi_2'^2(t)}{a_1^2(t)+a_2^2(t)} \quad (47)$$

$$\sigma_{\omega\,|\,t}^2 = \frac{a_1'^2(t)+a_2'^2(t)}{a_1^2(t)+a_2^2(t)} + \frac{a_1^2(t)a_2^2(t)(\varphi_1'(t)-\varphi_2'(t))^2}{\left(a_1^2(t)+a_2^2(t)\right)^2}. \quad (48)$$

These conditional moments are identical to (10), (32), and (33), respectively. In the next section, we give kernel constraints in the Cohen class to produce TFDs that yield the above expressions as time-conditional moments.

IV. TFD Kernel Constraints

It has been previously shown [15] that a spectrogram will yield the WAIF for its first conditional spectral moment if the signal components are well separated in the time–frequency plane and the spectrogram window is lowpass with cut-off frequency less than the smallest separation between the instantaneous frequencies of the components. (Note that this constraint is signal dependent.) We generalize this result for arbitrary kernel and consider the second conditional moment by giving the kernel constraints such that we obtain the moment candidates derived in the previous section.

Specifically, for a locally narrowband multicomponent signal with well-separated components and a TFD with a lowpass kernel such that

$$K_0(t) = \frac{2\theta_c}{\sqrt{2\pi}}\,\mathrm{Sa}(\theta_c t) \quad (49)$$

$$K_1(t) = 0 \quad (50)$$

$$K_2(t) = -\frac{\theta_c}{2\sqrt{2\pi}}\,\mathrm{Sa}''(\theta_c t) \quad (51)$$

where $\theta_c < |\varphi_1'(t) - \varphi_2'(t)|$, $K_i(t)$ was defined in (16) and $\mathrm{Sa}(x) = \sin(x)/x$, it follows by substitution into (17) and (18) that the first two conditional spectral moments and the spectral variance for this multicomponent signal are given by (for $N=2$)

$$\langle\omega\rangle_t = \frac{a_1^2(t)\varphi_1'(t) + a_2^2(t)\varphi_2'(t)}{a_1^2(t)+a_2^2(t)} \quad (52)$$

$$\langle\omega^2\rangle_t = \frac{a_1'^2(t)+a_2'^2(t)}{a_1^2(t)+a_2^2(t)} + \frac{a_1^2(t)\varphi_1'^2(t)+a_2^2(t)\varphi_2'^2(t)}{a_1^2(t)+a_2^2(t)} \quad (53)$$

$$\sigma_{\omega\,|\,t}^2 = \frac{a_1'^2(t)+a_2'^2(t)}{a_1^2(t)+a_2^2(t)} + \frac{a_1^2(t)a_2^2(t)(\varphi_1'(t)-\varphi_2'(t))^2}{\left(a_1^2(t)+a_2^2(t)\right)^2} \quad (54)$$

(see the Appendix). These results are identical to those obtained in the previous section.

The constraints given in (49)–(51) may be given in terms of the kernel as

$$\phi(\theta,0) = \begin{cases} 1, & \left|\frac{\theta}{\theta_c}\right| < 1 \\ 0, & \text{otherwise} \end{cases} \quad (55)$$

$$\left.\frac{\partial\phi(\theta,\tau)}{\partial\tau}\right|_{\tau=0} = 0 \quad (56)$$

$$\left.\frac{\partial^2\phi(\theta,\tau)}{\partial\tau^2}\right|_{\tau=0} = \begin{cases} \frac{\theta^2}{4}, & \left|\frac{\theta}{\theta_c}\right| < 1 \\ 0, & \text{otherwise.} \end{cases} \quad (57)$$

A kernel that satisfies these constraints and that yields (52)–(54) as conditional moments is given by

$$\phi(\theta,\tau) = (1 + c(\theta\tau)^2)e^{-\frac{(\theta\tau)^2}{\sigma}}\,\mathrm{rect}\left(\frac{\theta}{2\theta_c}\right) \quad (58)$$

where $c = (1/8) + (1/\sigma)$, and

$$\mathrm{rect}\left(\frac{\theta}{2\theta_c}\right) = \begin{cases} 1, & |\theta| \le \theta_c \\ 0, & \text{otherwise.} \end{cases} \quad (59)$$

This kernel is a modification (lowpass version) of the Cohen–Lee kernel [4], which was itself a modification of the

Choi–Williams kernel [1].[5] Other kernels that satisfy these constraints are also possible. For example, the kernel

$$\phi(\theta,\tau)=(1+c(\theta\tau)^2)e^{-\frac{(\theta\tau)^2}{\sigma}}\cos\left(\frac{\theta\tau}{2}\right)\text{rect}\left(\frac{\theta}{2\theta_c}\right) \quad (60)$$

(here, $c=(1/4)+(1/\sigma)$) also yields the conditional moments above. A RID-type kernel [10] can be designed to yield these moments as well, simply by using a RID kernel in lieu of the Choi–Williams kernel in the expressions above (with a different value for the constant c that depends on the particular RID kernel used).

A. Conditional Temporal Moments

Due to the time-domain and frequency-domain symmetry of the Cohen formulation of TFDs, i.e., because (12) can be equivalently expressed as

$$P(t,\omega)=\frac{1}{4\pi^2}\iiint S^*\left(u+\frac{\theta}{2}\right)S\left(u-\frac{\theta}{2}\right)\phi(\theta,\tau) \times e^{-j\theta t-j\tau\omega+j\tau u}\,d\theta\,d\tau\,du \quad (61)$$

where $S(\omega)=(1/\sqrt{2\pi})\int s(t)e^{-j\omega t}\,dt$, the above results hold also in the case of conditional temporal moments with a simple transcription of variables. That is, for a two-component signal with spectrum

$$S(\omega)=b_1(\omega)e^{j\psi_1(\omega)}+b_2(\omega)e^{j\psi_2(\omega)} \quad (62)$$

we maintain that the first- and second-order conditional temporal moments and conditional (local) duration are given by

$$\langle t\rangle_\omega=-\frac{b_1^2(\omega)\psi_1'(\omega)+b_2^2(\omega)\psi_2'(\omega)}{b_1^2(\omega)+b_2^2(\omega)} \quad (63)$$

$$\langle t^2\rangle_\omega=\frac{b_1'^2(\omega)+b_2'^2(\omega)}{b_1^2(\omega)+b_2^2(\omega)}+\frac{b_1^2(\omega)\psi_1'^2(\omega)+b_2^2(\omega)\psi_2'^2(\omega)}{b_1^2(\omega)+b_2^2(\omega)} \quad (64)$$

$$\sigma_{t|\omega}^2=\frac{b_1'^2(\omega)+b_2'^2(\omega)}{b_1^2(\omega)+b_2^2(\omega)}+\frac{b_1^2(\omega)b_2^2(\omega)(\psi_1'(\omega)-\psi_2'(\omega))^2}{(b_1^2(\omega)+b_2^2(\omega))^2}. \quad (65)$$

