

# Modified gravity and cosmology with nonminimal (derivative) coupling between matter and the Einstein tensor

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We construct new classes of modified theories in which the matter sector couples with the Einstein tensor, namely we consider direct couplings of the latter to the energy-momentum tensor, and to the derivatives of its trace. We extract the general field equations, which do not contain higher-order derivatives, and we apply them in a cosmological framework, obtaining the Friedmann equations, whose extra terms give rise to an effective dark energy sector. At the background level we show that we can successfully describe the usual thermal history of the universe, with the sequence of matter and dark-energy epochs, while the dark-energy equation-of-state parameter can lie in the phantom regime, tending progressively to  $-1$  at present and future times. Furthermore, we confront the theory with Cosmic Chronometer data, showing that the agreement is very good. Finally, we perform a detailed investigation of scalar and tensor perturbations, and extracting an approximate evolution equation for the matter overdensity we show that the predicted behavior is in agreement with observations.

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## I. INTRODUCTION

According to the concordance model of cosmology the universe is currently accelerating, while it entered this era after being in a long matter-dominated epoch. This behavior, alongside the early accelerated era of inflation, cannot be reproduced within the standard framework of general relativity and Standard Model of particles, and thus extra degrees of freedom should be introduced. One can attribute these extra degrees of freedom to new, exotic forms of matter, such as the inflaton field at early times (for reviews see [1, 2]) and/or the dark energy concept at late times (for reviews see [3, 4]). Alternatively, one can consider the extra degrees of freedom to have gravitational origin, i.e. to arise from a gravitational modification that possesses general relativity as a particular limit (see [5–7] and references therein).

In order to construct gravitational modifications one usually starts from the Einstein-Hilbert Lagrangian and extends it in various ways, resulting in  $f(R)$  gravity [8], in Gauss-Bonnet and  $f(G)$  gravity [9, 10], in Lovelock and  $f(\text{Lovelock})$  gravity [11, 12], etc (for a review see [13]). On the other hand, one may start from the equivalent, torsional formulation of gravity, and extend it in similar ways, obtaining  $f(T)$  gravity [14, 15],  $f(T, B)$  gravity [16],  $f(T, T_G)$  gravity [17] etc. Nevertheless, one can consider theories in which the geometric part of the action is coupled to the non-geometric sector. In the case of general relativity the simplest models are those with non-minimally coupled [18–21] and non-minimal-

derivatively coupled [22–27] scalar fields, or the general scalar-tensor [28–30] and Horndeski/Galileon theories [31–33]. In the case of torsional gravity one can similarly construct scalar-torsion theories [34, 35] or teleparallel Horndeski gravity [36, 37].

Inspired by these couplings between geometric and non-geometric sectors, one can proceed to the construction of theories in which the gravitational sector couples in a non-trivial way with the matter one, since there is no theoretical reason against such interactions. The simplest way is to consider that the matter Lagrangian  $\mathcal{L}_m$  is coupled to functions of the Ricci scalar [38–40], which can be extended to arbitrary functions of  $(R, L_m)$  [41–45]. Additionally, one can consider models where the Ricci scalar is coupled to the trace of the energy momentum tensor  $T$  and extend to arbitrary functions, such as in  $f(R, T)$  theory [46–52], or even consider terms of the form  $R_{\mu\nu}T^{\mu\nu}$  [53, 54]. Alternatively, one can follow the same path in the case of torsional gravity, and construct modifications in which the matter Lagrangian is coupled to functions of the torsion scalar [55–57], as well as theories where the torsion scalar is coupled to the trace of the energy momentum tensor [58–62]. We mention here that the above modifications, in which one handles the gravitational and matter sectors on equal footing, do not present any problem at the theoretical level, and one would only obtain observational constraints only in the case of baryonic matter due to non-geodesic motion.

In the present work, inspired by the coupling of the scalar fields to the Einstein tensor, as well as by the coupling of the matter sector to the Ricci scalar, we are in-

interested in constructing new classes of modified theories, in which the matter sector couples with the Einstein tensor. As one can see, we can directly couple the energy-momentum tensor to the Einstein tensor, i.e. consider a term  $G_{\mu\nu}T^{\mu\nu}$ , or we can couple the Einstein tensor to derivatives of the trace of the energy-momentum tensor, i.e. consider a term  $G_{\mu\nu}(\partial^\mu T)(\partial^\nu T)$ . Interestingly enough, similarly to the other matter-gravity couplings, in the cosmological applications of these theories the extra terms in the Friedmann equations, although originating from the matter sector, can lead to accelerated expansion.

The plan of the work is the following: In Section II we present the theoretical basis of the theories with couplings between matter sector and the Einstein tensor, extracting the general field equations. In Section III we proceed to the cosmological application, and in particular in subsection III A we investigate the background behavior, while in subsection III B we perform a detailed perturbation analysis. Finally, in Section IV we summarize the obtained results.

