Modified Teleparallel theories of gravity

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A thesis submitted in conformity with the requirements for the degree of PhD in Applied Mathematics

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September 06, 2018

Abstract

Teleparallel gravity is an alternative formulation of gravity which has the same field equations as General Relativity (GR), therefore, it also known as the Teleparallel equivalent of General Relativity (TEGR). This theory is a gauge theory of the translations with the torsion tensor being non-zero but with a vanishing curvature tensor, hence, the manifold is globally flat. An interesting approach for understanding the late-time accelerating behaviour of the Universe is called modified gravity where GR is extended or modified. In the same spirit, since TEGR is equivalent to GR, one can consider its modifications and study if they can describe the current cosmological observations. This thesis is devoted to studying several modified Teleparallel theories of gravity with emphasis on late-time cosmology. Those Teleparallel theories are in general different to the modified theories based on GR, but one can relate and classify them accordingly. Various Teleparallel theories are presented and studied such as Teleparallel scalar-tensor theories, quintom models, Teleparallel non-local gravity, and $f(T,B)$ gravity and its extensions (coupled with matter, extensions of new GR and Gauss-Bonnet) where T is the scalar torsion and B is the boundary term which is related with the Ricci scalar via $\overset{\circ}{R} = -T + B$.

This thesis was completed under the supervision of ***Christian Böhmer***.

Impact statement

The new results showed in this thesis are based on eleven different published scientific papers. Basically, this thesis aimed at two directions, establishing new research into some questions within modified gravity and cosmology, in particular, for theories based on Teleparallel gravity. There is no foreseen impact outside of academia at this stage.

One impact of this thesis was the possibility of determining a certain relationship between standard modified theories of gravity based on General Relativity (GR) and other modifications based on TEGR. Many of these theories were considered in isolation in the past and their relationship with other similarly looking theories was only made implicitly. Further, in this thesis, it was showed how two very popular modified theories coming from these two different approaches, $f(\overset{\circ}{R})$ (based on GR) and $f(T)$ gravity (based on TEGR), are related and could construct the fundamental theory underlying them both.

Another impact of this thesis is related to constructing a viable modified Teleparallel theory which could describe the evolution of the Universe without evoking any cosmological constant. This tells us that modifications of Teleparallel gravity could be good potential candidates for describing the dark energy problem.

An additional impact related to a quantum gravity approach was also shown in this thesis. There was proposed a new non-local Teleparallel theory of gravity which is the first attempt to integrate this fundamental physical concept from quantum gravity into Teleparallel gravity. Moreover, in different studies related to non-locality (based on GR), researchers found that exponential type of couplings were able to achieve a renormalisable theory of gravity. In all of those studies, researchers put those terms by hand. In one work presented in this thesis, it was shown that these exponential couplings appear naturally by using first physical principles with the symmetries of the Lagrangian. Therefore, this study had an impact in resolving this aspect about non-locality.

Preface

Acknowledgments

I would like to thank all the people who influenced and helped me throughout my PhD studies.

Firstly, I would like to thank my supervisor Christian Böhmer for his fantastic supervision over these four years. He has been an inspiration to me, not only as an excellent researcher but also as a person. It was very easy to be his student due to his supporting and easy going way of thinking. He has given me much good advice to become a better researcher and always been very objective and helpful. I also thank him for giving me the opportunity of doing independent research in other subjects and with other researchers.

Secondly, I would like to thank all my international collaborators who have helped me to learn many different subjects and gave me the opportunity to publish different papers. Explicitly, I would like to thank in alphabetical order, the following collaborators: Salvatore Capozziello, Mir Faizal, Eduardo I. Guendelman, Mubasher Jamil, Francisco Lobo, Sergei D. Odintsov and Muhammad Zubair.

Thirdly, I would like to thank Angela Cooper who has helped me a lot proofreading my thesis and for giving me very good feedback. I hope that she can be recover soon.

I would also like to thank my friends who have supported me in this journey. Special thanks to my friends: Konstantinos F. Dialektopoulos, Manuela Irarrázabal, Martin Krššák, Catalina Maturana, Mihai Marciu, Victor Montenegro, Gabriel Saavedra, Pietro Servini and Belgin Seymenoglu.

I would also like to thank all my family for supporting me not only in my PhD, but in all my studies, specially my parents Hernán and Pilar, my brothers Cristóbal and Joaquín and my grandparents Fridia, Ester and Moisés. Without them, I would not have been able to follow my dreams to study the subject that I love in a foreign country.

I would also like to thank the love of my life, Maria-José Vera who has been the most important person throughout this PhD. Without her love, support and caring, it would not have been possible to finish this PhD and also my life would not been so complete as it is. All the experiences shared with her gave me the power to continue and also to live the best years of my life.

Finally, I would like to acknowledge financial support from the Chilean government for supporting me through my doctoral scholarship Becas Chile (CONICYT).

To my family.

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1. S. Bahamonde, C. G. Boehmer, S. Carloni, E. J. Copeland, W. Fang and N. Tamanini, "Dynamical systems applied to cosmology: dark energy and modified gravity," [arXiv:1712.03107 [gr-qc]] (to appear in Physics Reports).

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3. S. Bahamonde, "Generalised nonminimally gravity-matter coupled theory," Eur. Phys. J. C **78** (2018) no.4, 326, [arXiv:1709.05319 [gr-qc]].

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1. M. Zubair, F. Kousar and S. Bahamonde, "Static spherically symmetric wormholes in generalized $f(R, \phi)$ gravity," [arXiv:1712.05699 [gr-qc]].

2. S. Bahamonde, D. Benisty and E. I. Guendelman, "Linear potentials in galaxy halos by Asymmetric Wormholes," [arXiv:1801.08334 [gr-qc]].

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Contents

1 Introduction

After the discovery that our Universe is expanding in an accelerating rate in 1998, our comprehension of it has dramatically changed. This scenario was in a completely opposite direction from what researchers expected and the responsible for this was labelled as dark energy. It turns out that General Relativity (GR) can describe this phenomenon by introducing a cosmological constant which acts like a fluid with a negative pressure and violates the energy conditions making effectively a kind of repulsive gravitational force. Due to the lack of theoretical motivations related to this constant, other approaches have been formulated to explain this late-time accelerating behaviour of the Universe. One approach which is important for this thesis is the one considered by the so-called modified theories of gravity. In this approach, one considers that GR can be extended or modified and such differences could explain the cosmological observations. There are other motivations on modifying GR such as: explaining dark matter, inflation, the theory being non-renormalisable, etc.

GR is based on a specific connection which is called the Levi-Civita connection which is symmetric and torsionless. This is of course, not the most general or unique way to describe gravity. This thesis will be focused on an alternative way to describe gravity which is called Teleparallel gravity. This theory is based on the Weitzenböck connection which is a skew-symmetric and curvatureless one with a non-zero torsion tensor. This theory is equivalent to GR on field equations, therefore, it is also called the Teleparallel equivalent of General Relativity (TEGR). Then, this is an alternative and equivalent way to describe gravity, having the same observational predictions as GR. However, the physical and mathematical interpretations are different. Since modified gravity has been very successful in cosmology, one can also consider modifications of Teleparallel gravity and explore its consequences. In general, modifications of TEGR are no longer equal to modifications of GR. This thesis will deal with modifications of TEGR and will try to make a comparison and classify them with respect to modified theories based on GR. Moreover, this study is also conducted in order to study some cosmological models that one can formulate from modifying TEGR. The emphasis will be given to the late-time accelerating behaviour of the Universe.

Even though modifications of GR are very popular in order to tackle some of the issues that appear in the theory, modifications to Teleparallel gravity were proposed only some years ago. Then, the following question arises: are modified Teleparallel theories of gravity good candidates to solve or alleviate some cosmological problems such as the dark energy of dark matter problem? In terms of cosmology, is there any advantage in modified Teleparallel gravity than standard modifications? Is there any way to connect or classify those theories? Several modified Teleparallel theories will be presented in this thesis in order to try to answer some of those questions.

This thesis is organised as follows: In Chap. 2, the Theory of General Relativity is presented. The physical and mathematical principles underlying this theory are described which give us the notion on how gravity can be described by geometrical quantities such as curvature or the metric. Then, a brief discussion about standard Friedman-Lemaître-Robertson-Walker (FLRW) cosmology is given with the ΛCDM model being presented as the easiest way to describe the cosmological observations. Chap. 3 is devoted to introducing Teleparallel gravity as an alternative and equivalent (in field equations) way to describe gravity. This theory can be written as a gauge theory of the translations where the torsion tensor is the field strength of the theory. Its equations and equivalence with GR is also shown in this chapter. Chap. 4 deals with the most popular modifications of TEGR, the so-called $f(T)$ gravity and Teleparallel scalar-tensor theories. The issue of the lack of Lorentz covariance in modified Teleparallel theories is also discussed and then, the "good tetrad" approach is presented as being a way to alleviate this issue. New Teleparallel scalar-tensor theories are presented and a extended quintom model is studied in the context of cosmology. In Chap. 5 there is presented and studied an interesting Teleparallel model labelled as $f(T,B)$ gravity which is directly connected to the so-called $f(\overset{\circ}{R})$ gravity which is one of the most popular modified theories based on GR. Then, flat FLRW cosmology will be studied using the Noether symmetry approach and the reconstruction method. After that, a generalised model which considers non-minimally couplings with matter is presented giving also a complete classification of how those generalised theories are related to theories based on GR. Chap. 7 is dedicated to present new classes of modified Teleparallel models based on the decomposition of the torsion scalar. Chap. 6 will introduce a generalisation of $f(T,B)$ by including Teleparallel Gauss-Bonnet terms and the trace of the energy-momentum tensor. As a final model, Chap. 8 presents a non-local theory of gravity based on Teleparallel gravity. This theory is introduced motivated by quantum gravity. The cosmology of those kind of models is presented and studied numerically and using the Noether symmetry approach. Finally, Chap. 9 finishes this thesis with some concluding remarks.

2

The Theory of General relativity

Chapter Abstract

This chapter is devoted to briefly introducing the most important conceptual and mathematical characteristics of General Relativity. A discussion related to the principles underlying the theory is presented. The most important mathematical and geometrical quantities are defined and the Einstein field equations are stated. A brief introduction to cosmology concludes this chapter.

2.1 A theory of curved spacetime

General Relativity (GR) was proposed by Einstein in 1915, changing the way we understand gravity. He introduced the idea of a dynamic spacetime, where gravity is explained as a geometric consequence, where matter and energy deform spacetime. Any source of energy curves the spacetime, which creates the notion that another particle is attracted by it. The idea of Newtonian gravity, based on the gravitational force is replaced by the notion of a deformable spacetime. It relies on several principles such as the equivalence principle which assumes an equivalence between accelerated frames and gravitational fields. It is assumed that the gravitational mass m_{g} is exactly the same as the inertial mass m_{i} (Roll et al., 1964).

GR has been a very successful theory. It has been tested with good precision at Solar System scales. GR also describes new phenomena that in standard Newtonian gravity are not present. Some of them are: gravitational lensing, gravitational redshift, gravitational waves, black holes, etc. Some of these effects have been measured in different scenarios. Further, gravitational waves were discovered in 2016 giving us the last untested verification of GR and a new window for astronomical observations.

In the following, the main mathematical and physical properties of GR will be introduced. The aim is to only briefly explain the most relevant concepts. For a more detailed descriptions of GR, see the classical books by Weinberg (1972); Misner et al. (1973) ; Wald (1984) and also more modern books D'Inverno (1992); Böhmer (2016).

2.2 Basic principles of the theory

Before going further in the mathematical description of General Relativity, the principles that are assumed to be true which are the basis of the theory will be described.

2.2.1 Equivalence principle

The equivalence principle is one of the most important ingredients of General Relativity. This principle states that there is an equivalence between the measurements obtained by an observer who is immersed in a gravitational field in an inertial reference frame, and an observer in the absence of a gravitational field in an accelerated reference frame. This principle can be separated into two parts: the *weak equivalence principle* and the *strong equivalence principle*.

2.2.1.1 Weak equivalence principle

The weak equivalence principle is important not only in General Relativity, but also in the Newtonian gravitational law. It formally states that:

> *"The movement of a test particle in a gravitational field is independent of its mass and composition."*

It should be clarified that a test particle is a particle which is affected by a gravitational field but it does not make any modification or contribution to it. Hence, if a body is free-falling and only gravitational forces are acting on it, its mass and composition will not play any role on its movement. Here is given an easy example of this effect. Suppose that an elephant and a mouse start falling down at a time t_0 from a tall building as is depicted in Fig. 2.1. Also, suppose that there is no air resistance so the only force acting on them is the gravitational force of the Earth. The weak equivalence principle states that even though the elephant has a much bigger mass than the mouse, they must reach the ground at the same time t_f. This principle is not something new since different physicists noticed this effect experientially in the past. Galileo Galilei in the 17th century observed experimentally this effect by dropping two different objects at the same time with different compositions and mass from the top of the Tower of Pisa. For this reason, this principle is also known as the *Galilean equivalence principle*. Later, Isaac Newton and Friedrich Wilhelm also noticed a similar effect by measuring the period of different pendulums with different masses and identical lengths. Even though these experiments were carried out with old experimental techniques, all of them suggested that the mass and composition of a body in a gravitational field do not contribute to its movement (assuming zero external forces other than gravity). For example, the astronauts from Apollo 11 who landed on the Moon in 1969 also carried out an experiment showing that a hammer and a feather dropped at the same height and time, reached the ground of the Moon at the exact same moment (see Shapiro et al. (1976) for details). This experiment was possible since the Moon has no atmosphere. More recent experiments have also observed this effect with great precision (see for example Baessler et al. (1999)).

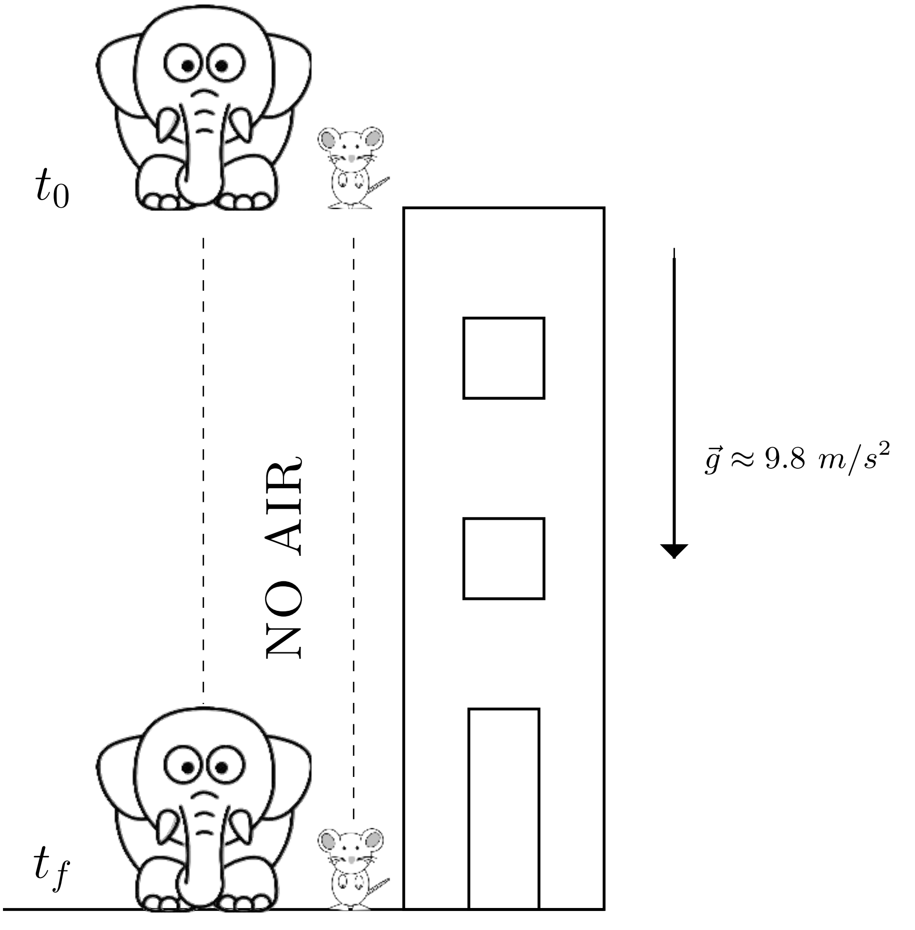


Figure 2.1: Representation of the weak equivalence principle.

Technically speaking, this principle states that the *inertial mass* will have exactly the same value as the *gravitational mass*. On the one hand, the inertial mass refers to the mass which appears in the second law of Newton. This quantity measures how resistant a certain body is to change its movement when a force acts on it. On the other hand, the gravitational mass is directly related to the gravitational forces and is a measure of how strong (or weak) a certain body will attract another. Physically speaking, there is not a fully well-explained reason on why these two quantities need to be equal. Since all the experiments suggest that these masses have the same value, then, this assumption is needed for a consistent theory of gravity. Moreover, modern experiments have shown that both masses are equal with a precision of 10^{-13} in order of magnitude (Baessler et al. (1999)). The question whether the gravitational mass and the inertial mass are always the same is somehow one of the most important and basis principles known in physics. Since this principle is based on observations, it will always be important to check if this principle is valid at all scales also with much higher precision. Further, there currently exist new projects such as the Satellite Test of the Equivalence Principle (STEP) and the CNES (*Sur le site du Centre national d'études spatiales*) micro-satellite, which are supposed to increase the precision of measurements up to around 10^{-17} of order in magnitude (Touboul et al. (2012)).

A brief description will show mathematically how this principle works using Newtonian laws. Consider a particle of inertial mass m_{i} and gravitational mass m_{g} which is moving in a certain gravitational field $\vec{g}$. If one assumes that there are no external forces, the force which acts on the particle will be given by

$$\vec{F} = m_{\mathrm{i}}\vec{a} = m_{\mathrm{g}}\vec{g}\,, \tag{2.1}$$

where $\vec{a}$ is the acceleration of the particle. Hence, this equation can be also expressed as

$$\vec{a} = \left(\frac{m_{\mathrm{g}}}{m_{\mathrm{i}}}\right)\vec{g}\,. \tag{2.2}$$

From here one can directly see that the trajectory of the particle depends on the gravitational field, its inertial mass and also on its gravitational mass. Consider that $\vec{x}(t)$ is the position vector of the particle measure with respect to a reference frame O, then the above equation will be

$$\frac{d^2\vec{x}(t)}{dt^2} = \left(\frac{\cancel{m_{\mathrm{g}}}}{\cancel{m_{\mathrm{i}}}}\right)\vec{g} = \vec{g}\,, \tag{2.3}$$

where it is assumed that the weak equivalence principle is always valid giving us $m_{\mathrm{i}} = m_{\mathrm{g}}$. Since both masses cancel in the above equation, the trajectory of the particle only depends on the gravitational field and does not depend on its mass nor composition. If one assumes that the gravitational field is constant and homogeneous, then the acceleration will also be constant. Since the acceleration is constant, the trajectory of the particle with respect to O will be either a straight line or a parabola.

2.2.1.2 Strong equivalence principle

Now, let us also infer another important result from Eq. (2.3). Consider another reference frame O' which is accelerating constantly with respect to the reference frame O. The reference coordinate systems are related as follows

$$t' = t\,, \quad \vec{x}' = \vec{x} - \frac{1}{2}\vec{a}\,t^2\,, \tag{2.4}$$

where $\vec{a}$ is the acceleration. By taking second derivatives with respect to time, it can easily be found that

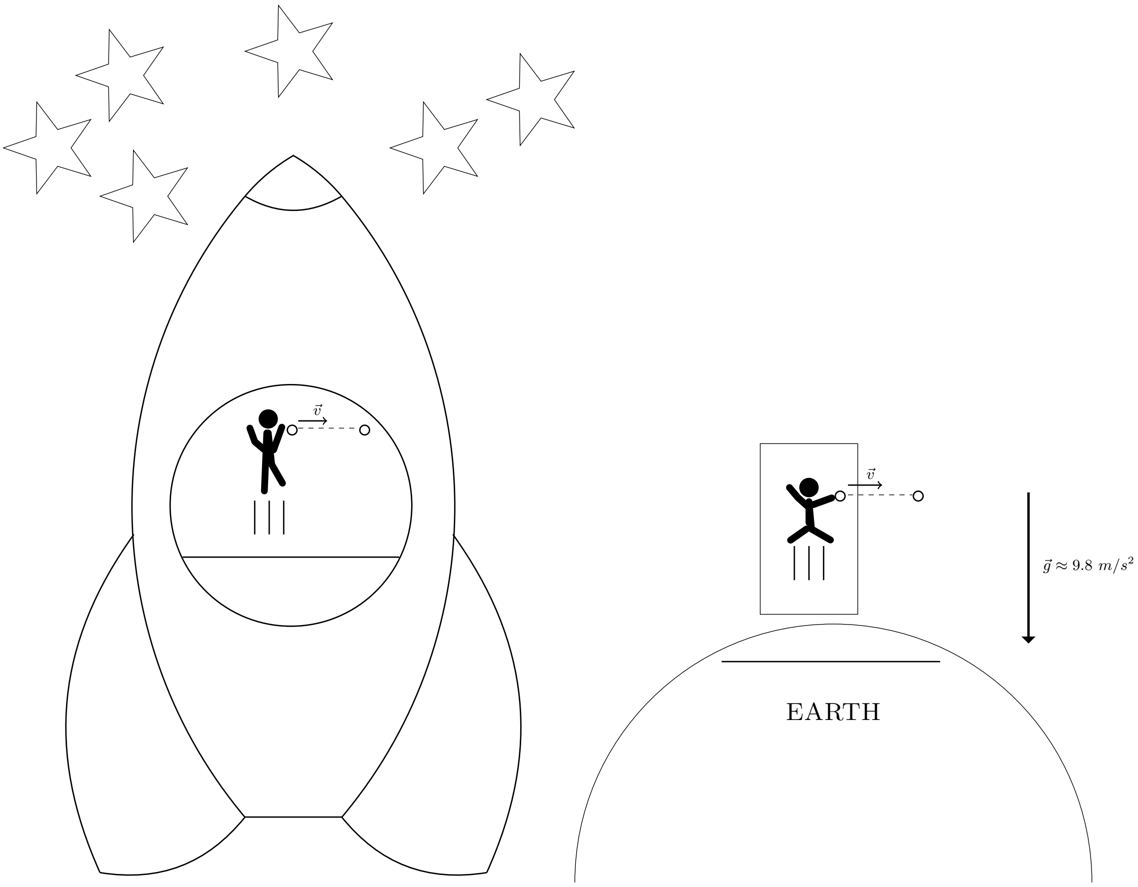
$$\frac{d^2\vec{x}'}{dt'^2} = \frac{d^2\vec{x}}{dt^2} - \vec{a}\,. \tag{2.5}$$