These results, along with the conditional spectral moments given previously, may be obtained from a Cohen-class TFD with a kernel such as

$$\phi(\theta,\tau)=(1+c(\theta\tau)^2)e^{-\frac{(\theta\tau)^2}{\sigma}}\text{rect}\left(\frac{\theta}{2\theta_c}\right)\text{rect}\left(\frac{\tau}{2\tau_c}\right) \quad (66)$$

where $\tau_c<|\psi_1'(\omega)-\psi_2'(\omega)|$, and $\theta_c<|\varphi_1'(t)-\varphi_2'(t)|$. It is straightforward to verify that this kernel satisfies the proposed constraints, namely

$$\phi(\theta,0)=\begin{cases}1, & \left|\frac{\theta}{\theta_c}\right|<1\\ 0, & \text{otherwise}\end{cases}$$

$$\phi(0,\tau)=\begin{cases}1, & \left|\frac{\tau}{\tau_c}\right|<1\\ 0, & \text{otherwise}\end{cases} \quad (67)$$

$$\left.\frac{\partial\phi(\theta,\tau)}{\partial\tau}\right|_{\tau=0}=0 \quad \left.\frac{\partial\phi(\theta,\tau)}{\partial\theta}\right|_{\theta=0}=0 \quad (68)$$

$$\left.\frac{\partial^2\phi(\theta,\tau)}{\partial\tau^2}\right|_{\tau=0}=\begin{cases}\frac{\theta^2}{4}, & \left|\frac{\theta}{\theta_c}\right|<1\\ 0, & \text{otherwise}\end{cases}$$

$$\left.\frac{\partial^2\phi(\theta,\tau)}{\partial\theta^2}\right|_{\theta=0}=\begin{cases}\frac{\tau^2}{4}, & \left|\frac{\tau}{\tau_c}\right|<1\\ 0, & \text{otherwise.}\end{cases} \quad (69)$$

Other kernels that yield these moments can also be designed. Indeed, any product kernel that yields the Cohen–Lee instantaneous bandwidth can be modified by multiplying it with the rectangle functions above to obtain a TFD with the proposed conditional moments in time and frequency. Additionally, the rectangle functions, which form an ideal, separable lowpass filter in the ambiguity domain, can be replaced by a nonseparable lowpass filter with finite rolloff. For example, a Butterworth-type lowpass kernel that (approximately) satisfies the kernel constraints is

$$\phi(\theta,\tau)=(1+c(\theta\tau)^2)e^{-\frac{(\theta\tau)^2}{\sigma}} \times\left(1+\left(\frac{\theta}{\theta_c}\right)^{2m}+\left(\frac{\tau}{\tau_c}\right)^{2n}\right)^{-1} \quad m,n\geq 1. \quad (70)$$

For this kernel, we have

$$\phi(\theta,0)=\left(1+\left(\frac{\theta}{\theta_c}\right)^{2m}\right)^{-1}$$

$$\phi(0,\tau)=\left(1+\left(\frac{\tau}{\tau_c}\right)^{2n}\right)^{-1} \quad (71)$$

$$\left.\frac{\partial\phi(\theta,\tau)}{\partial\tau}\right|_{\tau=0}=0 \quad \left.\frac{\partial\phi(\theta,\tau)}{\partial\theta}\right|_{\theta=0}=0 \quad (72)$$

$$\left.\frac{\partial^2\phi(\theta,\tau)}{\partial\tau^2}\right|_{\tau=0}=\frac{\theta^2}{4}\left(1+\left(\frac{\theta}{\theta_c}\right)^{2m}\right)^{-1}$$

$$\left.\frac{\partial^2\phi(\theta,\tau)}{\partial\theta^2}\right|_{\theta=0}=\frac{\tau^2}{4}\left(1+\left(\frac{\tau}{\tau_c}\right)^{2n}\right)^{-1} \quad (73)$$

where the degree of rolloff in the θ and τ directions is controlled by the parameters m and n, respectively. A comparison of this kernel to the rectangle-based kernel of (66) is shown in Fig. 1(a) and (b). Note that unlike the more conventional kernels, for example, the Choi–Williams kernel shown in Fig. 1(c), these new lowpass kernels peak away from the θ and τ axes.

One can also think of other signal-dependent modifications to add to the lowpass nature of the kernel, for example, such as incorporating the kernel design method of Baraniuk and Jones to align the kernel along the chirp direction of auto-components in the ambiguity plane [11]. As with kernel design in general, there

[5]Note that the σ parameter of the Choi–Williams kernel ($\phi(\theta,\tau)=e^{-((\theta\tau)^2/\sigma)}$) is retained in the modified Cohen–Lee kernel. Thus, the reduction of crossterms may also be controlled in the present formulation, with smaller values of σ producing a kernel that decays faster away from the τ-θ axes and, hence, greater cross-term attenuation.

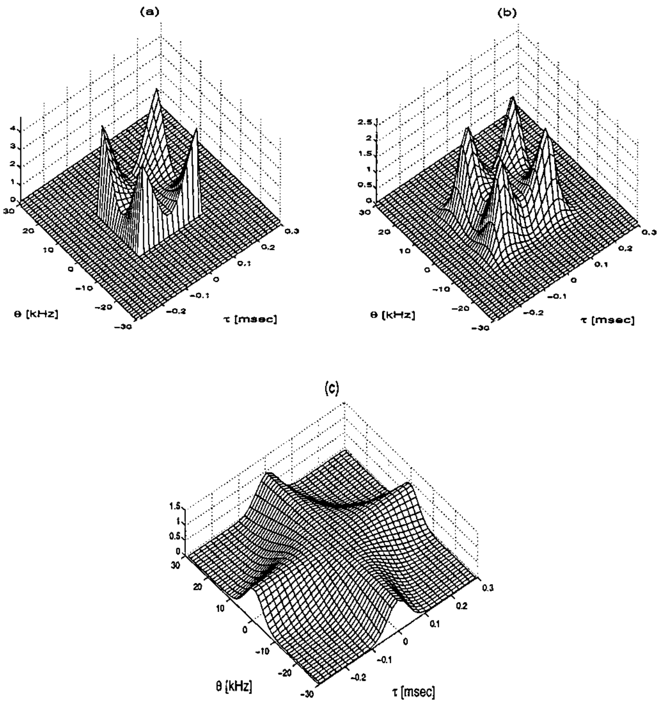

Fig. 1. (a) Separable (ideal lowpass) modified Cohen–Lee kernel [see (66)]. (b) Nonseparable (Butterworth-type) modified Cohen–Lee kernel [see (70)]. (c) Choi–Williams kernel $\phi(\theta,\tau)=e^{-((\theta\tau)^2/\sigma)}$. ($\sigma=100$ for all kernels).

is great flexibility in designing kernels to meet the new constraints proposed in (67)–(69) while simultaneously allowing the incorporation of additional desirable design criteria. In the examples that follow the next section, we focus on the satisfaction of the proposed conditional moment constraints and utilize the lowpass rectangle-based form of the Cohen–Lee kernel.