## II. NON-MINIMAL (DERIVATIVE) COUPLINGS BETWEEN MATTER AND EINSTEIN TENSOR

In this work we propose more general couplings between the geometric and the matter sectors, and in particular we study actions in which the energy-momentum tensor and its trace couple to the Einstein tensor. We consider actions of the form

$$S = \int \sqrt{-g} d^4x \left\{ \frac{1}{2\kappa^2} (R - 2\Lambda) + G_{\mu\nu} [\alpha T^{\mu\nu} + \beta (\partial^\mu T)(\partial^\nu T)] + L_m \right\}, \quad (1)$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $T^{\mu\nu}$  is the energy-momentum tensor defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \quad (2)$$

with  $L_m$  the matter Lagrangian, and  $T = T^{\mu\nu}g_{\mu\nu}$  is its trace. Furthermore,  $\kappa^2$  is the gravitational constant, while  $\alpha$  and  $\beta$  are the coupling parameters, while for completeness we consider the cosmological constant  $\Lambda$  too. Similarly to Horndeski construction [31–33] the use of the Einstein tensor ensures that the resulting field equations will not contain higher-order derivatives.

Variation of the action with respect to the metric leads to the following field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 \tilde{T}_{\mu\nu} = \kappa^2 [T_{\mu\nu} + \alpha T_{\mu\nu}^{(\alpha)} + \beta T_{\mu\nu}^{(\beta)}], \quad (3)$$

where we have defined

$$\begin{aligned} T_{\mu\nu}^{(\alpha)} &\equiv g_{\mu\nu} T_{\alpha\beta} G^{\alpha\beta} + R_{\mu\nu} T - 2G_\nu^\alpha T_{\mu\alpha} - 2G_\mu^\alpha T_{\nu\alpha} \\ &\quad - R T_{\mu\nu} - \square T_{\mu\nu} + \nabla_\alpha \nabla_\mu T_\nu^\alpha + \nabla_\alpha \nabla_\nu T_\mu^\alpha \\ &\quad - g_{\mu\nu} (\nabla_\alpha \nabla_\beta T^{\alpha\beta}) + g_{\mu\nu} \square T - \nabla_\mu \nabla_\nu T \\ &\quad - 2\Xi_{\mu\nu}, \end{aligned} \quad (4)$$

and

$$\begin{aligned} T_{\mu\nu}^{(\beta)} &\equiv g_{\mu\nu} G^{\alpha\beta} (\nabla_\alpha T) (\nabla_\beta T) + g_{\mu\nu} R^{\alpha\beta} (\nabla_\alpha T) (\nabla_\beta T) \\ &\quad + R_{\mu\nu} (\nabla_\alpha T) (\nabla^\alpha T) - 2 (\nabla_\alpha \nabla_\nu T) (\nabla^\alpha \nabla_\mu T) \\ &\quad + g_{\mu\nu} (\nabla_\alpha \nabla_\beta T) (\nabla^\alpha \nabla^\beta T) - g_{\mu\nu} (\square T)^2 \\ &\quad - 2R_{\mu\alpha\nu\beta} (\nabla^\alpha T) (\nabla^\beta T) - 2G_\nu^\alpha (\nabla_\alpha T) (\nabla_\mu T) \\ &\quad - 2G_\mu^\alpha (\nabla_\alpha T) (\nabla_\nu T) - R (\nabla_\mu T) (\nabla_\nu T) \\ &\quad + 2 (\nabla_\alpha \nabla^\alpha T) (\nabla_\mu \nabla_\nu T) \\ &\quad + 4G_{\alpha\beta} \nabla^\alpha \nabla^\beta T (T_{\mu\nu} + \Theta_{\mu\nu}). \end{aligned} \quad (5)$$

In the above expressions we have used that

$$\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = \frac{\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} \mathcal{L}_m + \frac{1}{2} g_{\alpha\beta} g_{\mu\nu} \mathcal{L}_m - \frac{1}{2} g_{\alpha\beta} T_{\mu\nu} - 2 \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}, \quad (6)$$

and we have introduced the tensors  $\Theta_{\mu\nu}$  and  $\Xi_{\mu\nu}$  as

$$\begin{aligned} \Theta_{\mu\nu} &\equiv g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} \\ &= g_{\mu\nu} \mathcal{L}_m - 2T_{\mu\nu} - 2g^{\alpha\beta} \frac{\delta^2 \mathcal{L}_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}}, \end{aligned} \quad (7)$$

and

$$\begin{aligned} \Xi_{\mu\nu} &\equiv G^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} \\ &= -G_{\mu\nu} \mathcal{L}_m + \frac{1}{2} G^{\alpha\beta} g_{\alpha\beta} (g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}) \\ &\quad - 2G^{\alpha\beta} \frac{\delta^2 \mathcal{L}_m}{\delta g^{\mu\nu} \delta g^{\alpha\beta}}. \end{aligned} \quad (8)$$

Note that due to the specific form of action (1), the above general field equations do not contain higher-order derivatives, and thus the theory does not suffer from ghost instabilities. Finally, taking the covariant derivative of (3), and using that  $\nabla^\mu G_{\mu\nu} = 0$ , we can obtain as usual the conservation equation  $\nabla^\mu G_{\mu\nu} = \kappa^2 \nabla^\mu \tilde{T}_{\mu\nu} = 0$ , namely

$$\nabla^\mu [T_{\mu\nu} + \alpha T_{\mu\nu}^{(\alpha)} + \beta T_{\mu\nu}^{(\beta)}] = 0. \quad (9)$$

Lastly, we mention that in the case where  $\alpha = \beta = 0$  we recover General Relativity and  $\Lambda$ CDM cosmology.

