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MODIFIED GRAVITY IN CONTEMPORARY UNIVERSE

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Astronomical data in favor of cosmological acceleration and possible explanations of accelerated expansion of the Universe are discussed. Main attention is paid to gravity modifications at small curvature which could induce accelerated cosmological expansion. It is shown that gravitating systems with mass density rising with time evolve to a singular state with infinite curvature scalar. The Universe evolution during the radiation-dominated epoch is studied in the R^2-extended gravity theory. Particle production rate by the oscillating curvature and the back reaction of particle production on the evolution of R are calculated in the one-loop approximation. Possible implications of the model for cosmological creation of nonthermal dark matter are discussed.

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A large set of independent, different-types astronomical data strongly indicate that our Universe today expands with acceleration. The data include the observation of the large scale structure of the Universe, the measurements of the angular fluctuations of the cosmic microwave background radiation, the determination of the Universe age (for a review, see [1]), and especially the discovery of the dimming of distant Supernovae [2]. It was established and unambiguously proved that the Universe expansion is accelerated, but the driving force behind this accelerated expansion is still unknown.

Among possible explanations, very popular is the assumption of a new (unknown) form of cosmological energy density with large negative pressure, $P < -\varrho/3$, the so-called dark energy (for a review, see, e.g., [3]). The latter can be either a small vacuum energy, which is identical to cosmological constant, or the energy density associated with an unknown, pressumably scalar field, which slowly varies in the course of the cosmological evolution. The problem of vacuum energy and possible ways to its solution are described in [4].

Soon after discovery of the accelerated expansion, theories with competing mechanism for producing cosmological acceleration have been proposed. These theories are based on the gravity modifications at large scales by introducing an additional term, $F(R)$, into the usual action of General Relativity (GR):

$$S = \frac{m_{\rm Pl}^2}{16\pi} \int d^4x \sqrt{-g}[R + F(R)] + S_m, \quad (1)$$

where $m_{\rm Pl} = 1.22 \cdot 10^{19}$ GeV is the Planck mass, and S_m is the matter action.

The usual Einstein–Hilbert action is linear in the curvature scalar R. This is the reason why the GR equations contain, as it is usual in other field theories, only second derivatives of metric despite the fact that the action also contains second derivatives. If the action differs from a simple linear GR form, the equation of motion would be higher than the second-order one. Such equations should contain some pathological features as an existence of tachyonic solutions or ghosts. However, the theories whose action depends only on a function of the curvature scalar, $F(R)$, are free of such pathologies because, as is known, they are equivalent to an addition of a scalar degree of freedom to the usual GR with the scalar field satisfying normal second-order field equation. That is why, modifications of gravity at large distances are mostly confined to $F(R)$ theories.

The pioneering work in this direction was done in [5], which was closely followed by [6]. In these works the singular in R action

$$F(R) = -\frac{\mu^4}{R} \quad (2)$$

has been explored with constant parameter μ chosen as $\mu^2 \sim R_c \sim 1/t_U^2$ to describe the observed cosmological acceleration.

The corresponding equation of motion reads

$$\left(\frac{1}{\mu^4} + \frac{1}{R^2}\right) R_{\alpha\beta} - \frac{R}{2}\left(\frac{1}{\mu^4} - \frac{1}{R^2}\right) g_{\alpha\beta} - D_{(\alpha}D_{\beta)}\left(\frac{1}{R^2}\right) + \\ + g_{\alpha\beta} D_\nu D^\nu \left(\frac{1}{R^2}\right) = \frac{8\pi T_{\alpha\beta}}{m_{\rm Pl}^2 \mu^4}. \quad (3)$$

Taking trace over α and β of this equation we obtain

$$D^2 R - 3\frac{(D_\alpha R)(D^\alpha R)}{R} = \frac{R^2}{2} - \frac{R^4}{6\mu^4} - \frac{T R^3}{6\mu^4}. \quad (4)$$

Here $T = 8\pi T_\nu^\nu / m_{\rm Pl}^2 > 0$. This equation has an evident solution in the absence of matter $R^2 = 3\mu^4$ which describes the accelerated de Sitter universe with a constant curvature scalar.

So far, so good but the small coefficient, μ^4, in front of the highest derivative or, what is the same, the large coefficient, $1/\mu^4$, in front of the nonderivative terms in the presented above form of the equation leads to a strong instability in the presence of matter [7] with the characteristic time:

$$\tau = \frac{\sqrt{6}\mu^2}{T^{3/2}} \sim 10^{-26}\ \mathrm{s} \left(\frac{\varrho_m}{\mathrm{g/cm^3}}\right)^{-3/2}, \tag{5}$$

where ϱ_m is the mass density of the body, and $\mu^{-1} \sim t_u \approx 3 \cdot 10^{17}$ s.