B. Global Averages

We have shown in the previous sections that the proposed second-order conditional moment candidate can be obtained in two different ways. Furthermore, we have given kernel constraints such that a Cohen-class TFD can yield these results for its conditional moments. We now show that the proposed conditional moments satisfy the necessary condition that their time average equals the global moments $\langle\langle\omega^n\rangle_t\rangle=\langle\omega^n\rangle$ [3].

The average of $\langle\omega^2\rangle_t$ is obtained from the TFD as

$$\begin{aligned}\langle\langle\omega^2\rangle_t\rangle &= \iint \omega^2 P(t,\omega)\,d\omega\,dt\\ &= \int \langle\omega^2\rangle_t P(t)\,dt. \qquad (74)\end{aligned}$$

For the modified Cohen-Lee kernel above (for which $K_0(t)$, $K_1(t)$ and $K_2(t)$ are given by (49)–(51)), we have (see Appendix)

$$\begin{aligned}&\int \langle\omega^2\rangle_t P(t)\,dt\\ &\quad= -\frac{1}{\sqrt{2\pi}}\int \Big[A^2(t)*K_2(t)\\ &\qquad -\frac{1}{2}(A'^2(t)+2A^2(t)\varphi'^2(t)-A(t)A''(t))*K_0(t)\Big]\,dt. \qquad (75)\end{aligned}$$

For an arbitrary function $f(t)$, it can be shown [using (49) and (51)] that

$$f(t)*K_2(t)=-\frac{1}{4}\frac{d^2}{dt^2}f(t)*K_0(t). \qquad (76)$$

Plugging this result into (75) gives

$$\begin{aligned}&\int \langle\omega^2\rangle_t P(t)\,dt\\ &\quad= -\frac{1}{\sqrt{2\pi}}\int\Big(-\frac{1}{4}\frac{d^2}{dt^2}A^2(t)-\frac{1}{2}(A'^2(t)+2A^2(t)\varphi'^2(t)\\ &\qquad -A(t)A''(t))\Big)*K_0(t)\,dt\\ &\quad= -\frac{1}{\sqrt{2\pi}}\int\Big(-\frac{1}{2}A(t)A''(t)-\frac{1}{2}A'^2(t)-\frac{1}{2}A'^2(t)\\ &\qquad -A^2(t)\varphi'^2(t)+\frac{1}{2}A(t)A''(t)\Big)*K_0(t)\,dt\\ &\quad= \frac{1}{\sqrt{2\pi}}\int (A'^2(t)+A^2(t)\varphi'^2(t))*K_0(t)\,dt\\ &\quad= \frac{\theta_c}{\pi}\iint (A'^2(u)+A^2(u)\varphi'^2(u))\mathrm{Sa}(\theta_c(t-u))\,du\,dt\\ &\quad= \int (A'^2(u)+A^2(u)\varphi'^2(u))\,du. \qquad (77)\end{aligned}$$

However, this last equation is exactly the second moment in frequency [3]

$$\begin{aligned}\langle\omega^2\rangle &= \int \omega^2|S(\omega)|^2\,d\omega\\ &= \int\left(\frac{A'(t)}{A(t)}\right)^2 A^2(t)\,dt+\int \varphi'^2(t)A^2(t)\,dt \qquad (78)\end{aligned}$$

and thus, we have that indeed $\langle\langle\omega^2\rangle_t\rangle=\langle\omega^2\rangle$ for the proposed second conditional moment [see (47)]. Similarly, the first conditional spectral moment $\langle\omega\rangle_t$ obtained from the modified Cohen–Lee distribution [i.e., the WAIF, (46)] averages to $\langle\omega\rangle$

$$\begin{aligned}\int \langle\omega\rangle_t P(t)\,dt &= \frac{1}{\sqrt{2\pi}}\int A^2(t)\varphi'(t)*K_0(t)\,dt\\ &= \frac{1}{\sqrt{2\pi}}\iint A^2(u)\varphi'(u)K_0(t-u)\,du\,dt\\ &= \frac{\theta_c}{\pi}\int A^2(u)\varphi'(u)\int \mathrm{Sa}(\theta_c(t-u))\,dt\,du\\ &= \int A^2(u)\varphi'(u)\,du=\langle\omega\rangle \qquad (79)\end{aligned}$$

where a derivation of the last equality can be found in Cohen's text [3]. Analogous results hold for the temporal conditional moments, as can be readily derived. We emphasize again that although the condition that the average of the local moments yields the correct global moments is not sufficient to uniquely define the local moments, it is a necessary condition for local moment candidates to satisfy [3].

V. Examples

In this section, we give examples illustrating the results obtained in the previous sections. We begin with simple examples of synthetic signals to compare our proposed instantaneous bandwidth candidate with previously proposed definitions. We then provide a few examples, for both synthetic and real signals, of the modified Cohen–Lee TFDs obtained via the kernel constraints given in the previous section and compute the conditional moments of these TFDs, with comparison to a Choi–Williams and a Cohen–Lee TFD.

A. Instantaneous Bandwidth Comparisons

Here, we consider the three signals:

$$s_1(t) = 2e^{j10\pi t} + e^{j20\pi t} \quad (80)$$

$$s_2(t) = e^{-j(5\pi t^2 - 10\pi t)} + e^{j(5\pi t^2 + 20\pi t)} \quad (81)$$

$$s_3(t) = e^{-5t^2} e^{j(5\pi t^2 + 10\pi t)} + e^{j(5\pi t^2 + 20\pi t)} \quad (82)$$

The Cohen–Lee instantaneous bandwidths [see (3)] for these signals are given by

$$\sigma_{\omega\,|\,t} = \frac{4\pi|\sin(10\pi t)|}{1 + \frac{4}{5}\cos(10\pi t)} \quad \text{[rad/s]} \quad (s_1(t)) \quad (83)$$

$$\sigma_{\omega\,|\,t} = \frac{5\pi|(2t+1)\sin(10\pi(t^2+t))|}{1 + \cos(10\pi(t^2+t))} \quad \text{[rad/s]} \quad (s_2(t)) \quad (84)$$

$$\sigma_{\omega\,|\,t} = \frac{e^{-5t^2}\left|5te^{-5t^2} + 10\pi\sin(10\pi t) + 10t\cos(10\pi t)\right|}{1 + e^{-10t^2} + 2e^{-5t^2}\cos(10\pi t)} \quad \text{[rad/s]} \quad (s_3(t)). \quad (85)$$