Now, by replacing Eq. (2.3) in the above equation, it gives us

$$\frac{d^2\vec{x}'}{dt'^2} = \vec{g} - \vec{a}\,. \tag{2.6}$$

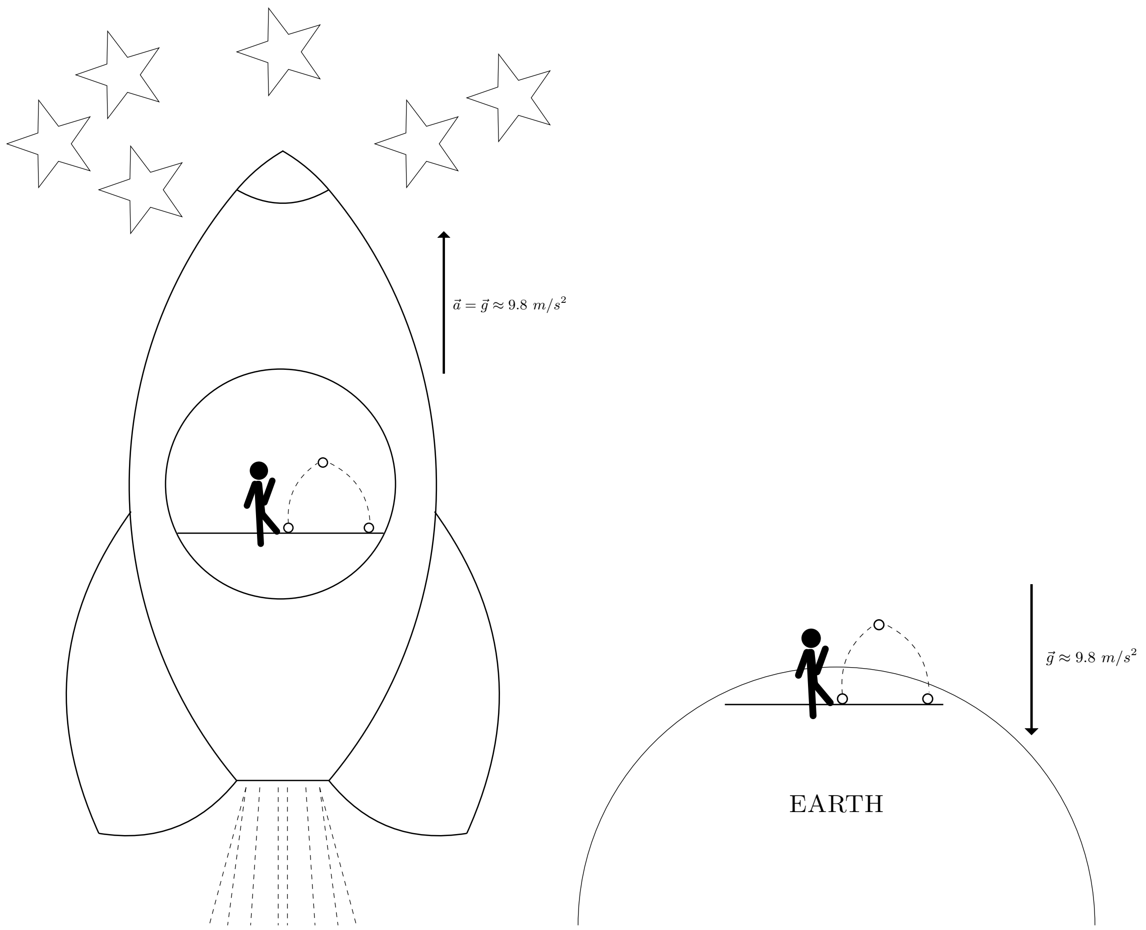
From this equation it can be noticed that there is a certain equivalence between an accelerated reference frame and a homogeneous constant gravitational field. For instance, if one takes $\vec{a} = \vec{g}$, i.e., if the reference frame O' is a free-falling reference frame, the trajectory which follows the particle with respect to the observer O' is a straight line with constant velocity with respect to O'. From here, one can infer that for any homogeneous and constant gravitational field, an observer who is free-falling will not experience any gravitational effects. Hence, it is possible to eliminate all the effects produced by a homogeneous constant gravitational field in a reference frame which is free-falling. Fig. 2.2b represents a free-falling frame where a person is inside a lift which is falling towards the Earth since its rope was cut. In this case, the term $\vec{a} = \vec{g}$ in Eq. (2.6) and then $d^2\vec{x}'/dt'^2 = 0$. If the person throws a ball, its trajectory must follow a straight line with constant velocity $\vec{v}$ with respect to the person. The movement of the ball will not experience any gravitational force. Fig. 2.2a represents a frame where a person is inside a rocket in the space in such a way that all the gravitational forces acting on it are zero. In this case $\vec{g} = \vec{a} = \vec{0}$ and then again one has $d^2\vec{x}'/dt'^2 = 0$. Therefore, if the person throws a ball, its trajectory will be identical to the other mentioned frame where the person is inside the elevator falling towards the Earth. Thus, there is an equivalence between the movement of the ball inside the lift and inside the rocket. Moreover, any experiment made in a frame as Fig. 2.2a will be equivalent to other experiments made in a frame as Fig. 2.2b. So, it could be said that both frames are indistinguishable from each other. This effect has been tested experimentally many times, see for example (Will, 2014). One interesting experiment is the *Zero Gravity facility* by NASA which uses a reduced gravity aircraft to test this effect. Since the 1960s, this special spacecraft has been used to experience a zero gravity environment by falling down for about 5 seconds. By doing this, the objects inside the vehicle are in free-fall due to the gravitational force of the Earth and hence, for some time, it is possible to eliminate all the gravitational effects inside the spacecraft. Another experiment is carried out by *ZARM* (the centre of applied zero technology and microgravity) located in Bremen. They have a special tower known as *Bremen Drop Tower*, where it is also possible to experiment highest-quality conditions of weightlessness, comparable to one millionth of the Earth's gravitational force.

This principle can also be understood the other way around. One in principle can replicate all the gravitational effects on its corresponding accelerating inertial frame. As an example, Fig. 2.3 shows two other physically equivalent reference frames. In the first frame, a person is inside a rocket in the space where all the gravitational effects are zero. In this frame, the term $\vec{g} = 0$ in Eq. (2.6). Now, consider that this rocket is accelerating constantly exactly as the gravitational acceleration in the Earth $\vec{a} = \vec{g}$ as it is drawn in Fig. 2.3a. Now, if this person shoots a ball, this ball will follow a parabola exactly as in the Earth. Moreover, the person inside the rocket cannot make any experiment to distinguish if he/she is on an accelerated rocket or he/she is on the surface of the Earth as Fig. 2.3b. Hence, physically speaking, the situations described in Figs. 2.3a and 2.3b are indistinguishable, making them equivalent.



(a) Person in a rocket. (b) Person in a lift falling towards the Earth.

Figure 2.2: Two equivalent reference frames showing the strong equivalence principle.



(a) Person in an accelerating rocket. (b) Person on the Earth.

Figure 2.3: Two equivalent reference frames showing the strong equivalence principle.

A similar thought experiment as is drawn in Fig. 2.2 was carried out by Einstein after he developed his Theory of Special Relativity. He thought about what would happen if someone is in an elevator and suddenly the rope is cut. If this happens, the elevator will start falling down towards the Earth and hence, a person inside the elevator will be in a free-falling reference frame. From the Einstein's perspective, he realised that since the person will not experience any gravitational effect when he/she is falling down, then there must be some connection between gravitational fields and accelerated reference frames. This thought experiment was important for him so he postulated the strong equivalence principle, which says:

> *"Locally, the behaviour of the matter in an accelerated reference frame can not be distinguished from the behaviour of it on its corresponding gravitational field."*

This principle states that if one considers a free-falling reference frame where the observers are constrained in a sufficiently small region where the inhomogeneities of a gravitational field are negligible, then this reference frame will be an inertial frame (locally). This kind of reference frame is called *local inertial reference frames.* All the known physical laws can be applied in inertial reference frames. Hence, using this principle, it is possible to connect all ideas of Special Relativity with systems which are immersed in gravitational fields. Therefore, when a certain gravitational field exists, it is possible to reproduce all the known physics for inertial reference frames if one considers a sufficiently small region which is in a free-falling reference frame. If one neglects all other forces, in these kinds of local reference frames, a test particle will be at rest or moving in a straight line with constant velocity.

An important consequence of this principle is related to the concept of gravity. The Newtonian concept of gravity states that a certain massive object with a certain mass produces an attractive gravitational force to another body. This results in producing that they experience movement (with respect to the mass centre of the system). In the next section, this idea will be discussed mathematically (see Sec. 2.3). Einstein changed this idea radically by saying that the matter (any kind of matter described by the energy-momentum tensor) would produce that the spacetime (an entity defined in Special Relativity which connects the notion of space and time) now will deform or curve in such a way that the objects/matter will experience this attraction force between them. Hence, gravity is now described as an effect due to the curvature of the spacetime and all kind of matter (even energy) can produce this kind of deformation. If one object contains more mass/energy, then the spacetime will be more curved. This idea can be summarised by using one of most famous John Wheeler's quotes: "Spacetime tells matter how to move; matter tells spacetime how to curve" (Wheeler & Ford, 2000). This will be discussed further in Sec. 2.4. Fig. 2.4 represents a pictorial view of the notion of gravity introduced by Einstein. In this picture, there is a big massive object (that for example this could be the Sun) and a smaller massive object (that could be the Earth). Both objects curve the spacetime but the more massive one curves it more. This deformation of the spacetime creates the smaller object to orbit around the bigger massive body (strictly speaking, it orbits around the centre of mass of the system).

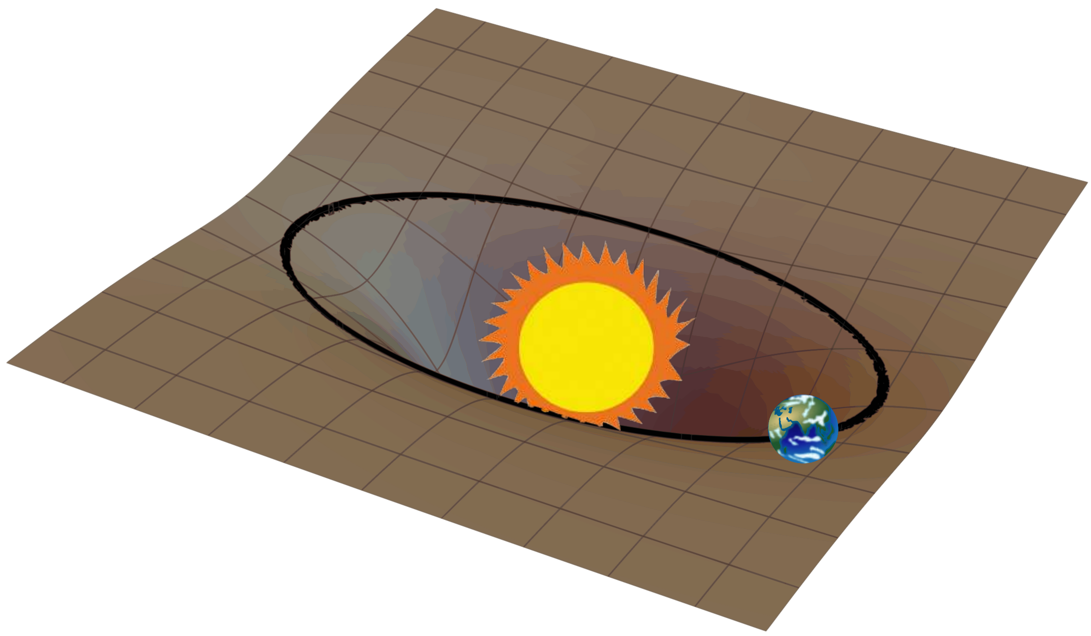


Figure 2.4: Representation of how a big massive object (the Sun) curves the spacetime making another small body (the Earth) orbits around the big massive object (or strictly speaking, orbits the centre of mass of the whole system).

A very important example of this mentioned effect is that light is deflected when it passes near a gravitational field, the so-called *light deflection*. This effect happens since the matter produced a curved spacetime so that when a ray of light passes near this curved region, it will experience a change in its direction (and even it can be magnified). This phenomenon was first observed by the astronomers Dyson, Eddington and Davidson in 1919 (Dyson et al. (1920)), only 4 years after Einstein proposed his Theory of General Relativity. To do this, they studied different stars located in the same direction as the Sun and during a Solar eclipse, they measured that the light coming from those stars were deflected by the gravitational field produced by the Sun. General Relativity predicts that these stars must appear in a different position due to the deformation of the spacetime of the Sun since their light must be influenced on their path to the Earth for being in the same direction as the Sun. This experiment was the first which verified the Theory of General Relativity. As pointed out before, this theory was a paradigm shift in physics since some researchers at that time were a little sceptical about the validity of GR. This experiment was crucial since gave a lot of credibility to the theory. More recent experiments have been carried out with high precision matching also with the GR's predictions (Will (2014)).

Another important prediction of General Relativity which directly emerges from the equivalence principle is the *gravitational time dilation*. GR says that for an observer who is closer to a matter source, its proper time will be slower. Hence, a person located on the surface of the Earth will measure a different proper time than an astronaut who is inside the International Space Station. Indeed, this effect has been measured several times and actually is one of the most important effects that needs to be taken into account for the correct operation of artificial satellites and GPS systems. Since the satellites which are used for GPS are not located on the Earth, their clocks will be slightly faster than the clocks on the Earth. The full phenomenon is a little more complicated since the satellites are also moving with respect to the Earth so another correction needs to be added to the clocks since Special Relativity predicts that the proper time also changes when a source is moving with respect to another (dilation of time). The later effects produce that the clocks of the satellites are slightly slower than a clock situated on the Earth. If one combines both effects, dilation of time due to the movement of the satellite (slow proper time) and the gravitational time dilation due to the fact that the satellite is a certain distance of the Earth (faster proper time), the clocks on-board are slightly faster than a clock located on the Earth. It has been measured that the combined effect is around 38 microseconds per day (Ashby, 2002). The error margin predicted by the GPS system is of the order of 15 metres, so it requires sensitivity on time of about 15 m/c (where c is the velocity of the light). This sensitivity is around 50 nano-seconds, so even though the delay time produced by relativistic effects is small, it is significant since the sensitivity of the instruments is very high. Further, if one does not take into account these relativistic effects, the clocks on-board would make the whole GPS system to make an incorrect position in only about 2 minutes and it would accumulate at a rate of about 10 kilometres per day. Therefore, the gravitational time dilation is an important effect that needs to be taken into account in a correct operating GPS system. This is one of the most important practical applications of General Relativity.

To conclude, the equivalence principle is one of the most important theoretical postulates in physics. Einstein used this principle as an initial point to develop his theory and by using it, all the kind of effects mentioned before appear naturally in the theory. Some of these consequences will be mathematically explained afterwards but some of them will not be shown since it is not the main aim of this thesis.

2.2.2 General Covariance principle

Another important principle which is assumed to be valid in GR is the covariance principle, which can be summarised as follows:

> *"Field equations must be written in a tensorial form so they must be covariant under an arbitrary but smooth transformation of the spacetime coordinates. In*

other words, they must be invariant under spacetime diffeomorphisms."

It is well-known that Special Relativity was founded on the basis of inertial observers where all of them are equivalent. In GR, Einstein incorporated non-inertial observers. As pointed out before, he introduced the idea of local inertial reference frames, where in a small region of the spacetime all the inhomogeneities of a gravitational field can be neglected. From the equivalence principle, accelerated reference frames (non-inertial reference frames) are equivalent locally to a frame which is not accelerating but immersed in its corresponding gravitational field. Since it is always possible to locally define a local inertial reference frame by choosing a small region of the spacetime in such a way that all the gravitational effects are neglected, Einstein incorporated all the well-known physics coming from inertial reference frames. Since the laws of physics cannot change from one observer to another, if one can find some laws with respect to a specific reference frame, they must be also valid for any other reference frame.

This principle also states that the field equations must be invariant under diffeomorphisms which requires a smooth (one-to-one) map between manifolds. Basically, if the equations are invariant under diffeomorphisms, this set of mapping preserves the structure of the manifold. This statement can be separated into two parts: *passive* and *active diffeomorphism invariant principles*. The former is intrinsically valid in all theories in physics. This principle says that a specific object in a theory (for example, a metric) can be represented in different coordinate systems but it cannot represent different physical consequences. The active diffeomorphism associates different objects in a certain manifold in the same coordinate system. Hence, a diffeomorphism map $f : x^{\mu} \rightarrow f^{\mu}(x^{\nu})$ associates one point to the manifold to another one. For more details regarding the diffeomorphism invariance, see Gaul & Rovelli (2000).

The covariant principle is relevant since it tells us that if one is able to formulate field equations in a tensorial form (quantities which are invariant under general transformations of spacetime coordinates), they must be also true in all other reference frames.

2.2.3 Lorentz covariance principle

The majority of the physics known assumes the validity of the Lorentz covariance principle which can be summarised as follows:

> *"Field equations must be valid in all inertial frames in such a way that experimental results are independent of the boost velocity or the orientation of a laboratory though space."*

This principle ensures that the experimental results must coincide in different inertial frames. This principle of course is related to the general covariance principle but it is more restricted since the quantities must be also covariant under Lorentz transformations. Mathematically speaking, this means that all the physical quantities must transform under the Lorentz group. As mentioned before, GR defined the locally inertial frames in a sufficiently small region of space. So, GR also assumes a locally Lorentz covariance principle which says that for any infinitesimal region, all the physical quantities must also be Lorentz covariant. In this sense, Einstein's field equations are invariant under a Lorentz transformation. This makes a consistency theory with respect to Special Relativity in the absence of gravitational fields. There are some modified theories of gravity which do not obey this principle. As will be seen in a forthcoming section, $f(T)$ gravity is an example of this (see Chap. 4). However, there is a way to alleviate this problem in this theory which will be discussed later.

2.2.4 Causality and relativity principles

Another two important principles which are implicit in GR are the causality and the relativity principles. The former one can be summarised as follows

> *Causality principle: "Each point in the spacetime must have a valid notion of past, present and future".*

The causality principle is of course implied by Special Relativity and light cones. This principle ensures that an effect in the future always occurs from a cause in the past from the light cone of the event. This is of course a consequence of the velocity of the light being finite. It can be understood as a cause-effect relationship always ensuring that the world line lies inside its light cone. There are some theories which are acasual, for example see Chap. 8.
Another important implicit principle is the relativity principle which tells us that

> *Relativity principle : "There is not a preferred inertial frame "*

The Relativity principles states that there are no special or preferred inertial frames. Hence, it is not possible to have a situation where a special frame exists where the experiments produce different results. All the physics laws must be equivalent in all inertial frames. Since it is always possible to have locally inertial frames, the principle of relativity must be also valid in GR (as it is valid in Special Relativity). From this principle comes the name of General Relativity since the movement is always related to a frame but neither of them should give rise different physical interpretations.

2.2.5 Mach's Principle

An important philosophical principle which inspired Einstein to develop GR is Mach's principle. Even though, this principle is not a fundamental assumption underlying GR, it is worth mentioning it since it inspired Einstein when he was formulating GR. Imagine that there is only one particle in the Universe. If this particle is moving and all the motion is always a relative concept (comparing different frames), how can one explain the inertia of this particle without having any other frame? Mach postulated the following philosophical statement

> *"The inertia felt by the bodies are the result of the interaction of all the matter of the Universe with them"*

Hence, this principle states that the large-scale structure of the Universe affects locally the behaviour of a certain body and hence the corresponding physical laws acting on it. Therefore, local inertial frames are affected by all the other distribution of matter in the Universe. The idea of inertia is something that Einstein questioned himself. Consider that someone is on a bus and this bus rapidly changes its direction. The person on the bus will feel a fictional force known as a centrifugal force. This kind of effect is considered to be an inertial one and Mach thought that the person on the bus actually is affected by the matter of the whole Universe (or distant stars) which creates these inertial effects. Physically speaking, this principle is vague and for that reason, some people believe that it is more a hypothesis than a principle or a conjecture. However, Einstein believed in this principle and even though it is more a philosophical point of view of the theory, it was important at that time. For a more detailed discussion about this principle, see Brans & Dicke (1961).

2.3 Important mathematical quantities

Since GR is a theory based on the assumption that matter/energy curves spacetime, Euclidean geometry is insufficient to fully describe the theory. For describing curved spaces, the geometry must be described by a more general kind of geometry allowing non-flat Euclidean spaces. The main aim of this section is to define some basic mathematical quantities that will be useful in all the thesis. The purpose of this section is to only formulate basic quantities and not be so mathematically rigorous. A deeper introduction to the important geometrical quantities can be found in the books Wald (1984); Böhmer (2016). GR is constructed in a Riemannian geometry which deals with the concept of Riemannian manifolds. Just after Einstein created his Theory of Special Relativity (1905), he wanted to incorporate the idea of gravity into it. Then, he proposed this notion of deformation of spacetime as a new way of thinking about gravitational effects. The problem was that he needed to spend a lot of time to formulate GR due to the fact that Riemannian geometry was needed. Up to that point, the concept of space in physics was always considered to be flat and non-dynamic. Hence, a new set of tools was needed for him to formulate his conceptions. So, he spent around 10 years formulating his ideas based on non-trivial spaces using the concepts only used by mathematicians at that time. GR is constructed as a $3+1$ dimensional continuum spacetime theory with three spatial dimensions and one temporal dimension. This spacetime is a special space known as a manifold $\mathcal{M}$ which is a set that has the local differential structure for $\mathbb{R}^4$ but not necessarily its global properties (see Wald (1984) for its mathematical definition and properties).