## III. COSMOLOGICAL APPLICATIONS

In the previous section we constructed theories with couplings between the Einstein tensor and the energy-momentum tensor and its trace. In the present section we proceed to the investigation of their cosmological applications. We consider a flat Friedmann-Robertson-Walker (FRW) spacetime metric of the form

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \quad (10)$$

where  $a(t)$  is the scale factor. For the matter sector we consider the standard perfect fluid, with energy-momentum tensor

$$T_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu}, \quad (11)$$

where  $u^\mu$  is the 4-velocity which satisfies  $u_\mu u^\mu = -1$ . Moreover, concerning the matter Lagrangian we assume the standard form  $\mathcal{L}_m = p_m$  [38–45]. Under these considerations, the tensors  $\Theta_{\mu\nu}$  and  $\Xi_{\mu\nu}$  become

$$\Theta_{\mu\nu} = -2(\rho_m + p_m)u_\mu u_\nu - p_m g_{\mu\nu}, \quad (12)$$

$$\Xi_{\mu\nu} = -G_{\mu\nu} p_m - \frac{1}{2} G^{\alpha\beta} g_{\alpha\beta} (\rho_m + p_m) u_\mu u_\nu. \quad (13)$$

### A. Background Evolution

We start our investigation by the examination of the background evolution. Substituting (10)–(13) into the general field equations (3) we obtain the Friedmann equations

$$3H^2 - \Lambda = \kappa^2(\rho_m + \rho_\alpha + \rho_\beta), \quad (14)$$

$$3H^2 + 2\dot{H} - \Lambda = -\kappa^2(p_m + p_\alpha + p_\beta), \quad (15)$$

with  $H = \dot{a}/a$  the Hubble function, and where we have introduced

$$\begin{aligned} \rho_\alpha &\equiv \alpha \left[ -6(p_m + \rho_m)\dot{H} + 3H^2(p_m - \rho_m) + 6H\dot{p}_m \right], \\ \rho_\beta &\equiv \beta \left\{ -12H(p_m + \rho_m)(3\dot{p}_m - \dot{\rho}_m)(3H^2 + 2\dot{H}) \right. \\ &\quad \left. - 3H^2 [4(p_m + \rho_m)(3\ddot{p}_m - \ddot{\rho}_m) \right. \\ &\quad \left. - 3(3\dot{p}_m - \dot{\rho}_m)^2] \right\}, \end{aligned} \quad (16)$$

and

$$\begin{aligned} p_\alpha &\equiv \alpha \left\{ -(3p_m + \rho_m)(3H^2 + 2\dot{H}) \right. \\ &\quad \left. - [2H(3\dot{p}_m + \dot{\rho}_m) + 2\ddot{p}_m] \right\}, \\ p_\beta &\equiv \beta \left\{ -(3\dot{p}_m - \dot{\rho}_m)^2(3H^2 + 2\dot{H}) \right. \\ &\quad \left. - 4H(3\dot{p}_m - \dot{\rho}_m)(3\ddot{p}_m - \ddot{\rho}_m) \right\}. \end{aligned} \quad (17)$$

Additionally, in the case of FRW metric, the general conservation (9) becomes

$$\dot{\rho}_m + \dot{\rho}_\alpha + \dot{\rho}_\beta + 3H(\rho_m + \rho_\alpha + \rho_\beta + p_m + p_\alpha + p_\beta) = 0. \quad (18)$$

We mention here that, as expected by the form of the general field equations (3)–(8), the Friedmann equations (14),(15) do not contain higher-order derivatives, and thus the theory is free from Ostrogradsky instabilities. Nevertheless, in the theory at hand, although the field equations are healthy, one can in principle have higher than second order derivatives in the general conservation equation (18) (in particular in the  $\beta$ -term). This is

typical in all theories of non-minimal matter couplings (see e.g. the relevant discussion in the well-known paper [46]). However, this is not a problem, since the conservation equation is not used in order to obtain the solutions (one uses only the two Friedmann equations), and after one has extracted the solutions he inserts them in the conservation equation which is trivially satisfied as expected.

As we observe, in the scenario of non-minimal coupling between the matter sector and the Einstein tensor, we obtain an effective dark energy sector with energy density and pressure

$$\rho_{DE} \equiv \frac{\Lambda}{\kappa^2} + \rho_\alpha + \rho_\beta \quad (19)$$

$$p_{DE} \equiv -\frac{\Lambda}{\kappa^2} + p_\alpha + p_\beta, \quad (20)$$

respectively, and thus with effective equation-of-state parameter

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}}. \quad (21)$$

Note that the above total conservation equation (18) can be further handled in two ways. The first is to consider that although the total energy is conserved the individual sectors do not, namely we obtain an effective interaction and a transfer of energy between matter and geometric sectors and vice versa, typical in all theories with matter-geometry couplings [38–62] (this effective interaction, apart from other features, has the advantage that it can alleviate the coincidence problem). The second choice is to additionally impose by hand the standard matter conservation equation, and thus to obtain also a separate conservation equation for the effective dark energy sector. In the following without loss of generality we make the first choice, and hence separately the matter sector is not conserved, however due to the imposed parameter values the effective interaction between dark-matter and dark energy is weak, and thus matter scales very close to  $a(t)^{-3}$  as required by observations.

