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Modified Piyavskii's Global One-Dimensional Optimization of a Differentiable Function

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ABSTRACT

Piyavskii's algorithm maximizes a univariate function satisfying a Lipschitz condition. We propose a modified Piyavskii's sequential algorithm which maximizes a univariate differentiable function f by iteratively constructing an upper bounding piece-wise concave function Φ of f and evaluating f at a point where Φ reaches its maximum. We compare the numbers of iterations needed by the modified Piyavskii's algorithm (n_C) to obtain a bounding piece-wise concave function Φ whose maximum is within ε of the globally optimal value f_{opt} with that required by the reference sequential algorithm (n_{ref}). The main result is that $n_C \le 2n_{ref} + 1$ and this bound is sharp. We also show that the number of iterations needed by modified Piyavskii's algorithm to obtain a globally ε-optimal value together with a corresponding point (n_B) satisfies $n_B \le 4n_{ref} + 1$ Lower and upper bounds for n_{ref} are obtained as functions of $f(x)$, ε, M_1 and M_0 where M_0 is a constant defined by $M_0 = \sup_{x\in[a,b]} - f''(x)$ and $M_1 \ge M_0$ is an evaluation of M_0.

Keywords: Global Optimization; Piyavskii's Algorithm

1. Introduction

We consider the following general global optimization problems for a function defined on a compact set $X \subseteq R \quad -0.9ex0.15ex1.4ex \quad 0.9ex^m$:

(P) Find

$$\left(x^*, f^*\right) \in X \times R \quad -0.9ex0.15ex1.4ex \quad 0.9ex$$

such that

$$f^* = f\left(x^*\right) \ge f\left(x\right) \quad \forall x \in X.$$

(P_1) Find $x_{opt} \in X$ such that $f_{opt} = f(x_{opt}) \ge f^* - \varepsilon$, where ε is a small positive constant.

Many recent papers and books propose several approaches for the numerical resolution of the problem (P) and give a classification of the problems and their methods of resolution. For instance, the book of Horst and Tuy [1] provides a general discussion concerning deterministic algorithms. Piyavskii [2,3] proposes a deterministic sequential method which solves (P) by iteratively constructing an upper bounding function F of f and evaluating f at a point where F reaches its maximum, Shubert [4], Basso [5,6], Schoen [7], Shen and Zhu [8] and Horst and Tuy [9] give a special aspect of its application by examples involving functions satisfying a Lipschitz condition and propose other formulations of the Piyavskii's algorithm, Sergeyev [10] use a smooth auxiliary functions for an upper bounding function, Jones *et al.* [11] consider a global optimization without the Lipschitz constant. Multidimensional extensions are proposed by Danilin and Piyavskii [12], Mayne and Polak [13], Mladineo [14] and Meewella and Mayne [15], Evtushenko [16], Galperin [17] and Hansen, Jaumard and Lu [18,19] propose other algorithms for the problem (P) or its multidimensional case extension. Hansen and Jaumard [20] summarize and discuss the algorithms proposed in the literature and present them in a simplified and uniform way in a high-level computer language. Another aspect of the application of Piyavskii's algorithm has been developed by Brent [21], the requirement is that the function is defined on a compact interval, with a bounded second derivative. Jacobsen and Torabi [22] assume that the differentiable function is the sum of a convex and concave functions. Recently, a multivariate extension is proposed by Breiman and Cutler [23] which use the Taylor development of f to build an upper bounding function of f. Baritompa and Cutler [24] give a variation and an acceleration of the Breiman and Cutler's method. In this paper, we suggest a modified Piyavskii's sequential algorithm which maximizes a univariate differentiable function f. The theoretical study of the number of iterations of Piyavskii's algorithm was initiated by Dani-

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lin [25]. Danilin's result was improved by Hansen, Jaumard and Lu [26]. In the same way, we develop a reference algorithm in order to study the relationship between n_B and n_{ref} where n_B denotes the number of iterations used by the modified Piyavskii's algorithm to obtain an ε-optimal value, and n_{ref} denotes the number of iterations used by a reference algorithm to find an upper bounding function, whose maximum is within ε of the maximum of f. Our main result is that $n_B \le 4n_{ref} + 1$. The last purpose of this paper is to derive bounds for n_{ref}. Lower and upper bounds for n_{ref} are obtained as functions of $f(x)$, ε, M_1 and M_0 where M_0 is a constant defined by

$$M_0 = \sup_{x\in[a,b]} - f''(x)$$

and $M_1 \ge M_0$ is an evaluation of M_0.

2. Upper Bounding Piecewise Concave Function

Theorem 1. *Let*

$$f \in C^2\left(\left[a,b\right], R \quad -0.9ex0.15ex1.4ex \quad 0.9ex\right)$$

and suppose that there is a positive constant M such that

$$M \ge \sup_{x\in[a,b]} - f''(x). \tag{1}$$

Let

$$P(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

and

$$\Phi(x) = P(x) + \frac{M}{2}(x-a)(b-x).$$

Then

$$\Phi(x) \ge f(x) \quad \text{for all } x \in [a,b].$$

Proof. Let

$$\Psi(x) = \Phi(x) - f(x),$$

we have

$$\Psi''(x) = -M - f''(x) \le 0.$$

Then Ψ is a concave function, the minimum of Ψ over $[a, b]$ occurs at a or b. Then

$$\Psi(x) \ge \Psi(a) = \Psi(b) = 0$$

for all $x \in [a,b]$.

This proves the theorem.

Remark 1. *Let*

$$u = \frac{a+b}{2} + \frac{1}{M}\frac{f(b)-f(a)}{b-a}.$$

Then the maximum Φ^* *of* $\Phi(x)$ *is*:

$$\Phi^* = \begin{cases} \frac{1}{2}\left[f(b) + f(a) + \frac{1}{M}\left(\frac{f(b)-f(a)}{b-a}\right)^2 \right. \\ \left. \quad + \frac{1}{4} M (a-b)^2 \right] \\ if \quad u \in [a,b] \\ f(a) \\ f(b) \end{cases}$$

If the function f is not differentiable, we generalize the above result by the following one:

Theorem 2. *Let f be a continuous on* $[a, b]$ *and suppose that there is a positive constant M such that for every* $h > 0$

$$f(u+h) - 2f(u) + f(u-h) \ge Mh^2 \quad \forall u \in [a+h, b-h]. \tag{2}$$

Then

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) + \frac{M}{2}(x-a)(b-x) \quad \forall x \in [a,b].$$

Proof. without loss of generality, we assume that $f(a) = f(b) = 0$ and $M = 0$.

Let us consider the function $f(x) - \Phi(x)$ instead of the function $f(x)$. It suffice to prove that

$$f^* = \max_{x\in[a,b]} f(x) \le 0.$$

Suppose by contradiction that $f^* > 0$, and let

$$u = \sup\left\{x \in [a,b] \middle| f(x) = f^*\right\}.$$

The function f is continuous, $f(u) = f^* > 0$, thus $a < u < b$, consequently $u \in [a+h, b-h]$ for $h > 0$ small and we have $f(u - h) \le f(u)$ and $f(u + h) < f(u)$.

Since $M = 0$, this contradicts the hypothesis (3). Hence $f^* \le 0$.

Remark 2. *Since the above algorithm is based entirely on the result of Theorem* 1, *it is clear that the same algorithm will be adopted for functions satisfying condition* (2).

If f is twice continuously differentiable, the conditions (1) and (2) are equivalents.

3. Modified Piyavskii's Algorithm

We call subproblem the set of information

$$P_i = \left[\Phi_i^*, x_i^*, a_i, b_i\right],$$

describing the bounding concave function

$$\Phi_i(x) = \frac{(b_i - x) f(a_i) + (x - a_i) f(b_i)}{b_i - a_i} + \frac{M}{2}(x - a_i)(b_i - x),$$

$$\Phi_i^* = \max_{x \in [a_i, b_i]} \Phi_i(x) \quad \text{and} \quad x_i^* = \arg\max_{x \in [a_i, b_i]} \Phi_i(x).$$

The algorithm that we propose memorizes all the subproblems and, in particular, stores the maximum Φ_i^* of each bounding concave function Φ_i over $[a_i, b_i]$ in a structure called Heap. A Heap is a data structure that allows an access to the maximum of the k values that it stores in constant time and is updated in $O(\mathrm{Log}_2 k)$ time. In the following discussion, we denote the Heap by H.