The corresponding proposed instantaneous bandwidths [see (33)] are

$$\sigma_\omega(t) = 4\pi \quad \text{[rads/s]} \quad (s_1(t)) \quad (86)$$

$$\sigma_\omega(t) = 5\pi|2t+1| \quad \text{[rad/s]} \quad (s_2(t)) \quad (87)$$

$$\sigma_\omega(t) = 10\left(\frac{t^2 e^{-10t^2}}{1 + e^{-10t^2}} + \frac{\pi^2 e^{-10t^2}}{(1 + e^{-10t^2})^2}\right)^{\frac{1}{2}} \quad \text{[rad/s]} \quad (s_3(t)). \quad (88)$$

Fig. 2(a) shows the Cohen–Lee bandwidth (dashed) plotted against the proposed instantaneous bandwidth (solid) for $s_1(t)$. Note the oscillatory nature of the former. The proposed instantaneous bandwidth, however, is constant for this signal, as expected, since the amplitudes and instantaneous frequencies of each component are constant.

The signal $s_2(t)$ is the sum of two symmetrically diverging chirps; the individual instantaneous frequencies of the two chirps are shown in Fig. 2(b). Since the instantaneous frequencies diverge linearly with time, we expect the spread in frequency at each time, i.e., the instantaneous bandwidth, to increase linearly with time as well. That is exactly the case with the proposed instantaneous bandwidth, plotted in Fig. 2(c), with comparison to the Cohen–Lee instantaneous bandwidth. Note that the Cohen–Lee instantaneous bandwidth has an oscillatory nature and is unbounded, making it difficult to interpret as the spread in frequency at each time for this signal.

For $s_3(t)$, which is a two-component signal comprised of parallel chirps (one with a constant amplitude and the other a Gaussian amplitude), the Cohen-Lee and proposed instantaneous bandwidths are shown in Fig. 2(d). As with the previous two examples, the Cohen–Lee bandwidth still exhibits behavior

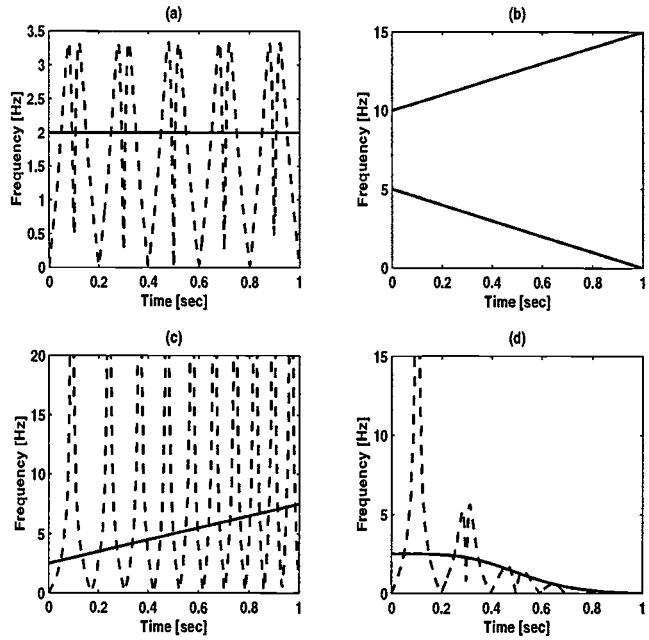

Fig. 2. (a) Cohen–Lee bandwidth (dashed) versus proposed instantaneous bandwidth (solid) for the two-tone signal given in (80). (b) Individual component instantaneous frequencies for diverging chirps given in (81). (c) Cohen–Lee bandwidth (dashed) versus proposed instantaneous bandwidth (solid) for the diverging chirps given in (81). (d) Cohen–Lee bandwidth (dashed) versus proposed instantaneous bandwidth (solid) for the chirp plus Gaussian-enveloped chirp two-component signal in (82).

that is difficult to interpret. The proposed instantaneous bandwidth, on the other hand, exhibits readily interpretable variations; as the amplitude on one of the components decays, the instantaneous bandwidth decreases steadily, approaching zero in the limit as $t \longrightarrow \infty$, for which the signal becomes a constant amplitude monocomponent signal.

B. Modified Cohen–Lee Distributions

We now give examples of the modified Cohen–Lee distribution for synthetic and real-world signals. We compare the conditional spectral variance of the modified Cohen–Lee distribution, which is equivalent to our proposed instantaneous bandwidth, to that of the Cohen–Lee and Choi–Williams distributions.

To compute the TFDs of the sampled signals, we first computed the discrete Wigner distribution and then took a 2-D inverse FFT to obtain a discrete ambiguity function $\mathcal{A}(\tau_m, \theta_l)$. This ambiguity function was then point-by-point multiplied with the appropriate kernel $\phi(\tau_m, \theta_l)$ and evaluated on a square grid sampling of

$$\left[-\frac{N}{2F_s}, \frac{N}{2F_s}\right] \times [-\pi F_s, \pi F_s] \quad \text{[s]} \times \text{[rad]} \quad (89)$$

where F_s is the sampling frequency, and N is the signal length. Finally, the 2-D FFT of the characteristic function $\mathcal{M}(\tau_m, \theta_l) = \mathcal{A}(\tau_m, \theta_l)\phi(\tau_m, \theta_l)$ gives the desired TFD $P(t_n, \omega_k)$.

1) Example 1—Chirps: Consider the two-component signal given by a sum of two linear FM chirps

$$s(t) = A\left(e^{-5(t-0.5)^2} e^{j20\pi(t-0.5)^2 + j100\pi t} + e^{-10(t-0.5)^2} e^{j10\pi(t-0.5)^2 + j180\pi t}\right) \quad (90)$$

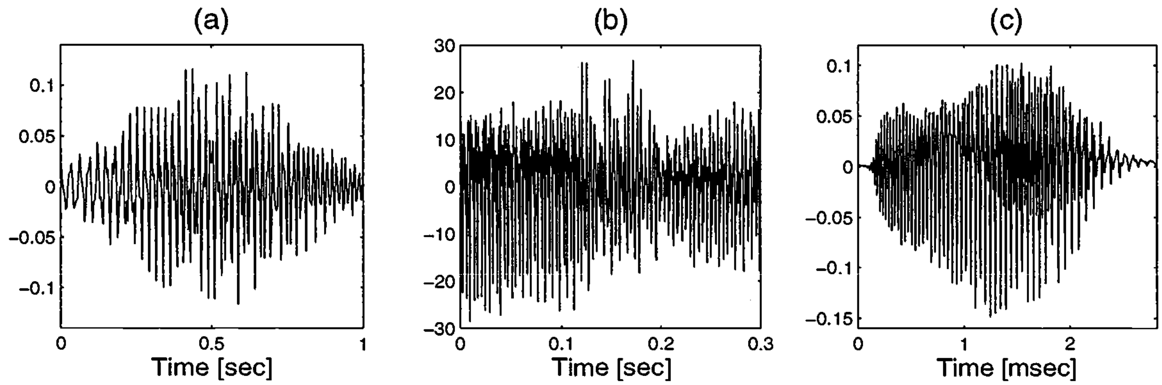

Fig. 3. (a) Chirp signal corresponding to results in Fig. 4. (b) Automobile signal corresponding to results in Fig. 5. (c) Bat echolocation signal corresponding to results in Fig. 6.