To avoid the problem of such instability, further modification of the modified gravity has been suggested. We will consider here some class of the models discussed in [8]. Some other forms of gravity modification are reviewed in [9]. The different actions suggested in works [8] have the form

$$F_{\mathrm{HS}}(R) = -\frac{R_{\mathrm{vac}}}{2} \frac{c(R/R_{\mathrm{vac}})^{2n}}{1 + c(R/R_{\mathrm{vac}})^{2n}}, \tag{6}$$

$$F_{\mathrm{AB}}(R) = \frac{\epsilon}{2} \log\left[\frac{\cosh\left(R/\epsilon - b\right)}{\cosh b}\right] - \frac{R}{2}, \tag{7}$$

$$F_S(R) = \lambda R_0 \left[\left(1 + \frac{R^2}{R_0^2}\right)^{-n} - 1\right]. \tag{8}$$

Despite different forms, these actions result in quite similar consequences. Below we essentially follow the analysis made in [10].

Introducing notation $f(R) = R + F(R)$, we can write the field equations as follows:

$$f'(R) R_\mu^\nu - \frac{f(R)}{2}\delta_\mu^\nu + (\delta_\mu^\nu D^2 - D_\mu D^\nu) f'(R) = \frac{8\pi T_\mu^\nu}{m_{\mathrm{Pl}}^2}. \tag{9}$$

Correspondingly, their trace is

$$3D^2 f'(R) + R f'(R) - 2 f(R) = T, \tag{10}$$

where prime denotes the ordinary derivative with respect to R.

The condition of accelerated expansion in the absence of matter is the existence of real positive root, $R = R_1 > 0$, of the equation

$$R f'(R) - 2 f(R) = 0, \tag{11}$$

where R_1 is (approximately) constant.

The following necessary conditions to avoid pathologies are to be satisfied:

1) future stability of cosmological solutions:

$$\frac{f'(R_1)}{f''(R_1) > R_1}; \tag{12}$$

2) classical and quantum stability (gravitational attraction and absence of ghosts):

$$f'(R) > 0; \tag{13}$$

3) absence of matter [7] instability:

$$f''(R) > 0; \tag{14}$$

4) existence of the stable Newtonian limit:

$$|f(R) - R| \ll R, \quad |f'(R) - 1| \ll 1, \quad Rf''(R) \ll 1. \tag{15}$$

Note that the effective scalaron mass squared is $M^2(R) = 1/(3f''(R))$, and the third condition (14) means that the scalaron is not a tachyon.

Despite considerable improvement, the models proposed in [8] possess another trouble — some feature, namely, in a cosmological situation they should evolve from a singular state with an infinite R in the past [11]. In other words, if we travel backward in time from a normal cosmological state, we come to a singular state with infinite curvature, while the energy density remains finite.

In cosmology, energy density drops down with time, and singularity does not appear in the future. However, systems with rising mass/energy density will evolve to a singularity, $R \to \infty$, in a finite time [12, 13]. Such future singularity is unavoidable, regardless of the initial conditions, and infinite value of R would be reached in time which is much shorter than the cosmological one.

Following [13], let us consider version (8) of $F(R)$ function in the case of large R. We analyze the evolution of R in massive objects with time varying mass density, $\varrho_m \gg \varrho_c$. The cosmological energy density at the present time is $\varrho_c \approx 10^{-29}$ g/cm^3, while matter density of, say, a dust cloud in a galaxy could be about $\varrho_m \sim 10^{-24}$ g/cm^3. Since the magnitude of the curvature scalar is proportional to the mass density of a nonrelativistic system, we find $R \gg R_0$. In this limit we can approximately take

$$F(R) \approx -\lambda R_0 \left[1 - \left(\frac{R_0}{R}\right)^{2n}\right]. \tag{16}$$

Gravitational field of such an object is supposed to be weak, so the background metric is approximately flat, and covariant derivatives can be replaced by the flat ones. Hence equation (10) takes the form

$$(\partial_t^2 - \Delta)R - (2n+2)\frac{\dot{R}^2 - (\nabla R)^2}{R} + \frac{R^2}{3n(2n+1)}\left[\frac{R^{2n}}{R_0^{2n}} - (n+1)\right] - \\ - \frac{R^{2n+2}}{6n(2n+1)\lambda R_0^{2n+1}}(R+T) = 0. \tag{17}$$

The equation is very much simplified if we choose another unknown function $w \equiv F' = -2n\lambda\,(R_0/R)^{2n+1}$ which satisfies

$$(\partial_t^2 - \Delta)w + U'(w) = 0. \qquad (18)$$

Here potential $U(w)$ is equal to

$$U(w) = \frac{1}{3}\,(T - 2\lambda R_0)\,w + \frac{R_0}{3}\left[\frac{q^\nu}{2n\nu} w^{2n\nu} + \left(q^\nu + \frac{2\lambda}{q^{2n\nu}}\right)\frac{w^{1+2n\nu}}{1+2n\nu}\right], \qquad (19)$$

where $\nu = 1/(2n+1)$ and $q = 2n\lambda$.