Modified Piyavskii's algorithm can be described as follows:

3.1. Step 1 (Initialization)

- $k \leftarrow 0$
- $[a_0, b_0] \leftarrow [a, b]$
- Compute $f(a_0)$ and $f(b_0)$
- $f_{opt}^0 \leftarrow \max\{f(a_0), f(b_0)\}$
- $x_{opt}^0 \leftarrow \mathrm{argmax}\{f(a_0), f(b_0)\}$
- Let Φ_0 be an upper bounding function of f over $[a_0, b_0]$
- Compute $\Phi_0^* = \max_{x \in [a_0, b_0]} \Phi_0(x)$
- $\Phi_{opt}^0 \leftarrow \Phi_0^*$
- if $\Phi_{opt}^0 - f_{opt}^0 \le \varepsilon$ Stop, f_{opt}^0 is an ε-optimal value
- $H \leftarrow \{P_0 = [\Phi_0^*, x_0^*, a_0, b_0]\}$

3.2. Current Step (k = 1, 2, $\cdots$)

While H is no empty do

- $k \leftarrow k + 1$
- Extract from H the subproblem $P_m = [\Phi_m^*, x_m^*, a_m, b_m]$ with the largest upper bound Φ_l^*
- $[a_k, b_k] \leftarrow [x_m^*, b_m]$
- $[a_l, b_l] \leftarrow [a_m, x_m^*]$

3.2.1 Update of the Best Current Solution

for $j = l, k$ **do**

3.2.1.1. Lower Bound

If $f(x_j^*) > f_{opt}^{k-1}$ **then**

$f_{opt}^k \leftarrow f(x_j^*)$

$x_{opt}^k \leftarrow x_j^*$

Else

$f_{opt}^k \leftarrow f_{opt}^{k-1}$

$x_{opt}^k \leftarrow x_{opt}^{k-1}$

End if

3.2.1.2. Upper Bound

- Build an upper bounding function Φ_j on $[a_j, b_j]$
- Compute $\Phi_j^* = \max_{x \in [a_j, b_j]} \Phi_j(x)$
- Add $P_j = [\Phi_j^*, x_j^*, a_j, b_j]$ to H

End for

- Delete from H all subproblems

$$P = [\Phi_p^*, x_p^*, a_p, b_p]$$

with $\Phi_p^* \le f_{opt}^k$

3.2.2. Optimality Test

- Let Φ_{opt}^k be the maximum of all Φ_j^*
- **If** $\Phi_{opt}^k - f_{opt}^k \le \varepsilon$, **then** Stop, f_{opt}^k is an ε-optimal value

End While

Let $[a_n, b_n]$ denote the smallest subinterval of $[a, b]$ containing x_n, whose end points are evaluation points. Then the partial upper bounding function $\Phi_{n-1}(x)$ spanning $[a_n, b_n]$ is deleted and replaced by two partial upper bounding functions, the first one spanning $[a_n, x_n]$ denoted by $\Phi_{nl}(x)$ and the second one spanning $[x_n, b_n]$ denoted by $\Phi_{nr}(x)$ (**Figure 1**).

Proposition 1. *For $n \ge 2$ the upper bounding function $\Phi_n(x)$ is easily deduced from $\Phi_1(x), \cdots, \Phi_{n-1}(x)$ as follows*:

$$\Phi_{nl}^* = \frac{\left(\Phi_{n-1}^* + f(x_n)\right)^2 - 4f(a_n)(x_n)}{4\left(\Phi_{n-1}^* - f(a_n)\right)}$$

and

$$\Phi_{nr}^* = \frac{\left(\Phi_{n-1}^* + f(x_n)\right)^2 - 4f(b_n)f(x_n)}{4\left(\Phi_{n-1}^* - f(b_n)\right)}.$$

Proof. In the case where

$$x' = \frac{a_n + x_n}{2} + \frac{1}{M}\frac{f(x_n) - f(a_n)}{x_n - a_n}$$

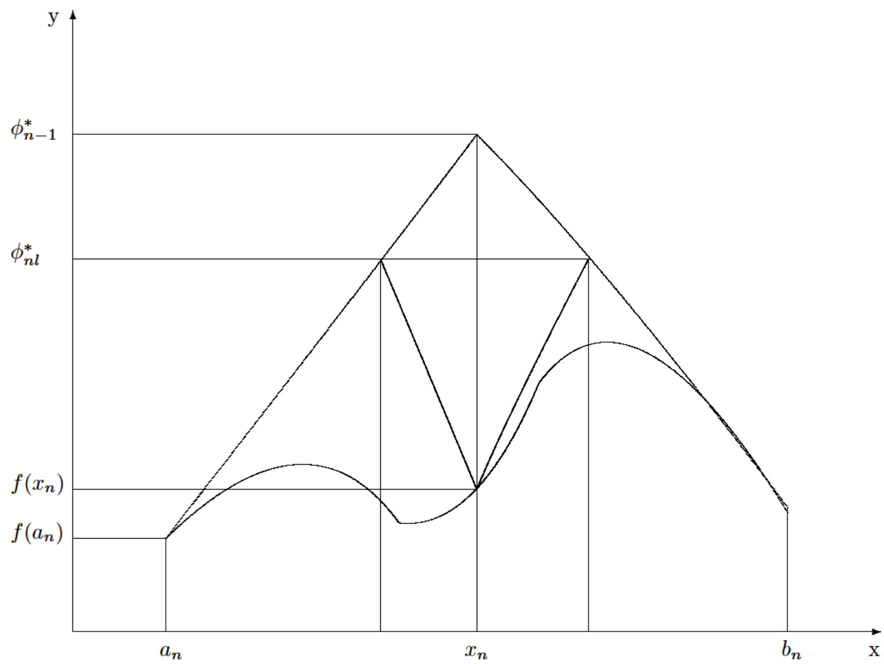

Figure 1. Upper bounding piece-wise concave function.

is in $[a_n, x_n]$, then from remark 1, the maximum of $\Phi_{nl}(x)$ is given as follows

$$\Phi^*_{nl} = \frac{1}{2}\left[f(x_n) + f(a_n) + \frac{1}{M}\left(\frac{f(x_n) - f(a_n)}{x_n - a_n}\right)^2 + \frac{1}{4}M(x_n - a_n)^2 \right] \quad (3)$$

The maximum of $\Phi_{n-1}(x)$ spanning $[a_n, b_n]$ is reached at the point

$$x_n = \frac{a_n + b_n}{2} + \frac{1}{M}\frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

We deduce that

$$(x_n - a_n)^2 = \frac{1}{4}(b_n - a_n)^2 + \frac{1}{M}\left(f(b_n) - f(a_n)\right) + \frac{1}{M^2}\left(\frac{f(b_n) - f(a_n)}{b_n - a_n}\right)^2,$$

then

$$\frac{1}{4}M(x_n - a_n)^2 = \frac{1}{2}\left(\Phi^*_{n-1} - f(a_n)\right).$$

By substitution in expression (3), we have the result. We show in the same way that the maximum Φ^*_{nr} of $\Phi_{nr}(x)$ defined in $[x_n, b_n]$ is given by:

$$\Phi^*_{nr} = \frac{\left(\Phi^*_{n-1} + f(x_n)\right)^2 - 4f(b_n)f(x_n)}{4\left(\Phi^*_{n-1} - f(b_n)\right)}.$$

Remark 3. *Modified Piyavskii's algorithm obtains an upper bound on f within ε of the maximum value of f(x) when the gap* $\Phi^*_n - f^*$ *is not larger than ε, where*

$$f^* = \max_{x\in[a,b]} f(x).$$

But f^* *and*

$$x^* = \arg\max_{x\in[a,b]} f(x)$$

are unknown. So modified Piyavskii's algorithm stops only after a solution of value within ε of the upper bound has been found, i.e., when the error $\Phi^*_n - f^*_n$ *is not larger than ε. The number of iterations needed to satisfy each of these conditions is studied below.*

4. Convergence of the Algorithm

We now study the error and the gap of Φ_n in modified Piyavskii's algorithm as a function of the number *n* of iterations. The following proposition shows how they decrease when the number of iterations increases and provides also a relationship between them.

Proposition 2. *Let* $\delta_n = \Phi^*_n - f^*_n$ *and* $\delta'_n = \Phi^*_n - f^*$ *denote respectively the error and the gap after n iterations of modified Piyavskii's algorithm. Then, for* $n \geq 2$ *1)* $\delta_{2n-1} \leq \delta_n/4$, *2)* $\delta'_{2n-1} \leq \delta'_n/4$, *3)* $\delta_{2n-1} \leq \delta'_n/2$.

Proof. First, notice that (Φ^*_n), (δ_n) and (δ'_n) are non increasing sequences and (f^*_n) is nondecreasing. After *n* iterations of the modified Piyavskii's algorithm, the upper bounding piece-wise concave function $\Phi_n(x)$ contains $(n-1)$ partial upper bounding concave functions, which we call first generation function. We call the partial upper bounding concave functions obtained by splitting these second generation function. Φ_m must belong to the second generation for the first time after no more than another $n-1$ iterations of the modified Piyavskii's algorithm, at iteration m ($m \le 2n-1$).

Let k $(n+1 \le k \le m)$ be the iteration after which the maximum Φ_m^* of $\Phi_m(x)$ has been obtained, by splitting the maximum Φ_{k-1}^* of $\Phi_{k-1}(x)$. Then, in the case where

$$x_k' = \frac{a_k + x_k}{2} + \frac{f(x_k) - f(a_k)}{M(x_k - a_k)}$$

is in $[a_k, x_k]$, and from proposition 1, we get:

$$\Phi_m^* = \frac{\left(\Phi_{k-1}^* + f(x_k)\right)^2 - 4f(a_k)f(x_k)}{4\left(\Phi_{k-1}^* - f(a_k)\right)}$$

$$= \frac{1}{4}\left\{2f(x_k) + f(a_k) + \Phi_{k-1}^* + \frac{\left(f(x_k) - f(a_k)\right)^2}{\Phi_{k-1}^* - f(a_k)}\right\},$$

where

$$x_k = \arg\max_{x\in[a,b]}\Phi_{k-1}(x)$$

and a_k is one of the endpoints of the interval over which the function Φ_{k-1} is defined.