2.3.1 Metric tensor

In order to measure angles between curves and distances between points in non-trivial spacetimes one needs to define the *metric tensor*. This tensor is a rank 2 symmetric tensor defined on a smooth manifold $\mathcal{M}$, labelled as $g_{\mu\nu}$, which is directly related to the line element (or the interval in Special Relativity) ds as follows

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \,. \tag{2.7}$$

The line element measures the distance between two infinitesimal points located at x^μ and $x^\mu + dx^\mu$. The metric tensor is usually called *the metric* for simplicity. Moreover, it is also usual (but incorrect) to also refer to the line element as the metric. In general, the metric depends on the coordinates. A non-degenerate spacetime is the one which satisfies $g \equiv \det(g_{\mu\nu}) \neq 0$. The inverse of the metric $g^{\mu\nu}$ is always well-defined for non-degenerated spacetimes, which of course satisfies $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$. Since $dx^\mu dx^\nu$ is symmetric, only the symmetric part of $g_{\mu\nu}$ plays a role in the line element. A symmetric metric tensor in n dimensions contains $n(n+1)/2$ functions. Therefore, in 4 dimensions as in GR, the metric tensor has 10 functions. The metric in n dimensions is a $n \times n$ matrix with positive, negative or even mixed sign eigenvalues. General Relativity is a 4 dimensional theory (one time dimension and three space dimensions) which assumes a Lorentzian metric whose posses three positive eingenvalues and one negative eigenvalue. Hence, it is usual to consider two kind of signature notations, namely $(-+++)$ or $(+---)$. The metric and its inverse allow us also to map contravariant (defined in the tangent space) and covariant vectors (defined in the cotangent space). For any arbitrary covariant vector v_μ, its corresponding contravariant vector v^μ can be obtained by contracting it with the metric as $v^\mu = g^{\mu\nu}v_\nu$. Hence, one can also define the scalar product of two arbitrary vectors v^μ and w^μ via

$$v \cdot w = g_{\mu\nu}v^\mu w^\nu \,, \tag{2.8}$$

where both vectors are evaluated at the same point of the manifold. From this definition, one can also define the norm of a vector on the manifold which reads

$$v \cdot v = |v|^2 = g_{\mu\nu} v^\mu v^\nu \,. \tag{2.9}$$

When vectors are orthogonal to themselves, they are called *null vectors* and satisfy $g_{\mu\nu} v^\mu v^\nu = 0$.

Further, one can easily then define that the angle $\theta_{v,w}$ between two arbitrary non-null vectors as follows

$$\cos\theta_{v,w} = \frac{v \cdot w}{|v||w|} = \frac{g_{\mu\nu} v^\mu w^\nu}{\sqrt{(g_{\lambda\rho} v^\lambda v^\rho)(g_{\alpha\beta} w^\alpha w^\beta)}} \,. \tag{2.10}$$

From this definition it can be seen that two vectors are orthogonal if $g_{\mu\nu} v^\mu w^\nu = 0$. Furthermore, the metric and its inverse also allows us to raise and lower indices for tensors. For example, for a $\binom{1}{3}$ tensor $T_{\mu\nu\lambda}{}^\alpha$, one can also define its corresponding $\binom{2}{2}$ tensor as $T_{\mu\nu}{}^{\lambda\alpha} = g^{\beta\lambda} T_{\mu\nu\beta}{}^\alpha$.

Let us now consider an arbitrary curve $\mathcal{C}$ on a manifold parametrised by $x^\mu = x^\mu(\lambda)$ in a certain coordinate system x^μ. If this curve starts at a point P_i and finishes at a point P_f (the end points of the curves), the length of this curve is defined as

$$L = \int_{P_i}^{P_f} ds = \int_{P_i}^{P_f} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int_{P_i}^{P_f} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \,. \tag{2.11}$$

Moreover, one can also define the n-volume of a space of dimension n as

$$V = \int \det(g_{\mu\nu}) d^n x \,. \tag{2.12}$$

As can be seen, the definitions of angles, distances, longitudes and volume depend on the metric. Hence, this geometric object is one of the main ingredients in General Relativity.

2.3.2 Geodesic equation

A *geodesic equation* is defined as a particular curve which joins two points P_i and P_f making its length (defined by (2.11)) stationary under small variations and vanish at the end points. Basically, this curve represents the curve which minimises (or maximises) the distance between the points P_i and P_f. This definition comes from the principle of least action or Hamilton's principle which states that the dynamics of any physical system is determined by minimising the action $\mathcal{S}$ of the system. Mathematically speaking, this is equivalent to taking variations of the action and then assuming that for all possible perturbations, that variation is $\delta\mathcal{S} = 0$. In this case, to find the shortest length which joins the points, one can consider that L and ds in (2.11) are the action and the Lagrangian of the system respectively. Therefore, by taking variations in (2.11) it can directly be found that geodesics satisfy the Euler-Lagrange equation,

$$\frac{\delta ds}{\delta x^{\mu}} = \frac{\partial ds}{\partial x^{\mu}} - \frac{d}{d\lambda}\frac{\partial ds}{\partial\left(\frac{dx^{\mu}}{d\lambda}\right)} = 0\,, \tag{2.13}$$

where $ds = \sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}}$. It is easy to check that the first and second term on the above equation can be written as

$$\frac{\partial ds}{\partial x^{\mu}} = \frac{1}{2}\frac{\partial g_{\nu\beta}}{\partial x^{\mu}}\left(\frac{dx^{\nu}}{d\lambda}\right)\left(\frac{dx^{\beta}}{d\lambda}\right), \tag{2.14}$$

$$\frac{d}{d\lambda}\frac{\partial ds}{\partial\left(\frac{dx^{\mu}}{d\lambda}\right)} = \frac{d}{d\lambda}\left(\frac{1}{ds}g_{\mu\nu}\frac{dx^{\nu}}{d\lambda}\right) = g_{\mu\nu}\frac{d^2x^{\nu}}{d\lambda^2} + \left(\frac{\partial g_{\mu\beta}}{\partial x^{\nu}}\right)\left(\frac{dx^{\nu}}{d\lambda}\right)\left(\frac{dx^{\beta}}{d\lambda}\right), \tag{2.15}$$

where it was used the fact that $ds = 1$ for the affine parametrisation. By replacing the above expression in the Euler-Lagrange equation (2.13) and then multiplying it with the inverse of the metric $g^{\alpha\mu}$, one finds the geodesic equation given by

$$\frac{d^2x^\alpha}{d\lambda^2}+\left\{{\alpha \atop \mu\nu}\right\}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=0\,, \tag{2.16}$$

where the Christoffel symbol of the second kind $\left\{{\alpha \atop \mu\nu}\right\}$ has been defined as

$$\left\{{\alpha \atop \mu\nu}\right\}=\frac{1}{2}g^{\alpha\beta}\Big(\partial_\mu g_{\nu\beta}+\partial_\nu g_{\mu\beta}-\partial_\beta g_{\mu\nu}\Big)\,. \tag{2.17}$$

It is easy to see that when one is considering an Euclidean space in Cartesian coordinates, the Christoffel symbol is zero for all the components and then from the geodesic equation, one obtains $d^2x^\alpha/d\lambda^2=0$ which gives us that a straight line is the shortest curve which joins the points P_i and P_f. It is important to mention that this quantity does not transform as a tensor under a general coordinate transformations, so, it is not a tensor. Since the metric is symmetric, one can directly notice that the Christoffel symbol is symmetric on its lower indices giving us that for a n-dimensional space, it contains $n^2(n+1)/2$ components. Thus, for GR which is a 4-dimensional theory, this quantity contains 40 components.

In conclusion, the geodesic equation is the equation which describes the motion for any kind of body (or particle) when gravitational effects are taking into account and it is one of the most important ingredients in GR.

2.3.3 Covariant derivative, connections and parallel transportation

Let us consider an arbitrary vector v_μ and perform a coordinate transformations from $x_\mu \to x'_\mu$. By taking partial derivatives of the new vector v'_μ with respect to the second coordinate system $\partial v'_\mu/\partial x'^\nu$, it can be seen that this quantity is related to $\partial v_\mu/\partial x^\nu$ as follows

$$\partial'_\nu v'_\mu = \frac{\partial v'_\mu}{\partial x'^\nu} = \Big(\frac{\partial x^\alpha}{\partial x'^\nu}\Big)\Big(\frac{\partial x^\beta}{\partial x'^\mu}\Big)\frac{\partial v_\alpha}{\partial x^\beta} + \Big(\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}\Big)v_\alpha\,, \tag{2.18}$$

so that due to the second term on the right hand side of the above equation, derivatives of vectors do not transform as tensors under a general coordinate transformation. This quantity depends on the coordinates chosen and therefore it is not a convenient quantity to consider on the manifold. One can also make a similar transformation and compute partial derivatives in two different coordinate systems with respect to an arbitrary tensor, for example $A_{\mu\nu}$. If one computes $\partial'_\nu A'_{\mu\lambda}$, one can also obtain the same conclusion that partial derivatives do not transform as tensors under general coordinate transformations. Therefore, one can not know if a vector (or a tensor) is constant on a manifold without introducing another kind of derivative. Thus, one needs to define another type of derivative which transforms covariantly under general coordinate transformations.

Let us first define a geometrical object defined on a manifold known as *connection* labelled as $\Gamma^\alpha_{\mu\nu}$. This geometrical object is the one which transforms from one coordinate system to another as

$$\Gamma'^\alpha_{\mu\nu} = \Big(\frac{\partial x'^\alpha}{\partial x^\beta}\Big)\Big(\frac{\partial x^\lambda}{\partial x'^\mu}\Big)\Big(\frac{\partial x^\epsilon}{\partial x'^\nu}\Big)\Gamma^\beta_{\lambda\epsilon} + \Big(\frac{\partial x'^\alpha}{\partial x^\beta}\Big)\Big(\frac{\partial^2 x^\beta}{\partial x'^\mu x'^\nu}\Big)\,. \tag{2.19}$$

In general, a connection defined on a certain manifold does not necessary depend on the metric. Using this definition, it is possible to define a *covariant derivative* ∇_μ on a manifold which has the property that it transforms covariantly under a general coordinate transformation. The covariant derivative of covariant and contravariant vectors can be defined as

$$\nabla_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\beta_{\mu\nu} v_\beta\,, \tag{2.20}$$

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\beta\mu} v^\beta \,. \tag{2.21}$$

It is easy to check that $\nabla_\mu v_\nu$ and $\nabla_\mu v^\nu$ transform as tensors as expected. For any arbitrary $\binom{n}{m}$ rank tensor $A_{\mu\nu\ldots}{}^{\alpha\beta\cdots}$, one can then define that the covariant derivative with respect to it is given by

$$\begin{aligned} \nabla_\lambda A_{\mu\nu\ldots}{}^{\alpha\beta\cdots} &= \partial_\lambda A_{\mu\nu\ldots}{}^{\alpha\beta\cdots} - \Gamma^\epsilon_{\mu\lambda} A_{\epsilon\nu\ldots}{}^{\alpha\beta\cdots} - \Gamma^\epsilon_{\nu\lambda} A_{\mu\epsilon\ldots}{}^{\alpha\beta\cdots} - \ldots \\ &\quad + \Gamma^\alpha_{\epsilon\lambda} A_{\mu\nu\ldots}{}^{\epsilon\beta\cdots} + \Gamma^\beta_{\epsilon\lambda} A_{\mu\nu\ldots}{}^{\alpha\epsilon\cdots} + \ldots \quad . \end{aligned} \tag{2.22}$$

Again, it is easy to check that this definition is covariant under coordinate transformations. Hence, the connection allows us to define derivatives which transform covariantly under general coordinate transformations. One can also notice that for any function f, the covariant derivative is exactly the same as the partial derivative $\nabla_\mu f = \partial_\mu f$. The covariant derivative maps a $\binom{n}{m}$ rank tensor to a $\binom{n}{m+1}$ rank tensor and it is linear. Additionally, covariant derivatives satisfy the Leibniz rule and commute with contraction. From the latter properties, it can be clearly seen that

$$\nabla_\mu (g_{\alpha\beta} g^{\beta\gamma}) = 0 \to \quad \nabla_\mu g_{\alpha\beta} = -g_{\alpha\nu} g_{\beta\gamma} \nabla_\mu g^{\nu\gamma} \,. \tag{2.23}$$

Let us now define a tensor $Q_{\mu\alpha\beta}$, to rewrite the above equation as

$$\nabla_\mu g_{\alpha\beta} = Q_{\mu\alpha\beta} \equiv -g_{\alpha\nu} g_{\beta\gamma} \nabla_\mu g^{\nu\gamma} \,. \tag{2.24}$$

Clearly, $Q_{\mu\alpha\beta}$ is a 3 rank symmetric tensor on its last two indices. From here, it is necessary to distinguish between two kind of geometries which depend on the choice of the connection defined on the manifold:

1. *Metricity or metric compatibility*: when $Q_{\mu\alpha\beta} = 0$

2. *Non-metricity*: when $Q_{\mu\alpha\beta} \neq 0$.

A connection which satisfies the metric compatibility condition is labelled as *Lorentz connection*. General Relativity is a theory which assumes the first kind of connection where $\nabla_\mu g_{\alpha\beta} \equiv 0$. Moreover, another important condition for the connection is established in GR. It is assumed that the connection in GR is torsion-less (assumes that torsion tensor is identically zero, see Sec. 2.3.4), which is equivalent as having a symmetric connection

$$\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}_{\nu\mu} \,. \tag{2.25}$$

This torsion-less connection is also named the *Levi-Civita connection* and it is one of the main ingredients in constructing GR. Hence, GR is a geometric theory based on a specific connection which satisfies the metric compatibility condition and it is torsion-less. From hereafter, $\circ$ will be used to denote mathematical quantities computed with the Levi-Civita connection (GR). This notation will be useful in other sections dealing with Teleparallel gravity where the connection and then covariant derivatives are different.
One can verify that the unique connection which satisfies the metric compatibility condition which is also symmetric, is given by the Christoffel symbol (2.17), namely

$$\overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu} = \left\{ {}^{\alpha}_{\mu\nu} \right\} = \frac{1}{2} g^{\alpha\beta} \Big(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \Big) \,. \tag{2.26}$$

Therefore, the torsion-less and compatibility conditions give us the above connection which matches with the Christoffel symbol defined in the geodesic equation.

In standard flat spacetimes, a vector v^i remains constant along a line if it satisfies

$$\frac{dv^i}{d\lambda} = 0 \,, \tag{2.27}$$

where λ is an affine parameter used to characterised the curve. Since GR is based on curved spacetimes, the notion of transporting vectors parallel along a line needs to be changed. In this context, one needs to introduce the *parallel transportation* that allows us to define this kind of transportation in curved spacetimes. It is then said that any vector v^i is parallel transported along a curve parametrised by λ if

$$T^i \nabla_i v^j = 0\,, \tag{2.28}$$

where ∇_i is the covariant derivative and T^i is a tangent vector of the curve. According to this definition, it can be proved that the final transported vector v'^i resulting by parallel transporting the vector v^i from x^i to an infinitesimally close point located at $x^i + dx^i$ will be

$$v'^i(x+dx) = v^i(x) - \Gamma^i_{jk} v^j(x) dx^k = v^i(x) + \delta v^i(x)\,. \tag{2.29}$$

Additionally, for a covariant vector v_i, after a parallel transportation along the curve, one can also obtain the vector v'_i located at $x + dx^i$ via

$$v'_i(x+dx) = v_i(x) + \Gamma^k_{il} v_k(x) dx^l = v_i(x) + \delta v_i(x)\,. \tag{2.30}$$

These definitions will be useful in forthcoming sections where the geometrical interpretation of some important mathematics quantities will be explained.

2.3.4 Curvature tensor, Ricci scalar, scalar curvature and torsion tensor

In general, unlike partial differentiation, a covariant derivative defined on a manifold does not necessary commute, i.e., in general, $\nabla_\gamma \nabla_\lambda A_{\mu\nu\cdots}{}^{\alpha\beta\cdots} \neq \nabla_\lambda \nabla_\gamma A_{\mu\nu\cdots}{}^{\alpha\beta\cdots}$. Let us first study the commutator relationship considering an arbitrary smooth function

f by computing its second covariant derivative using (2.21) which gives us

$$\nabla_\mu \nabla_\nu f = \partial_\mu \partial_\nu f - \Gamma^\lambda_{\mu\nu} \partial_\lambda f \,, \tag{2.31}$$

$$\nabla_\nu \nabla_\mu f = \partial_\nu \partial_\mu f - \Gamma^\lambda_{\nu\mu} \partial_\lambda f \,. \tag{2.32}$$

Now, using $\partial_\nu \partial_\mu f = \partial_\mu \partial_\nu f$, one finds that the commutator relation for covariant derivative for a function is given by

$$2\nabla_{[\mu} \nabla_{\nu]} f = \nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = -T^\lambda{}_{\mu\nu} \partial_\lambda f \,, \tag{2.33}$$

where it has been defined the *torsion tensor* $T^\lambda{}_{\mu\nu}$ as

$$T^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \,. \tag{2.34}$$

This quantity will be important in the forthcoming chapter where Teleparallel theories of gravity are considered. As pointed out before, GR is based on a torsion-free connection, so that in that theory it is assumed that $T^\lambda{}_{\mu\nu} \equiv 0$ or equivalently the connection is symmetric $\mathring{\Gamma}^\alpha_{\nu\mu} = \mathring{\Gamma}^\alpha_{\mu\nu}$. The torsion tensor is skew-symmetric on its last two indices. Hence, for a n-dimensional space, it contains $n^2(n-1)/2$ components and therefore in 4 dimensions contains 24 components. It can be proved that even though the connection Γ does not transform as a tensor under a general coordinate transformation, the torsion tensor does transform as a tensor.

Even though this quantity is assumed to be zero in GR, let us try to understand better what this quantity is measuring. To do that, let us consider a 2 dimensional geometry and then define three different points P_0, P_1 and P_2 separated infinitesimally from each other. Let us suppose that P_0 is located at x^i and P_1 and P_2 are located at $x^i + dx_1^i$ and $x^i + dx_2^i$ respectively. Hence, the difference between the points P_0 and P_1 is just dx_1^i and the difference between P_0 and P_2 is just dx_2^i. Then, let us transport the vectors dx_1^i and dx_2^i defined at the point P_0 towards the points P_3 and P_4. By doing this, one is able to define the two new vectors as dx_3^i and dx_4^i respectively. These two vectors defined at the points P_1 and P_2 can be obtained with the parallel transportation equation obtained in (2.29), giving us

$$dx_3^i(P_1) = dx_2^i - \Gamma^i_{jk}(x)dx_1^k dx_2^j\,, \quad dx_4^i(P_2) = dx_1^i - \Gamma^i_{jk}(x)dx_1^j dx_2^k\,. \tag{2.35}$$

Using these two new vectors, one can then define two new points P_3 and P_4 with coordinates being equal to

$$x^i(P_3) = x^i(P_1) + dx_3^i(P_1)\,, \quad x^i(P_4) = x^i(P_2) + dx_4^i(P_2)\,. \tag{2.36}$$

Now, by subtracting the coordinate of these points, it is found the following expression

$$\begin{aligned} x^i(P_3) - x^i(P_4) &= \Big(x^i + dx_1^i + dx_2^i - \Gamma^i_{jk}(x)dx_1^k dx_2^j\Big) \\ &\quad -\Big(x^i + dx_2^i + dx_1^i - \Gamma^i_{jk}(x)dx_1^j dx_2^k\Big) &(2.37)\\ &= T^i{}_{jk}(x)dx_1^j dx_2^k\,, &(2.38)\end{aligned}$$

where Eq. (2.34) was used. From here, one can understand the geometrical meaning of the torsion tensor. The difference of the coordinates of the points P_3 and P_4 is proportional to this tensor. Hence, this quantity measures the failure to close an infinitesimal parallelogram since if the torsion tensor is different from zero, those points will not coincide. Fig. 2.5 represents more or less the situation described here, starting from a point P_0 and then defining new points P_1 and P_2 infinitesimally separated. As can be seen from the figure, the points P_3 and P_4 do not coincide, making an "open" parallelogram. Obviously, when the torsion tensor is zero, the points coincide and the standard kind of geometry is recovered where a closed parallelogram is obtained. GR is a theory which assumes that the torsion tensor is zero so that, the final parallelogram always will be closed. However, in Teleparallel theories of gravity when the torsion tensor is different from zero, this geometrical effect plays a role (see Chap. 3).

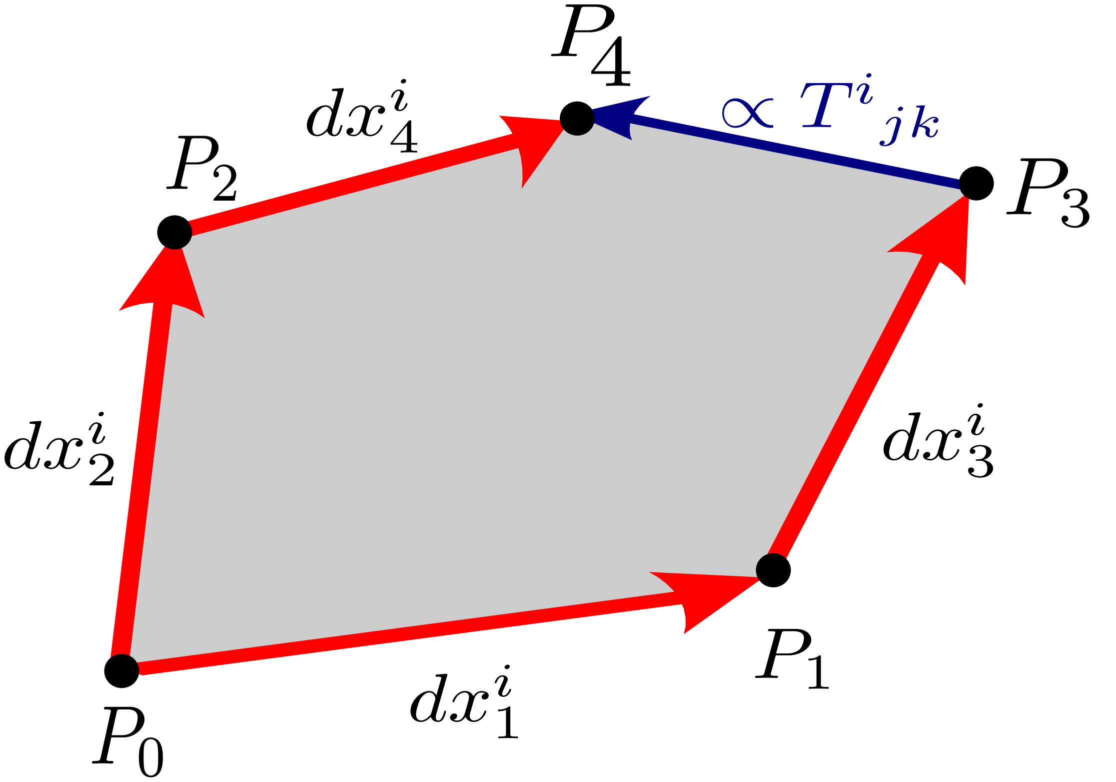


Figure 2.5: Representation of a 2 dimensional manifold with torsion. This representation shows that the parallelogram will not be closed on a manifold with torsion.