Let us proceed to the numerical examination of the above Friedmann equations. Without loss of generality we focus on the case of dust matter ( $p_m \approx 0$ ). Moreover, we introduce the density parameters as

$$\Omega_m \equiv \frac{\kappa^2}{3H^2} \rho_m \quad (22)$$

$$\Omega_{DE} \equiv \frac{\kappa^2}{3H^2} \rho_{DE}, \quad (23)$$

for the matter and effective dark energy sector respectively. Finally, as usual we use the redshift  $1+z = a_0/a$  as the independent variable, setting the present scale factor to  $a_0 = 1$  (from now on the subscript “0” denotes the value of a quantity at present).

We solve equations (14),(15) numerically, imposing the conditions  $\Omega_{DE}(z=0) \equiv \Omega_{DE0} \approx 0.7$  and therefore  $\Omega_m(z=0) \equiv \Omega_{m0} \approx 0.3$  in agreement with observations [63], which then determines the relation between  $\alpha$ ,

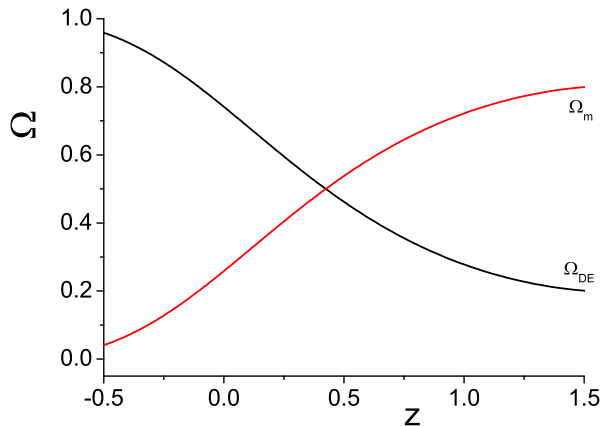


FIG. 1: The evolution of the effective dark energy density parameter  $\Omega_{DE}$  (black-solid) and of the matter density parameter  $\Omega_m$  (red-dashed), as a function of the redshift  $z$ , for the scenario of non-minimal (derivative) coupling between matter and Einstein tensor, with  $\alpha = -0.05$  and  $\beta = 0.01$ , in units where  $\kappa^2 = 1$ . We have imposed  $\Omega_{m0} \approx 0.3$  at present time.

$\beta$  and  $\Lambda$ . In Fig. 1 we draw the resulting evolution of  $\Omega_{DE}(z)$  and  $\Omega_m(z)$ . As we can see the scenario at hand can describe the thermal history of the universe successfully, namely the sequence of matter and late-time acceleration epochs. Additionally, in Fig. 2 we depict the corresponding evolution of the effective equation-of-state parameter. As we can see,  $w_{DE}$  is algebraically smaller in the past, while it is closer to  $-1$  at present time, as required by observations, before going asymptotically to  $-1$  in the future where the cosmological constant dominates. Note that in this example  $w_{DE}$  lies in the phantom regime despite the fact that the effective dark-energy sector constitutes from dust matter terms. This is typical also in other scenarios of couplings between matter and geometry mentioned in the Introduction, and reveals the capabilities of such interactions [38–52, 55–62].

For consistency purposes, we close this subsection by a brief confrontation with the Cosmic Chronometer datasets, which are based on the  $H(z)$  measurements through the relative ages of massive and passively evolving galaxies and the corresponding estimation of  $dz/dt$  [64]. In Fig. 3 we compare the  $H(z)$  evolution predicted from our scenario, with the  $H(z)$  Cosmic Chronometer data from [65] at  $3\sigma$  confidence level, while for completeness we present the  $\Lambda$ CDM curve too. As we observe the agreement is very good, and the  $H(z)$  evolution lies within the errors of the direct measurements of  $H(z)$ , exhibiting a slightly higher accelerating behavior in the past due to the phantom nature of dark energy in these examples.

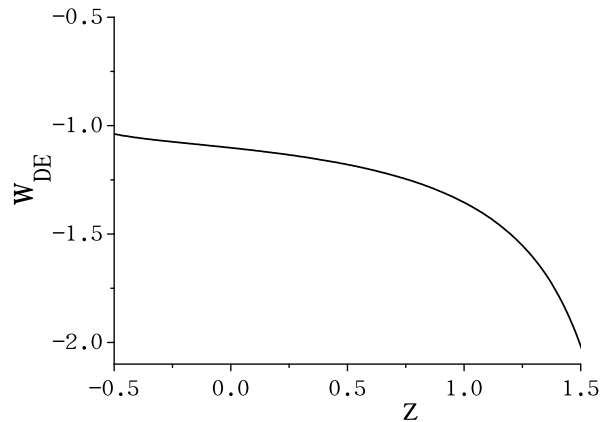


FIG. 2: The evolution of the effective dark-energy equation-of-state parameter  $w_{DE}$ , as a function of the redshift  $z$ , for the scenario of non-minimal (derivative) coupling between matter and Einstein tensor, with  $\alpha = -0.05$  and  $\beta = 0.01$ , in units where  $\kappa^2 = 1$ . We have imposed  $\Omega_{m0} \approx 0.3$  at present time.