1) We have

$$\delta_{2n-1} = \Phi_{2n-1}^* - f_{2n-1}^* \le \Phi_m^* - f_m^*,$$

where

$$f_m^* = \max_{i=0,\cdots,m} f(x_i) = \max\left(f_{k-1}^*, f(x_k)\right).$$

If $f_m^* = f(x_k)$, therefore, we have

$$\delta_{2n-1}$$

$$\le \frac{\left(\Phi_{k-1}^* + f(x_k)\right)^2 - 4f(a_k)f(x_k)}{4\left(\Phi_{k-1}^* - f(a_k)\right)} - f(x_k)$$

$$\le \frac{\left(\Phi_{k-1}^* - f(x_k)\right)^2}{4\left(\Phi_{k-1}^* - f(a_k)\right)}$$

$$\le \frac{\Phi_{k-1}^* - f(x_k)}{4}$$

$$\le \frac{\Phi_n^* - f_n^*}{4}$$

$$= \frac{\delta_n}{4}.$$

If $f_m^* = f_{k-1}^*$, we will consider two cases:

• **case 1**

If $x_k' \le a_k$, then $\Phi_m^* = f(a_k) = f_m^*$ and if $x_k' \ge x_k$, then $\Phi_m^* = f(x_k) = f_m^*$ and the algorithm stops.

• **case 2**

$x_k' \in [a_k, x_k]$, then we have

$$x_k' = \frac{a_k + x_k}{2} + \frac{1}{M}\frac{f(x_k) - f(a_k)}{x_k - a_k} \ge a_k,$$

hence

$$f(a_k) - f(x_k) \le \frac{M}{2}(x_k - a_k)^2$$

$$= \Phi_{k-1}^* - f(a_k)$$

(proposition 1).

Moreover

$$f(x_k) - f(a_k) \le \Phi_{k-1}^* - f(a_k).$$

We deduce from these two inequalities that

$$\left|f(x_k) - f(a_k)\right| \le \Phi_{k-1}^* - f(a_k).$$

Therefore

$$\delta_{2n-1}$$

$$\le \frac{1}{4}\left\{2f(x_k) + f(a_k) + \Phi_{k-1}^* + \frac{\left(f(x_k) - f(a_k)\right)^2}{\Phi_{k-1}^* - f(a_k)}\right\} - f_{k-1}^*$$

$$\le \frac{1}{4}\left\{2f(x_k) + \Phi_{k-1}^* + f(a_k) + \left|f(x_k) - f(a_k)\right|\right\} - f_{k-1}^*$$

$$\le \frac{1}{4}\left(3f_{k-1}^* + \Phi_{k-1}^*\right) - f_{k-1}^*$$

$$\le \frac{\Phi_{k-1}^* - f_{k-1}^*}{4}$$

$$\le \frac{\Phi_n^* - f_n^*}{4} = \delta_n / 4.$$

2) We have

$$\delta_{2n-1}' = \Phi_{2n-1}^* - f^* \le \Phi_m^* - f^*.$$

We follow the same steps to prove the second point of 1) and show that $\delta_{2n-1}' \le \delta_n'/4$.

3)

• **case 1**

If $x_k' \le a_k$ or $x_k' \ge x_k$, then $\Phi_m^* = f_m^* = f^*$ and the algorithm stops.

• **case 2** $x_k' \in [a_k, x_k]$. First, we have

$$\Phi_m^* = \frac{1}{4}\left\{2f(x_k) + f(a_k) + \Phi_{k-1}^* + \frac{\left(f(x_k) - f(a_k)\right)^2}{\Phi_{k-1}^* - f(a_k)}\right\}$$

$$\le \frac{f(x_k) + \Phi_{k-1}^*}{2},$$

and

$$f^* \le \Phi_m^* \le \frac{f(x_k)+\Phi_{k-1}^*}{2} \le \frac{f^*+\delta_n'+f(x_k)}{2},$$

therefore $f(x_k) \ge f^* - \delta_n'$. Hence

$$\delta_{2n-1} \le \frac{\Phi_n^* - f(x_k)}{4} \le \delta_n'/2.$$

This proves the proposition.

Proposition 3. *Modified Piyavskii's algorithm* (*with* $\varepsilon = 0$) *is convergent*, *i.e. either it terminates in a finite number of iterations or*

$$\lim_{n\to\infty}\Phi_n^* = \lim_{n\to\infty} f_n^* = f^* = \max_{x\in[a,b]} f(x).$$

Proof. This is an immediate consequence of the definition of δ_n and i) of proposition 2.

5. Reference Algorithm

Since the number of necessary function evaluations for solving problem (*P*) measures the efficiency of a method, Danilin [25] and Hansen, Jaumard and Lu [26] suggested studying the minimum number of evaluation points required to obtain a guaranteed solution of problem (*P*) where the functions involved are Lipschitz-Continuous on $[a,b]$.

In the same way, we propose to study the minimum number of evaluation points required to obtain a guaranteed solution of problem (*P*) where the functions involved satisfy the condition (1).

For a given tolerance ε and constant *M* this can be done by constructing a reference bounding piece-wise concave function Φ_{ref} such that

$$\max_{x\in[a,b]} \Phi_{ref}(x) = f^* + \varepsilon,$$

with a smallest possible number n_{ref} of evaluations points $y_1, y_2, \cdots, y_{n_{ref}}$ in $[a,b]$.

Such a reference bounding piece-wise concave function is constructed with $f^* = \max_{x\in[a,b]} f(x)$ which is assumed to be known. It is of course, designed not to solve problem (*P*) from the outset, but rather to give a reference number of necessary evaluation points in order to study the efficiency of other algorithms. It is easy to see that a reference bounding function $\Phi_{ref}(x)$ can be obtained in the following way:

Description of the Reference Algorithm

1). Initialization

- $k \leftarrow 1$
- $y_1 \leftarrow$ root of the equation

$$f^* + \varepsilon - \frac{M}{2}(a-x)^2 - f(x) = 0.$$

2). Reference bounding function

While $y_k < b$ **do**

- compute $f(y_k)$
- $x_{p_k} \leftarrow y_k + \sqrt{\frac{2}{M}\left(f^* + \varepsilon - f((y_k))\right)}$
- $y_{k+1} \leftarrow$ root of the equation

$$f^* + \varepsilon - f(x) - \frac{M}{2}\left(x_{p_k} - x\right)^2 = 0$$

- $k \leftarrow k+1$

End While.

3). Minimum number of evaluations points

- $n_{ref} \leftarrow k$

A reference bounding function Φ_{ref} is then defined as follows (see **Figure 2**)

$$\Phi_{ref}(x) = \begin{cases} f^* + \varepsilon - \frac{M}{2}(x-a)^2 & \text{if } x \in [a, y_1] \\ f^* + \varepsilon - \frac{M}{2}\left(x - x_{p_k}\right)^2 & \text{if } x \in [y_k, y_{k+1}], \; k = 1, \cdots, n_{ref} - 1 \\ f^* + \varepsilon - \frac{M}{2}\left(x - x_{p_{n_{ref}}}\right)^2 & \text{if } x \in \left[y_{n_{ref}}, b\right] \end{cases}$$

6. Estimation of the Efficiency of the Modified Piyavskii's Algorithm

Proposition 4. *For a given a tolerance* ε, *let* Φ_{ref} *be a reference bounding function with* n_{ref} *evaluations points* $y_1, \cdots, y_{n_{ref}}$ *and* $\Phi_{ref}^* = f^* + \varepsilon$. *Let* Φ_C *be a bounding function obtained by the modified Piyavskii's algorithm with evaluation points* $x_1, \cdots, x_{n_C}$, *where* n_C *is the smallest number of iterations such that*

$$\Phi_C^* = \max_{x\in[a,b]} \Phi_C(x) \le f^* + \varepsilon.$$

Then we have

$$n_C \le 2n_{ref} + 1, \tag{4}$$

and the bound is sharp for all values of n_{ref}

Proof. The proof is by induction on n_{ref} We initialize the induction at $n_{ref} = 1$.