Notice that the infinite R singularity corresponds to $w = 0$.

If only the dominant terms are retained and if the space derivatives are neglected, Eq. (18) simplifies to

$$\ddot{w} + \frac{T}{3} - \frac{q^\nu(-R_0)}{3w^\nu} = 0. \qquad (20)$$

Potential U would depend upon time if the mass density of the object changes with time. We parameterize it as

$$T = T(t) = T_0(1 + \kappa\tau), \qquad (21)$$

where τ is dimensionless time introduced below.

With dimensionless quantities $t = \gamma\tau$ and $w = \beta z$, where

$$\gamma^2 = \frac{3q}{(-R_0)}\left(-\frac{R_0}{T_0}\right)^{2(n+1)}, \quad \beta = \gamma^2 T_0/3 = q\left(-\frac{R_0}{T_0}\right)^{2n+1}, \qquad (22)$$

the equation further simplifies:

$$z'' - z^{-\nu} + (1 + \kappa\tau) = 0. \qquad (23)$$

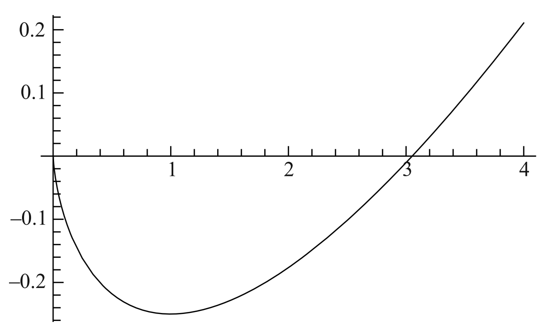

Fig. 1. Potential $U(z) = z(1+\kappa\tau) - z^{1-\nu}/(1-\nu),\ \nu = \frac{1}{5}, \tau = 0$

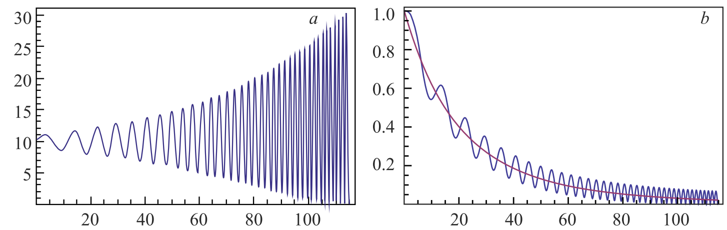

Fig. 2. Ratio $z(\tau)/z_{\min}(\tau)$ (a) and functions $z(\tau)$ and $z_{\min}(\tau)$ (b) for $n = 2$, $\kappa = 0.01$, $\varrho_m/\varrho_c = 10^5$. The initial conditions are $z(0) = 1$ and $z'(0) = 0$

Minimum of potential $U(z)$ (Fig. 1) sits at $z_{\min} = (1 + \kappa\tau)^{-1/\nu}$. When the mass density rises, the minimum moves towards zero and becomes more and more shallow. If at the process of «lifting» of the potential, $z(\tau)$ happens to be at $U > 0$, it would overjump the potential which is equal to zero at $z = 0$. In other words, $z(\tau)$ would reach zero, which corresponds to infinite R, and so the singularity can be reached in finite time (Fig. 2).

The aforementioned problems can be cured by adding to the action quadratic in curvature term $R^2/(6m^2)$ [14], which prevents from the singular behavior both in the past and in the future.

In the homogeneous case and in the limit of large ratio R/R_0, the addition of R^2 term leads to the following modification of the equation of motion:

$$\left[1 - \frac{R^{2n+2}}{6\lambda n(2n+1)R_0^{2n+1}m^2}\right]\ddot{R} - (2n+2)\frac{\dot{R}^2}{R} - \frac{R^{2n+2}(R+T)}{6\lambda n(2n+1)R_0^{2n+1}} = 0. \quad (24)$$

With dimensionless curvature and time

$$y = -\frac{R}{T_0}, \quad \tau_1 = t\left[-\frac{T_0^{2n+2}}{6\lambda n(2n+1)R_0^{2n+1}}\right]^{1/2} \quad (25)$$

the equation for R is transformed into

$$\left(1 + gy^{2n+2}\right)y'' - 2(n+1)\frac{(y')^2}{y} + y^{2n+2}\left[y - (1 + \kappa_1\tau_1)\right] = 0, \quad (26)$$

where prime now means derivative with respect to τ_1.