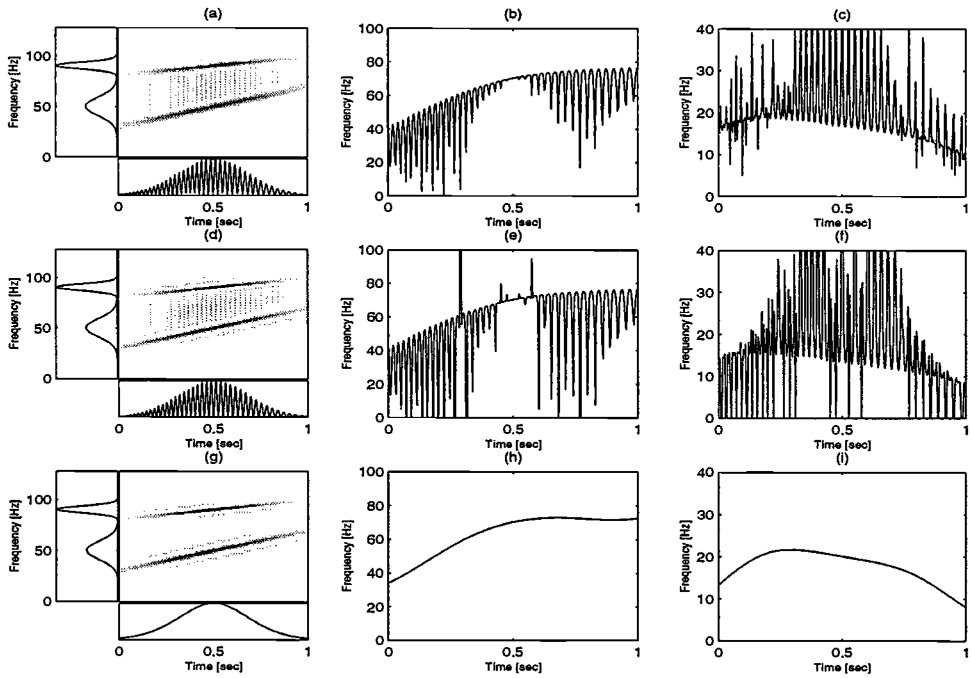

Fig. 4. (a)–(c) Choi–Williams TFD of signal given in (90), conditional mean frequency ($\langle\omega\rangle_t$), and (magnitude of) the instantaneous bandwidth of the Choi–Williams TFD ($\sigma_{\omega\mid t}$). (d)–(f) Cohen–Lee distribution, its first conditional moment, and its instantaneous bandwidth. (g)–(i) Corresponding plots for the modified Cohen–Lee distribution. The frequency and time marginals of each TFD are shown on the left and below the TFD plot, respectively, in (a), (d), and (g).

where A is a normalization constant. The real part of this signal is shown in Fig. 3(a). Fig. 4 shows a Choi–Williams distribution, a Cohen–Lee distribution, and the proposed modified Cohen–Lee distribution, along with their corresponding conditional moments. [Note that the instantaneous bandwidth of the Choi–Williams distribution is complex; we therefore plot its magnitude in Fig. 4(c)]. As designed, the conditional moments of the modified Cohen–Lee distribution do not range outside the spectral support of the signal and are readily interpreted as conditional averages in the usual sense. For this example, we chose $\theta_c = 40\pi$ rad/s ($N = 256, F_s = 256$ Hz).

2) Example 2—Automobile Signal: In this example, we compute TFDs for an acoustic recording of the racing engine of a Formula 1 race car, shown in Fig. 3(b).[6] The Choi–Williams, Cohen–Lee, and modified Cohen–Lee distributions for this signal, along with the corresponding conditional mean frequencies and instantaneous bandwidths, are shown in Fig. 5. The kernel used for the modified Cohen–Lee TFD is given by (58), where $\theta_c = 110\pi$ rad/s ($N = 700, F_s \approx 11$ kHz).

3) Example 3—Bat Echolocation Signal: We next compute a modified Cohen–Lee distribution of a bat echolocation signal.[7] The signal is shown in Fig. 3(c). Fig. 6 shows the Choi–Williams distribution, the Cohen–Lee distribution, and the modified Cohen–Lee distribution, along with the corresponding conditional means and instantaneous bandwidths. As in the previous examples, the moments of the modified Cohen–Lee distribution do not range beyond the spectral support of the signal and can be interpreted as true conditional averages. The kernel used for the modified Cohen–Lee TFD in this example is given by (66), where $\theta_c = 16\,000\pi$ rad/s and $\tau_c = 0.22$ ms ($N = 400, F_s \approx 142$ kHz).

[6]The authors thank G. Rizzoni at The Ohio State University for the F1 data.

[7]The authors thank C. Condon, K. White, and A. Feng of the Beckman Institute, University of Illinois, for making the bat data publicly available.