- **Case 1** $y_1 = b$. (see **Figure 3**)

We have to show that $\Phi_C(x) \le \Phi_R(x), \quad \forall x \in [a,b]$. We have

$$\Phi_C(x) = \frac{(b-x)f(a) + (x-a)f(b)}{b-a} + \frac{M}{2}(x-a)(b-x)$$

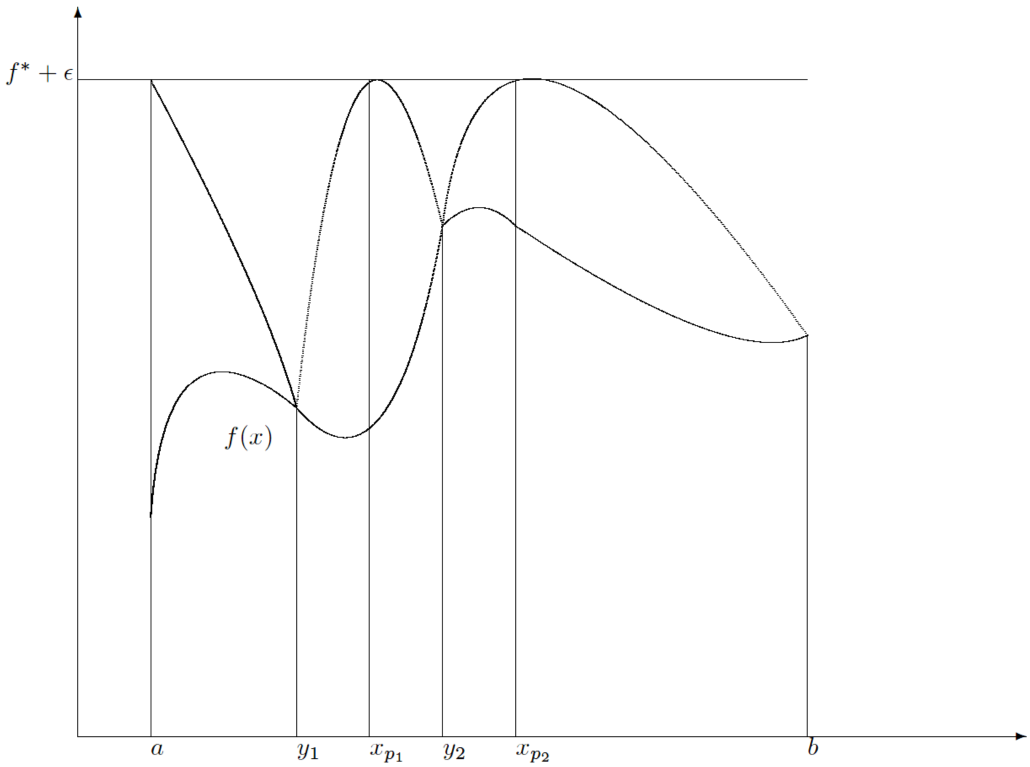

Figure 2. Reference bounding function.

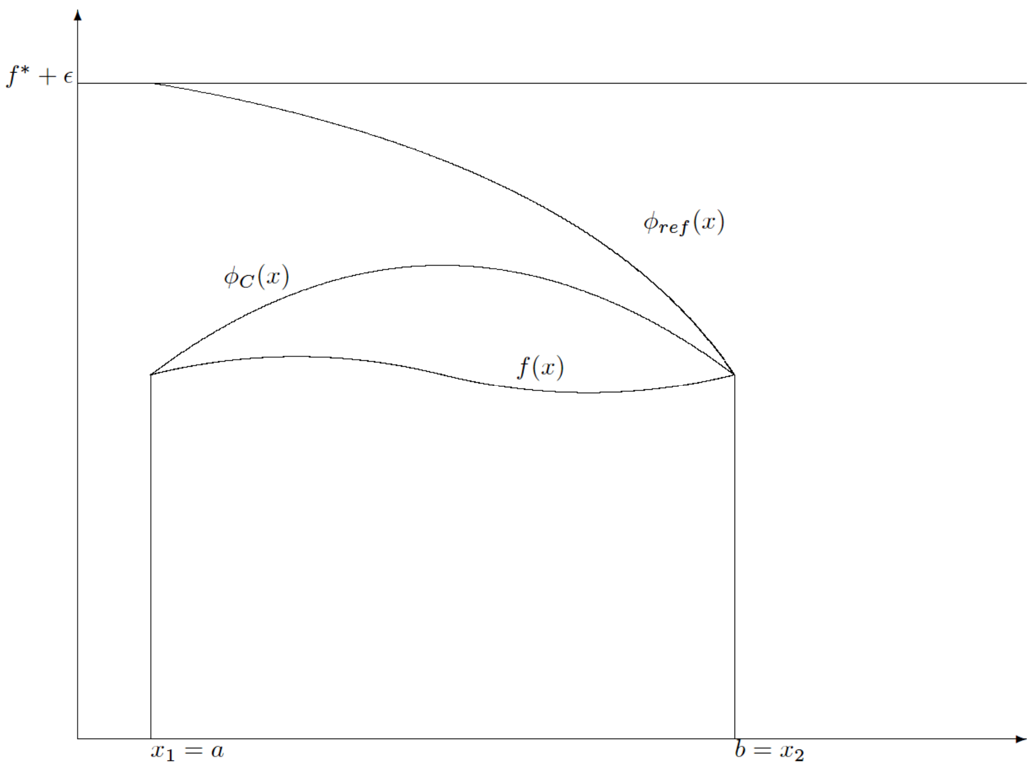

Figure 3. Case $y_1 = b$.

and $\Phi_{ref}(x) = f^* + \varepsilon - \frac{M}{2}(x-a)^2$. But $f(b) = \Phi_{ref}(b) = f^* + \varepsilon - \frac{M}{2}(b-a)^2$, hence

$$\Phi_C(x)=\frac{(b-x)f(a)+(x-a)(f^*+\varepsilon)}{b-a}-\frac{M}{2}(x-a)(b-a)+\frac{M}{2}(x-a)(b-x)$$
$$=\frac{(b-x)f(a)+(x-a)(f^*+\varepsilon)}{b-a}-\frac{M}{2}(x-a)^2.$$

Therefore

$$\Phi_C(x)-\Phi_{ref}(x)=\frac{b-x}{b-a}\left(f(a)-(f^*+\varepsilon)\right)\le 0,$$

hence the result holds.

• **case 2** $y_1<b$. (see **Figure 4**)

If $n_c=2$, (4) holds. In this case, we have

$$\Phi_{ref}(x)=\begin{cases} f^*+\varepsilon-\frac{M}{2}(x-a)^2 & si\ x\in[a,y_1] \\ f^*+\varepsilon-\frac{M}{2}\left(x-x_{p_1}\right)^2 & si\ x\in[y_1,b], \end{cases}$$

where

$$x_{p_1}=y_1+\sqrt{\frac{2}{M}\left(f^*+\varepsilon-f(y_1)\right)}.$$

Assuming that $\Phi_1^*>f^*+\varepsilon$ and $x_3\in[a,y_1]$ (the proof is similar for $x_3\in[y_1,b]$). We denote by $\Phi_l(x)$ and $\Phi_r(x)$ the partial bounding functions defined respectively on $[a,x_3]$ and $[x_3,b]$ and by Φ_r^* a maximum value of $\Phi_r(x)$. We show that

$$\Phi_l(x)\le\Phi_{ref}(x)\quad \forall x\in[a,x_3]$$

and that $\Phi_r^*\le f^*+\varepsilon$. Indeed, we have

$$\Phi_l(x)=\frac{(x_3-x)f(a)+(x-a)f(x_3)}{x_3-a}+\frac{M}{2}(x-a)(x_3-x)$$

and

$$f(x_3)\le\Phi_{ref}(x_3)=f^*+\varepsilon-\frac{M}{2}(a-x_3)^2,$$

thus

$$\Phi_l(x)-\Phi_{ref}(x)\le\frac{x_3-x}{x_3-a}\left(f(a)-(f^*+\varepsilon)\right)\le 0\quad \forall x\in[a,x_3].$$

If

$$x''=\frac{x_3+b}{2}+\frac{f(b)-f(x_3)}{M(b-x_3)}$$

is in $[x_3,b]$, then

$$\Phi_r^*=\frac{1}{2}\left\{f(x_3)+\frac{1}{2}\left(f(b)+\Phi_1^*\right)+\frac{\left(f(b)-f(x_3)\right)^2}{2\left(\Phi_1^*-f(b)\right)}\right\}.$$

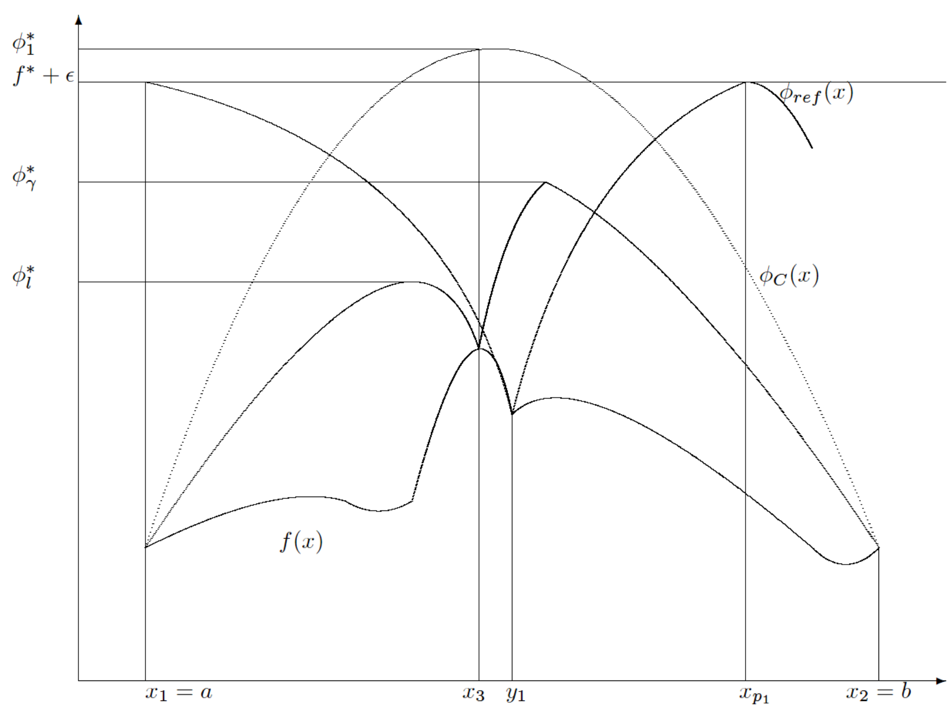

Figure 4. Case $y_1<b$.