Let us now consider an arbitrary vector v^μ and again use the definition of covariant derivative acting on contravariant vectors given in (2.21). By taking second covariant derivatives of this vector, one can easily find that

$$\nabla_\gamma \nabla_\lambda v^\mu = \partial_\gamma(\partial_\lambda v^\mu + \Gamma^\mu_{\alpha\lambda} v^\alpha) - \Gamma^\alpha_{\lambda\gamma}(\partial_\alpha v^\mu + \Gamma^\mu_{\beta\alpha} v^\beta) + \Gamma^\mu_{\alpha\gamma}(\partial_\lambda v^\alpha + \Gamma^\alpha_{\beta\lambda} v^\beta)\,, \quad (2.39)$$

$$\nabla_\lambda \nabla_\gamma v^\mu = \partial_\lambda(\partial_\gamma v^\mu + \Gamma^\mu_{\alpha\gamma} v^\alpha) - \Gamma^\alpha_{\gamma\lambda}(\partial_\alpha v^\mu + \Gamma^\mu_{\beta\alpha} v^\beta) + \Gamma^\mu_{\alpha\lambda}(\partial_\gamma v^\alpha + \Gamma^\alpha_{\beta\gamma} v^\beta)\,, \quad (2.40)$$

so that by assuming $\partial_\lambda \partial_\gamma v^\mu = \partial_\gamma \partial_\lambda v^\mu$ and subtracting the above equations, one finds

$$2\nabla_{[\gamma}\nabla_{\lambda]} v^\mu = \nabla_\gamma \nabla_\lambda v^\mu - \nabla_\lambda \nabla_\gamma v^\mu = R^\mu{}_{\alpha\gamma\lambda} v^\alpha + T^\rho{}_{\gamma\lambda} \nabla_\rho v^\mu \,, \tag{2.41}$$

where the *curvature tensor* $R^\mu{}_{\alpha\gamma\lambda}$ was defined as

$$R^\mu{}_{\alpha\gamma\lambda} \equiv \partial_\gamma \Gamma^\mu_{\alpha\lambda} - \partial_\lambda \Gamma^\mu_{\alpha\gamma} + \Gamma^\mu_{\beta\gamma}\Gamma^\beta_{\alpha\lambda} - \Gamma^\mu_{\beta\lambda}\Gamma^\beta_{\alpha\gamma} \,. \tag{2.42}$$

This geometric object is skew-symmetric on its last two lower indices $R^\mu{}_{\alpha\gamma\lambda} = -R^\mu{}_{\alpha\lambda\gamma}$. Hence, it contains $n^3(n-1)/2$ components in a n-dimensional space, giving rise to 96 components in a 4 dimensional space. Exactly as with the definition of the torsion tensor, it can be proved that the curvature tensor transforms as a tensor under a general coordinate transformations. It is interesting to note that a sufficient condition for finding a coordinate system where all the components of the connection $\Gamma^\alpha_{\mu\nu} = 0$, is by having that in that coordinate system, $R^\mu{}_{\alpha\gamma\lambda} = 0$ and $T^\lambda{}_{\mu\nu} = 0$.

Let us now try to understand geometrically what this quantity means. To see this, let us take a 2 dimensional example with an initial vector v^i_0 (where $i = 1, 2$) defined on a certain point $P_0 = x^i$ on a manifold which possesses curvature. Let us now parallelly transport the vector v^i_0 around a small closed curve. If one parallelly transports the vector v^i_0 from the point P_0 located at x^i to a point P_1 located at a point $x^i + dx^i_1$, from (2.29), one obtains

$$v^i_0(x) - v^i_1(P_1) = \Gamma^i_{jk}(x) v^j_0(x) dx^k_1 \,. \tag{2.43}$$

Now, one can parallelly transport the vector v^i_1 from the point P_1 to the point P_2 located at $x^i + dx^i_1 + dx^i_2$. Since the new vector obtained by transporting v^i_1 is different, it will be labelled as v^i_2. Then, one closes the curve by transporting the vector from P_2 to P_3 (located at $x^i + dx^i_2$) and then from P_3 to our initial point P_0

with the vectors labelled as v_3^i and v_4^i respectively. Therefore, as written above, the difference between the vectors for that circuit is given by

$$
\begin{aligned}
v_1^i(P_1) - v_2^i(P_2) &= \Gamma^i_{jk}(P_1) v_1^j(P_1) dx_2^k \,, &(2.44)\\
v_2^i(P_2) - v_3^i(P_3) &= -\Gamma^i_{jk}(P_2) v_2^j(P_2) dx_1^k \,, &(2.45)\\
v_3^i(P_3) - v_4^i(x) &= -\Gamma^i_{jk}(P_3) v_3^j(P_3) dx_2^k \,. &(2.46)
\end{aligned}
$$

It should be remarked that it was used that $x^i(P_3) - x^i(P_2) = -dx_1^i$ and $x^i(P_0) - x^i(P_3) = -dx_2^i$. One is interested to find how the initial vector $v_0^i(x)$ is related with the final vector $v_4^i(x)$. The coordinates of the points P_i are known so that it is easy to find that the connections evaluated at each point are

$$
\begin{aligned}
\Gamma^i_{jk}(P_1) &= \Gamma^i_{jk}(x + dx_1) = \Gamma^i_{jk}(x) + dx_1^l (\partial_l \Gamma^i_{jk})(x) \,, &(2.47)\\
\Gamma^i_{jk}(P_2) &= \Gamma^i_{jk}(x + dx_1 + dx_2) = \Gamma^i_{jk}(x) + (dx_1^l + dx_2^l)(\partial_l \Gamma^i_{jk})(x) \,, &(2.48)\\
\Gamma^i_{jk}(P_3) &= \Gamma^i_{jk}(x + dx_2) = \Gamma^i_{jk}(x) + dx_2^l (\partial_l \Gamma^i_{jk})(x) \,. &(2.49)
\end{aligned}
$$

Using the above relationships and (2.43) and (2.44), one can find that the vector $v_2^i(P_2)$ is equal to

$$
\begin{aligned}
v_2^i(P_2) &= v_0^i(x) - \Gamma^i_{jk}(x) v_0^j(x) dx_1^k - \Gamma^i_{jk}(x) v_0^j(x) dx_2^k \\
&\quad + \Gamma^i_{jk}(x) \Gamma^j_{lm}(x) v_0^l(x) dx_1^m dx_2^k - (\partial_l \Gamma^i_{jk})(x) v_0^j(x) dx_1^l dx_2^k \,, &(2.50)
\end{aligned}
$$

where terms proportional to $dx_1^l dx_1^m dx_1^k$ were neglected. Then, by using (2.45) and (2.49) and after some simplifications, the vector $v_3^i(P_3)$ becomes

$$
v_3^i(P_3) = v_0^i(x) - \Gamma^i_{jk}(x) v_0^j(x) dx_2^k + R^i{}_{jkl}(x) v_0^j dx_1^k dx_2^l \,, \qquad (2.51)
$$

where (2.42) was used to incorporate $R^i{}_{jkl}(x)$, which is the curvature tensor evaluated at the initial point P_0. Finally, from (2.46) one can find that the difference between the final vector $v_4^i(x)$ and the initial vector $v_0^i(x)$ at the point P_0 located at x^i becomes

$$v_4^i(x) - v_0^i(x) = R^i{}_{jkl}(x) v_0^j dx_1^k dx_2^l \,. \tag{2.52}$$

Hence, the difference between those vectors is proportional to the curvature tensor. From here, one can directly understand geometrically what this tensor represents. If one starts with an initial vector $v_0^i(x)$ and one parallelly transports it around a closed curve, one will end up with another vector $v_4^i(x)$, and the difference between them is proportional to the curvature tensor evaluated at that point. Hence, the curvature tensor measures the failure of this vector to return to its original value when it is parallelly transported. This procedure is depicted in Fig. 2.6 where it can be seen that the final vector $v_4^i(x)$ differs by the initial vector $v_0^i(x)$ after parallelly transporting it around a closed curved.

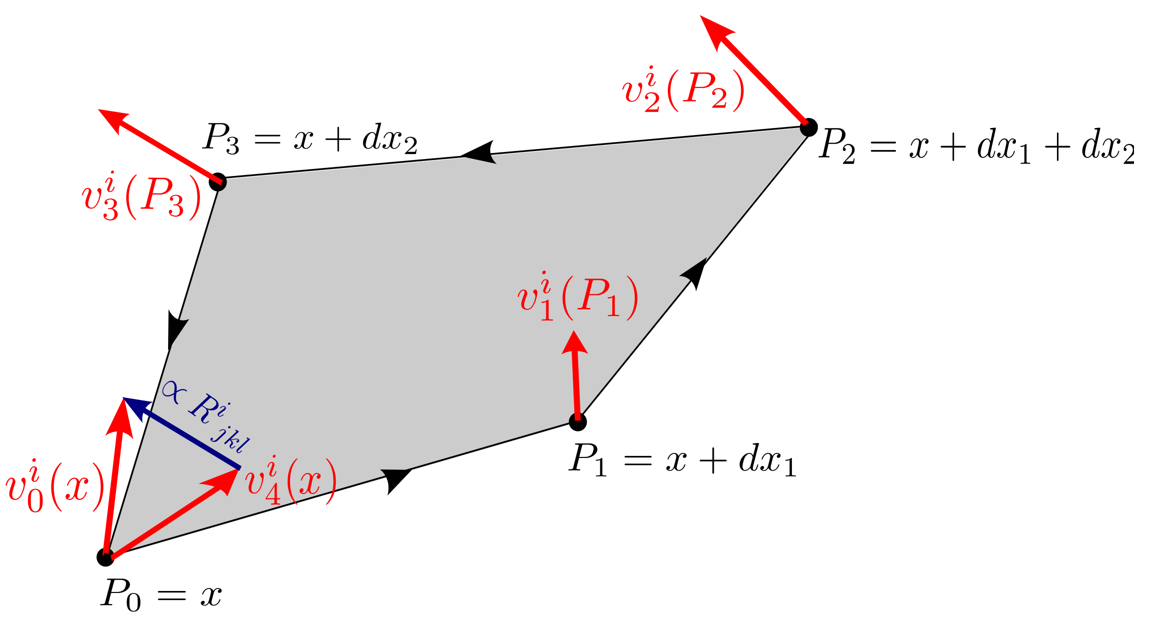


Figure 2.6: Representation of a 2 dimensional manifold endorsed with curvature. If the curvature tensor is different than zero, the final vector $v_4^i(x)$ would not coincide with the initial one $v_0^i(x)$.

Let us now state the *Ricci theorem* which says that any Lorentz connection can be written as a combination of torsion, known as *contortion tensor* $K_{\mu}{}^{\lambda}{}_{\nu}$, and the Levi-Civita connection $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$, which reads as

$$\Gamma^{\lambda}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} + K_{\mu}{}^{\lambda}{}_{\nu}\,, \tag{2.53}$$

where the contortion tensor is defined as

$$K_{\mu}{}^{\lambda}{}_{\nu} = \frac{1}{2}(T^{\lambda}{}_{\mu\nu} - T_{\nu\mu}{}^{\lambda} + T_{\mu}{}^{\lambda}{}_{\nu})\,. \tag{2.54}$$

This theorem is valid for any connection. If one uses this relationship in the curvature (2.42), one obtains

$$R^{\lambda}{}_{\mu\sigma\nu} = \overset{\circ}{R}{}^{\lambda}{}_{\mu\sigma\nu} + \overset{\circ}{\nabla}_{\sigma}K_{\nu}{}^{\lambda}{}_{\mu} - \overset{\circ}{\nabla}_{\nu}K_{\sigma}{}^{\lambda}{}_{\mu} + K_{\sigma}{}^{\lambda}{}_{\rho}K_{\nu}{}^{\rho}{}_{\mu} - K_{\sigma}{}^{\rho}{}_{\mu}K_{\nu}{}^{\lambda}{}_{\rho}\,, \tag{2.55}$$

where $R^{\lambda}{}_{\mu\sigma\nu}$ is a generic curvature and $\overset{\circ}{R}{}^{\lambda}{}_{\mu\sigma\nu}$ is a curvature computed with the Levi-Civita connection. Hence, curvature can be always split as a combination of a torsion piece which depends on the contortion tensor and, the curvature computed with the Levi-Civita connection. In GR, torsion is zero so that $K_{\sigma}{}^{\lambda}{}_{\mu} = 0$ and then $R^{\lambda}{}_{\mu\sigma\nu} = \overset{\circ}{R}{}^{\lambda}{}_{\mu\sigma\nu}$. It is important to mention that by doing some computations, one can also relate the curvature tensor with the torsion tensor yielding

$$R^{\mu}{}_{[\alpha\gamma\lambda]} = -T^{\beta}{}_{[\alpha\gamma}T^{\mu}{}_{\lambda]\beta} - \nabla_{[\alpha}T^{\mu}{}_{\gamma\lambda]}\,. \tag{2.56}$$

It is also possible to define some contractions of the curvature tensor. One important contraction of the curvature tensor is the *Ricci tensor* which is obtained by contracting the first and third indices of the curvature tensor, giving us

$$R_{\alpha\lambda} \equiv R^{\mu}{}_{\alpha\mu\lambda} = \partial_{\mu}\Gamma^{\mu}_{\alpha\lambda} - \partial_{\lambda}\Gamma^{\mu}_{\alpha\mu} + \Gamma^{\mu}_{\beta\mu}\Gamma^{\beta}_{\alpha\lambda} - \Gamma^{\mu}_{\beta\lambda}\Gamma^{\beta}_{\alpha\mu}\,, \tag{2.57}$$

which in general is not symmetric. Again, by using the Ricci theorem, one obtains

$$R_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} + \overset{\circ}{\nabla}_\lambda K_\nu{}^\lambda{}_\mu - \overset{\circ}{\nabla}_\nu K_\lambda{}^\lambda{}_\mu + K_\lambda{}^\lambda{}_\rho K_\nu{}^\rho{}_\mu - K_\lambda{}^\rho{}_\mu K_\nu{}^\lambda{}_\rho\,. \tag{2.58}$$

Moreover, it is also possible to express the above quantity as

$$2R_{[\alpha\lambda]} = -3\nabla_{[\alpha} T^\beta{}_{\beta\lambda]} + T^\beta{}_{\beta\gamma} T^\gamma{}_{\alpha\lambda}\,. \tag{2.59}$$

It is easy to see that for GR, the Ricci tensor is always symmetric since the torsion tensor is zero. It should be remarked that one can also define the contraction $R^\gamma{}_{\gamma\mu\lambda} = \bar{R}_{\mu\lambda}$ which is a totally skew-symmetric tensor and it vanishes for a symmetric connection (as in GR). This quantity is known as *homothetic curvature*. The other contraction $R^\gamma{}_{\mu\lambda\gamma} = -R_{\mu\lambda}$ is just minus the curvature tensor due to its skew-symmetric property on its last two indices. Therefore, there are only two possible independent contractions for the curvature tensor and for the GR case, only the Ricci tensor is non-zero.

Further, one can also consider a contraction with the metric and the Ricci tensor, which gives us the so-called *curvature scalar* or *Ricci scalar* given by

$$R \equiv g^{\alpha\lambda} R_{\alpha\lambda} = g^{\alpha\lambda}\partial_\mu \Gamma^\mu_{\alpha\lambda} - \partial^\alpha \Gamma^\mu_{\alpha\mu} + g^{\alpha\lambda}\Gamma^\mu_{\beta\mu}\Gamma^\beta_{\alpha\lambda} - g^{\alpha\lambda}\Gamma^\mu_{\beta\lambda}\Gamma^\beta_{\alpha\mu}\,. \tag{2.60}$$

A Riemannian space is the one which possesses curvature but the torsion tensor is zero and then the connection is equal to the Levi-Civita one. As pointed out before, GR is based on this kind of spaces. For this case, the curvature tensor has also two additional symmetry conditions

$$\overset{\circ}{R}_\alpha{}^\mu{}_{\gamma\lambda} = -\overset{\circ}{R}{}^\mu{}_{\alpha\gamma\lambda}\,, \quad \overset{\circ}{R}{}^\mu{}_{\alpha\gamma\lambda} = \overset{\circ}{R}_{\gamma\lambda}{}^\mu{}_\alpha\,, \tag{2.61}$$

which gives us that the curvature tensor contains only $n^2(n^2-1)/12$ components in a torsion-less space. Hence, in GR, the curvature tensor contains 20 components.

2.4 Einstein field equations

2.4.1 Newton's law of Universal Gravitation

This section will briefly review some basic properties of the Newtonian Gravitational law. The aim is to have a picture of those laws in order to then study the Einstein field equations which are a generalisation of them. First of all, Newtonian gravity is of course based on Newtonian laws. After Einstein proposed Special Relativity, it was known that Newton's laws are valid only in some situations. Surprisingly, for the most part of the phenomena at the scales of the human are within the range of Newton laws. Mainly all the physics employed on Earth is based on these laws and even though they are incomplete, they cannot be discarded. It is now known that classically, these laws are valid when:

- A body is moving with a certain velocity v which is very small comparable with the velocity of light ($v/c \ll 1$).
- Is valid when the gravitational field is in a weak field regime ($GM/r \ll 1$).
- They are valid only in inertial reference frames.
- Is based on the Galilean transformations (not Lorentz transformations as Special and General Relativity)

In 1687, Newton proposed the Universal gravitational laws which in some sense, governed the notion of gravity from that time until Einstein proposed GR. Let us consider a case where there are two particles whose masses m_1 and m_2 are separated by a vector $\vec{r} = \vec{x}_2 - \vec{x}_1$. For observational reasons, Newton realised that the force which exerts the particle m_2 to the particle with mass m_1 must be proportional to its masses and inversely proportional to the square of the distance between the vector which separates them, namely

$$\vec{F} = G\,\frac{m_1 m_2}{|\vec{r}|^2}\,\frac{\vec{r}}{|\vec{r}|}\,, \tag{2.62}$$

where $|\vec{r}|$ is the norm of the vector $\vec{r}$ and G is a gravitational constant that *a priori* needs to be determined by experimental measurements. Fig. 2.7 shows a representation of this situation. Cavendish in 1789 was the first who measured this constant using a torsion balance. To be precise, Cavendish was not trying to calculate this constant. Rather, he was trying to calculate the density of the Earth. Moreover, the real value of the gravitational constant G was not important at that time. However, if one can measure the density of the Earth, using Newtonian laws, it is also possible to estimate the value of G. Cavendish did not compute G but using the value of the density of the Earth that he found, it is trivial to find the correct value of G. Therefore, mostly all authors said that Cavendish was the first who actually measured the value of G, even though in his notes he did not mention G at all. Hence, one can say that Cavendish was the first who measured implicitly the value of G. Currently, we know that the value of G in the international system of units is approximately $G \approx 6.67408(31) \cdot 10^{-11}\mathrm{Nm}^2/\mathrm{kg}^2$ (Mohr et al., 2016).

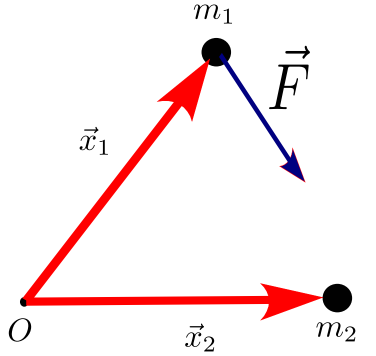


Figure 2.7: Representation of how the Newtonian Gravitational law works for two particle with masses m_1 and m_2 located at $\vec{x}_1$ and $\vec{x}_2$ with respect to an observer O.

Since Newton knew that the gravitational force acts even at very long distances, he realised that it was convenient to also define the gravitational field $\vec{g}$. A gravitational field $\vec{g}_1$ of a particle with mass m_1 is defined as the gravitational force which this mass generates to a test particle (m_2 in this case), divided by the mass of this test particle m_2, which reads as

$$\vec{g}_1 = -\frac{\vec{F}}{m_2} = -G\frac{m_1}{|\vec{r}|^2}\frac{\vec{r}}{|\vec{r}|}\,. \tag{2.63}$$

Notice that the minus sign comes from the third Newtonian law since the force appearing on the above equation is the force which exerts m_1 to m_2 and hence it is a minus of difference with respect to the force given in Eq. (2.62). Hence, a certain particle m_1 will generate a gravitational field $\vec{g}_1$ for having a mass and its value will be given by the above expression. Now, if one generalises this idea by changing the particle to a continuous body, one can express its gravitational field by

$$\vec{g} = -G\int_V \rho(\vec{x}')\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3}dV'\,, \tag{2.64}$$

where V is the volume of the body, dV' is an infinitesimal volume of the body and now the vectors $\vec{x}'$ and $\vec{x}$ describe the position where the gravitational field is computed and the origin vector of the coordinate system respectively. In this equation, $\rho(\vec{x}')$ is the density distribution of the matter which of course in general can be non-constant. From this definition, it is easy to check that $\vec{\nabla}\times\vec{g} = 0$ which tells us that the Newtonian gravitational field is irrotational. Hence, there exists a gravitational potential ϕ such that

$$\vec{g} = -\vec{\nabla}\phi\,, \tag{2.65}$$

where ϕ is defined as

$$\phi = G\int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}dV' \,. \tag{2.66}$$

Thus, the gravitational potential satisfies the Poisson equation given by

$$\nabla^2\phi = 4\pi G\rho \,, \tag{2.67}$$

and also the gravitational field satisfies the Gauss equation $\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$. The above equation can be used to calculate the gravitational potential for any kind of configuration as, for example, a spherically symmetric body. An important consequence can be seen from this equation. First of all, it tells us that the gravitational effects act instantly since the time is not present in this equation. This means that Newtonian gravity says that if the Sun suddenly disappears, all the planets orbiting around it, will automatically feel that the gravitational force of the Sun disappeared. This is of course not consistent with Special Relativity which says that the maximum speed which information can travel is the velocity of the light. The latter thought experiment is known as the *cosmic catastrophe*. Today, it is known that the basics of Newtonian laws are valid only for some specific cases as mentioned earlier.

Using GR, it is possible to check that the gravitational field propagates at exactly the velocity of the light so it is not instant as is proposed by Newtonian gravity. Even though the Newtonian gravitational laws have some inconsistencies, the Poisson equation for the gravitational field could describe mostly all the phenomena in the Planetary system. Hence, it is a good tool for astronomers and engineers to estimate some specific gravitational scenarios such as the movement of comets, asteroids or even the movement of the Earth around the Sun. This is true since the gravitational field in those situations can be considered as being weak and then Newtonian law can be used. However, for example, when one wants to understand how Mercury is moving around the Sun, one cannot use Newtonian gravity. This problem happens since the gravitational field is much stronger near the Sun and since Mercury is the closest planet to the Sun, the gravitational effects cannot be estimated using Newtonian gravity. This problem was actually observed before Einstein came out with GR, when astronomers did not understand why the orbit of Mercury and its movement around the Sun were not the one expected using Newtonian gravity. Moreover, at that time, astronomers needed to introduce a new planet near the Sun to more or less match the observations. This planet of course was never found. This problem is known as the *Mercury perihelion shifting*. Using GR, one can study how Mercury moves around the Sun and actually one can measure and see that the orbit of Mercury changes around 43 arcseconds per century on its perihelion in every cycle, matching with a high precision with astronomical observations. For the interested reader about confrontation between GR and experiment, see Will (2014).