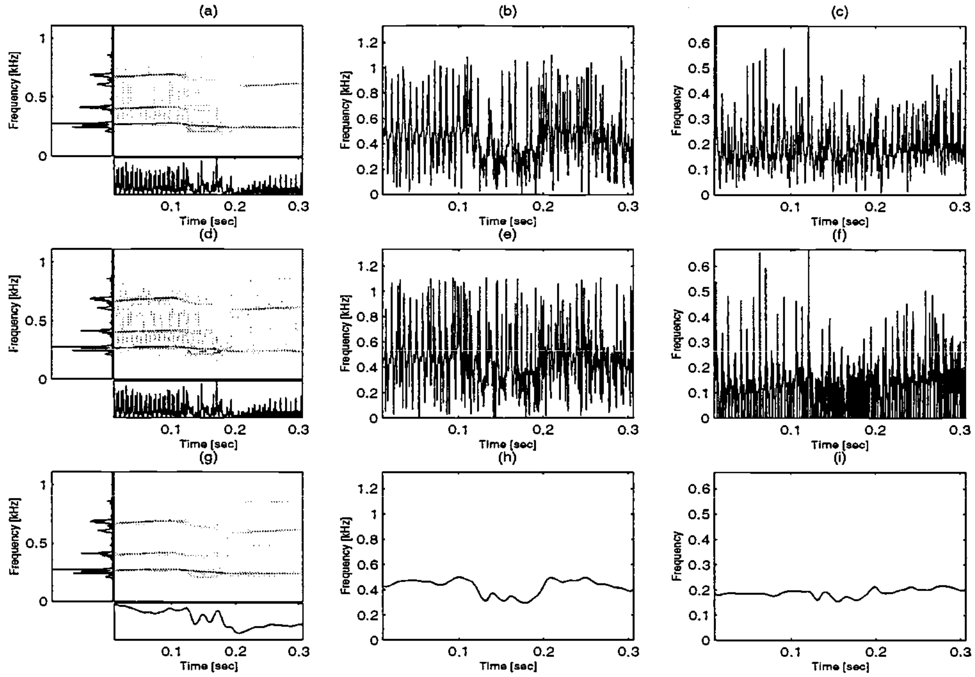

Fig. 5. (a)–(c) Choi–Williams TFD of automobile signal, conditional mean frequency ($\langle\omega\rangle_t$), and (magnitude of) the instantaneous bandwidth of the Choi–Williams TFD ($\sigma_{\omega\mid t}$). (d)–(f) Cohen–Lee distribution, its first conditional moment, and its instantaneous bandwidth. (g)–(i) Corresponding plots for the modified Cohen-Lee distribution.

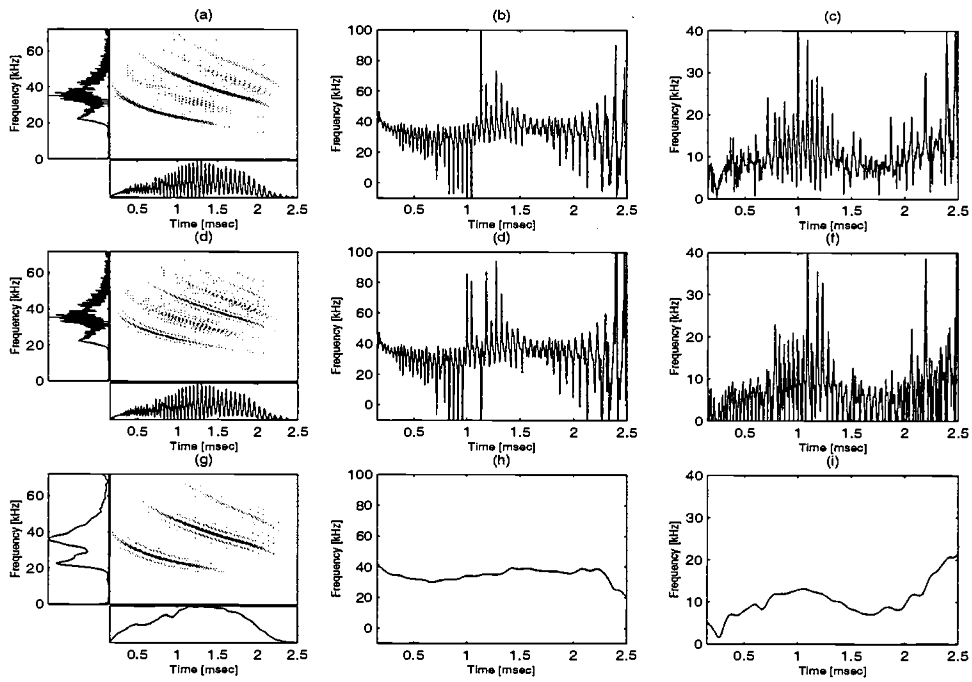

Fig. 6. (a)–(c) Choi–Williams TFD of bat echolocation signal, conditional mean frequency ($\langle\omega\rangle_t$), and (magnitude of) the instantaneous bandwidth of the Choi–Williams TFD ($\sigma_{\omega\mid t}$). (d)–(f) Cohen–Lee distribution, its first conditional moment, and its instantaneous bandwidth. (g)–(i) Corresponding plots for the modified Cohen–Lee distribution.

VI. Conclusion

Instantaneous bandwidth is interpreted as the spread in frequency at a given time, which derives from time–frequency distribution theory. However, the definition of instantaneous bandwidth given by Cohen and Lee does not usually conform to this interpretation, and most TFDs do not yield second conditional moments that can be interpreted as the spread in frequency at a given time. We have obtained a new candidate for the instantaneous bandwidth of multicomponent signals that can be interpreted as the spread in frequency at each time and have provided the kernel constraints required for a Cohen-class TFD to yield this result. TFDs that satisfy these new constraints were shown to be modified Cohen–Lee distributions. As with the smoothed-pseudo Wigner distribution, where the correct marginals are sacrificed for other properties, the marginals of the modified Cohen–Lee distribution do not equal the true marginals since the kernel is not one along the entire θ and τ axes [see (58) and (66)], in order to obtain interpretable first and second conditional moments.[8] The new instantaneous bandwidth does not exhibit erratic oscillatory behavior for multicomponent signals and does not range beyond the global bandwidth of the signal. In addition, the new instantaneous bandwidth was shown to be consistent with the global spectral bandwidth in that the time-average of the conditional second moment equals the global second moment. Examples were provided to demonstrate the new TFD and conditional spectral moments. Analogous results hold for the conditional temporal moments and the "local duration" of a signal.