As

$$\left|f(b)-f(x_3)\right| \le \Phi_1^* - f(b)$$

(see the proof of proposition 2), we have

$$\Phi_r^* \le \frac{1}{2}\left\{f(x_3)+\Phi_1^*\right\}$$

$$\le \frac{1}{2}\left\{f^* + \varepsilon - \frac{M}{2}(a-x_3)^2 + \Phi_1^*\right\}$$

$$\le \frac{1}{2}\left\{f^* + \varepsilon - \Phi_1^* + f(a) + \Phi_1^*\right\} \le f^* + \varepsilon.$$

If $x'' \ge b$, then $\Phi_r^* = f(b) \le f^* + \varepsilon$.

If $x'' \le x_3$ then $\Phi_r^* = f(x_3) \le f^* + \varepsilon$ and (4) holds.

Assume now that (4) holds for $n_{ref} \le k \ (k \ge 1)$, and consider $n_{ref} = k + 1$. The relation (4) holds for $n_c = 2$, therefore we may assume that

$$\Phi_1^* > f^* + \varepsilon, x_1 = a \text{ and } x_2 = b.$$

If we let

$$x_3 = \arg\max_{x\in[a,b]} \Phi_1(x),$$

the modified Piyavskii's algorithm has to be implemented on two subintervals $[x_1, x_3]$ and $[x_3, x_2]$ There are two cases which need to be discussed separately:

• case 1 (see **Figure 5**)

There is a subinterval containing all the n_{ref} evaluation points of the best bounding function $\Phi_{ref}(x)$. Without loss of generality, we may assume that $[x_1, x_3]$ contains all n_{ref} evaluation points of Φ_{ref}. Let x' and x'' be the solution of $\Phi_1(x) = f^* + \varepsilon$. By symmetry, we have

$$x_3 = \frac{x' + x''}{2}.$$

Let $y'_{n_{ref}}$ be the abscissa of the maximum of the last partial bounding function of $\Phi_{ref}(x)$. Due to our assumption, we have

$$y'_{n_{ref}} = y_{n_{ref}} + \sqrt{\frac{2}{M}\left(f^* + \varepsilon - f\left(y_{n_{ref}}\right)\right)} \ge x'',$$

since otherwise there is an another evaluation point of f on the interval

$$\left[y_{n_{ref}}, b\right].$$

Let $y'_{n_{ref}-1}$ be the abscissa of the maximum of the partial bounding function of $\Phi_{ref}(x)$. preceding $y'_{n_{ref}}$. In this case, we have

$$x' > y'_{n_{ref}-1} = y_{n_{ref}} - \sqrt{\frac{2}{M}\left(f^* + \varepsilon - f\left(y_{n_{ref}}\right)\right)}.$$

But $y'_{n_{ref}-1} < x'$ implies that $n_{ref} = 1$, which is not the case according to the assumption ($n_{ref} \ge 2$). Therefore, this case will never exist.

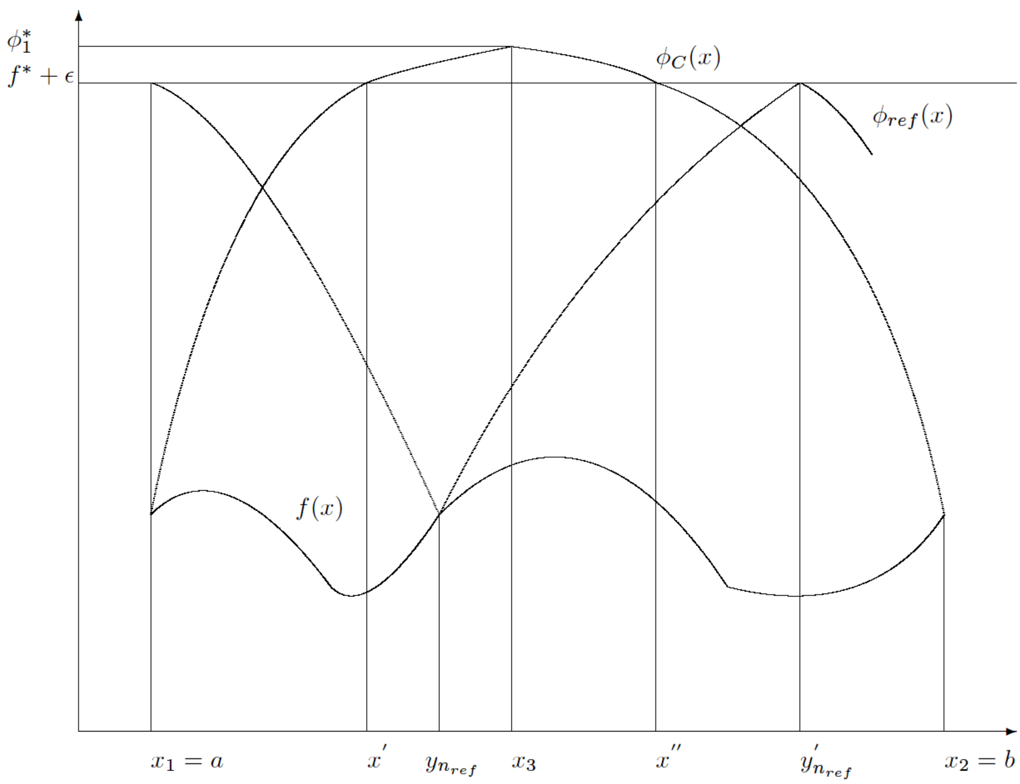

Figure 5. The subinterval contains all the n_{ref} evaluation points.

• case 2

None of the subintervals contains all the n_{ref} evaluation points of $\Phi_{ref}(x)$. We can then apply induction, reasoning on the subintervals $[x_1, x_3]$ and $[x_3, x_2]$ obtained by modified Piyavskii's algorithm. If they contain n_c^1 and n_c^2 evaluation points respectively, $[x_1, x_2]$ contains $n_c^1 + n_c^2 - 1$ evaluation points as x_3 belongs to both subintervals. Then (4) follows by induction on n_{ref}.

Consider the function $f(x) = x$ defined on [0, 1], the constant M is equal to 4. For $\varepsilon = 0$ we have $n_C = n_{ref} + 1$ (see **Figure 6**), and the bound (4) is sharp.

As noticed in remark 3, the modified Piyavskii's algorithm does not necessarily stop as soon as a cover is found as described in proposition 4, but only when the error does not exceed ε. We now study the number of evaluation points necessary for this to happen.

Proposition 5. *For a given tolerance* ε, *let* n_{ref} *be the number of evaluation points in a reference bounding function of f and* n_B *the number of iterations after which the modified Piyavskii's algorithm stops. We then have*

$$n_B \le 4n_{ref} + 1. \quad (5)$$

Proof. Let $n_{C'} = 2n_{ref} + 1$. Due to proposition 4, after $n_{C'}$ iterations, we have $\Phi^*_{C'} - f^* \le \varepsilon$. Due to 3) of proposition 2, after $2n_{C'} - 1$ iterations, we have

$$\Phi^*_{2n_{C'}-1} - f^*_{2n_{C'}-1} \le \Phi^*_{n_{C'}} - f^*,$$

which shows that the termination of modified Piyavskii's algorithm is satisfied. This proves (5).

7. Bounds on the Number of Evaluation Points

In the previous section, we compared the number of evaluation functions in the modified Piyavskii's algorithm and in the reference algorithm. We now evaluate the number n_B of evaluation functions of the modified Piyavskii's algorithm itself. To achieve this, we derive bounds on n_{ref}, from which bounds on n_B are readily obtained using the relation $n_{ref} \le n_B \le 4n_{ref} + 1$.