2.4.2 Einstein field equations

Einstein was an intuitive person formulating his theories. Based on the principles mentioned in a Sec. 2.2, he was able to formulate the field equations which rules the movement of bodies when gravitational fields are presented. In order to find these field equations, Einstein used some conditions that those equations must satisfied and then he formulated the equations intuitively. First, he realised that the field equations should have a similar form as Maxwell's equations. One can proceed as follows. Let us take the metric $g_{\mu\nu}$ in a slightly curved spacetime in such a way that one can express it as a flat Minkowski metric plus an additional small metric term. This perturbation can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \text{with } |h_{\mu\nu}| \ll 1\,. \tag{2.68}$$

Now, consider the Newtonian limit where the gravitational fields are small and also that the bodies move with velocities much smaller than the velocity of the light. For coordinates $x^\mu = (t, x^i)$, the geodesic equation (2.16) for this case will be given by

$$\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} \approx 0\,. \tag{2.69}$$

Then, for the 0-component, one obtains

$$\frac{d^2t}{d\tau^2} = 0\,, \tag{2.70}$$

which gives us $t = c_1\tau + c_2$. So, without loosing generality, one has $d\tau = dt$. Then, the i-th component in (2.69) becomes

$$\frac{d^2\vec{x}}{dt^2} \approx -\frac{1}{2}\vec{\nabla}h_{00}\,. \tag{2.71}$$

Now, for the 2nd Newtonian law $d^2\vec{x}/dt^2 = \vec{g}$ and by comparing the Newtonian Poisson equation (2.67) with the above equation, it is easy to see that

$$\nabla^2 h_{00} = 8\pi G\rho\,, \tag{2.72}$$

so that one directly finds that the Newtonian gravitational potential is related to the perturbed metric as

$$h_{00} \approx -2\phi\,, \tag{2.73}$$

so that the 00 component of the metric will be

$$g_{00} \approx -(1+2\phi)\,. \tag{2.74}$$

Hence, the first ingredient needed to construct the field equations is that the 00 component of the metric must satisfy the above equation for the cases where the gravitational field is weak.

For non-relativistic matter, the 00 component of the energy-momentum tensor is $\mathcal{T}_{00} \approx -\rho$ and then by using (2.68) and (2.72) one finds that

$$\nabla^2 g_{00} = 8\pi G \mathcal{T}_{00} \,. \tag{2.75}$$

The above equation could describe the Newtonian Gravitational law for the specific case where the gravitational field is not sufficiently large to deform sufficiently the spacetime. The above equation is known as the *Newtonian gravitational limit*. The above equation is just a guess of the final field equation for any kind of gravitational field since it is based on non-relativistic assumptions. However, one can notice that the form of the Einstein field equations for any distribution kind of matter, must relate the energy-momentum tensor to gravity. Therefore, let us take the following form of the possible field equations,

$$\overset{\circ}{G}_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu} \,, \tag{2.76}$$

where κ^2 is a unknown coupling constant and $\overset{\circ}{G}_{\mu\nu}$ is a tensor that it is unknown *a priori*, but it must satisfy some specific properties to ensure the validity of all the principles studied in Sec. 2.2. Therefore this tensor must satisfy the following conditions

- It must be formed from the metric tensor and also its derivatives. $\overset{\circ}{G}_{\mu\nu}$ must have dimensions of a second derivative. Any other terms with $N \neq 2$ number of derivatives would appear multiplied with a constant of dimensions of length to the power of $N-2$. GR assumes that the field equations are uniform in scale, so that $\overset{\circ}{G}_{\mu\nu}$ must have only terms with second derivatives of the metric.

Hence, this tensor only contains either terms quadratic in the first derivatives or linear in the second derivatives of the metric.

- From Eq. (2.76), it can be noticed that the right-hand side of the equation appears the energy-momentum tensor which always is a symmetric tensor. Hence, $\overset{\circ}{G}_{\mu\nu}$ must also be a symmetric tensor.

- Since the matter must be conserved in the sense of covariant differentiation, i.e., $\overset{\circ}{\nabla}_\mu \mathcal{T}^{\mu\nu} = 0$, from Eq. (2.76), one notices that this new tensor $\overset{\circ}{G}_{\mu\nu}$ must also be covariantly conserved, i.e., $\overset{\circ}{\nabla}_\mu \overset{\circ}{G}^{\mu\nu} = 0$. This condition ensures that matter and energy are conserved and also that particles follow geodesics.

- As one can see from Eq. (2.75), for weak gravitational fields (Newtonian limit), the 00 component of the tensor $\overset{\circ}{G}_{\mu\nu}$ must be approximately equal to $\overset{\circ}{G}_{00} \approx \nabla^2 g_{00}$.

As discussed in Sec. 2.3.4, the curvature tensor (2.42) is constructed with the metric and its derivatives. Moreover, this tensor contains up to second derivatives in the metric so that it seems like a plausible quantity to consider in the final form of the field equations. Further, it can be seen that a linear combination of its contraction (Ricci tensor and scalar curvature) is the most general form that $\overset{\circ}{G}_{\mu\nu}$ must have in order to full fill the first condition written above. Therefore, one can write down that the most general $\overset{\circ}{G}_{\mu\nu}$ should take the following form

$$\overset{\circ}{G}_{\mu\nu} = C_1 \overset{\circ}{R}_{\mu\nu} + C_2\, g_{\mu\nu} \overset{\circ}{R}\,, \tag{2.77}$$

which of course is a symmetric tensor and C_1 and C_2 are arbitrary constants. These constants can be set directly by using the Newtonian limit and also the fact that this tensor must satisfy $\overset{\circ}{\nabla}_\mu G^{\mu\nu} = 0$. A simple analysis tells us that using these two conditions, one has that $C_1 = -2C_2$. It should be noted again that the symbol $\circ$

was introduced to label quantities computed with the Levi-Civita connection since in forthcoming chapters, it will be dealt with other kinds of connections. Therefore, a tensor which satisfies all the requirements established above is the one which is given by

$$\overset{\circ}{G}_{\mu\nu} \equiv \overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R} \,. \tag{2.78}$$

This tensor is known as the *Einstein tensor*. Moreover, from the Newtonian limit one can also notice that the coupling constant must be equal to

$$\kappa^2 = 8\pi G \,. \tag{2.79}$$

Hence, from Eq. (2.76), the final form of the famous Einstein field equation is the following

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2} \overset{\circ}{R} \, g_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu} \,. \tag{2.80}$$

The above equations are known as the *Einstein field equations*. The left-hand side of this equation describes the geometry of spacetime, characterised by the Ricci tensor, scalar curvature and the metric. The right-hand side of this equation contains all the information related to the matter content, which is in the energy-momentum tensor $\mathcal{T}_{\mu\nu}$. The constant κ^2 is just a coupling constant that needs to be equal to $8\pi G$ in order to match the Newtonian limit given by the Poisson equation (2.67). In this form, one can directly notice that the concept of gravity is directly related to the geometry. This consequence comes directly from the principles assumed in the theory. In general, this equation is difficult to solve since it is a system of 10 non-linear partial differential equations. For that reason, Einstein thought that nobody will be able to solve his equation. Surprisingly, only one year after Einstein proposed its equations, Karl Schwarzschild found the first exact solution of GR for a vacuum spherically symmetric geometry. Currently, there are more than 4000 known exact solutions in GR (Stephani et al., 2009). Basically, the equations are complicated to solve but they can be treated by assuming some symmetries in the spacetime studied.

Soon after Einstein proposed this famous equation, in 1917 he proposed its first modification. With the aim of having a stationary Universe, he realised that one needed to introduce an additional term to Eq. (2.80). He thought that the Universe must be static and that also that it must satisfy the Mach's principle. Moreover, he also thought that in order to avoid the gravitational collapse, one needed to introduce this new term to his equation. He said: "The term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars". This term is known as the *cosmological constant* and he noticed that it is possible to modify his field equations to also satisfy the condition of having a static Universe. This modification it is known as the *Einstein field equations with cosmological constant*, explicitly given by

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu}\,, \tag{2.81}$$

where Λ is the famous cosmological constant. Some years later Einstein introduced this cosmological constant, astronomical observations proved that the Universe is expanding (not static). Then, he thought that adding this term to his equation was the most important failure of his career. Today, it is known that the Universe is expanding but also in an accelerated way. Trying to understand the acceleration of the Universe is one of the most important goals for the physics today. Actually, one of the most important models for modern cosmology is based on this cosmological constant since it can mimic (quite well) the behaviour of the accelerated expansion of the Universe (see Sec. 2.5.2). One can say that the motivational idea of introducing this constant was incorrect (since Einstein wanted to have a static Universe), however, in the end, this constant is one of the most important ingredients in modern cosmology.

Another alternative way to find the Einstein field equations is by using the standard Lagrangian approach. Soon after Einstein proposed its equation, the mathematician Hilbert realised that it is also possible to define an action which describes the Einstein field equations. Then, one can vary this action with respect to the metric using the standard variational method. The action principle tells us that for any given action $\mathcal{S}$, its variation $\delta\mathcal{S}$ (in this case with respect to the metric) must be zero. This principle comes from the idea of the minimization of the action. This action is known as the Einstein-Hilbert action and it is explicitly given by

$$\mathcal{S}_{\rm GR} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\overset{\circ}{R} + \mathcal{S}_{\rm m}\,, \tag{2.82}$$

where $g = \det(g_{\mu\nu})$ and $\mathcal{S}_{\rm m}$ is any matter action. In this action the scalar curvature $\overset{\circ}{R}$ is the main ingredient. By varying this action with respect to the metric and equating it to zero, $\delta\mathcal{S}_{\rm GR} = 0$, one directly finds that the integrand of the first term becomes

$$\sqrt{-g}\,\delta\overset{\circ}{R} + \overset{\circ}{R}\,\delta\sqrt{-g} = \Big[\frac{\delta\overset{\circ}{R}}{\delta g^{\mu\nu}} - \frac{1}{2}\overset{\circ}{R}\,g_{\mu\nu}\Big]\delta g^{\mu\nu}\sqrt{-g}\,, \tag{2.83}$$

where it has been used that $\delta\sqrt{-g} = -(1/2)\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}$. From the definition of the scalar curvature $\overset{\circ}{R} = \overset{\circ}{R}_{\mu\nu}g^{\mu\nu}$ given in Eq. (2.60), now with $\Gamma^{\lambda}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$, it can be directly found that the first term in the above equation becomes

$$\begin{aligned}\sqrt{-g}\frac{\delta\overset{\circ}{R}}{\delta g^{\mu\nu}} &= \sqrt{-g}\overset{\circ}{R}_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\,\overset{\circ}{\nabla}_{\sigma}\Big[g^{\mu\nu}\delta\Gamma^{\sigma}_{\nu\mu} - g^{\mu\sigma}\delta\Gamma^{\lambda}_{\lambda\mu}\Big] \\ &= \sqrt{-g}\overset{\circ}{R}_{\mu\nu}\delta g^{\mu\nu} + \text{b.t.}\,,\end{aligned} \tag{2.84}$$

where $\sqrt{-g}\,\overset{\circ}{\nabla}_{\mu}V^{\mu} = \partial_{\mu}(\sqrt{-g}\,V^{\mu})$ was used (where V^{μ} is any vector). By using the latter identity, one can directly see that the whole last term on the above equation is a boundary term (b.t.). Since this boundary term does not affect the field equations, it can be neglected. Hence, from Eq. (2.83) and the action (2.82) one directly finds that

$$\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}\, g_{\mu\nu} = \kappa^2 \mathcal{T}_{\mu\nu}\,, \tag{2.85}$$

where $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor defined as

$$\mathcal{T}_{\mu\nu} \;\; = \;\; -\frac{2}{\sqrt{-g}}\Big(\frac{\delta \mathcal{S}_{\rm m}}{\delta g^{\mu\nu}}\Big)\delta g^{\mu\nu}\,. \tag{2.86}$$

As can be seen, Eq. (2.85) matches with (2.80) so that, the Einstein field equations obtained from the Einstein-Hilbert action match with the ones proposed by Einstein. One can also add a cosmological constant to the Einstein-Hilbert action by adding the term

$$\mathcal{S}_{\Lambda} = -\frac{1}{\kappa^2}\int d^4x\,\sqrt{-g}\,\Lambda\,, \tag{2.87}$$

and then find the corresponding Einstein field equation with a cosmological constant. For modified theories of gravity, the Lagrangian approach is extremely important. In future chapters, this approach will be used to obtain the field equations of modified theories of gravity. This thesis will be focusing on modified theories of gravity, explicitly speaking, modified Teleparallel models. As pointed out, it will focus on studying some cosmological models. To understand this better, the basis concepts of cosmology will be introduced in the next section.

2.5 Friedmann-Lemaître-Robertson-Walker cosmology

Cosmology is facing an important challenge today. The discovery of the accelerated expansion of the Universe changed radically the notion of how our Universe is evolving. This section is devoted to describing some basic properties of cosmology, starting from Newtonian cosmology and then finishing with the well-known Friedmann-Lemaître-Robertson-Walker cosmologies starting from GR. These sections will be useful to also understand future sections related to modified theories of gravity where different models will be proposed to mimic the dark energy problem.

Cosmology tries to address and model the Universe at great scales and understand how it evolves and how it will continue evolving in time. Hence, its study is difficult and challenging but also it is an extremely interesting and on-going subject. One can say that cosmology is quite a new field on physics but that is not true at all. From the very beginning of times, mankind has asked about what is our place in the Universe and how it all started. Even though those questions have always been on the table, before the 20th century, cosmology was thought to be more a subject related to philosophy rather than physics. One can say that modern cosmology started in 1929, when the astronomer Edwin Hubble discovered that the Universe is expanding. The expansion of the Universe has been one of the most important discoveries in the XX century since it changed the vision and comprehension of the Universe. Hubble used distant galaxies and realised that if a galaxy is further away from the Earth, they tended to move faster with respect to the Earth. So, distant galaxies are moving further away from Earth than the nearest ones. The most accepted model which deals with those observations is that the Universe itself is expanding and hence, with respect to an observer on the Earth, a further away galaxy will move faster than nearer galaxies. An easy way to understand the expansion of the Universe is by thinking that the Universe is similar to a balloon which is inflated at each moment as it is depicted in Fig. 2.8. Consider that initially at t_0, the Earth is located at a point A of this balloon, and there are two galaxies: a nearer galaxy located at the point B and a further away galaxy located at a point C. Now, if the balloon is inflated at t', from the perspective of an observer located on the Earth (comoving with the expansion), the galaxies will be displaced towards two new points B' and C'. Now the distance between the galaxies are greater due to the expansion, i.e., $\Delta x'_{AB} > \Delta x_{AB}$ and $\Delta x'_{AC} > \Delta x_{AC}$. Hence, from a comoving observer located on the Earth, the galaxies would be moving further away with respect to the Earth. Since, there are more space contained from the Earth and the further away galaxy located initially at B, the increment in the $\Delta x_{AC} \to \Delta x'_{AC}$ is greater than the increment in the distance $\Delta x_{AB} \to \Delta x'_{AB}$. Therefore, the quotient $\frac{\Delta x'_{AC} - \Delta x_{AC}}{t' - t_0} > \frac{\Delta x'_{AB} - \Delta x_{AB}}{t' - t_0}$, or in other words, the velocities of the galaxies with respect to the Earth satisfy $v_{AC} > v_{AB}$. Hence, a further away galaxy will move faster than a closer galaxy with respect to the comoving observer located in the Earth. From the perspective of the Earth, one can think that the galaxies are moving further away from it, but it is the balloon (the Universe) which is creating this effect since it is inflating in all directions making the points become displaced. This is actually the same as Hubble measured: distant galaxies were moving faster away with respect to the Earth. Therefore, exactly as the balloon is expanding one can understand those measurements as the Universe itself is expanding.

Hubble measured the recession velocity v for different galaxies and noticed that depending on its distance D with respect to the Earth, one can establish the following relationship

$$v = H_0 D\,, \tag{2.88}$$

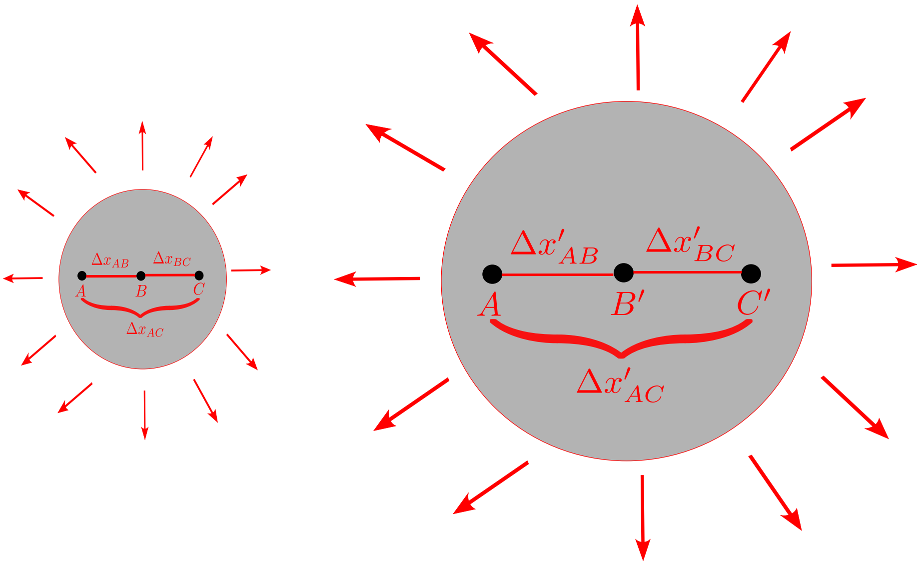


Figure 2.8: The Universe is represented as an expanding balloon where an observer located at the Earth is said to be comoving with the expansion. The further away galaxy (located initially at C) will move faster than the closer galaxy (located initially at B) with respect to the Earth.

where H_0 is known as the *Hubble constant*. The above equation is known as the *Hubble law*. Hence, the recession velocity of a further away galaxy will be greater than a near galaxy. Hubble measured this constant but it is known that his measurements were not good enough to find a good approximation of its real value. Actually, from his measurements, the relationship between v and D was not linear. Since he developed this theory, many researchers have tried to estimate this constant with higher precision. This is one of the most important constants in physics and with different modern techniques and astronomical measurements, it is believed to be approximately $H_0 \approx 67,4 \pm 0.5$ km s^{-1}Mpc^{-1} (Aghanim et al., 2018). However, there is a discrepancy of around 3.6σ about this value with local measurements of the Hubble constant, which suggest that its value is slightly bigger; $H_0 \approx 73,24 \pm 1,74$ km s^{-1}Mpc^{-1} (Riess et al., 2016). Then, those values and error bars are non-overlapping. The Hubble constant is also important to estimate the age of the Universe since it is directly related to it via (Hawley & Holcomb, 2005)

$$t_{\rm Universe} = \frac{1}{H_0} \approx 13.8 \text{ Gyrs}\,. \tag{2.89}$$

Thus, the Universe is more or less 13.8 billion years old. The Hubble law can be seen as the beginning of modern cosmology and because of that, Hubble is regarded as one of the fathers of cosmology.

A more recent important discovery which changed all the notion of modern cosmology was discovered in 1998. Supernovae type Ia have a consistent peak luminosity so they can be used as standard candles in the Universe. Two independent groups of researchers studied these kinds of astrophysical objects and they noticed that the Universe is not only expanding, but also it is expanding in an accelerated rate (see Riess et al. (1998); Perlmutter et al. (1999)). This is an extremely important discovery since it was in the opposite direction as physicists believed. Further, the researchers who discovered the accelerated expansion of the Universe won the Nobel prize in 2011. If the Universe is accelerating, there must be something unknown that is creating this behaviour. Since this ingredient was not understood, it was labelled as *dark energy*. Even today researchers do not understand well this quantity and it is an on-going field to study. One of the most important goals is to understand this quantity better and hence, some people believe that GR must be modified to understand this problematic issue in a better way. This thesis will be focus on some modifications of GR with an emphasis on studying the dark energy problem.

In the next sections, it will briefly describe the basic notions of Newtonian cosmology and then the standard Friedmann-Lemaître-Roberton-Walker cosmology starting in GR with a cosmological constant. The latter model gives us one of the most important models of modern cosmology describing the current observations.