We introduced here the new parameter, g, which can prevent from the approach to infinity and is equal to

$$g = -\frac{T_0^{2n+2}}{6\lambda n(2n+1)m^2R_0^{2n+1}} > 0. \quad (27)$$

For very large m, or small g, when the second term in the coefficient of the second derivatives in Eqs. (24) and (26) can be neglected, numerical solution

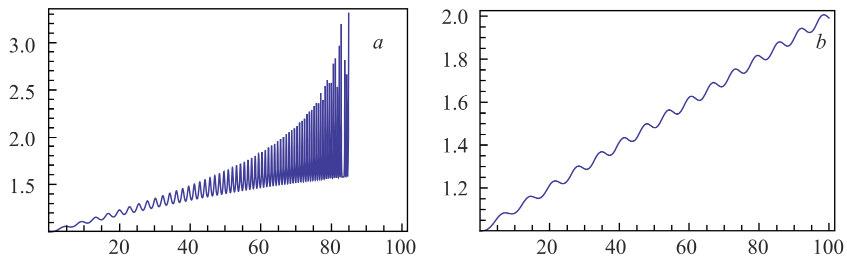

Fig. 3. Numerical solutions of Eq. (26) for $n = 3$, $\kappa_1 = 0.01$, $y(\tau_{\rm in}) = 1 + \kappa_1\tau_{\rm in}$, $y'(\tau_{in}) = 0$. *a*) $g = 0$; *b*) $g = 1$

demonstrates that R would reach infinity in finite time in accordance with the results presented above (see Fig. 3, *a*). Nonzero g would terminate the unbounded rise of R. To avoid too large deviation of R from the usual gravity, coefficient g should be larger than or of the order of unity. In Fig. 3, *b*, it is clearly seen, that for $g = 1$ the amplitude of oscillations remains constant whereas the average value of R increases with time.

As follows from Eq. (26), the frequency of small oscillations of y around $y_0 = 1 + \kappa_1\tau_1$ in dimensionless time τ_1 is

$$\omega_\tau^2 = \frac{1}{g}\frac{g y_0^{2n+2}}{1 + g y_0^{2n+2}} \leqslant \frac{1}{g}. \tag{28}$$

It means that in physical time the frequency would be

$$\omega \sim \frac{1}{t_U}\left(\frac{T_0}{R_0}\right)^{n+1}\frac{y_0^{n+1}}{\sqrt{1 + g y_0^{2n+2}}} \leqslant m. \tag{29}$$

In particular, for $n = 5$ and for a galactic gas cloud with $T_0/R_0 = 10^5$, the oscillation frequency would be 10^{12} Hz $\approx 10^{-3}$ eV. Higher density objects, e.g., those with $\rho = 1$ g/cm^3, would oscillate with much higher frequency, saturating bound (29), i.e., $\omega \sim m$. All kinds of particles with masses smaller than m might be created by such an oscillating field.

As more detailed analysis shows (our work in progress with L. Reverberi), the pattern of oscillations in the presence of matter is more complicated. They very much differ from the harmonic ones and consist of high narrow spikes with characteristic frequency of the order of m, separated by wide low amplitude periods.

An important effect which is not taken into account in Eq. (24) and which also inhibits unbounded rise of R is the particle production by oscillating curvature R. The technique for calculations of particle production applicable to the case of modified gravity (6)–(8) was worked out in [15] for the case of R^2 gravity in cosmological situation (in the early Universe), where the classical results [16] for particle production were reproduced.

As was done in [15], let us consider the cosmological evolution of the Universe in a theory with only an additional R^2 term in the action, neglecting other terms which have been introduced to generate the accelerated expansion in the contemporary Universe. The impact of such terms is negligible in the limit of sufficiently large curvature, $|R| \gg |R_0|$, where R_0 is the cosmological curvature at the present time.

In other words, we study below the cosmological evolution of the early and not so early Universe in the model with the following action:

$$S = -\frac{m_{\rm Pl}^2}{16\pi} \int d^4x \sqrt{-g} \left(R - \frac{R^2}{6m^2} \right) + S_m, \tag{30}$$

with the account of the back reaction of particle production.