Proposition 6. *Let f be a* C^2 *function defined on the interval* $[a, b]$. *Set*

$$M_0 = \sup_{x\in[a,b]} - f''(x)$$

and let

$$f^* = \max_{x\in[a,b]} f(x).$$

Then the number n_{ref} *of evaluation points in a reference cover* Φ_{ref} *using the constant* $M_1 \ge M_2$ *is bounded by the following inequalities*:

$$n_{ref} \le \frac{\sqrt{(M_1 + M_0)(f^* + \varepsilon)} \int_a^b \frac{\mathrm{d}x}{f^* + \varepsilon - f(x)}}{\sqrt{2}\log\left(1 + \sqrt{\frac{M_0}{M_1 + M_0}}\right)}. \quad (6)$$

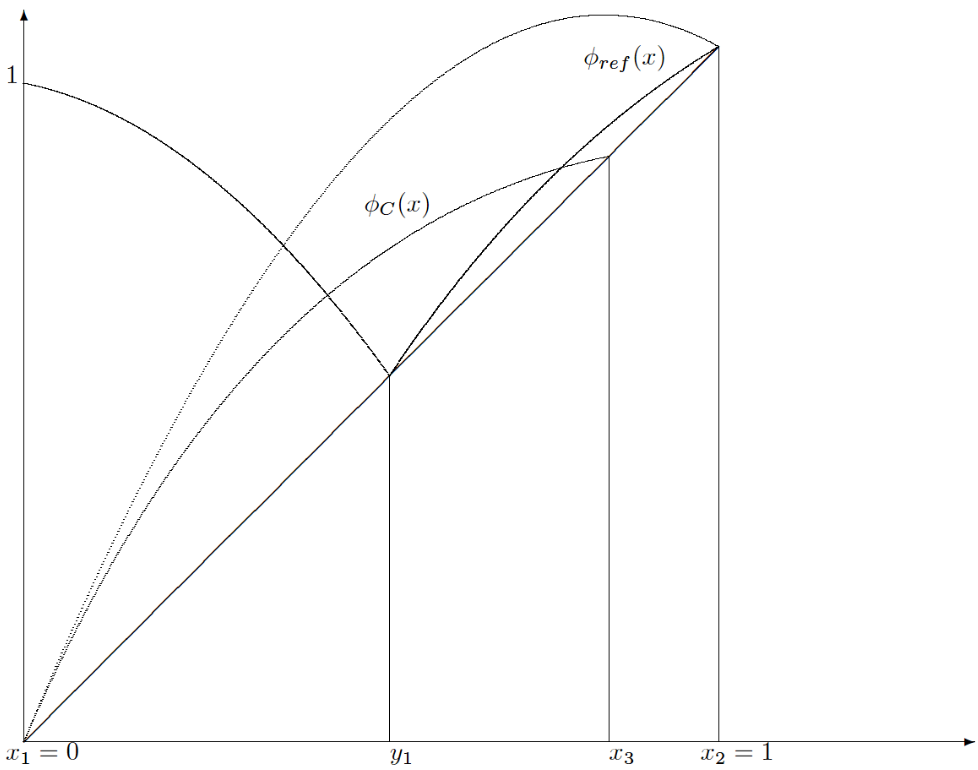

Figure 6. None of the subintervals contains all the n_{ref} evaluation points.

$$n_{ref} \geq \begin{cases} \frac{\sqrt{M_0\varepsilon}}{\pi\sqrt{2}}\int_a^b \frac{\mathrm{d}x}{f^* + \varepsilon - f(x)} \\ .2in \quad .2inif \left(\frac{M_1 - M_0}{2M_1}\right)^2 \left(f^* + \varepsilon\right) \geq \varepsilon, \\ \frac{\sqrt{M_0\varepsilon}}{4\sqrt{2}}\int_a^b \frac{\mathrm{d}x}{f^* + \varepsilon - f(x)} \\ .2in \quad .2inif \quad M_0 = M_1. \end{cases} \tag{7}$$

Proof. We suppose that the reference cover Φ_{ref} has $n_{ref} - 1$ partial upper bounding functions defined by the evaluation points

$$a = y_1, y_2, \cdots, y_{n_{ref}} = b.$$

We consider an arbitrary partial upper bounding function and the corresponding subinterval $[y_i, y_{i+1}]$ for$i$ $i = 1, \cdots, n_{ref} - 1$. To simplify, we move the origin to the point $(y_i, f(y_i))$. Let $d = y_{i+1} - y_i$, $z = f(y_{i+1}) - f(y_i)$ and $h = f^* + \varepsilon$. We assume $z \geq 0$ (See **Figure 7**).

Let Φ^r_{ref} be the partial upper bounding function defined on $[y_i, y_{i+1}]$ and x_{p_i} the point where

$$\max_{x\in[y_i, y_{i+1}]} \Phi^r_{ref}(x) = f^* + \varepsilon$$

is reached, then

$$x_{p_i} = y_i + \sqrt{\frac{2}{M}\left(f^* + \varepsilon - f(y_i)\right)}$$
$$= y_{i+1} - \sqrt{\frac{2}{M}\left(f^* + \varepsilon - f(y_{i+1})\right)}.$$

We deduce that

$$h = \frac{1}{2M_1}\left(\frac{z}{d} + \frac{\mathrm{d}M_1}{2}\right)^2.$$

Let g_1 be the function defined on $[y_i, y_{i+1}]$ by:

$$g_1(x)$$
$$= \frac{(y_{i+1} - x) f(y_i) + (x - y_i) f(y_{i+1})}{y_{i+1} - y_i}$$
$$- \frac{M_0}{2}(x - y_i)(y_{i+1} - x)$$
$$= (\frac{z}{d} - \frac{M_0 d}{2})x + \frac{M_0}{2}x^2.$$

We have

$$f(x) \geq g_1(x) \quad \forall x \in [a,b].$$

Thus we have

$$\int_0^d \frac{\mathrm{d}x}{f^* + \varepsilon - f(x)}$$
$$\geq \int_0^d \frac{\mathrm{d}x}{f^* + \varepsilon - g_1(x)}$$
$$= \int_0^d \frac{\mathrm{d}x}{\frac{1}{2M_1}\left(\frac{z}{d} + \frac{\mathrm{d}M_1}{2}\right)^2 - \left(\frac{z}{d} - \frac{M_0 d}{2}\right)x - \frac{M_0}{2}x^2}.$$

Consider the function

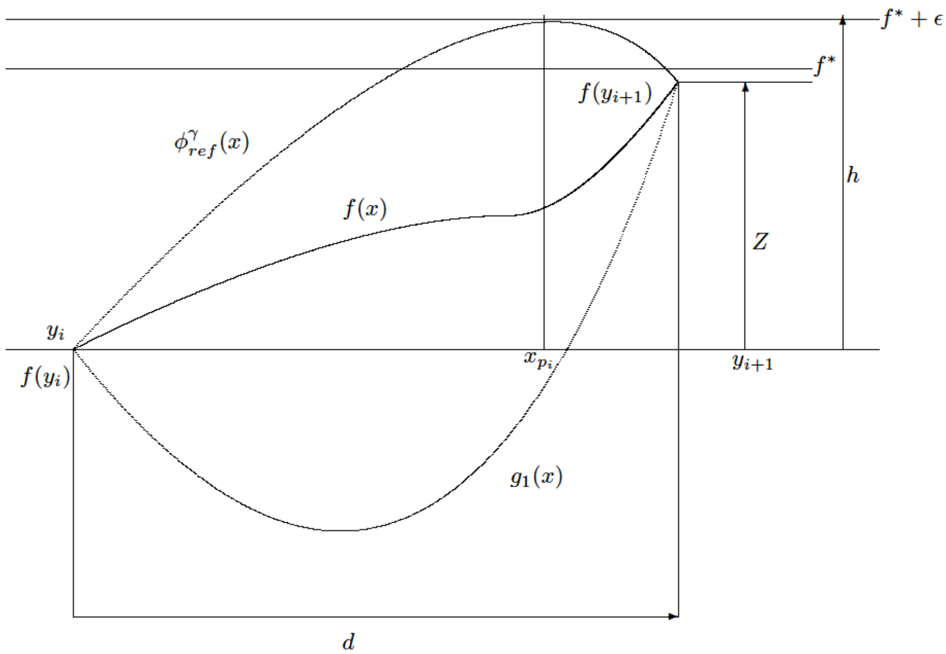

Figure 7. The reference cover Φ_{ref} has $n_{ref} - 1$ partial upper bounding functions.

$$F_1(x)=\frac{1}{2M_1}\left(\frac{z}{d}+\frac{\mathrm{d}M_1}{2}\right)^2-\left(\frac{z}{d}-\frac{M_0d}{2}\right)x-\frac{M_0}{2}x^2.$$

Let x_1 and x_2 be the roots of the equation $F_1(x) = 0$; they are given by

$$\begin{cases} x_1=\frac{1}{M_0}\left\{\left(\frac{M_0d}{2}-\frac{z}{d}\right)-\sqrt{\left(\frac{M_0d}{2}-\frac{z}{d}\right)^2+\frac{M_0}{M_1}\left(\frac{M_1d}{2}+\frac{z}{d}\right)^2}\right\} \\ x_2=\frac{1}{M_0}\left\{\left(\frac{M_0d}{2}-\frac{z}{d}\right)+\sqrt{\left(\frac{M_0d}{2}-\frac{z}{d}\right)^2+\frac{M_0}{M_1}\left(\frac{M_1d}{2}+\frac{z}{d}\right)^2}\right\} \end{cases}$$

Then F_1 is written in the following way

$$F_1(x)=-\frac{M_0}{2}(x-x_1)(x-x_2).$$

Let

$$\alpha=\frac{M_0d}{2}+\frac{z}{d},$$

$$\theta=\frac{M_0d}{2}-\frac{z}{d},$$

$$\beta=\sqrt{\left(\frac{M_0d}{2}-\frac{z}{d}\right)^2+\frac{M_0}{M_1}\left(\frac{M_1d}{2}+\frac{z}{d}\right)^2}.$$