2.5.1 Newtonian Cosmology

The basic ideas of cosmology will be introduced using standard Newtonian gravity and then GR will be used to understand cosmology in a more rigorous way. An expanding Universe can be naively understood using the ideas of Newtonian gravity and Newton's laws. Let us suppose that the Universe is homogeneous and spherically symmetric. Consider a comoving spherically symmetric surface located in a certain position of that Universe. Let us also suppose that the radius of this surface is much less than the radius of the Universe. Therefore, all the galaxies inside the comoving surface will only by influenced by their own gravitational fields. In this way, one can neglect all the other gravitational fields from the rest of the Universe. Thus, this situation can be understood if one imagines that the Universe is a Newtonian gas expanding homogeneously and isotropically in all directions. Any particle of this gas (which can be seen as a cluster of galaxies) will have the following trajectory

$$x(t) = x(t_0)\frac{a(t)}{a(t_0)}\,, \tag{2.90}$$

where $a(t)$ is a scale factor of the whole gas and t_0 is an initial time where the expansion started. This function will tell us how the gas is expanding (or contracting). If one assumes a coordinate system where the centre of the gas is located at its origin, the gravitational energy potential of a particle of the gas $U(t)$ located at a distance equal to the radius of the comoving sphere $|x(t)|$ can be found by using Eq. (2.66), giving us

$$U(t) = M\phi(t) = -Gm\frac{\rho(t)V(t)}{|x(t)|} = -\frac{4}{3}Gm\pi\rho|x(t_0)|^2\frac{a(t)^2}{a(t_0)^2}\,, \tag{2.91}$$

where M is the mass inside the sphere of radius $|x(t)|$ and $V(t) = (4/3)\pi|x(t)|^3$ is the volume of the sphere. The kinetic energy of this particle will be

$$K(t) = \frac{1}{2}m|x(t_0)|^2\frac{\dot{a}(t)^2}{a(t_0)^2}\,, \tag{2.92}$$

where dots represent differentiation with respect to the time t. Then, the total energy of the particle will be the sum of the energy potential plus the kinetic energy, namely

$$E = \frac{1}{2}m\frac{|x(t_0)|^2}{a(t)^2}\left[\dot{a}(t)^2 - \frac{8\pi G}{3}\rho a(t)^2\right]\,. \tag{2.93}$$

On the other hand, from the 2nd Newtonian law, one directly finds that the acceleration $\ddot{x}$(t) of the gas will be

$$\ddot{x}(t) = -G\frac{M}{|x(t)|^2} = -\frac{4}{3}\pi G\rho|x(t)|\,. \tag{2.94}$$

Now, if one substitutes (2.90) into the above equation, one can find a differential equation for the scale factor:

$$\ddot{a}(t) = -\frac{4}{3}\pi G\rho(t)a(t)\,. \tag{2.95}$$

Now, the energy density can be written as

$$\rho(t) = \frac{M}{V(t)} = \frac{3}{4\pi}M\frac{a^3(t_0)}{x(t_0)}\,a(t)^{-3} = \rho_0 a(t)^{-3}\,, \tag{2.96}$$

where initial energy density $\rho_0 = 3Ma^3(t_0)/(4\pi x(t_0))$ was defined here. Therefore, one can rewrite (2.95) as follows

$$\frac{d}{dt}(\dot{a}^2(t)) = \frac{8}{3}\pi G\rho_0\frac{d}{dt}\left(\frac{1}{a(t)}\right)\,. \tag{2.97}$$

Finally, by integrating this equation, one finds

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a(t)^2}\,, \tag{2.98}$$

where k is an integration constant. The above equation is known as the *first Friedmann-Lemaître-Robertson-Walker* (FLRW) equation, with the scale factor $a(t)$ being the function which measures distances in a expanding or contracting universe. Using GR, the constant k can be identified as the spatial curvature of the universe. Using Newtonian gravity, it is not possible to identify k with the spatial curvature of the Universe. This equation is extremely important in cosmology since as it will be seen later on, this equation also appears naturally using GR. The model described above using Newtonian gravity to find the FLRW equations was firstly introduced in McCrea & Milne (1934), soon after the FLRW model was introduced in GR.

Let us now briefly study what one can say about this model. As pointed out before, this model visualizes the expanding Universe as a Newtonian gas which is expanding in time. Each particle of the gas can be interpreted as a galaxy in an expanding Universe. From observational measurements, at these scales, it has been tested that the Universe is homogeneous and isotropic (see Sec. 2.5.2 for further details). Hence, in principle one can choose any supercluster of galaxies (a particle in our model) to draw the spherical surface which contains it on this boundary in such a way that one can neglect all the other gravitational effects by the rest of the Universe. By using (2.93) and (2.98), one can find that the total energy of a particle on the boundary of this sphere will be given by

$$E = -\frac{1}{2}m\frac{|x(t_0)|^2}{a(t)^2}k\,. \tag{2.99}$$

From here, one can notice three important cases which depend on the sign of the constant k. Depending on this sign, different kinds of expanding universes can be obtained. From the above equation, one can notice that this model tells us that for $k < 0$ the universe expands infinitely, for $k = 0$ the universe expands until a late-time $t \to \infty$ and for $k > 0$ the universe expands and then it stops to begin a new contracting era (known as *Big Crunch*). The model used using the Newtonian laws is known as *Newtonian Cosmology.* Even though, Newtonian Cosmology might reproduce some viable characteristics from (2.98) and (2.99), to fully understand the dynamics of our Universe, one needs to use GR. The Newtonian interpretation is essentially incomplete for various reasons, for instance:

- The Newton laws do not work for situations where the particles move at velocities comparable to the velocity of the light.
- To have a better understanding of what it means the scale factor $a(t)$, one needs to use GR.
- To justify that one can neglect all the effects of the matter outside the sphere one needs to assume the Birkhoff Theorem. To probe this theorem, one of course needs GR.
- To understand that the constant of integration k is a spatial curvature one needs to use GR.
- Newtonian gravity uses the concept of absolute time which is incorrect.

There are other several important problems coming from this approach. However, the Newtonian laws can give us an intuitive and initial idea of the behaviour of an expanding universe. Therefore, to understand it with more precision and in a more rigorous physical way, GR needs to be used.

2.5.2 FLRW Cosmology and ΛCDM model

Cosmology tries to describe the structure, history and the dynamic of the Universe considering it as an entire whole entity. When one is referring to Einstein cosmology, one is referring to the cosmology which assumes that GR is valid. Therefore, it implies that also the GR principles and the Einstein field equations are assumed to be correct. As will be seen in following sections, it is also possible to obtain more general equations by taking some modified theories of gravity. In mostly all those models, the standard GR case can be recovered in a a specific limit. Mainly all the cosmological models assumed that at cosmological scales, the Universe is homogeneous and isotropic. Therefore, any location in the Universe does not occupy a special position. The latter is known as the *cosmological principle*, which can be summarised as follows

> *"Over very large scales and statistically, the Universe is homogeneous and isotropic on its spatial part, always it has been like that and it will be like this for ever ".*

Homogeneous means that the Universe must be uniformly distributed and isotropic means that there is not any preferred direction. The cosmological principle also says that those characteristics must be valid from the beginning of the Universe and moreover, that it will be like that for ever. Hence, the Universe has translational and rotational symmetries. It also states that it is valid on very large scales. Currently, there is observational evidence that the Universe satisfies the cosmological principle for scales at around $l \gtrsim 200h^{-1}[Mpc]$[1] ($\sim 10^8$ light years). Observational observations such as the 2dF Galaxy Redshift Survey (Loveday et al., 1995) shows that at that mentioned scale, the Universe is homogeneous and that the galaxies are somehow non-randomly associated or organized. It is important to mention that one of the most important evidences which suggests that our Universe is isotropic comes from CMB observations (Ade et al., 2014b).

[1] $h^{-1}[Mpc]$ is a galaxy distance unit which depends on the Hubble constant H_0 and hence it depends on the cosmological model assumed. In this case $h = H_0/100[km/s/Mpc]$

2.5.2.1 FLRW cosmology

The assumption that the Universe is homogeneous and isotropic, one can find a metric which describes this kind of spacetime. The metric which describes a universe which satisfies the cosmological principle is the FLRW metric. In this geometry, the space contains the greatest number of symmetries so it is the maximally symmetric spacetime. Technically speaking, a maximally symmetric spacetime is a spacetime which has $N(N+1)/2$ Killing vectors for N spatial dimensions of the spacetime. For the GR case $N = 3$, there are 6 independent Killing vectors. They are associated with the invariance of rotations and translations in the FLRW spacetime. Using pseudo-spherical coordinates (t, r, θ, ϕ) centred at any point of the universe, the most general spacetime with this property is given by the FLRW metric which is

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2\right], \tag{2.100}$$

where $a(t) > 0$ is the *scale factor of the universe* which has dimensions of length and of course only depends on the coordinate t since re-scale distances through the expansion. This function is the same one introduced in the previous Newtonian cosmology section. The constant $k = \{-1, 0, 1\}$ is the *spatial curvature* representing different kind of universes. For $k = -1$ the universe is an open universe (hyperbolic space), hence if for example one draws a triangle, the sum of their angles is less than 180°. For the $k = 0$ case, the universe is flat (Euclidean space) and finally for the case $k = +1$ the universe is closed (3-sphere space). For the last case, the sum of the angles of a triangle is always greater than 180°. Cosmological observations tell us that the Universe is almost flat $k \approx 0$ (Aghanim et al., 2018). This metric was first derived by Alexander Friedmann in 1924 but his results were not regarded important at that time since the Universe was assumed to be static. Soon after, Georges Lemaître also derived the same result independently and later in the 30s

Howard Roberton and Arthur Walker showed rigorously that actually the above spacetime is the most general one satisfying the cosmological principle. For these historical reasons, some people only name this spacetime as Friedmann-Roberton (FR) or Friedmann-Roberton-Walker (FRW). However, being rigorous, all of them somehow contributed to this important metric so throughout this thesis, this metric will be labelled as the FLRW metric.

To fully describe the dynamics of the Universe, one also needs to assume the matter/energy which describes the content of the Universe. In cosmology, it is usual to model all the matter/energy content of the Universe as a *perfect fluid* whose energy-momentum tensor is given by

$$\mathcal{T}_{\mu\nu} = (\rho + p)u_\mu u_\nu - p\, g_{\mu\nu}\,, \tag{2.101}$$

where $p = p(t)$ and $\rho = \rho(t)$ are the pressure and energy density of the fluid and u^μ is the 4-velocity of an observer comoving with the expansion $u^\mu = (1, 0, 0, 0)$ satisfying $u^\mu u_\mu = 1$. It should be noted that a perfect fluid is the one which that has an equation of state which relates its pressure with its energy density as $\rho = \rho(p)$ or $p = p(\rho)$. Now, there are all the ingredients to obtain the FLRW equations in a GR setting. For the FLRW metric (2.100) and the matter content described by a perfect fluid as (2.101), the standard FLRW equations can be obtained using the Einstein field equations (2.80), giving us

$$3H^2 + \frac{3k}{a^2} = \kappa^2\rho\,, \tag{2.102}$$

$$3H^2 + 2\dot{H} + \frac{k}{a^2} = -\kappa^2 p\,, \tag{2.103}$$

where dots denote differentiation with respect to time and the so-called *Hubble pa-*

rameter or *Hubble rate* has been introduced,

$$H = \frac{\dot{a}}{a}\,. \tag{2.104}$$

Eq. (2.102) is called the first Friedmann equation of the Friedmann constraint and Eq. (2.103) is called the second Friedmann equation or the acceleration equation. The first FLRW equation is the same one described in the Newtonian Cosmology section (see Eq. (2.98)).

Sometimes it is also convenient to derive the so-called *Raychaudhuri equation* which can be directly found from the FLRW equations. By subtracting (2.102) with (2.103) one arrives at

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(3p+\rho)\,, \tag{2.105}$$

which tells us about the acceleration of the universe. The universe is accelerating(decelerating) when $\ddot{a} > 0(\ddot{a} < 0)$ and moreover expanding(contracting) if $\dot{a} > 0(\dot{a} < 0)$. From this equation one can notice that if $3p+\rho > 0$ the universe will be decelerating whereas for $3p+\rho < 0$ the universe is accelerating. By using the fact that the energy-momentum tensor is covariantly conserved ($\overset{\circ}{\nabla}_\mu \mathcal{T}^{\mu\nu} = 0$), one can obtain the *conservation equation for the fluid* which is given by

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{2.106}$$

This equation is not an independent equation since it can also be obtained directly from the FLRW equations. There are two independent equations and there are three unknown variables ($a(t), \rho$ and p), so the system must be closed with an additional equation. To do this, one usually assumes that the fluid relates its energy density with its pressure as a certain function $p = p(\rho)$. Hence, there is a certain *equation* *of state* (EoS) which relates those quantities. Moreover, a *barotropic fluid* is usually assumed for standard cosmology. This fluid has the following equation of state

$$p = w\,\rho\,, \tag{2.107}$$

where w is a constant known as the *equation of state parameter*. Depending on its value, it could represent different kind of fluids, namely:

- Cosmological constant fluid $w = -1$.
- Dark energy fluid $w = -1/3$.
- Dust (non-relativistic fluid) $w = 0$.
- Radiation (relativistic fluid) $w = 1/3$.

Fluids are known to be always in the regime where $w = [0, 1]$ but negative values of w could be also considered. However, up to now, fluids with negative w have not been discovered yet. Using the EoS equation, the system of equations is closed. Then, by using the conservation equation of the fluid (2.106) it can be directly found that the energy density behaves as

$$\rho(t) = \rho_0\, a(t)^{-3(1+w)}\,. \tag{2.108}$$

Hence, the energy density for matter-like, radiation-like and a cosmological-like fluids behave respectively as

$$\rho_{\rm m} = \rho_{0,\rm m}\, a(t)^{-3}\,, \quad \rho_{\rm rad} = \rho_{0,\rm rad}\, a(t)^{-4}\,, \quad \rho_\Lambda = \rho_0\,. \tag{2.109}$$

Here $\rho_{0,\rm m}, \rho_{0,\rm rad}$ and ρ_0 are integration constants. If one uses the equation of state

(2.107) into Eq. (2.105), one finds

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(3w+1)\rho\,. \tag{2.110}$$

From this equation one can see that the only way to achieve an accelerating scenario would be when $w < -1/3$. Hence, one needs negative values of the state parameter to obtain accelerating universes. Therefore, dark energy needs to be described by a negative value of the state parameters which gives rise to repulsive gravitational effects. These kind of fluids violate the standard energy conditions. A fluid described by a $w < -1$ state parameter is known as *phantom fluid*. Fluids with $w < 0$ have not been found yet in the laboratory.

Assuming a flat universe ($k = 0$), if one replaces (2.108) in the Friedmann constraint (2.102), one can find that the scale factor of the universe takes a power-law form when $w \neq -1$

$$a(t) \propto t^{\frac{2}{3(1+w)}}\,, \tag{2.111}$$

and an exponential form

$$a(t) \propto e^{\pm\sqrt{\frac{\kappa^2\rho_0}{3}}t}\,, \tag{2.112}$$

when $w = -1$. From (2.111), one can see that for a radiation-like fluid the scale factor behaves as $a(t) \propto t^{1/2}$ and for a matter-like fluid as $a(t) \propto t^{2/3}$. It should be remarked that (2.111) could describe an expanding or contracting universe depending on the sign of the square root. The solution described by (2.111) is known as a *power-law solution* and the solution behaving as (2.112) is known as a *de-Sitter solution*. The latter solution corresponds to a never-ending accelerating behaviour of the universe.

2.5.2.2 A very brief history of the Universe

In this section, it will be very briefly discussed what observations have told us about the history of the Universe. Roughly speaking, from different observations, it has been noticed that the Universe has faced different eras. It started at around 13.8 billion years ago in a *Big Bang* in a *cosmological singularity* where all the evolution of the Universe started. The Universe was really hot and small and then during the expansion became colder and bigger. It can be said that GR is not a sufficiently powerful theory to fully describe what happened from times from 0 to the Planck time (around $< 10^{-43}$ seconds after the Big Bang). To understand this era better, a quantum theory of gravity is needed since quantum effects are important at those scales. Up to now, there is not a final accepted quantum gravity theory so this is a challenging open question in physics. After this, the Universe experienced an *inflation era*, which was an extremely rapid accelerating expansion of the primordial Universe. The Universe expanded around 10^{26} times in a very small fraction of time of around 10^{-33} to 10^{-32} seconds (Guth, 1981). This is the same as more or less expanding the Universe from a nanometre to approximately 10 light years in only that extremely small period of time! After that, the Universe passed an era dominated by radiation (around 10 seconds after the Big Bang) where photons dominated the evolution of the Universe (Ryden, 2016). At this stage, the Universe became transparent and then the Cosmic Background radiation was created (CMB) (Spergel et al., 2007). This epoch finished at around 400.000 years after the Big Bang (Ryden, 2016). As seen, in GR, $a(t) \propto t^{1/2}$ in this era. Cosmology at early-times tries to deal with those epochs of the Universe. After that, the Universe faced a matter dominated era, that according to GR, the scale factor behaves as $a(t) \propto t^{2/3}$. It is believed that in this epoch, the galaxies were formed. Finally, at approximately 10 billion years after the Big Bang, the Universe started experiencing a dark energy era.

The physics of the era comprehending from matter to the dark energy dominated era is known as late-time eras. In this moment, we are living in the transition between matter-dark energy eras. This history of course, is just a very brief description about what actually has happened to the Universe and how it has been evolved. However, a good cosmological model should be able to at least describe those kinds of different dominated eras. One of the most challenging and important questions in physics is how to try to describe the evolution of the Universe from the Big Bang. This naive model of our Universe is constructed by different observational measurements (see for example Aghanim et al. (2018); Spergel et al. (2007)).

2.5.2.3 ΛCDM model

The simplest model which mimics the dark energy is known as the Λ-*cold-dark-matter model* or ΛCDM. In this model, the cosmological constant Λ plays an important role and it is assumed that the Einstein field equations with the cosmological constant are valid. For the FLRW metric, the Einstein field equations with a cosmological constant (2.81) takes the following form

$$3H^2 + \frac{3k}{a^2} - \Lambda = \kappa^2 \rho \,, \tag{2.113}$$

$$3H^2 + 2\dot{H} + \frac{k}{a^2} - \Lambda = -\kappa^2 p \,. \tag{2.114}$$

These equations are called the FLRW with cosmological constant. It should be noted that Λ can be written either on the left hand side or the right hand side of the above equations, but to emphasize that it is related to the geometry (or a modification of the Einstein field equations), it was written on the left hand side. This is of course just a matter of convention; one can also think that Λ is a kind of matter and then, it should appear on the right hand side of the above equations. If one assumes that $\rho = p = k = 0$ and $\Lambda \neq 0$ one can directly find $a(t) \propto e^{\sqrt{\Lambda/3}t}$ which according to

(2.112), is the same solution as a $w = -1$ fluid with

$$\Lambda = \kappa^2 \rho_0 \,. \tag{2.115}$$

Then, a flat universe without any source will give rise a de-Sitter universe. Hence, the cosmological constant is related to the vacuum energy density $\rho_0 = \rho_{\rm vac}$ and behaves as a fluid with a negative pressure. Since a constant energy density ($w = -1$) give rise to the same evolution of a universe as only having a cosmological constant, this kind of fluid is sometimes known as a cosmological-like fluid.

Let us now consider the standard flat ΛCDM model where it is assumed that $k = 0$ and the matter is described by a energy density $\rho = \rho_{\rm m}$ and a barotropic pressure $p = p_{\rm m} = w\rho_{\rm m}$. For this case, the energy density will be given by Eq. (2.108) and from the FLRW with a cosmological constant (2.113)-(2.114), one can directly find that the scale factor behaves as

$$a(t) \propto \sinh\left[\frac{1}{2}\sqrt{3\Lambda}(1+w)t\right]^{\frac{2}{3(w+1)}} \,, \tag{2.116}$$

which behaves as $a(t) \propto t^{\frac{2}{3(1+w)}}$ (matter dominated) for $t \to 0$ and $a(t) \propto e^{\sqrt{\Lambda/3}}$ (cosmological constant dominated) for $t \to \infty$. Hence, this model has the correct asymptotic behaviour starting with a matter-dominated era and then finishing in dark energy dominated era with Λ being the responsible of this epoch. The latter epoch is then dominated by the cosmological constant. For early-times, this model can replicate a radiated dominated era by choosing $w = 1/3$ and a matter dominated era by choosing $w = 0$. However, this model cannot describe both early dominated eras at the same time. To achieve a universe which faces the correct history of our Universe; radiation era $\to$ matter era $\to$ cosmological constant era; an additional fluid must be introduced. However, by doing that, it is not possible to find an exact solution for the scale factor. One can then use for example, dynamical systems techniques to analyse and model this more reliable representation.

If one introduces more fluids, in general, these fluids must satisfy the conservation equation given by (2.106). So, for example let us assume that there are two fluids characterised by their two energy-momentum tensor $\mathcal{T}^{(1)}_{\mu\nu}$ and $\mathcal{T}^{(2)}_{\mu\nu}$ respectively. Since the total energy-momentum tensor $\mathcal{T}_{\mu\nu} = \mathcal{T}^{(1)}_{\mu\nu} + \mathcal{T}^{(2)}_{\mu\nu}$ satisfies the conservation equation, one can rewrite that each fluid satisfies

$$\overset{\circ}{\nabla}{}^{\mu}\mathcal{T}^{(1)}_{\mu\nu} = Q_{\nu}\,, \quad \overset{\circ}{\nabla}{}^{\mu}\mathcal{T}^{(2)}_{\mu\nu} = -Q_{\nu}\,, \tag{2.117}$$

where Q_ν is a vector which measures a *transfer of energy-momentum* between the fluids. If there is no interaction or transfer of energy-momentum between those fluids, one obtains that $Q_\nu = 0$ and therefore each fluid, independently satisfy its own conservation equation $\overset{\circ}{\nabla}{}^{\mu}\mathcal{T}^{(1)}_{\mu\nu} = 0$ and $\overset{\circ}{\nabla}{}^{\mu}\mathcal{T}^{(2)}_{\mu\nu} = 0$. This procedure of course can be generalised for N fluids but of course the model will be more complicated to analyse.

Today, it is also known that the matter content described by ρ_m is mainly composed by another unknown quantity labelled as *dark matter* in 1933 by Fritz Zwicky. This quantity was introduced with the aim to understand different astronomical observations that did not match with the theoretical gravity models. Examples for those observations are weak gravitational lensing (Clowe et al., 2006), CMB (Spergel et al., 2007), galaxy rotation curves (Rubin et al. (1980); Navarro et al. (1996); Corbelli & Salucci (2000)) and also observations from galactic halos and clustering (Moore et al. (1999); Cooray & Sheth (2002)). It is believed that this kind of matter does not interact electromagnetically, and thus it is not possible to directly observe it with standard telescopes. Usually, it is modelled as a cold kind of matter with an EoS $w = 0$.