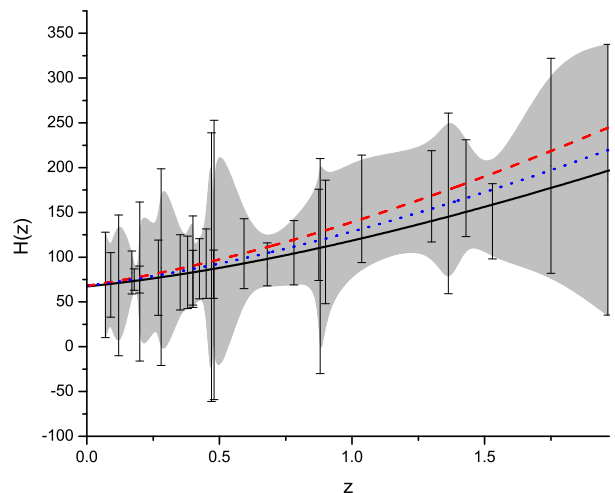


FIG. 3: The  $H(z)$  in units of  $\text{Km/s/Mpc}$  as a function of the redshift, for or the scenario of non-minimal (derivative) coupling between matter and Einstein tensor, with  $\alpha = -0.05$  and  $\beta = 0.01$  (red-dashed), and with  $\alpha = -0.01$  and  $\beta = 0.001$  (blue-dotted) in units where  $\kappa^2 = 1$ , on top of the Cosmic Chronometers data points from [65] at  $3\sigma$  confidence level. For comparison we also present the  $\Lambda$ CDM curve (black - solid). We have imposed  $\Omega_{m0} \approx 0.3$ .

## B. Cosmological perturbations

In this subsection we perform a detailed perturbation analysis of the theory at hand in the linear regime [66–72]. Concerning the scalar perturbations we start from the standard perturbed metric of isentropic perturba-

tions in the Newtonian gauge:

$$ds^2 = -(1 + 2\phi) dt^2 + a^2 \delta_{ij} (1 - 2\psi) dx^i dx^j,$$

and as usual we consider the expressions

$$\begin{aligned} \delta T^0_0 &= -\delta\rho_m \\ \delta T^i_j &= \delta p_m \delta_{ij} \\ \delta T^i_0 &= -\frac{(\rho_m + p_m)}{a} \partial_i V \\ \delta T^0_i &= a(\rho_m + p_m) \partial_i V, \end{aligned} \quad (24)$$

where  $V$  is defined through  $u_\mu = a(-\phi, \partial_i V)$ . Hence, inserting these in the general field equations (3), and using additionally the background Friedmann equations (14),(15) to eliminate terms, we finally obtain the time-time and the space-diagonal equations respectively given by