We have

$$\int_0^d\frac{\mathrm{d}x}{F_1(x)}=\frac{1}{\beta}\left\{\log\left|\frac{\alpha+\beta}{\alpha-\beta}\right|+\log\left|\frac{\theta+\beta}{\theta-\beta}\right|\right\}$$
$$=\frac{1}{\beta}\left\{\log\left(\frac{\alpha+\beta}{\beta-\alpha}\right)+\log\left(\frac{\theta+\beta}{\beta-\theta}\right)\right\}.$$

Since $\theta \geq 0$ and g_1 reaches its minimum at the point θ, then

$$\frac{\theta+\beta}{\beta-\theta}\geq 1$$

and

$$\int_0^d\frac{\mathrm{d}x}{F_1(x)}\geq\frac{1}{\beta}\log\frac{\alpha+\beta}{\beta-\alpha}.$$

Since

$$\log\frac{\alpha+\beta}{\beta-\alpha}\geq 2\log\left(1+\sqrt{\frac{M_0}{M_1+M_0}}\right),$$

and

$$\frac{1}{\beta}\geq\frac{1}{\sqrt{2(M_1+M_0)(f^*+\varepsilon)}},$$

then

$$\int_0^d\frac{\mathrm{d}x}{f^*+\varepsilon-f(x)}$$
$$\geq\sqrt{\frac{2}{(M_1+M_0)(f^*+\varepsilon)}}\log\left(1+\sqrt{\frac{M_0}{M_1+M_0}}\right),$$

and

$$\int_a^b\frac{\mathrm{d}x}{f^*+\varepsilon-f(x)}$$
$$\geq n_{ref}\sqrt{\frac{2}{(M_1+M_0)(f^*+\varepsilon)}}\log\left(1+\sqrt{\frac{M_0}{M_1+M_0}}\right).$$

This proves (6).

Now let us consider the function G defined and continuous on $[y_i, y_{i+1}]$ such that

- $f(x)\leq G(x)\quad \forall x\in[y_i,y_{i+1}]$,
- $G(x)\leq f^*$.

Two cases are considered

case 1: If

$$\frac{1}{2M_1}\left(\frac{z}{d}+\frac{M_1d}{2}\right)^2-\frac{1}{2M_0}\left(\frac{z}{d}+\frac{M_0d}{2}\right)^2\geq\varepsilon, \quad (8)$$

then, the function G is given by

$$G(x)=g_2(x)=\left(\frac{z}{d}+\frac{M_0d}{2}\right)x-\frac{M_0}{2}x^2.$$

Hence

$$\int_0^d\frac{\mathrm{d}x}{f^*+\varepsilon-f(x)}\leq\int_o^d\frac{\mathrm{d}x}{f^*+\varepsilon-g_2(x)}.$$

We consider

$$F_2(x)=\frac{1}{2M_1}\left(\frac{z}{d}+\frac{M_1d}{2}\right)^2-\left(\frac{z}{d}+\frac{M_0d}{2}\right)x+\frac{M_0}{2}x^2,$$

its derivative is

$$F_2'(x)=-\left(\frac{z}{d}+\frac{M_0d}{2}\right)+M_0x,$$

$$F_2'(x)=0\Leftrightarrow x^*=\frac{1}{M_0}\left(\frac{z}{d}+\frac{M_0d}{2}\right).$$

Thus F_2 is expressed as follows

$$F_2(x)=F_2(x^*)+\frac{M_0}{2}(x-x^*)^2,$$

hence

$$\int_0^d \frac{\mathrm{d}x}{F_2(x)} = \sqrt{\frac{2}{M_0F_2(x^*)}}\left\{\arctan\left(\sqrt{\frac{M_0}{2F_2(x^*)}}(d-x^*)\right) + \arctan\left(\sqrt{\frac{M_0}{2F_2(x^*)}}x^*\right)\right\}.$$

Since

$$F_2(x^*) = \frac{1}{2M_1}\left(\frac{M_1 d}{2}+\frac{z}{d}\right)^2 - \frac{1}{2M_0}\left(\frac{M_0 d}{2}+\frac{z}{d}\right)^2 \geq \varepsilon$$

and $\arctan x \leq \frac{\pi}{2}$,

then $\int_0^d \frac{\mathrm{d}x}{F_2(x)} \leq \pi\sqrt{\frac{2}{M_0\varepsilon}}$,

and

$$\int_a^b \frac{\mathrm{d}x}{f^*+\varepsilon-f(x)} \leq n_{ref}\pi\sqrt{\frac{2}{M_0\varepsilon}}.$$

Moreover, the inequality

$$\left(\frac{M_1-M_0}{2M_1}\right)^2\left(f^*+\varepsilon\right) \geq \varepsilon$$

implies (8), this proves the first inequality of (7).

Case 2: Suppose that:

$$\frac{1}{2M_1}\left(\frac{M_1 d}{2}+\frac{z}{d}\right)^2 - \frac{1}{2M_0}\left(\frac{M_0 d}{2}+\frac{z}{d}\right)^2 \leq \varepsilon. \quad (9)$$

Let X_1 et X_2 be the roots of the equation

$$\frac{1}{2M_1}\left(\frac{z}{d}+\frac{M_1 d}{2}\right)^2 - \varepsilon - \left(\frac{z}{d}+\frac{M_0 d}{2}\right)x + \frac{M_0}{2}x^2 = 0,$$

X_1 and X_2 are given by

$$\begin{cases} X_1 = \frac{1}{M_0}\left\{\left(\frac{z}{d}+\frac{M_0 d}{2}\right) - \sqrt{\left(\frac{z}{d}+\frac{M_0 d}{2}\right)^2 - \frac{M_0}{M_1}\left(\frac{z}{d}+\frac{M_1 d}{2}\right)^2 + 2M_0\varepsilon}\right\}, \\ X_2 = \frac{1}{M_0}\left\{\left(\frac{z}{d}+\frac{M_0 d}{2}\right) + \sqrt{\left(\frac{z}{d}+\frac{M_0 d}{2}\right)^2 - \frac{M_0}{M_1}\left(\frac{z}{d}+\frac{M_1 d}{2}\right)^2 + 2M_0\varepsilon}\right\}. \end{cases}$$

In this case, G is given by

$$G(x) = g_3(x) = \begin{cases} \left(\frac{z}{d}+\frac{M_0 d}{2}\right)x - \frac{M_0}{2}x^2 \\ \text{if } \ x\in[0,X_1]\cup[X_2,d] \\ \frac{1}{2M_1}\left(\frac{z}{d}+\frac{M_1 d}{2}\right)^2 - \varepsilon \\ \text{if } \ x\in[X_1,X_2]. \end{cases}$$

We have

$$\begin{aligned} &\int_0^d \frac{\mathrm{d}x}{f^*+\varepsilon-f(x)} \\ &\leq \int_0^d \frac{\mathrm{d}x}{f^*+\varepsilon-g_3(x)} \\ &= \int_0^{X_1} \frac{\mathrm{d}x}{F_2(x^*)+\frac{M_0}{2}(x-x^*)^2} \\ &+ \int_{X_1}^{X_2}\frac{\mathrm{d}x}{\varepsilon} + \int_{X_2}^d \frac{\mathrm{d}x}{F_2(x^*)+\frac{M_0}{2}(x-x^*)^2}. \end{aligned}$$

Moreover, $M_0 = M_1$ implies the condition (9) and we have

$$\begin{cases} F_2(x^*) = 0 \\ X_1 = \frac{1}{M_0}\left\{\left(\frac{z}{d}+\frac{M_0 d}{2}\right) - \sqrt{2M_0\varepsilon}\right\} \\ X_2 = \frac{1}{M_0}\left\{\left(\frac{z}{d}+\frac{M_0 d}{2}\right) + \sqrt{2M_0\varepsilon}\right\} \end{cases}$$

Therefore

$$\int_0^{X_1} \frac{\mathrm{d}x}{\frac{M_0}{2}(x-x^*)^2} £\sqrt{\frac{2}{M_0\varepsilon}},$$

$$\int_{X_1}^{X_2}\frac{\mathrm{d}x}{\varepsilon} = \frac{4}{\sqrt{2M_0\varepsilon}},$$

$$\int_{X_2}^{d} \frac{\mathrm{d}x}{\frac{M_0}{2}(x-x^*)^2} \leq \sqrt{\frac{2}{M_0\varepsilon}},$$

and

$$\int_0^d \frac{\mathrm{d}x}{f^*+\varepsilon-g_3(x)} \leq 4\sqrt{\frac{2}{M_0\varepsilon}}.$$

Hence

$$n_{ref} \geq \frac{\sqrt{M_0\varepsilon}}{4\sqrt{2}}\int_a^b \frac{\mathrm{d}x}{f^*+\varepsilon-f(x)}.$$

Table 1. Computational results with $\varepsilon = 10^{-2}$.