The modified Einstein equations for this theory read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{3m^2} \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} + g_{\mu\nu} \mathcal{D}^2 - \mathcal{D}_\mu \mathcal{D}_\nu \right) R = \frac{8\pi}{m_{\rm Pl}^2} T_\mu \nu, \tag{31}$$

where $\mathcal{D}^2 \equiv g^\mu \nu \mathcal{D}_\mu \mathcal{D}_\nu$ is the covariant D'Alembert operator.

The curvature scalar R is expressed through the Hubble parameter $H = \dot{a}/a$ as

$$R = -6\dot{H} - 12H^2. \tag{32}$$

Therefore, the time–time component of Eq. (31) reads

$$\ddot{H} + 3H\dot{H} - \frac{\dot{H}^2}{2H} + \frac{m^2 H}{2} = \frac{4\pi m^2}{3m_{\rm Pl}^2 H} \varrho, \tag{33}$$

where overdots denote derivative with respect to physical time t.

Taking trace of Eq. (31) yields

$$\ddot{R} + 3H\dot{R} + m^2(R + T) = 0. \tag{34}$$

This equation is a sort of Klein–Gordon equation for a homogeneous scalar field, the «scalaron», of mass m, with a source term proportional to the trace of the energy–momentum tensor of matter. The General Relativity case may be recovered when $m \to \infty$. In this limit we expect to obtain the usual algebraic relation between the curvature scalar and the trace of the energy–momentum tensor of matter:

$$m_{\rm Pl}^2 R_{\rm GR} = -8\pi T_\mu^\mu. \tag{35}$$

However, unlike the usual GR, in higher-order theories, curvature and matter are related to each other through a differential equation, but not simply algebraically. Therefore, the theory may approach GR as $m \to \infty$ in a nontrivial way or even not approach it at all.

For a perfect fluid with relativistic equation of state, $P = \varrho/3$, the trace of the energy–momentum tensor of matter T^μ_μ vanishes, and R satisfies the homogeneous equation. The GR solution $R = 0$ satisfies this equation, but if one assumes that neither R nor $\dot{R}$ vanish initially, the general solution for R will be an oscillating function with a decreasing amplitude. The decrease of the amplitude is induced by the cosmological expansion (the second term in Eq. (34)) and by particle production by the oscillating gravitational field $R(t)$. The latter is not included in this equation and will be taken into account below.

In what follows, we study the cosmological evolution in the R^2 theory assuming rather general initial conditions for R and H and dominance of relativistic matter which is red-shifted according to

$$\dot{\varrho}_R + 4H\varrho_R = 0. \tag{36}$$

We will use either the set of Eqs. (33) and (36) or the set (32) and (34) as the basic equations. They are, of course, equivalent, but their numerical treatment may be somewhat different.

There is a possibility of gravitational particle production, which may nontrivially affect the solutions of the above equations. In the first approximation, however, we neglect such contributions, which will be dealt with later on in the final part of this paper.

It is convenient to rewrite the equations in terms of the dimensionless quantities $\tau = H_0 t$, $h = H/H_0$, $r = R/H_0^2$, $y = 8\pi\varrho/(3m_{\rm Pl}^2 H_0^2)$, and $\omega = m/H_0$, where H_0 is the value of the Hubble parameter at some initial time t_0. Thus, the following two equivalent systems of equations are obtained:

$$\begin{cases} h'' + 3hh' - \dfrac{h'^2}{2h} + \dfrac{\omega^2}{2}\dfrac{h^2 - y}{h} = 0, \\ y' + 4hy = 0, \end{cases} \tag{37}$$

and

$$\begin{cases} r'' + 3hr' + \omega^2 r = 0, \\ r + 6h' + 12h^2 = 0. \end{cases} \tag{38}$$

Here prime indicates derivative with respect to dimensionless time τ.

First, we assume that the deviations from GR are small and expand $h = 1/(2\tau) + h_1$ and $y = 1/(4\tau^2) + y_1$, assuming that $h_1/h \ll 1$ and $y_1/y \ll 1$, and linearize the system of equations.

The complete asymptotic solution for h has the form

$$h(\tau) \simeq \frac{1}{2\tau} + \frac{c_1 \sin(\omega\tau + \varphi)}{\tau^{3/4}}, \tag{39}$$

and describes oscillations of the Hubble parameter around the GR value $1/(2\tau)$. Moreover, the amplitude of such oscillations decreases more slowly than $1/\tau$, so for sufficiently large τ, the second term would start to dominate, the oscillations will become large, and the condition $h_1 \ll h$ will no longer be satisfied. After this stage is reached, the linear approximation becomes invalid.