APPENDIX A
COHEN-CLASS CONDITIONAL SPECTRAL MOMENTS

By (12) and (13), we have

$$\begin{aligned}\langle\omega^n\rangle_t P(t) &= \int \omega^n P(t,\omega)\,d\omega\\
&= \int \omega^n \left[\frac{1}{4\pi^2}\iiint s^*\left(u-\frac{\tau}{2}\right)s\left(u+\frac{\tau}{2}\right)\right.\\
&\quad\left.\times\,\phi(\theta,\tau)e^{-j\theta t - j\omega\tau + j\theta u}\,du\,d\theta\,d\tau\right]d\omega\\
&= \frac{1}{2\pi}\iiint s^*\left(u-\frac{\tau}{2}\right)s\left(u+\frac{\tau}{2}\right)\phi(\theta,\tau)\\
&\quad\times\left[\frac{1}{2\pi}\int\omega^n e^{-j\omega\tau}\,d\omega\right]e^{-j\theta(t-u)}\,du\,d\theta\,d\tau\\
&= \frac{1}{2\pi}\iiint s^*\left(u-\frac{\tau}{2}\right)s\left(u+\frac{\tau}{2}\right)\phi(\theta,\tau)\\
&\quad\times\left[\frac{1}{(-j)^n}\delta^{(n)}(\tau)\right]e^{-j\theta(t-u)}\,du\,d\theta\,d\tau\\
&= \frac{(-j)^n}{2\pi}\iint\left(\frac{\partial}{\partial\tau}\right)^n\\
&\quad\left\{s^*\left(u-\frac{\tau}{2}\right)s\left(u+\frac{\tau}{2}\right)\phi(\theta,\tau)\right\}\Big|_{\tau=0}\\
&\quad\times\,e^{-j\theta(t-u)}\,du\,d\theta\\
&= \frac{(-j)^n}{2\pi}\iint\left[\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\partial}{\partial\tau}\right)^k\right.\\
&\quad\left\{s^*\left(u-\frac{\tau}{2}\right)s\left(u+\frac{\tau}{2}\right)\right\}\\
&\quad\left.\times\left(\frac{\partial}{\partial\tau}\right)^{n-k}\phi(\theta,\tau)\right]\Bigg|_{\tau=0}e^{-j\theta(t-u)}\,du\,d\theta\\
&= \frac{(-j)^n}{2\pi}\iint\left[\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{l=0}^{k}\binom{k}{l}\left(\frac{d}{d\tau}\right)^l\right.\right.\\
&\quad\left. s^*\left(u-\frac{\tau}{2}\right)\left(\frac{d}{d\tau}\right)^{k-l}s\left(u+\frac{\tau}{2}\right)\right)\\
&\quad\left.\times\left(\frac{\partial}{\partial\tau}\right)^{n-k}\phi(\theta,\tau)\right]\Bigg|_{\tau=0}e^{-j\theta(t-u)}\,du\,d\theta\end{aligned}$$

$$\begin{aligned}&= \frac{(-j)^n}{\sqrt{2\pi}}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}\\
&\quad\times\int\left\{\left(\frac{d}{d\tau}\right)^l s^*\left(u-\frac{\tau}{2}\right)\left(\frac{d}{d\tau}\right)^{k-l}\right.\\
&\quad\left.\times\,s\left(u+\frac{\tau}{2}\right)\right\}\Bigg|_{\tau=0}\\
&\quad\times\left[\frac{1}{\sqrt{2\pi}}\int\left\{\left(\frac{\partial}{\partial\tau}\right)^{n-k}\phi(\theta,\tau)\right\}\right.\\
&\quad\left.\times\,e^{-j\theta(t-u)}\,d\theta\Big|_{\tau=0}\right]du\\
&= \frac{(-j)^n}{\sqrt{2\pi}}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}\int\left(\frac{-1}{2}\right)^l\\
&\quad\times\,s^{*(l)}(u)\left(\frac{1}{2}\right)^{k-l}s^{(k-l)}(u)\\
&\quad\times\left[\frac{1}{\sqrt{2\pi}}\int\left\{\left(\frac{\partial}{\partial\tau}\right)^{n-k}\phi(\theta,\tau)\right\}\right.\\
&\quad\left.\times\,e^{-j\theta(t-u)}\,d\theta\Big|_{\tau=0}\right]du.\end{aligned}\tag{91}$$

Substituting $K_i(t)$ from (16) into (91) gives

$$\begin{aligned}\langle\omega^n\rangle_t P(t) &= \frac{(-j)^n}{\sqrt{2\pi}}\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}\int\left(\frac{-1}{2}\right)^l\\
&\quad\times\,s^{*(l)}(u)\left(\frac{1}{2}\right)^{k-l}s^{(k-l)}(u)K_{n-k}(t-u)\,du\\
&= \frac{(-j)^n}{\sqrt{2\pi}}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k\sum_{l=0}^{k}\binom{k}{l}(-1)^l\\
&\quad\times\int s^{*(l)}(u)s^{(k-l)}(u)K_{n-k}(t-u)\,du\\
&= \frac{(-j)^n}{\sqrt{2\pi}}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^k\\
&\quad\times\left\{\sum_{l=0}^{k}\binom{k}{l}(-1)^l s^{*(l)}(t)s^{(k-l)}(t)\right\}\\
&\quad *\,K_{n-k}(t).\end{aligned}\tag{92}$$

A similar calculation gives the time marginal $P(t) = (1/\sqrt{2\pi})|s(t)|^2 * K_0(t)$. We then have (93), shown at the top of the next page. In particular, we have

$$\langle\omega\rangle_t = \frac{-j|s(t)|^2 * K_1(t) + \mathrm{Im}\{s^*(t)s'(t)\} * K_0(t)}{|s(t)|^2 * K_0(t)}\tag{94}$$

and

$$\begin{aligned}\langle\omega^2\rangle_t &= \frac{-|s(t)|^2 * K_2(t) - 2j\mathrm{Im}\{s^*(t)s'(t)\} * K_1(t)}{|s(t)|^2 * K_0(t)}\\
&\quad-\frac{\frac{1}{2}\left(\mathrm{Re}\{s^*(t)s''(t)\} - |s'(t)|^2\right) * K_0(t)}{|s(t)|^2 * K_0(t)}\end{aligned}\tag{95}$$

where $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ denote the real and imaginary parts, respectively.

[8] We note that it is possible, using an iterative weighted least-squares method, to generate Cohen–Posch (positive) TFDs [7] that yield the WAIF and the proposed instantaneous bandwidth for the first and second conditional moments, respectively [8]. These TFDs satisfy the marginals and have signal-dependent kernels (and therefore are not in the bilinear class of TFDs) [7].

$$\langle \omega^n \rangle_t = \frac{(-j)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k \left\{ \sum_{l=0}^{k} \binom{k}{l} (-1)^l s^{*(l)}(t) s^{(k-l)}(t) \right\} * K_{n-k}(t)}{|s(t)|^2 * K_0(t)}. \tag{93}$$

$$\langle \omega \rangle_t = \frac{\left\{ \sum_{i=1}^{N} a_i^2 \varphi_i' + \sum_{\substack{i,k=1\\ i\neq k}}^{N} [a_i(a_k\varphi_k' \cos \Delta\varphi_{ik} - a_k' \sin \Delta\varphi_{ik})] \right\} * K_0}{\left\{ \sum_{i=1}^{N} a_i^2 + \sum_{\substack{i,k=1\\ i\neq k}}^{N} a_i a_k \cos \Delta\varphi_{ik} \right\} * K_0} \tag{104}$$

$$\langle \omega^2 \rangle_t = \frac{\left\{ \begin{array}{c} \sum_{i=1}^{N} \left(a_i'^2 + a_i^2 \varphi_i'^2\right) - \frac{1}{4} \sum_{\substack{i,k=1\\ i\neq k}}^{N} \left[a_i a_k \Delta^2 \varphi_{ik}' - \left(\frac{d}{dt}\right)^2 (a_i a_k) - 2a_i' a_k' \right. \\ \left. - 2a_i \left(a_k \varphi_k' (\varphi_k' + \varphi_i') - a_k''\right) \right] \cos \Delta\varphi_{ik} - \frac{1}{4} \sum_{\substack{i,k=1\\ i\neq k}}^{N} \left[a_i a_k \Delta\varphi_{ik}'' \right. \\ \left. -2\Delta\varphi_{ik}' \frac{d}{dt} a_i a_k - 4a_i a_k' \varphi_k' - 2a_i a_k \varphi_k'' - 2a_i a_k' \varphi_i' + 2a_i' a_k \varphi_k' \right] \sin \Delta\varphi_{ik} \end{array} \right\} * K_0}{\left\{ \sum_{i=1}^{N} a_i^2 + \sum_{\substack{i,k=1\\ i\neq k}}^{N} a_i a_k \cos \Delta\varphi_{ik} \right\} * K_0} \tag{105}$$