Function number	Function $f(x)$	Constant M	Interval [a, b]	Optimal value f_{opt}	Optimal point(s) x_{opt}	N_C	n_{ref}	$\frac{N_C}{n_{ref}}$
1	$(-3x+1.4)\sin(18x)$	626.4	[0, 1]	1.48907	0.96609	13	7	1.85
2	$\sum_{k=1}^{5} k\sin((k+1)x+k)$	350	[−10, 10]	12.03125	−6.7745761 −0.49139 5.791785	76	44	1.72
3	$10x\sin 10x$	2100	[0, 1]	7.9197273	0.79785	16	10	1.6
4	$-\sin x-\sin 10x/3$	12	[2.7, 7.5]	1.899599	5.145735	11	6	1.83
5	$\sum_{k=1}^{5} k\cos((k+1)x+k)$	350	[−10, 10]	14.508	−7.0835 −0.8003 5.48286	69	39	1.76
6	$e^{-x}\sin 2\pi x$	51	[0, 4]	0.788685	0.224885	21	13	1.61
7	$(16x^2-24x+5)e^{-x}$	24	[1.9, 3.9]	3.85045	2.868	16	11	1.45
8	$25x-128x^2+282.5x^3$ $-278.7x^4+100.9x^5$	370	[0, 1]	1.72866	0.18916	33	24	1.37
9	$-\sin x-\sin 2x/3$	9	[3.1, 20.4]	1.90596	17.029	30	19	1.57
10	$x-\sin 3x+1$	10	[0, 6.5]	7.81567	5.87287	11	6	1.83
11	$x\sin x$	31	[0, 10]	7.91673	7.9787	22	12	1.83
12	$-2\cos x-\cos 2x$	15	[−1.57, 6.28]	1.5	2.094 4.189	26	17	1.52
13	$-\sin^3 x-\cos^3 x$	18	[0, 6.28]	1	3.142	28	18	1.55
14	$-\sin x-\sin\frac{10x}{3}$ $-\log x+0.84x-3$	12.1	[2.7, 7.5]	1.6013	5.19978	11	6	1.83

Table 2. The number of function evaluations of modified Piyavskii's algorithm for different values of *ε* and *M*.

	Test function No 1					Test function No 2					Test function No 3				
$M\varepsilon$	626.4	800	975	1500	2000	350	400	500	600	1000	2100	2150	2200	2300	3000
0.5	10	11	15	18	18	67	68	72	75	103	11	11	12	12	13
10^{-2}	13	17	21	24	26	76	78	85	86	122	16	16	17	17	18
10^{-4}	17	22	25	28	32	88	89	95	103	141	22	22	22	22	23
10^{-6}	20	26	31	36	38	96	98	108	118	160	27	27	28	28	29
10^{-8}	24	29	36	41	46	106	111	121	132	177	31	32	34	33	36

8. Computational Experiences

In this section, we report the results of computational experiences performed on fourteen test functions (see **Tables 1** and **2**). Most of these functions are test functions drawn from the Lipschitz optimization literature (see Hansen and Jaumard [20]).

The performance of the Modified Piyavskii's algorithm is measured in terms of N_C, the number of function evaluations. The number of function evaluations N_C is compared with (n_{ref}), the number of function evaluations required by the reference sequential algorithm. We observe that N_C is on the average only 1.35 larger than (n_{ref}). More precisely, we have the following estimation

$$1.35 \le \frac{N_C}{n_{ref}} \le 1.85 .$$

For the first three test functions, we observe that the influence of the parameter M is not very important, since the number of function evaluations increase appreciably for a same precision ε.

REFERENCES

[1] R. Horst and H. Tuy, "Global Optimization-Deterministic Approaches," 2nd Edition, Springer, Berlin, 1992.

[2] S. A. Piyavskii, "An Algorithm for Finding the Absolute Minimum of a Function," *USSR Computational Mathematics and Mathematical Physic*s, Vol. 12, No. 4, 1967, pp. 13-24.

[3] S. A. Piyavskii, "An Algorithm for Finding the Absolute Minimum of a Function," *USSR Computational Mathematics and Mathematical Physic*s, Vol. 12, No. 4, 1967, pp. 57-67. doi:10.1016/0041-5553(72)90115-2

[4] B. O. Shubert, "A Sequential Method Seeking the Global Maximum of a Function," *SIAM Journal on Numerical Analysis*, Vol. 9, No. 3, 1972, pp. 379-388. doi:10.1137/0709036

[5] P. Basso, "Iterative Method for the Localization of the Global Maximum," *SIAM Journal on Numerical Analysis*, Vol. 19, No. 4, 1982, pp. 781-792. doi:10.1137/0719054

[6] P. Basso, "Optimal Search for the Global Maximum of Function with Bounded Seminorm," *SIAM Journal on Numerical Analysis*, Vol. 22, No. 5, 1985, pp. 888-903. doi:10.1137/0722053

[7] F. Schoen, "On a Sequential Search Strategy in Global Optimization Problems," *Calcolo*, Vol. 19, No. 3, 1982, pp. 321-334. doi:10.1007/BF02575808

[8] Z. Shen and Y. Zhu, "An Interval Version of Shubert's Iterative Method for the Localization of the Global Maximum," *Computing*, Vol. 38, No. 3, 1986, pp. 275-280. doi:10.1007/BF02240102

[9] R. Horst and H. Tuy, "On the Convergence of Global Methods in Multi extremal Optimization," *Journal of Optimization Theory and Applications*, Vol. 54, No. 2, 1987, pp. 253-271. doi:10.1007/BF00939434

[10] Y. D. Sergeyev, "Global One-dimensional Optimization Using Smooth Auxiliary Functions," *Mathematical Programming*, Vol. 81, No. 1, 1998, pp. 127-146. doi:10.1007/BF01584848

[11] D. R. Jones, C. D. Perttunen and B. E. Stuckman, "Lipschitzian Optimization without the Lipschitz Constant," *Journal of Optimization Theory and Applications*, Vol. 79, No. 1, 1993, pp. 157-181. doi:10.1007/BF00941892

[12] Y. M. Danilin and S. A. Piyavskii, "An Algorithm for Finding an Absolute Minimum," *USSR Computational Mathematics and Mathematical Physic*s, Vol. 12, No. 4, 1967, pp. 25-37.

[13] D. Q. Mayne and E. Polak, "Outer Approximation Algorithm for Non Differentiable Optimization Problems," *Journal of Optimization Theory and Applications*, Vol. 42, No. 1, 1984, pp. 19-30. doi:10.1007/BF00934131

[14] R. H. Mladineo, "An Algorithm for Finding the Global Maximum of Multimodal, Multivariate Function," *Mathematical Programming*, Vol. 34, No. 2, 1986, pp. 188-200. doi:10.1007/BF01580583

[15] C. C. Meewella and D. Q. Mayne, "An Algorithm for Global Optimization of Lipschitz Functions," *Journal of Optimization Theory and Applications*, Vol. 57, No. 2, 1988, pp. 307-323. doi:10.1007/BF00938542

[16] Y. G. Evtushenko, "Numerical Methods for Finding the Global Extremum of a Function," *USSR Computational Mathematical Physics*, Vol. 11, No. 6, 1971, pp. 38-54. doi:10.1016/0041-5553(71)90065-6

[17] E. A. Galperin, "The Cubic Algorithm," *Journal of Mathematical Analysis and Application*s, Vol. 112, No. 2, 1985, pp. 635-640. doi:10.1016/0022-247X(85)90268-9

[18] P. Hansen, B. Jaumard and S. H. Lu, "Global Optimization of Univariate Lipschitz Functions: I. Survey and Properties," *Mathematical Programming*, Vol. 55, No. 1-3, 1992, pp. 251-272. doi:10.1007/BF01581202

[19] P. Hansen, B. Jaumard and S. H. Lu, "Global Optimization of Univariate Lipschitz Functions: II. New Algorithms and Computational Comparaison," *Mathematical Programming*, Vol. 55, No. 1-3, 1992, pp. 273-292. doi:10.1007/BF01581203

[20] P. Hansen and B. Jaumard, "Lipschitz Optimization," In: R. Horst and P. M. Pardalos, Eds., *Handbook of Global Optimization*, Kluwer Academis Publishers, Dordrecht, 1994, pp. 407-493.

[21] R. P. Brent, "Algorithms for Minimization without Derivatives," Prentice-Hall, Englewood Cliffs, 1973.

[22] S. E. Jacobsen and M. Torabi, "A Global Minimization Algorithm for a Class of One-Dimensional Functions," *Journal of Mathematical Analysis and Applications*, Vol. 62, No. 2, 1987, pp. 310-324. doi:10.1016/0022-247X(78)90128-2

[23] L. Breiman and A. Cutler, "A Deterministic Algorithm for Global Optimization," *Mathematical Programming*, Vol. 58, No. 1-3, 1993, pp. 179-199. doi:10.1007/BF01581266

[24] W. Baritompa and A. Cutler, "Accelerations for Global Optimization Covering Methods Using Second Derivatives," *Journal of Global Optimization*, Vol. 4, No. 3, 1994, pp. 329-341. doi:10.1007/BF01098365

[25] Y. M. Danilin, "Estimation of the Efficiency of an absolute Minimum Finding Algorithm," *USSR Computational Mathematical Physics*, Vol. 11, No. 4, 1971, pp. 261-267. doi:10.1016/0041-5553(71)90020-6

[26] P. Hansen, B. Jaumard and S. H. Lu, "On the Number of Iterations of Piyavskii's Global Optimization Algorithm," *Mathematics of Operations Research*, Vol. 16, No. 2, 1991, pp. 334-350. doi:10.1287/moor.16.2.334