For completeness, let us finish this briefly description of ΛCDM by introducing three different fluids: a cold dark matter-like fluid $\rho_{\rm cdm}$, a baryonic matter-like fluid

$\rho_{\rm b}$ and a radiation-like fluid $\rho_{\rm rad}$. The first two matter-like fluids can be described as a fluid with an EoS $w = 0$ behaving as $\rho_{\rm m} \propto a(t)^{-3}$. Then, the first FLRW equation (2.113) reads

$$3H^2 \;=\; \kappa^2(\rho_{\rm cdm} + \rho_{\rm b} + \rho_{\rm rad}) + \Lambda - \frac{3k}{a^2}\,. \tag{2.118}$$

Assuming that there is not interaction between the fluids, the energy densities behave as Eq. (2.109). Now, by dividing the above equation by $3H^2$, one can rewrite it as follows

$$1 = \Omega_{\rm m} + \Omega_{\rm rad} + \Omega_\Lambda + \Omega_k\,, \tag{2.119}$$

where *dimensionless density parameters* were defined in the following way

$$\Omega_{\rm m} = \Omega_{\rm cdm} + \Omega_{\rm b} = \frac{\kappa^2}{3H^2}(\rho_{\rm cdm} + \rho_{\rm b})\,, \quad \Omega_{\rm rad} = \frac{\kappa^2\rho_{\rm rad}}{3H^2}\,, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2}\,, \tag{2.120}$$

and then $\Omega_k = -3k/\dot{a}^2 = 1 - \Omega_{\rm m} - \Omega_{\rm rad} - \Omega_\Lambda$. Now, by using the solutions (2.109) and then by replacing them in (2.119) one finds

$$H^2 = H_0^2\Big[(\Omega_{{\rm cdm},0} + \Omega_{{\rm b},0})\,a^{-3} + \Omega_{{\rm rad},0}\,a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}\,a^{-2}\Big]\,, \tag{2.121}$$

where it was assumed that the integration constants (related to $\rho_{0,x}$) are $a_0 = 1$ and $\Omega_{0,x}$, and H_0 is the Hubble constant. Using this simple model, one can mimic different epochs of the Universe as matter, radiation and cosmological constant eras. From the above equation, one can directly see that at early-times, the radiation term will dominate, then matter will dominate and finally the Universe will be dominated by dark energy represented by the cosmological constant. From observations we know that the best way to describe our Universe is by having the following parameters (Aghanim et al., 2018)

$$\Omega_{\mathrm{cdm},0}h^2 = 0.11933 \pm 0.00091\,, \quad \Omega_{\mathrm{b},0}h^2 = 0.02242 \pm 0.00014\,, \quad \Omega_{\mathrm{rad}} \approx 10^{-4}\,, \tag{2.122}$$

$$\Omega_{\mathrm{m},0} = 0.3111 \pm 0.0056\,, \quad \Omega_{\Lambda,0} = 0.6889 \pm 0.0056\,, \quad \Omega_{k,0} = 0.001 \pm 0.002\,. \tag{2.123}$$

The above values tells us that our Universe is mainly composed by the dark energy component. Hence, it is extremely important to understand its behaviour and its physical properties. Observational measurements suggest that our Universe is almost flat ($k \approx 0$). This will be considered in almost all the models studied throughout this thesis. One can also notice that non-baryonic matter (cold dark matter) consists of around 84% of all the matter content ρ_m. Thus, one can say that cosmology is facing the most exciting and interesting time on its history since the most important ingredients, dark energy and dark matter, are not very well understood yet. One of the most important motivations of introducing modified theories of gravity is to try to understand those quantities better. Using Combining Planck data with Pantheon supernovae and BAO data, it has been measured that the dark energy state parameter is $w = -1.03 \pm 0.003$ which is consistent with having a cosmological constant as a responsible of dark energy (Aghanim et al., 2018). Hence, one can say that the ΛCDM described by Eq. (2.121), is the simplest model which describes the different epochs that the Universe has experienced. Of course, this is not the final model and there are certain problems that will be discussed better in the next section.

2.5.2.4 The Cosmological constant problem

Despite the good precision with which ΛCDM predicts the current cosmological observations, theoretically there are some issues and one of the most important ones is called *the cosmological constant problem.* In this section, this problem will be briefly explained. For more details, see the reviews by Weinberg (1989, 2000); Martin (2012). The cosmological constant is introduced in GR as a parameter with the aim of describing the accelerating expansion of the Universe. Then, using observations, one can constrain its value. According to what was shown in the previous section, a cosmological constant acts like a fluid with a negative pressure with an equation of state equal to $p_\Lambda = -\rho_\Lambda$ (with $w_\Lambda = -1$). From observations, the cosmological constant needs to take a very small value of approximately (Clifton et al., 2012)

$$\Lambda \approx 10^{-52}\,\mathrm{m}^{-2}\,. \tag{2.124}$$

However, it is not possible to consider a fundamental theoretical reason on why this value needs to take this very small value. In quantum field theory, the energy density of a vacuum $\rho_{\rm vac}$ is given by (Sakharov, 1968)

$$\langle 0|T_{\mu\nu}|0\rangle = -\rho_{\rm vac}\, g_{\mu\nu}\,, \tag{2.125}$$

where $|0\rangle$ is the vacuum state which, classically, is the state of minimum energy. At the quantum level, due to the Heisenberg principle, the kinetic and the potential energy cannot be zero at the same time. Then, an additional source of energy coming from quantum effects (vacuum state) contributes to the Einstein field equations. Thus, there are two types of cosmological constants, one related to classical effects and the other related to quantum ones. Classically, it is interpreted as the value of matter fields when the kinetic energy is zero and the potential is minimum, then $\rho_{\rm vac} = V(\phi_{\rm min})$, where ϕ is a classical scalar field which represents a perfect fluid. By doing this, one obtains that classically,

$$\rho_\Lambda \simeq 10^{-47}\,\mathrm{GeV}^4\,. \tag{2.126}$$

On the other hand, the quantum effects due to fluctuations in the vacuum state can be computed by considering a massive scalar field ϕ with mass m and a potential $V(\phi) = m^2\phi^2/2$. The energy density of this quantum state is given by (Martin et al., 2014)

$$\rho_{\rm vac}^{\rm QM} = \langle 0| \rho_\phi |0\rangle = \frac{1}{2(2\pi)^3}\int \sqrt{k^2+m^2}\, d^3k = \frac{m^4}{64\pi^2}\log\left(\frac{m^2}{\mu^2}\right) \simeq -10^8\,{\rm GeV}^4 \ , \tag{2.127}$$

where $k = |k^i|$ is the norm of the 3-dimensional momentum k^i. It is clear that the above integral diverges (UV divergence), therefore a cut-off scale $k_{\rm max}$ was introduced to obtain the above approximated value. Quantum field theory would be valid until this cut-off and GR is supposed to be valid just below the Planck scale. It should be noted here that the zero-point energy density is positive for bosons and negative for fermions. Both contributions are equal in absolute value and then the final result is zero. Fermions are anti-commuting objects, giving rise to a negative vacuum energy density. This is the reason why in the above equation, a negative sign appears. For more details about this, see Weinberg (1989); Martin (2012).

The problem arises here. The Standard Model of particles predicts two symmetry breaking phase transitions: Electro-Weak and QCD phase transitions. In those phase transitions, the vacuum energy plays an important role being non-zero and from experiments and observations, it has been measured that the energy density of vacuum in those transitions are at the order of

$$\rho_{\rm vac}^{\rm EW} \simeq 10^8\,{\rm GeV}^4\,, \quad \rho_{\rm vac}^{\rm QCD} \simeq 10^{-2}\,{\rm GeV}^4\,. \tag{2.128}$$

Then, the total energy density of vacuum is the sum of each quantity

$$\rho_{\rm vac} = \rho_{\rm vac}^{\rm EW} + \rho_{\rm vac}^{\rm QCD} + \rho_{\rm vac}^{\rm QM} \simeq 10^8{\rm GeV}^4\,. \tag{2.129}$$

Now, if one compares the above resulting value with the observed value of the cosmological constant (2.126), one notices that there is a huge difference of around 45 orders of magnitude! This is known as the cosmological constant problem and theoretically, is still an open problem in modern physics. One approach on trying to solve this issue is by modifying the Einstein field equations, which is known as modified theories of gravity. This thesis will deal with modifications of gravity but considering an alternative and equivalent formulation of gravity, known as Teleparallel gravity.

3

Teleparallel gravity

Chapter Abstract

This chapter is devoted to presenting a translation gauge theory of gravity known as Teleparallel gravity. The most important mathematical quantities are defined and it is demonstrated that their field equations are equivalent to the Einstein field equations. This theory represents an alternative description of gravity where the spacetime is globally flat but not trivial since the torsion tensor is non-zero and appears as the field strength of the theory.

3.1 An alternative description of gravity: Teleparallel gravity

General Relativity (GR) is a very successful theory accurately describing the dynamics of the Solar System. All predictions of General Relativity, including gravitational waves, have now been experimentally verified. Nonetheless, when applied to the entire Universe, we are faced with conceptual and observational challenges that are sometimes simply summarised as the dark energy and the dark matter problems. When considering the total matter content of the Universe, it turns out that approximately 95% is made up of these two components we do not fully understand yet. This, together with developments in other fields of physics has motivated a variety of models which can be seen as extensions or modifications of General Relativity.

Perhaps surprisingly, alternative formulations of General Relativity were constructed and discussed shortly after the formulation of the Einstein field equations. One such description, which is of particular interest here, is the so-called *Teleparallel gravity* or *Teleparallel equivalent of General Relativity* (TEGR). Its equations of motion are identical to those of General Relativity and their actions only differing by a total derivative term. While both theories are conceptually different, experimentally these two theories are indistinguishable. In TEGR, there is not a geodesic equation as in GR. Instead, similar to electromagnetism, force equations describe the movement of particles under the influence of gravity. Additionally, the dynamical variable is the tetrad instead of the metric as in GR. With the aim to unifying electromagnetism with gravity, Einstein introduced this approach in the 1920s with the tetrads being the dynamical fundamental variables. This quantity has 16 degrees of freedom and the metric only 10, so Einstein thought that the remaining degrees of freedom could be used to describe electromagnetism. Of course, he did not succeed this unification due to the fact the extra degrees of freedom, in the end, were related to the Lorentz invariance of the theory. These ideas were first introduced by Einstein (Einstein, 1928).

The key mathematical result to this approach goes back to Weitzenböck in 1923 (Weitzenböck, 1923) who noted that it is indeed possible to choose a connection such that the curvature vanishes everywhere but being non-trivial by having non-zero torsion. The Teleparallel name stems from the fact that the notion of parallelism is global instead of local on flat manifolds, see for instance Møller (1961); Hayashi & Nakano (1967); Cho (1976); Aldrovandi & Pereira (2013); Maluf (2013) and reference therein. Additionally, parallel transportation does not depend on the path chosen, hence, angles and lengths are invariant under parallel transport.

This chapter is devoted to introducing the basis foundations of TEGR, which is derived as a gauge theory of the translations. The field strength is given by torsion tensor and due to the nature of the spin connection, it gives a globally flat spacetime with zero curvature. For a more detailed description of TEGR, see the book Aldrovandi & Pereira (2013), which will be used in this thesis to review the most important properties of this theory.

3.2 Tetrads and linear frames

Any gravitational theory is constructed in a manifold which at any point, a tangent space is defined as a Minkowski spacetime with a metric $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$. In this thesis, Greek indices $\alpha, \beta, \gamma, ..$ denote spacetime indices whereas Latin indices $a, b, c, ..$ denote tangent space indices. Then, local bases for vector fields are defined as $\{\partial_\mu\} = \{\frac{\partial}{\partial x^\mu}\}$ for a spacetime coordinate x^μ, and $\{\partial_a\} = \{\frac{\partial}{\partial x^a}\}$ for a tangent space coordinate x^a. The fundamental variable in Teleparallel gravity is the so-called *tetrad field* or *vierbein* which are the basis for vectors on the tangent space. In this thesis, the tetrad is labelled as e^a_μ and its inverse as E^μ_a. Hence, the basis can be written as $e^a = e^a_\mu dx^\mu$ and $e_a = E^\mu_a \partial_\mu$. It should be noted that tetrads can be always defined if the manifold is assumed to be differentiable. The tetrads satisfy the orthonormal conditions, namely

$$E^\mu_m e^n_\mu = \delta^n_m \,, \tag{3.1}$$

$$E^\nu_m e^m_\mu = \delta^\nu_\mu \,. \tag{3.2}$$

Additionally, the linear basis satisfies the following commutation relationship,

$$[e_a, e_b] = e_a e_b - e_b e_a = E^\mu_a \partial_\mu (E^\nu_b \partial_\nu) - E^\mu_b \partial_\mu (E^\nu_a \partial_\nu) \,. \tag{3.3}$$

After some simplifications, one gets that the above equation becomes

$$[e_a, e_b] = E^\mu_a E^\nu_b (\partial_\mu e^c_\nu - \partial_\nu e^c_\mu) e_c = f^c{}_{ab} e_c \,, \tag{3.4}$$

where the *structure coefficients* or *coefficient of anholonomy* $f^c{}_{ab}$ was defined as the curl of the basis $\{e_a\}$. Holonomic frames or inertial frames are the ones where $f^c{}_{ab}$ vanishes at every point of the manifold. Linear frames provide a relation between Minkowski tangent spacetime metric η_{ab} and Minkowski spacetime metric $\eta_{\mu\nu}$, namely,

$$\eta_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \,, \tag{3.5}$$

which are known as *trivial frames.* Let us now define linear frames where $f^c{}_{ab}$ is related to gravitation and inertia. In those frames, a pseudo-riemannian metric and its inverse can be written with the tetrads as follows

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \,, \quad g^{\mu\nu} = E^\mu_a E^\nu_b \eta^{ab} \,. \tag{3.6}$$

It should be remarked that in those frames, it is always possible to locally have $f^c{}_{ab} = 0$ but globally they will not be zero since now they represent both inertia and gravity. It is easy to verify that the determinant of the tetrad is related to the determinant of the metric by

$$e = \det(e^a_\mu) = \sqrt{-g} \,. \tag{3.7}$$

Then, in Teleparallel gravity, the fundamental variable is the tetrads and metric can be reconstructed via (3.6).

3.3 Gravitational gauge theory of the translations

Teleparallel theories of gravity are based on the idea of working within a geometrical framework where the notion of parallelism is globally defined. In the standard formulation of General Relativity this is only possible for spacetimes which are flat and hence are completely described by the Minkowski metric $\eta_{\mu\nu}$. When working on manifolds with torsion, it is possible to construct geometries which are globally flat but have a non-trivial geometry. It turns out that Teleparallel gravity can be written as a gauge theory of the translations. A gauge transformation is defined as a local translation of the tangent space coordinates,

$$x^a \to x'^a + \epsilon^a(x^\mu)\,, \tag{3.8}$$

where ϵ^a is a transformation parameter and then the infinitesimal transformation takes the form $\delta x^a = \epsilon^b P_b x^a$ with $P_a = \partial_a$ being the infinitesimal generator that satisfies $[P_a, P_b] = \partial_a\partial_b - \partial_b\partial_a = 0$. Now, a general source field $\varphi = \varphi(x^a(x^\mu))$ will transform under the transformation (3.8) as $\delta\varphi = \epsilon^a\partial_a\varphi$. A global translation will lead that those derivatives to transform according to

$$\delta(\partial_\mu\varphi) = \epsilon^a\partial_a\partial_\mu\varphi + (\partial_a\varphi)\partial_\mu\epsilon^a\,, \tag{3.9}$$

which is not covariant. The above expression will be covariant only if ϵ^a is constant. Then, one needs to introduce an auxiliary gauge potential B_μ with the aim to recover covariance when derivatives are acting on any generic source field φ. If one introduces the translational gauge potential

$$B_\mu = B^a_\mu P_a \,, \tag{3.10}$$

with P_a being the infinitesimal generators of translations, one can construct a gauge covariant derivative

$$e_\mu \varphi = \partial_\mu \varphi + B^a_\mu \partial_a \varphi \,, \tag{3.11}$$

with $\delta B^a_\mu = -\partial_\mu \epsilon^a$, in such a way that for any source field φ, one obtains

$$\delta(e_\mu \varphi) = \epsilon^a \partial_a \partial_\mu \varphi \,, \tag{3.12}$$

which transforms covariantly under gauge transformations. Then, the gravitational coupling prescription is achieved by replacing $\partial_\mu \varphi \to e_\mu \varphi$, giving

$$e_\mu \varphi = e^a_\mu \partial_a \varphi \,, \tag{3.13}$$

and the tetrad field now is given by

$$e^a_\mu = \partial_\mu x^a + B^a_\mu \,. \tag{3.14}$$

It should be noted that the above tetrad is non-trivial, then $B^a_\mu \neq \partial_\mu \epsilon^a$.

Let us now perform a Lorentz transformation $x^a \to \Lambda^a_b x^b$ which changes the frame from $e^a_\mu \to \Lambda^a_b e^b_\mu$ and then the gauge potential transforms as $B^a_\mu \to \Lambda^a_b B^b_\mu$. Then, the covariant translation derivative transforms as $e_\mu \varphi = e^a_\mu \partial_a \varphi$ as before but with the tetrads being equal to

$$e^a_\mu = \partial_\mu x^a + \overset{\bullet}{w}{}^a{}_{b\mu} x^b + B^a_\mu \,, \tag{3.15}$$

where

$$\overset{\bullet}{w}{}^a{}_{b\mu} = \Lambda^a_d \partial_\mu \Lambda^d_b \tag{3.16}$$

is a purely gauge quantity known as *spin connection* which represents inertial effects in the new rotated frame. Since this connection is different from the ones computed in GR, bullets will be used to differentiate between them. Here, one can find that the gauge covariant derivative acting on a vector x^a is equivalent to

$$\overset{\bullet}{\mathcal{D}}_\mu x^a \equiv \partial_\mu x^a + \overset{\bullet}{w}{}^a{}_{b\mu} x^b \,, \tag{3.17}$$

and notice that this term represents the non-gravitational part of the tetrad. In these frames, one finds $\delta B^a_\mu = -\overset{\bullet}{\mathcal{D}}_\mu \epsilon^a$.

Let us remember here that any general connection can be split into two parts,

$$\Gamma^\lambda_{\mu\nu} = E^\lambda_a \partial_\nu e^a_\mu + E^\lambda_a w^a{}_{b\nu} e^b_\mu \equiv E^\rho_a \mathcal{D}_\mu e^a_\nu \,, \tag{3.18}$$

where the first part depends on tetrads and the second part on the spin connection. Here $\mathcal{D}_\mu$ is a covariant derivative where the generators act on tangent space only. Conversely, one can find that the spin connection can be written as

$$w^a{}_{b\mu} = e^a_\nu \partial_\mu E^\nu_b + e^a_\nu \Gamma^\nu_{\rho\mu} E^\rho_b = e^a_\nu \nabla_\mu E^\nu_b \,, \tag{3.19}$$

where ∇_μ is a general covariant derivative acting on the spacetime indices. The above relationship is valid for any theory of gravity.

If one computes the commutation relation of gauge covariant derivatives, one obtains

$$\begin{aligned} [e_\mu, e_\nu] &= e_\mu e_\nu - e_\nu e_\mu = e^a_\mu \partial_a e^b_\nu \partial_b - e^a_\nu \partial_a e^b_\mu \partial_b \\ &= \left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \overset{\bullet}{w}{}^a{}_{b\mu} B^b_\nu - \overset{\bullet}{w}{}^a{}_{b\nu} B^b_\mu\right) P_a \,, \end{aligned} \tag{3.20}$$

where Eq. (3.15) was used. In gauge theories, the field strength is given exactly by the above quantity, which in this case is

$$T^{a}{}_{\mu\nu} \equiv \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \overset{\bullet}{w}{}^{a}{}_{b\mu} B^b_\nu - \overset{\bullet}{w}{}^{a}{}_{b\nu} B^{b}{}_{\mu} = \overset{\bullet}{\mathcal{D}}_\mu B^a_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a_\mu \,. \tag{3.21}$$

Now, if one replaces (3.15) in the above equation and uses $[\overset{\bullet}{\mathcal{D}}_\mu, \overset{\bullet}{\mathcal{D}}_\nu] x^a = 0$, one obtains that the field strength can be rewritten as

$$T^{a}{}_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \overset{\bullet}{w}{}^{a}{}_{b\mu} e^b_\nu - \overset{\bullet}{w}{}^{a}{}_{b\nu} e^b_\mu = \overset{\bullet}{\mathcal{D}}_\mu e^a_\nu - \overset{\bullet}{\mathcal{D}}_\nu e^a_\mu \equiv \overset{\bullet}{\Gamma}{}^a_{\mu\nu} - \overset{\bullet}{\Gamma}{}^a_{\nu\mu} \,, \tag{3.22}$$

which is skew-symmetric and the so-called *Weitzenböck connection* was defined as

$$\overset{\bullet}{\Gamma}{}^a_{\mu\nu} = \overset{\bullet}{\mathcal{D}}_\mu e^a_\nu = \partial_\mu e^a_\nu + \overset{\bullet}{w}{}^{a}{}_{b\mu} e^b_\nu \,. \tag{3.23}$$

If one compares (2.34) with (3.22), one notices that the field strength $T^{a}{}_{\mu\nu}$ is exactly the torsion tensor. This means that in Teleparallel gravity, the field strength is the torsion tensor and the connection is the Weitzenböck connection which is skew-symmetric on its lower indices. It can be proved that the torsion tensor transforms as a tensor and it is covariant under local Lorentz transformations, i.e., if $x'^a \to \Lambda^a_b x^b$, the torsion tensor $T'^a_{\mu\nu} \to \Lambda^a_b T^b_{\mu\nu}$. Moreover, since tetrads are invariant under gauge transformations, the torsion tensor is also invariant under gauge transformations. It should be noted again that quantities with bullets mean that they are computed in terms of the Weitzenböck connection. To avoid writing bullets in all quantities, only connections, covariant derivatives and terms computed from curvature will be labelled with them and all the other quantities as torsion will not have a bullet.

It is worthy mentioning here that $\overset{\bullet}{w}{}^{a}{}_{b\mu}$ is a purely inertial connection which represents inertial effects. Then, one can choose a specific frame where globally, this term vanishes. Using (2.42), one can rewrite the curvature tensor as

$$R^a{}_{b\mu\nu} = \partial_\mu w^a{}_{b\nu} - \partial_\nu w^a{}_{b\mu} + w^a{}_{c\mu} w^c{}_{b\nu} - w^a{}_{c\nu} w^c{}_{b\mu}\,, \tag{3.24}$$

then, if $w_a{}^{b\nu} = \overset{\bullet}{w}_a{}^{b\nu}$, the connection is purely inertial (see (3.16)) and the curvature is identically zero:

$$\overset{\bullet}{R}{}^a{}_{b\mu\nu} = \partial_\mu \overset{\bullet}{w}{}^a{}_{b\nu} - \partial_\nu \overset{\bullet}{w}{}^a{}_{b\mu} + \overset{\bullet}{w}{}^a{}_{c\mu} \overset{\bullet}{w}{}^c{}_{b\nu} - \overset{\bullet}{w}{}^a{}_{c\nu} \overset{\bullet}{w}{}^c{}_{b\mu} = 0\,, \tag{3.25}$$

and for a non-trivial tetrad, $T^\mu{}_{\nu\lambda} \neq 0$ since the potential $B^b_\mu \neq 0$. Therefore, Teleparallel gravity is a theory where the curvature is zero (globally flat spacetime) but torsion is non-zero. Moreover, Teleparallel gravity is a gauge theory of translations where the torsion tensor appears as the strength field of the theory. On the other hand, in GR, the spin connection $\overset{\circ}{w}{}^a{}_{b\mu}$ gives a zero torsion but a non-vanishing curvature. Let us finish this section by using (3.19) which in Teleparallel gravity will read as

$$\overset{\bullet}{w}{}^a{}_{b\mu} = e^a_\nu \partial_\mu E^\nu_b + e^a_\nu \overset{\bullet}{\Gamma}{}^\nu_{\rho\mu} E^\rho_b = e^a_\nu \overset{\bullet}{\nabla}_\mu E^\nu_b\,, \tag{3.26}$$

which in a specific frame where the spin connection vanishes $\overset{\bullet}{w}{}^a{}_{b\mu} = 0$, one obtains the *Teleparallel condition*:

$$e^a_\nu \partial_\mu E^\nu_b + e^a_\nu \overset{\bullet}{\Gamma}{}^\nu_{\rho\mu} E^\rho_b = e^a_\nu \overset{\bullet}{\nabla}_\mu E^\nu_b = 0\,. \tag{3.27}$$

From the above equation, one can understand why the theory is called Teleparallel gravity. It turns out that this equation is a distant parallelism condition since tetrad is parallel-transported by $\overset{\bullet}{\Gamma}{}^\nu_{\rho\mu}$. This however, is no longer true for other frames where the spin connection is non-zero. This thesis will be focused on modifications of Teleparallel gravity in the frames where the spin connection vanishes since the majority of the literature, historically, has been written in those frames. It should be remarked again that spin connection is a pure gauge quantity which measures inertial effects so that it does not modify the equations of motion.