However, we can proceed further using a sort of truncated Fourier expansion which allows one to take into account the nonlinearity of the system in the limit $\omega\tau \gg 1$. As a result, we have found that $h_1/h \to \mathrm{const}$. In other words, the amplitude of the oscillating part of h asymptotically behaves as $1/\tau$, i.e., in the same way as the slowly varying part of h, but the oscillation center is shifted above the GR value $1/2$.

Numerical solutions of Eqs. (37) with the initial conditions $h_0 = 1 + \delta h_0$, $h_0' = -2$, $y_0 = 1$ presented in Figs. 4, 5 demonstrate good agreement with the analytical results. In the linear regime (Fig. 4) function $h\tau$ oscillates around the

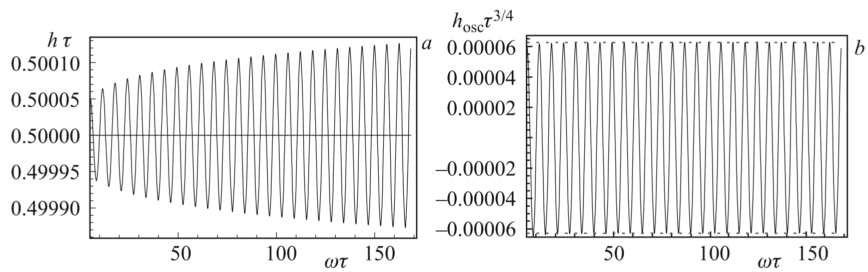

Fig. 4. Small deviations from GR: $\delta h_0 = 10^{-4}$, $\omega = 10$. $h\tau$ rises, $h\tau^{3/4}$ remains constant

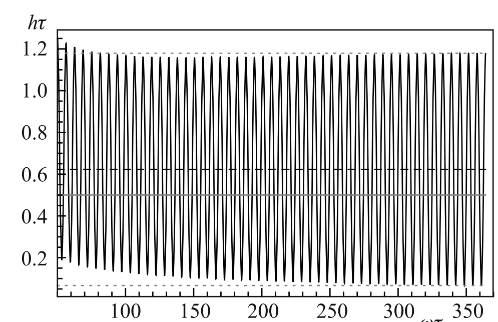

Fig. 5. Nonlinear regime: $\delta h_0 = 1.5$, $\omega = 100$. The central value of oscillations of Hubble parameter is about 0.6 and is shifted from the GR value 0.5

central value $h = 1/2$ with amplitude $h_1 \sim \tau^{-3/4}$. As the deviation from the ideal GR behavior increases, the amplitudes of the oscillating terms of both h and r decrease faster than $\tau^{-3/4}$ (linear regime), and rather close to τ^{-1}. Furthermore, the Hubble parameter does not oscillate around the GR value $h\tau = 1/2$, but around a larger value (Fig. 5).

However, oscillations are damped due to particle production, which can be evaluated as follows. We start from a massless scalar field ϕ minimally coupled to gravity:

$$S_\phi = \frac{1}{2} \int d^4x \sqrt{-g}\, g^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi. \tag{40}$$

In spatially-flat Friedmann–Robertson–Walker background it leads to the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\Delta\phi = 0. \tag{41}$$

Field ϕ enters the equation of motion for R (34) via the trace of its energy–momentum tensor:

$$T^\mu_\mu(\phi) = -g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi \equiv -(\partial\phi)^2.$$

In terms of the conformally rescaled field, $\chi \equiv a(t)\phi$, and conformal time η, such that $a\, d\eta = dt$, we can rewrite the equations of motion as

$$R'' + 2\frac{a'}{a}R' + m^2a^2R = 8\pi\frac{m^2}{m_{\rm Pl}^2}\frac{1}{a^2}\left[\chi'^2 - (\boldsymbol{\nabla}\chi)^2 + \frac{a'^2}{a^2}\chi^2 - \frac{a'}{a}(\chi\chi' + \chi'\chi)\right], \tag{42}$$

$$R = -6a''/a^3, \tag{43}$$

$$\chi'' - \Delta\chi + \frac{1}{6}a^2R\chi = 0, \tag{44}$$

while action (40) takes the form

$$S_\chi = \frac{1}{2}\int d\eta\, d^3x \left(\chi'^2 - (\boldsymbol{\nabla}\chi)^2 - \frac{a^2R}{6}\chi^2\right). \tag{45}$$

Here and above, prime denotes derivative with respect to conformal time.

We derive a closed equation for R taking the average value of the χ-dependent quantum operators in the r.h.s. of Eq. (42) over vacuum in the presence of an external classical gravitational field R following the procedure described in [17], where similar equation was obtained in the one-loop approximation.