APPENDIX B
MODIFIED COHEN–LEE KERNEL

Throughout this work, we have used the functions $K_i(t)$ defined by

$$K_i(t) = \frac{1}{\sqrt{2\pi}} \int \left\{ \left(\frac{\partial}{\partial \tau}\right)^i \phi(\theta, \tau) \right\} \Bigg|_{\tau=0} e^{-j\theta t} \, d\theta \tag{96}$$

where $\phi(\theta, \tau)$ is the kernel in the Cohen class of bilinear TFDs. We see that $\{(\partial/\partial\tau)^i \phi(\theta, \tau)\}|_{\tau=0}$ and $K_i(t)$ are Fourier transform pairs. Here, we detail the derivation of $K_0(t)$–$K_2(t)$ for the modified Cohen–Lee kernel

$$\phi(\theta, \tau) = (1 + c\,(\theta\tau)^2) e^{-\frac{(\theta\tau)^2}{\sigma}} \operatorname{rect}\left(\frac{\theta}{2\theta_c}\right) \tag{97}$$

where σ and θ_c are positive free parameters, and $c = (1/8) + (1/\sigma)$.

For this kernel, we have

$$\phi(\theta, \tau)|_{\tau=0} = \operatorname{rect}\left(\frac{\theta}{2\theta_c}\right) \tag{98}$$

$$\begin{aligned} &\left\{ \left(\frac{\partial}{\partial \tau}\right) \phi(\theta, \tau) \right\} \Bigg|_{\tau=0} \\ &\quad = \left\{ \left[(1 + c(\theta\tau)^2) \left(-\frac{2\theta^2 \tau}{\sigma}\right) + 2c\theta^2 \tau \right] \right. \\ &\qquad \left. \times e^{-\frac{(\theta\tau)^2}{\sigma}} \operatorname{rect}\left(\frac{\theta}{2\theta_c}\right) \right\} \Bigg|_{\tau=0} = 0 \end{aligned} \tag{99}$$

$$\begin{aligned} &\left\{ \left(\frac{\partial}{\partial \tau}\right)^2 \phi(\theta, \tau) \right\} \Bigg|_{\tau=0} \\ &\quad = \left\{ \left[2\theta^2 \tau c - \frac{4\theta^4 \tau^2}{\sigma} + 2\theta^2 c - \frac{2\theta^2}{\sigma} (\tau + 1)(1 + c(\theta\tau)^2) \right] \right. \\ &\qquad \left. \times e^{-\frac{(\theta\tau)^2}{\sigma}} \operatorname{rect}\left(\frac{\theta}{2\theta_c}\right) \right\} \Bigg|_{\tau=0} \\ &\quad = \frac{\theta^2}{4} \operatorname{rect}\left(\frac{\theta}{2\theta_c}\right). \end{aligned} \tag{100}$$

Finally, taking Fourier transforms of the right sides of the above equations yields

$$K_0(t) = \frac{2\theta_c}{\sqrt{2\pi}} \operatorname{Sa}(\theta_c t) \tag{101}$$

$$K_1(t) = 0 \tag{102}$$

$$\begin{aligned} K_2(t) &= -\frac{1}{4} \delta''(t) * K_0(t) \\ &= -\frac{\theta_c}{2\sqrt{2\pi}} \int \operatorname{Sa}(\theta_c u) \delta''(t-u)\, du \end{aligned} \tag{103}$$

which were given in (49)–(51). We see that $K_0(t)$ is the impulse response of an ideal lowpass filter with cut-off frequency θ_c. Similarly, $K_2(t)$ is the impulse response of such a lowpass filter followed by a second-order differentiator.

APPENDIX C
CONDITIONAL SPECTRAL MOMENTS: MULTICOMPONENT SIGNALS

In this section, we generalize the results obtained in Section III and give an instantaneous bandwidth expression for the general N-component signal.

Substituting the multicomponent signal $s(t) = \sum_{i=1}^{N} a_i(t) e^{j\varphi_i(t)}$ into (94) and (95) and noting that $K_1(t) = 0$ and $f(t) * K_2(t) = -(1/4) f''(t) * K_0(t)$ for the modified Cohen-Lee kernel, we have (104) and (105), shown at the top of the page, where $\Delta\varphi_{ik} = \varphi_i - \varphi_k$, and the dependence on t has been dropped for convenience in notation. Recall that $K_0(t)$ acts as an ideal lowpass filter with cutoff frequency $\theta_c < |\Delta\varphi_{ik}'(t)|$ for each i, k and all t. Thus, (104) and (105) reduce to

$$\langle \omega \rangle_t = \frac{\sum_{i=1}^{N} a_i^2 \varphi_i'}{\sum_{i=1}^{N} a_i^2} \tag{106}$$

and

$$\langle \omega^2 \rangle_t = \frac{\sum_{i=1}^{N} a_i'^2}{\sum_{i=1}^{N} a_i^2} + \frac{\sum_{i=1}^{N} a_i^2 \varphi_i'^2}{\sum_{i=1}^{N} a_i^2}. \tag{107}$$

It then follows that

$$\sigma^2_{\omega\,|\,t} = \frac{\sum_{i=1}^N \left(a_i'^2 + a_i^2 \varphi_i'^2\right)}{\sum_{i=1}^N a_i^2} - \left(\frac{\sum_{i=1}^N a_i^2 \varphi_i'}{\sum_{i=1}^N a_i^2}\right)^2 \tag{108}$$

or equivalently

$$\sigma^2_{\omega\,|\,t} = \frac{\sum_{i=1}^N a_i'^2}{\sum_{i=1}^N a_i^2} + \frac{\sum_{\substack{i,k=1\\ i<k}}^N a_i^2 a_k^2 \left(\varphi_i' - \varphi_k'\right)^2}{\left(\sum_{i=1}^N a_i^2\right)^2}. \tag{109}$$

Note that $\langle \omega^2 \rangle_t$ and $\sigma^2_{\omega\,|\,t}$ are positive for all multicomponent signals and that (106), (107) and (109) generalize (52)–(54), respectively.

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