$$\begin{aligned} 6H(H\phi + \dot{\psi}) - 2\frac{\nabla^2}{a^2}\psi &= -\kappa^2 \left\{ \delta\rho_m + \alpha \left\{ 3(H^2 - 2\dot{H})\delta p_m + 6H\delta\dot{p}_m - 3(H^2 + 2\dot{H})\delta\rho_m + 6(\rho_m + p_m)\ddot{\psi} \right. \right. \\ &+ 6 \left[ H^2(\rho_m - p_m) - 2H\dot{p}_m + 2\dot{H}(\rho_m + p_m) \right] \phi + 6H(\rho_m + p_m)\dot{\phi} + 6[H(\rho_m - p_m) - \dot{p}_m]\dot{\psi} \\ &+ \frac{\nabla^2}{a^2} \left[ 4H(\rho_m + p_m)aV + 2(\rho_m + p_m)\phi - 2(\rho_m - p_m)\psi - 2\delta p_m \right] \left. \right\} \\ &+ \beta \left\{ -36H^2(\rho_m + p_m)\delta\ddot{p}_m - 18[6H^3(\rho_m + p_m) + 4H\dot{H}(\rho_m + p_m) + 3H^2(\dot{\rho}_m - 3\dot{p}_m)]\delta\dot{p}_m \right. \\ &+ 12H \left[ (2\dot{H} + 3H^2)(\dot{\rho}_m - 3\dot{p}_m) + H(\ddot{\rho}_m - 3\ddot{p}_m) \right] (\delta\rho_m + \delta p_m) \\ &+ 6H[(4\dot{H} + 6H^2)(\rho_m + p_m) + 3H(\dot{\rho}_m - 3\dot{p}_m)]\delta\dot{\rho}_m + 12H^2(\rho_m + p_m)\delta\ddot{\rho}_m \\ &+ \left\{ (144H^3 + 96H\dot{H})[3p_m\dot{p}_m + \rho_m(3\dot{p}_m - \dot{\rho}_m) - p_m\dot{\rho}_m] \right. \\ &+ 12H^2 \left[ 12p_m\ddot{p}_m - 3(3\dot{p}_m - \dot{\rho}_m)^2 + 4\rho_m(3\ddot{p}_m - \ddot{\rho}_m) - 4p_m\ddot{\rho}_m \right] \left. \right\} \phi \\ &+ 36H^2[3p_m\dot{p}_m + \rho_m(3\dot{p}_m - \dot{\rho}_m) - p_m\dot{\rho}_m]\dot{\phi} + \left\{ 4(27H^2 + 6\dot{H})[3p_m\dot{p}_m + \rho_m(3\dot{p}_m - \dot{\rho}_m) - p_m\dot{\rho}_m] \right. \\ &+ 6H \left[ 12p_m\ddot{p}_m - 3(3\dot{p}_m - \dot{\rho}_m)^2 + 4\rho_m(3\ddot{p}_m - \ddot{\rho}_m) - 4p_m\ddot{\rho}_m \right] \left. \right\} \dot{\psi} + 3H[24p_m\dot{p}_m + 8\rho_m(3\dot{p}_m - \dot{\rho}_m) - 8p_m\dot{\rho}_m]\ddot{\psi} \\ &+ 4 \left[ (2\dot{H} + 3H^2)(\rho_m + p_m) + H(\dot{\rho}_m - 3\dot{p}_m) \right] \frac{\nabla^2}{a^2} (3\delta p_m - \delta\rho_m) + 8H(\rho_m + p_m)(3\dot{p}_m - \dot{\rho}_m) \frac{\nabla^2}{a^2} \phi \\ &+ 2 \left[ (3\dot{p}_m - \dot{\rho}_m)^2 - 12p_m\ddot{p}_m + 4H(\rho_m + p_m)(\dot{\rho}_m - 3\dot{p}_m) + 4p_m\ddot{\rho}_m + 4\rho_m(\ddot{\rho}_m - 3\ddot{p}_m) \right] \frac{\nabla^2}{a^2} \psi \left. \right\}, \end{aligned} \quad (25)$$

$$\begin{aligned} 2(3H^2 + 2\dot{H})\phi + 2H(\dot{\phi} + 3\dot{\psi}) + 2\ddot{\psi} + \frac{\nabla^2}{a^2}(\phi - \psi) &= \kappa^2 \left\{ \delta p_m \right. \\ &- \alpha \left\{ 2\delta\ddot{p}_m - 4\ddot{p}_m\phi + (3H^2 + 2\dot{H})[3\delta p_m + \delta\rho_m - 2(\rho_m + 3p_m)\phi] - 2\dot{\rho}\dot{\psi} - 2\ddot{\psi}(\rho_m + 3p_m) \right. \\ &+ 2H \left[ 3\delta\dot{p}_m + \delta\dot{\rho}_m - 2(\dot{\rho}_m + 3\dot{p}_m)\phi - (\rho_m + 3p_m)(\dot{\phi} + 3\dot{\psi}) \right] - 2\dot{p}_m(\dot{\phi} + 3\dot{\psi}) \\ &+ \frac{\nabla^2}{a^2} \left\{ 2[\dot{\rho}_m + \dot{p}_m + 2H(\rho_m + p_m)]aV + 2(\rho_m + p_m)a\dot{V} - \delta p_m + \delta\rho_m + (\rho_m - p_m)(\phi - \psi) \right\} \left. \right\} \\ &- \beta \left\{ 2(2\dot{H} + 3H^2)[3\dot{p}_m(3\delta\dot{p}_m - \delta\dot{\rho}_m) - 2\phi(9\dot{p}_m^2 + \dot{\rho}_m^2) + \dot{\rho}_m(\delta\dot{\rho}_m - 3\delta\dot{p}_m + 12\dot{p}_m\phi)] \right. \\ &- 12\dot{p}_m(3\ddot{p}_m - \ddot{\rho}_m)\dot{\psi} + 4H \left\{ 3\ddot{p}_m(3\delta\dot{p}_m - \delta\dot{\rho}_m) + \ddot{\rho}_m(\delta\dot{\rho}_m - 3\delta\dot{p}_m + 12\dot{p}_m\phi) + \dot{p}_m(9\delta\ddot{p}_m - 3\delta\ddot{\rho}_m - 36\ddot{p}_m\phi) \right. \\ &- \frac{3}{2}(9\dot{p}_m^2 + \dot{\rho}_m^2)(\dot{\phi} + \dot{\psi}) + \dot{\rho}_m[\delta\ddot{\rho}_m - 3\delta\ddot{p}_m + 4(3\ddot{p}_m - \ddot{\rho}_m)\phi + 9\dot{p}_m(\dot{\phi} + \dot{\psi})] \left. \right\} \\ &- 2(9\dot{p}_m^2 + \dot{\rho}_m^2)\ddot{\psi} + 4\dot{\rho}_m[(3\ddot{p}_m - \ddot{\rho}_m)\dot{\psi} + 3\dot{p}_m\ddot{\psi}] \\ &+ 2\frac{\nabla^2}{a^2} \left\{ [H(\dot{\rho}_m - 3\dot{p}_m) + \ddot{\rho}_m - 3\ddot{p}_m](3\delta p_m - \delta\rho_m) - (3\dot{p}_m - \dot{\rho}_m)^2(\phi + \psi) \right\} \left. \right\}, \end{aligned} \quad (26)$$