Equation (44) has the formal solution

$$\chi(x) = \chi^{(0)}(x) - \frac{1}{6}\int d^4y\, G(x,y)\, a^2(y)\, R(y)\, \chi(y) \equiv \chi^{(0)}(x) + \delta\chi(x), \tag{46}$$

where the massless Green function is

$$G(x,y) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\delta\left((x_0-y_0)-|\mathbf{x}-\mathbf{y}|\right) \equiv \frac{1}{4\pi r}\delta(\Delta\eta - r). \tag{47}$$

We assume that the particle production effects slightly perturb the free solution, so $\delta\chi$ can be considered small and the Dyson-like series can be truncated at first order, yielding

$$\chi(x) \simeq \chi^{(0)}(x) - \frac{1}{6}\int d^4y\, G(x,y)\, a^2(y)\, R(y)\, \chi^{(0)}(y) \equiv \chi^{(0)}(x) + \chi^{(1)}(x). \tag{48}$$

We calculate the vacuum expectation values of the various terms in the right-hand side of Eq. (42), keeping only first-order terms in $\chi^{(1)}$. All terms containing only $\chi^{(0)}$ and its derivatives are not related to the particle production and can be reabsorbed by the renormalization procedure into parameters of the theory.

The dominant contribution of particle production is therefore given by the right-hand side of the integro-differential equation:

$$\ddot{R} + 3H\dot{R} + m^2 R \simeq -\frac{1}{12\pi}\frac{m^2}{m_{\rm Pl}^2}\frac{1}{a^4}\int_{\eta_0}^{\eta} d\eta' \frac{(a^2(\eta')R(\eta'))''}{\eta-\eta'} \simeq$$

$$\simeq -\frac{1}{12\pi}\frac{m^2}{m_{\rm Pl}^2}\int_{t_0}^{t} dt' \frac{\ddot{R}(t')}{t-t'}. \tag{49}$$

This equation is naturally nonlocal in time since the impact of particle production depends upon all the history of the evolution of the system.

Using again the procedure of truncated Fourier expansion including the back-reaction effects in the form of Eq. (49), we obtaine the decay rate

$$\Gamma_R = \frac{m^3}{48 m_{\rm Pl}^2}. \tag{50}$$

This result is in agreement with [16]. Correspondingly, the oscillating part of R or H behaves as

$$\cos m_1 t \to {\rm e}^{-\Gamma_R t}\, \cos m_1 t, \tag{51}$$

where m_1 is equal to m plus radiative corrections.

We use this result in the calculation of the energy density influx of the produced particles into the primeval plasma.

From Eq. (44) follows that the amplitude of gravitational production of two identical χ particles with momenta p_1 and p_2 in the first order in perturbation theory is given by

$$A(p_1,p_2) \simeq \int d\eta\, d^3x \frac{a^2 R}{6}\langle p_1, p_2|\chi\chi|0\rangle, \tag{52}$$

where

$$\langle p_1, p_2|\chi\chi|0\rangle = \sqrt{2}\, \mathrm{e}^{i(E_{p_1}+E_{p_2})\eta - i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{x}}. \tag{53}$$

Since the produced particles are massless and on-mass-shell, $E_k^2 = \mathbf{k}^2$, and function a^2R has the form

$$a^2(\eta)R(\eta) = D(\eta)\sin(\tilde{\omega}\eta),$$

where $D(\eta)$ is a slowly varying function of (conformal) time; $\tilde{\omega}$ is the frequency conjugated to conformal time.

Taking $E_{p_i} \geqslant 0$ and neglecting at this stage the variation of D with time, we obtain

$$A(p_1, p_2) \simeq -\frac{i}{6\sqrt{2}}\, D(\eta)(2\pi)^4\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2)\,\delta(E_{p_1} + E_{p_2} - \tilde{\omega}). \tag{54}$$

The particle production rate per unit comoving volume and unit conformal time is therefore given by

$$n' = \int \frac{d^2p_1\, d^3p_2}{(2\pi)^6 4E_{p_1}E_{p_2}} \frac{|A(p_1, p_2)|^2}{V\Delta\eta} \simeq \frac{D^2(\eta)}{576\pi}, \tag{55}$$

where V is the total volume; $\Delta\eta$ is the interval of conformal time when production takes place; n is the number density of the produced particles, and prime denotes derivative with respect to conformal time.

The rate of gravitational energy transformation into a certain species of elementary particles in conformal variables is

$$\varrho' = \frac{n'\tilde{\omega}}{2} = \frac{D^2(\eta)\tilde{\omega}}{1152\pi}, \tag{56}$$

and so the rate of variation of the physical energy density of the produced χ particles is

$$\dot{\varrho}_\chi = \frac{m\langle R^2\rangle}{1152\pi}. \tag{57}$$

Here $\langle R^2\rangle$ is the square of the amplitude of the oscillations of R and we substituted $\tilde{\omega} = am$.

The total rate of the gravitational energy transformation into elementary particles is obtained by multiplying the above result by the number of the produced particle species, N_{eff}:

$$\dot{\varrho}_{\mathrm{PP}} = N_{\mathrm{eff}}\,\dot{\varrho}_\chi. \tag{58}$$

Now we can calculate the evolution of the cosmological energy density of matter, which is determined by the equation

$$\dot{\varrho} = -4H\varrho + \dot{\varrho}_{\mathrm{PP}}. \tag{59}$$

We assumed here that the produced matter is relativistic, and so the first term in the r.h.s. describes the usual cosmological red shift, while the second term is the particle source from the oscillations of R. Since ϱ is not oscillating but a smoothly varying function of time, its red shift is predominantly determined by the nonoscillating part of the Hubble parameter, $H_c \simeq \alpha/2t$.

Parameterizing the oscillating part of the curvature as

$$R \simeq -\frac{6\beta m \sin mt}{t}\,\mathrm{e}^{-\Gamma_R t}, \tag{60}$$

we find that the energy density of matter obeys the equation

$$\dot{\varrho} = -\frac{2\alpha}{t}\varrho + \frac{\beta^2 m^3 N_{\mathrm{eff}}}{32\pi t^2}\mathrm{e}^{-2\Gamma_R t}. \tag{61}$$

The characteristic decay time of the oscillating curvature is

$$\tau_R = \frac{1}{2\Gamma_R} = \frac{24 m_{\mathrm{Pl}}^2}{m^3} \simeq 2\left(\frac{10^5\ \mathrm{GeV}}{m}\right)^3 \mathrm{s}. \tag{62}$$

The contribution of the produced particles into the total cosmological energy density reaches its maximum value at approximately this time. The ratio of the energy density of the newly produced energetic particles and that of those already existing in the plasma is given by the expression

$$\frac{\varrho_{\mathrm{hi}}}{\varrho_{\mathrm{therm}}} = \frac{8\beta^2 N_{\mathrm{eff}}}{\kappa(2\alpha_1 - 1)}\,\frac{1-(2\Gamma_R t_{\mathrm{in}})^{2\alpha_1 - 1}}{(2\Gamma_R t_{\mathrm{in}})^{2\alpha_1 - 2}}. \tag{63}$$

Parameter κ is arbitrary, and depends upon the thermal history of the Universe before t_{in}. In particular, $\kappa = 0$ is possible and does not contradict our picture, since the equations of motion have nontrivial oscillating solutions even if $\varrho = 0$.

Depending upon the cosmological history and the values of the parameters of the theory, the role of nonthermal particles may vary from negligible up to very significant. It is worth noting that even initially small contribution of the oscillations of R into the total cosmological energy density could rise due to a weak decrease of the oscillation amplitude. Moreover, in R-dominated universe, the Hubble parameter could be different from the GR one, $H = \alpha/(2t)$ with $\alpha > 1$, and hence the energy density of relativistic cosmological matter drops faster than $1/t^2$. This also amplifies possible nonthermal contribution into the cosmological energy density.

The influx of energetic protons and antiprotons produced by the oscillations of R could have an impact on BBN if such protons were not thermalized at BBN era. Their effect would either allow one to obtain some bounds on m or even to improve the agreement between the theoretical predictions for BBN and the measurements of primordial light nuclei abundances.

The oscillating curvature might also be a source of dark matter in the form of heavy supersymmetric (SUSY) particles. Since the expected light SUSY particles have not yet been discovered at LHC, to some people supersymmetry somewhat lost its attractiveness. The contribution of the stable lightest SUSY particle into the cosmological energy is proportional to

$$\Omega \sim m_{\rm SUSY}^2/m_{\rm Pl}, \tag{64}$$

and for $m_{\rm SUSY}$ in the range 100–1000 GeV the cosmological fraction of these particles would be of order of unity. It is exactly what is necessary for dark matter. However, it excludes thermally produced LSPs if they are much heavier. If LSPs came from the decay of R and their mass is larger than the «mass» of R, i.e., m, but not too much larger, the LSP production could be sufficiently suppressed to make a reasonable contribution to dark matter.

These and some other manifestations of the considered modified gravity models will be discussed elsewhere.

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