while the space non-diagonal equation reads as

$$\begin{aligned} \psi - \phi = & -\kappa^2 \left\{ \alpha \left\{ \delta p_m - \delta \rho_m \right. \right. \\ & -2a [2H (\rho_m + p_m) + \dot{p}_m + \dot{\rho}_m] V \\ & \left. -2a (\rho_m + p_m) \dot{V} + (\rho_m - p_m) (\psi - \phi) \right\} \\ & + \beta \left\{ 2 [H (3\dot{p}_m - \dot{\rho}_m) + 3\ddot{p}_m - \ddot{\rho}_m] (3\delta p_m - \delta \rho_m) \right. \\ & \left. + (3\dot{p}_m - \rho_m)^2 (\phi + \psi) \right\}. \end{aligned} \quad (27)$$

Note that the last equation implies that the anisotropic stress in the scenario at hand is non-zero, and it becomes zero in the case  $\alpha = \beta = 0$ .

For completeness, we investigate the tensor perturbations, too. As usual we consider

$$ds^2 = -dt^2 + a^2 (\delta_{ij} + 2h_{ij}) dx^i dx^j,$$

where  $h_{ij}$  is transverse and traceless. Hence, we finally obtain the tensor equation

$$\begin{aligned} \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2} h_{ij} = & \kappa^2 \left\{ \alpha \left\{ (\rho_m + 3p_m) \ddot{h}_{ij} \right. \right. \\ & + [3H (\rho_m + 3p_m) + \dot{\rho}_m + 3\dot{p}_m] \dot{h}_{ij} \\ & \left. + (\rho_m - p_m) \frac{\nabla^2}{a^2} h_{ij} \right\} \\ & + \beta \left\{ (3\dot{p}_m - \rho_m)^2 \ddot{h}_{ij} + (3\dot{p}_m - \rho_m)^2 \frac{\nabla^2}{a^2} h_{ij} \right. \\ & + [3H (3\dot{p}_m - \rho_m)^2 + 6\dot{p}_m (3\ddot{p}_m - \ddot{\rho}_m) \\ & \left. + 2\dot{\rho}_m (\ddot{\rho}_m - 3\ddot{p}_m)] \dot{h}_{ij} \right\}. \end{aligned} \quad (28)$$

$$\ddot{\delta} + 2H\dot{\delta} + \frac{\kappa^2 \rho_m \delta \left\{ 6\beta \rho_m \dot{\rho}_m H^2 + (k^2 + 8\beta \rho_m \dot{H}) \dot{\rho}_m + H [4\beta \dot{\rho}_m^2 + 3\rho_m (\alpha - 2\beta \ddot{\rho}_m)] \right\}}{2k^2 \dot{\rho}_m [-1 + \kappa^2 \beta (3H \rho_m \dot{\rho}_m + \dot{\rho}_m^2 + 4\rho_m \ddot{\rho}_m)]} = 0.$$

The above equation determines the evolution of matter overdensity in the scenario at hand. We can bring it in the more convenient form [68–72]

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}} \rho_m \delta = 0,$$

$$8\pi G_{\text{eff}} = \kappa_{\text{eff}}^2 \equiv \frac{\kappa^2 \left\{ 6\beta \rho_m \dot{\rho}_m H^2 + (k^2 + 8\beta \rho_m \dot{H}) \dot{\rho}_m + H [4\beta \dot{\rho}_m^2 + 3\rho_m (\alpha - 2\beta \ddot{\rho}_m)] \right\}}{k^2 \dot{\rho}_m [1 - \kappa^2 \beta (3H \rho_m \dot{\rho}_m + \dot{\rho}_m^2 + 4\rho_m \ddot{\rho}_m)]}.$$

As expected, when the nonminimal derivative coupling

Let us now focus on the scalar perturbations, and introduce as usual the important quantity  $\delta \equiv \frac{\delta \rho_m}{\rho_m}$ , namely the matter overdensity. Inserting  $\delta$  into (25)–(27), considering dust matter, and transforming as usual to the Fourier space, with  $k$  the wavenumber, we extract the evolution equation for  $\delta$ , which can be solved numerically. However, in order to further simplify the expressions, we assume small deviations from  $\Lambda$ CDM cosmology, namely  $\kappa^2(\rho_\alpha + \rho_\beta) \ll \Lambda$ , which implies small values for  $\alpha$  and  $\beta$  ( $\alpha, \beta \ll (\kappa^2 H^2)^{-1}$ ), and moreover we focus on sub-horizon scales, i.e. with  $k \gg aH$ , and for the sound speed we consider  $c_{\text{eff}}^{(m)2} \equiv \delta p_m / \delta \rho_m \ll 1$ . Hence, we result in the following equation:

by defining an effective gravitational constant

switches off, i.e. in the case  $\alpha = \beta = 0$ , we obtain  $\kappa_{\text{eff}}^2 =$

$\kappa^2$  and thus  $G_{\text{eff}} = G$  and hence we recover the results of  $\Lambda$ CDM cosmology.

In order to provide a more transparent picture of the behavior of perturbations and the features of the large scale structure, in Fig. 4 we depict the evolution of  $\delta$  as a function of the redshift. As we observe, the theories with nonminimal (derivative) coupling between matter and the Einstein tensor can describe the evolution of the large scale structure in agreement with observations. Furthermore, this evolution is sensitive to the coupling parameters, hence one can use  $f\sigma_8$  data in order to extract constraints on them and break possible degeneracies that may appear at the background level.

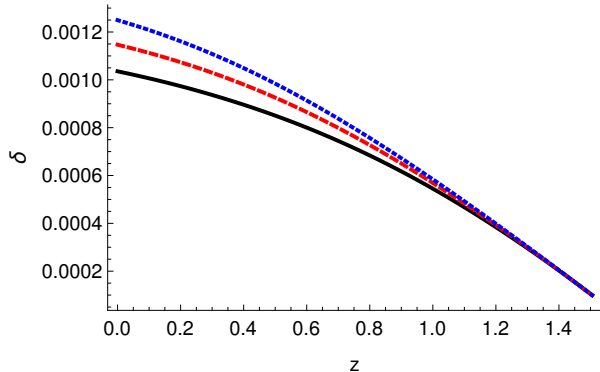


FIG. 4: The evolution of the matter overdensity  $\delta \equiv \frac{\delta\rho_m}{\rho_m}$ , as a function of the redshift  $z$ , for the scenario of non-minimal (derivative) coupling between matter and Einstein tensor, with  $\alpha = -0.1$ ,  $\beta = 0.1$  (black-solid),  $\alpha = -0.2$ ,  $\beta = 0.2$  (red-dashed), and  $\alpha = -0.5$ ,  $\beta = 0.4$  (blue-dotted), in units where  $\kappa^2 = 1$ , at scale  $k = 10^{-3} h\text{Mpc}^{-1}$ . We have imposed  $\Omega_{m0} \approx 0.3$  at present time.

#### IV. CONCLUSIONS

In this work, inspired by the coupling of scalar fields to the Einstein tensor, as well as by the coupling of the matter sector to the Ricci scalar, we constructed new classes of modified theories in which the matter sector couples with the Einstein tensor. In particular, we considered a direct coupling of the energy-momentum tensor to the Einstein tensor, and a coupling of the Einstein tensor to the derivatives of the trace of the energy-momentum tensor.

Firstly, we extracted the general field equations, which comparing to General Relativity include corrections depending on the two coupling parameters of the theory. Then we proceeded to the cosmological application around a flat FRW background, and at the background level we extracted the Friedmann equations from the general field equations, whose extra terms can be absorbed in an effective dark energy sector.

Assuming the matter sector to be dust we elaborated the equations numerically, and we saw that the scenario

at hand can successfully describe the usual thermal history of the universe, with the sequence of matter and dark-energy epochs. Additionally, we examined the dark-energy equation-of-state parameter and we showed that it can lie in the phantom regime, tending progressively to  $-1$  at present and future times. It is interesting to mention that this behavior is obtained although the effective dark-energy sector constitutes from matter terms, nevertheless it is not uncommon in theories with couplings between matter and geometry. Finally, for completeness we confronted the theory with Cosmic Chronometer data, showing that the agreement is very good, and that the predicted  $H(z)$  evolution lies within the errors of the direct measurements of  $H(z)$ , exhibiting a slightly higher accelerating behavior in the past due to the phantom nature of dark energy.

We proceeded to the detailed investigation of the perturbations, both scalar and tensor ones. Focusing on scalar perturbations, we extracted the evolution equation of the matter overdensity, which is a crucial observable since it quantifies the matter clustering and the large scale structure. Elaborating the equation numerically, we saw that the predicted evolution of the matter overdensity is in agreement with observations.

It would be both interesting and necessary to perform a full observational confrontation with joined datasets from Cosmic Microwave Background (CMB), Baryonic Acoustic Oscillations (BAO), Supernovae Type Ia (SNIa), and Redshift space distortion (RSD)  $f\sigma_8$  observations, in order to extract constraints on the new coupling parameters. Additionally, one could perform a detailed phase-space analysis in order to investigate the asymptotic behavior of the scenario, independently of the initial conditions and the specific evolution of the Universe. Moreover, it would be interesting to examine the possible alleviation of the  $H_0$  and  $S_8$  tensions [73] in these theories (the fact that we obtain effective phantom behavior is a promising feature). Finally, apart from the linear perturbation analysis, whose equations have been presented in this work, we should investigate the the non-linear perturbations too, since they play a crucial role when gravitational instability grows enough to understand the formation of Large Scale Structure. Such studies lie beyond the scope of the present work and are left for future projects.

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