3.4 Action, field equations and equivalence with GR

In this section, the Teleparallel field equations will be derived and their equivalence with GR will be stated. For Teleparallel gravity, one has that the relationship between the Weitzenböck connection and the Levi-Civita connection (used in GR) is given by the Ricci theorem (2.53),

$$\overset{\bullet}{\Gamma}{}^{\lambda}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} + K_{\mu}{}^{\lambda}{}_{\nu}\,, \tag{3.28}$$

or in terms of spin connections

$$\overset{\bullet}{w}{}^{a}{}_{bc} = \overset{\circ}{w}{}^{a}{}_{bc} + K_{b}{}^{a}{}_{c}\,. \tag{3.29}$$

As seen in Sec. 2.3.4, from Eq. (2.55), the curvature tensor can be split into two pieces: one part depending only on a combination of torsion, known as the contortion tensor defined as (2.54), and the Riemannian curvature $\overset{\circ}{R}{}^{\lambda}{}_{\alpha\mu\nu}$ computed with the Levi-Civita connection. Then, by contracting $R^{\lambda}{}_{\alpha\lambda\nu} = R_{\alpha\nu}$, one obtains the Ricci tensor which can be also split into two pieces as in (2.58). Now, if one contracts the Ricci tensor given by (2.58), with the inverse of the metric as $g^{\alpha\beta}R_{\alpha\beta}$, one obtains the Ricci scalar (or scalar curvature) which can be written as

$$R = \overset{\circ}{R} + T - \frac{2}{e}\partial_{\mu}(eT^{\mu}) = \overset{\circ}{R} - T + B\,, \tag{3.30}$$

where the *boundary term*

$$B = \frac{2}{e}\partial_\mu(eT^\mu) = 2\overset{\bullet}{\nabla}_\mu T^\mu\,, \tag{3.31}$$

and the so-called *torsion scalar* T

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\nu\mu\lambda} - T^\mu T_\mu\,, \tag{3.32}$$

were defined. Here, the *torsion vector* $T_\mu = T^\lambda{}_{\lambda\mu}$ was also defined. Later, it will be remarked why B is called the boundary term. For simplicity, it is convenient to introduce the *superpotential*

$$S_\sigma{}^{\mu\nu} = \frac{1}{4}(T_\sigma{}^{\mu\nu} - T^\mu{}_\sigma{}^\nu - T^\nu{}_\sigma{}^\mu) + \frac{1}{2}(\delta^\nu_\sigma T^\mu - \delta^\mu_\sigma T^\nu) = \frac{1}{2}(K_\sigma{}^{\mu\nu} - \delta^\mu_\sigma T^\nu + \delta^\nu_\sigma T^\mu)\,, \tag{3.33}$$

in order to rewrite the scalar torsion in a compact way as follows

$$T = S_\lambda{}^{\mu\nu}T^\lambda{}_{\mu\nu}\,. \tag{3.34}$$

For GR, one has $T^\lambda{}_{\mu\nu} \equiv 0$ and hence $T = B = 0$ which gives us that the Riemannian curvature $\overset{\circ}{R} = R$. On the other hand, for Teleparallel gravity, the Weitzenböck connection gives us a globally flat spacetime with zero curvature and then with zero scalar curvature (see (3.25)). By choosing the TEGR case, one needs to replace quantities with the bullets, then $R^\lambda{}_{\alpha\mu\nu} = \overset{\bullet}{R}{}^\lambda{}_{\alpha\mu\nu}$ and also $R = \overset{\bullet}{R}$. Therefore, in Teleparallel gravity, Eq. (3.30) becomes

$$R \overset{!}{=} \overset{\bullet}{R} = \overset{\circ}{R} + T - \frac{2}{e}\partial_\mu(eT^\mu) = \overset{\circ}{R} - T + B = 0\,, \tag{3.35}$$

which can be arranged to obtain

$$\overset{\circ}{R} = -T + \frac{2}{e}\partial_\mu(eT^\mu) = -T + B\,. \tag{3.36}$$

Let us emphasise here that T and B are computed with the Weitzenböck connection whereas $\overset{\circ}{R}$ is a scalar computed from the Levi-Civita connection. Then, (3.36) is a fundamental equation which relates GR with Teleparallel gravity and it will appear frequently in the following chapters. It should be remarked again that this equation is only valid in Teleparallel gravity.

The Lagrangian of Teleparallel gravity needs to be constructed as any gauge theory, with the strength field which is the torsion tensor. To do this, one needs to consider the following action

$$\mathcal{S}_{\rm TEGR} = \frac{1}{2\kappa^2}\int \mathrm{tr}(T\wedge \star T) + \mathcal{S}_{\rm m}\,, \tag{3.37}$$

where tr represents the trace, $T = (1/2)T^a{}_{\mu\nu}P_a dx^\mu \wedge dx^\nu$ is a 2-form and $\star T = (1/2)\star T^a{}_{\mu\nu}P_a dx^\mu \wedge dx^\nu$ is its Hodge dual ($\wedge$ is the exterior product). The constant $\kappa^2 = 8\pi G$ is added in the action in order to match it with the Newtonian limit (and GR). Additionally, a matter action $\mathcal{S}_{\rm m}$ was added for increased generality. It is important to mention that this Hodge dual is not the standard one. In this case, this Hogde dual is more general since TEGR is constructed on the soldered bundle giving the opportunity to transform internal and spacetime indices with the tetrad fields. Therefore, this dual is sometimes called as generalised Hodge dual. Involving all the possible index contractions, one has

$$\star T^\lambda{}_{\mu\nu} = e\,\epsilon_{\mu\nu\rho\sigma}(aT^{\lambda\rho\sigma} + bT^{\rho\lambda\sigma} + cT^{\alpha\rho}{}_\alpha g^{\lambda\sigma})\,, \tag{3.38}$$

where a, b and c are constants and $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol. Since the torsion tensor is skew-symmetric on its last two indices, $T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu}$, only two contractions are needed to construct the generalised Hodge dual, then one needs to set $b = 2a$. Moreover, in 4 dimensions one has $\star\star T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu}$ which gives us that $a = 1/4$ and $c = -1$. Further, if one uses some identities from the form calculus, one obtains that the action (3.37) becomes

$$\mathcal{S}_{\rm TEGR} = \frac{1}{2\kappa^2}\int T e\, d^4x + \mathcal{S}_{\rm m}\,, \tag{3.39}$$

with T being the scalar torsion (see (3.34)). From here one can notice a very important property of Teleparallel gravity that it is one of the reasons why it is a fundamental theory. According to (3.36), the Ricci scalar $\overset{\circ}{R}$ and the torsion scalar T differ by the term B. Since the Einstein-Hilbert action, which is the action which produces the Einstein equations (see (2.82)), is constructed with the Lagrangian $\mathcal{L}_{\rm GR} = \sqrt{-g}\overset{\circ}{R} = e\overset{\circ}{R}$ and the Teleparallel Lagrangian is $\mathcal{L}_{\rm TEGR} = eT$, they will differ by

$$\mathcal{L}_{\rm GR} - \mathcal{L}_{\rm TEGR} = e(\overset{\circ}{R} + T) = eB = \partial_{\mu}(eT^{\mu})\,. \tag{3.40}$$

Clearly, the Teleparallel Lagrangian and the Einstein-Hilbert Lagrangian differs only by a term which is a boundary term. That is the reason why B was labelled as a boundary term before. This means that the Teleparallel action (3.39) differ only by a boundary term with respect to the Einstein-Hilbert action (2.82). Thus, if one takes variations with respect to the tetrads in the Teleparallel action, one obtains the same equations as taking variations of the Einstein-Hilbert action (2.82) with respect to the metric. Therefore, action (3.39) gives the Einstein field equations. This is the reason why in the literature, Teleparallel gravity is also known as the Teleparallel equivalent of General Relativity or TEGR. Although they have the same field equations, they are not completely equivalent theories since they have different actions and different physical interpretations of the same physical effect. Since Teleparallel gravity has the same equations as GR, all the experiments that have confirmed its validity can be also understood as confirming that Teleparallel gravity is correct also. Then, one can conclude that it is a matter of interpretation whether gravity is described by a zero torsion theory with non-zero curvature (GR) or a theory with zero curvature and non-zero torsion (TEGR).

It should be mentioned here that Einstein used another approach to construct the Teleparallel action based on scalars constructed by all the possible contractions of the torsion tensor: $T^{\mu\nu\lambda}T_{\mu\nu\lambda}, T^{\mu\nu\lambda}T_{\nu\mu\lambda}$ and $T_{\lambda\mu}{}^{\lambda}T^{\nu\mu}{}_{\nu}$. By doing this, one has a three parameter theory, that Einstein set in a very specific way in order to recover a theory which is equivalent to GR in field equations. A theory without setting these three parameters was later studied by Hayashi & Shirafuji (1979) and called *New General Relativity*. In Chap. 6, this theory and its modifications will be discussed.

To conclude this section, let us find the Teleparallel field equations which are equivalent to the Einstein field equations. Since this thesis will deal with the frames where the spin connection is zero, this will be assumed here, giving us that the torsion tensor (3.22) can be written as

$$T^{a}{}_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu = \overset{\bullet}{\mathcal{D}}_\mu e^a_\nu - \overset{\bullet}{\mathcal{D}}_\nu e^a_\mu = \overset{\bullet}{\Gamma}{}^a_{\mu\nu} - \overset{\bullet}{\Gamma}{}^a_{\nu\mu}\,. \tag{3.41}$$

One can label this approach as the *pure tetrad formalism* since the spin connection does not appear. It should be noted that TEGR was first formulated in this formalism and historically, it has remained like this over time. An important consequence arises here. The torsion tensor is no longer covariant under Lorentz transformations. This, however, it is not an important issue with TEGR since it turns out that the theory with zero spin connection is quasi-local Lorentz covariant due to the fact that the Teleparallel action (3.39) is invariant under local Lorentz transformations by a boundary term (Cho, 1976). For more details about this, see Sec. 4.2. When one considers modifications of Teleparallel gravity, one needs to be careful with the

Lorentz covariance. This will be discussed in the forthcoming chapters (see Sec. 4.2).

Let us now take variations of the action (3.39) with respect to the tetrads, giving us

$$\delta\mathcal{S}_{\rm TEGR} = \frac{1}{2\kappa^2}\int (e\delta T + T\delta e)d^4x + \int \delta(\mathcal{L}_{\rm m}\, e)d^4x\,, \tag{3.42}$$

where $\mathcal{L}_{\rm m}$ is a density matter Lagrangian. If one uses $g^{\mu\nu} = \eta^{ab}E_a^{\mu}E_b^{\nu}$, it is easy to show that

$$\delta g^{\mu\nu} = -\Big(g^{\nu\lambda}E_a^{\mu} + g^{\mu\lambda}E_a^{\nu}\Big)\delta e^a_{\lambda}\,, \tag{3.43}$$

$$\delta e = eE_a^{\lambda}\delta e^a_{\lambda}\,. \tag{3.44}$$

Then, the second term in (3.42) is equal to

$$T\delta e = eTE_a^{\lambda}\delta e^a_{\lambda}\,. \tag{3.45}$$

The first term in (3.42) requires more computations. By expanding that term one obtains

$$e\delta T = e\left(\frac{1}{4}\delta(T^{\mu\nu\lambda}T_{\mu\nu\lambda}) + \frac{1}{2}\delta(T^{\mu\nu\lambda}T_{\nu\mu\lambda}) - \delta(T^{\mu}T_{\mu})\right). \tag{3.46}$$

By varying and taking derivatives in (3.2) one finds respectively the following relations

$$\delta E_m^{\sigma} = -E_n^{\sigma}E_m^{\mu}\delta e^n_{\mu}\,, \quad \partial_{\nu}E_m^{\sigma} = -E_n^{\sigma}E_m^{\mu}\partial_{\nu}e^n_{\mu}\,. \tag{3.47}$$

Then, using Eqs (3.43) and (3.47), it is easy to show that

$$\delta T^{\lambda}{}_{\mu\nu} = -E_a^{\lambda}T^{\beta}{}_{\mu\nu}\delta e^a_{\beta} + E_a^{\lambda}(\partial_{\mu}\delta e^a_{\nu} - \partial_{\nu}\delta e^a_{\mu})\,, \tag{3.48}$$

$$\delta T^{\mu} = -\Big(E_a^{\mu}T^{\lambda} + g^{\mu\lambda}T_a + T^{\lambda}{}_a{}^{\mu}\Big)\delta e^a_{\lambda} + g^{\mu\nu}E_a^{\lambda}\Big(\partial_{\lambda}\delta e^a_{\nu} - \partial_{\nu}\delta e^a_{\lambda}\Big)\,. \tag{3.49}$$

Using the latter equations, it is possible to find the following identities:

$$\begin{aligned}
\delta(T_\mu T^\mu) &= -2\left(T^\beta T^\alpha{}_{\beta\mu} + T^\alpha T_\mu\right) E_a^\mu \delta e_\alpha^a + 2\left(T^\alpha E_a^\mu - T^\mu E_a^\alpha\right)\partial_\alpha \delta e_\mu^a\,, && (3.50)\\
\delta(T_{\alpha\mu\nu}T^{\mu\alpha\nu}) &= 2\left(T^{\beta\mu\alpha} - T^{\alpha\mu\beta}\right) T_{\mu\alpha\nu} E_a^\nu \delta e_\beta^a + \left(T^\alpha{}_\mu{}^\beta - T^\beta{}_\mu{}^\alpha\right) E_a^\mu \partial_\alpha \delta e_\beta^a\,, && (3.51)\\
\delta(T_{\alpha\mu\nu}T^{\alpha\mu\nu}) &= -4T^{\alpha\mu\nu}T_{\alpha\mu\beta}E_a^\beta \delta e_\nu^a + 4T_\alpha{}^{\mu\nu}E_a^\alpha \partial_\mu \delta e_\nu^a\,. && (3.52)
\end{aligned}$$

By replacing (3.50)-(3.52) in (3.46), integrating by parts and neglecting boundary terms, one finds

$$e\delta T = 4\Big[-\partial_\mu(eS_a{}^{\mu\lambda}) + eT^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu}\Big]\delta e_\lambda^a\,. \tag{3.53}$$

Finally, by replacing (3.45) and (3.53) in (3.42) and setting $\delta\mathcal{S}_{\mathrm{TEGR}} = 0$, one obtains the Teleparallel field equations given by

$$\frac{4}{e}\partial_\mu(eS_a{}^{\mu\lambda}) - 4T^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu} - TE_a^\lambda = 2\kappa^2\mathcal{T}_a^\lambda\,, \tag{3.54}$$

where the energy-momentum tensor was defined as

$$\mathcal{T}_a^\lambda = \frac{1}{e}\frac{\delta(e\mathcal{L}_m)}{\delta e_\lambda^a}\,. \tag{3.55}$$

If ones introduces the following current vector

$$j_a{}^\mu = E_a^\sigma T^\rho{}_{\nu\sigma}S_\rho{}^{\mu\nu} - \frac{1}{4}TE_a^\mu\,, \tag{3.56}$$

one finds that the conservation law is valid,

$$\partial_\mu\Big(e(j_a{}^\mu - 2\kappa^2\mathcal{T}_a{}^\mu)\Big) = 0\,. \tag{3.57}$$

Using (2.58), (3.28) and (3.36), one finds that the Einstein tensor is equivalent to

$$\overset{\circ}{G}{}^{\lambda}_{\nu} = \frac{2}{e} e^a_{\nu} \partial_\mu (e S_a{}^{\mu\lambda}) - 2T^{\sigma}{}_{\mu\nu} S_\sigma{}^{\lambda\mu} - \frac{1}{2} T \delta^{\lambda}_{\nu} \,, \tag{3.58}$$

hence, the Teleparallel field equations (3.54) are equivalent to the Einstein field equations (2.80). This is of course expected as it was discussed before since their Lagrangians differ only by a boundary term, hence, giving the same field equations.

3.5 The Teleparallel force equation

In the previous section, it was demonstrated that the Teleparallel field equations are the same as the Einstein field equations. In this section, in the context of Teleparallel gravity, the equation which describes how particles move in the presence of gravity will be presented. The motion of a spinless particle of mass m immersed in a gravitational field given by the gauge potential B^μ_a is described by the following action

$$\mathcal{S} = m \int_a^b ds = m \int_a^b g_{\mu\nu} \frac{dx^\mu}{ds} dx^\nu = m \int_a^b g_{\mu\nu} u^\mu dx^\nu = m \int_a^b u_a e^a \,, \tag{3.59}$$

where $u^\mu = dx^\mu/ds$ and $u^a = u^\mu e^a_\mu$. Then, by using $e^a = dx^a + B^a_\mu dx^\mu$ (see (3.14)), one obtains

$$\mathcal{S} = m \int_a^b u_a (dx^a + B^a_\mu dx^\mu) \,. \tag{3.60}$$

It should be noted again that the spin connection was set to be zero (special frame). Now, by taking variations in the above action, using $\delta x^a = (\partial_\mu x^a)\delta x^\mu$, $\delta B^a_\mu = (\partial_\lambda B^a_\mu)\delta x^\lambda$ and $e^a \delta u_a$, and finally by simplifying the expression, one arrives at

$$\delta \mathcal{S} = m \int_a^b \Big(e^a_\mu \frac{du_a}{ds} - (\overset{\bullet}{\mathcal{D}}_\mu B^a_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a_\mu) u_a u^\nu \Big) \delta x^\mu ds \,. \tag{3.61}$$

By using (3.41), one notices that the term $\overset{\bullet}{\mathcal{D}}_\mu B^a_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a_\mu = T^a{}_{\mu\nu}$ is the torsion tensor. Therefore, by setting $\delta\mathcal{S} = 0$, one finds

$$e^a_\mu \frac{du_a}{ds} = T^\lambda{}_{\mu\nu} u_\lambda u^\nu \,, \tag{3.62}$$

that can be rewritten as

$$\frac{du_\mu}{ds} + \left(K^\lambda{}_{\mu\nu} - \overset{\bullet}{\Gamma}{}^\lambda_{\mu\nu}\right) u_\lambda u^\nu = 0 \,, \tag{3.63}$$

where $T^\lambda{}_{\mu\nu} u_\lambda u^\nu = -K^\lambda{}_{\mu\nu} u_\lambda u^\nu$ and the Teleparallel condition (3.27) were used. The above equation is the *Teleparallel force equation* which is the equation of motion of a particle m in a gravitational field. Here, contortion plays the role of a force. Here, one can notice an important difference between GR and Teleparallel gravity. In GR, free-falling particles follow geodesics given by (2.16). On the other hand, in Teleparallel gravity, particles follow the force equation where contortion (or torsion) acts as the force. From the Ricci theorem $K^\lambda{}_{\mu\nu} - \overset{\bullet}{\Gamma}{}^\lambda_{\mu\nu} = \overset{\circ}{\Gamma}{}^\lambda_{\mu\nu}$, one finds that the Teleparallel force equation is equivalent to the dynamics described by the geodesic equation (see (2.16)). Then, spinless particles will follow the same trajectories in the presence of gravity in GR than Teleparallel gravity. Then, TEGR is also consistent with the equivalence principle. An interesting feature can be interpreted here. As discussed in Sec. 2.2.1, GR is based on the weak equivalence principle which says that the gravitational mass is equal to the inertial mass, i.e., $m_{\rm g} = m_{\rm i}$. This principle is based on experiments and a priori, it could be violated at some scales such as quantum scales (Damour, 2001; Lammerzahl, 1996). If new experiments show that this principle is violated at some scales, then GR will be ruled out at those scales. So far, all the experiments have measured with great precision the equivalence between those masses but future experiments with better precision could find a certain difference between them. On the other hand, Teleparallel gravity does not have this principle as one of the bases of the theory. This principle was not used to construct Teleparallel gravity as a gauge theory of the translations. Then, Teleparallel gravity dispenses with the weak equivalence principle. Hence, it can comply with universality, but remains a consistent theory in its absence. Moreover, in Aldrovandi et al. (2004a,b), it was found that the Teleparallel force equation (3.62) would be modified if one does not assume that $m_{\rm i} = m_{\rm g}$, giving

$$\left(\partial_\mu x^a + \alpha B^a_\mu\right)\frac{du_a}{ds} = \alpha T^\lambda{}_{\mu\nu} u_\lambda u^\nu \,, \tag{3.64}$$

where $\alpha = m_{\rm g}/m_{\rm i}$ is the ratio between the gravitational mass and the inertial mass. If $\alpha = 1$, one recovers the standard Teleparallel force equation. This equations could be extremely important if the weak equivalence principle is violated. Let us emphasise again here that this formalism cannot be achieved in the context of GR. This is one of the great advantages of Teleparallel gravity with respect to GR and it could be one of the unique ways to distinguish between these equivalent gravitational theories.

4 $f(T)$ gravity and Teleparallel scalar-tensor theories

Chapter Abstract

This chapter discusses the most popular modifications of Teleparallel gravity, the so-called $f(T)$ gravity and Teleparallel scalar-tensor theories. The physical motivations for introducing modifications of gravity will be discussed, giving emphasis to cosmology. The issue of the breaking of the Lorentz transformations is also discussed and then how one can alleviate the issue using "good tetrads". Some cosmological predictions or models are also provided and a new quintom scalar field model is presented.

4.1 Why do we want to modify TEGR (or GR)?

Since GR (or TEGR) describes many effects that have been observed and it is a successful theory, why are we interested in modifying it? In this section, there will be discussed some motivations and physical reasons on why modified gravity could be a reasonable way to address some issues that cannot be avoided with standard GR. For a more detailed description of modified gravity, see the reviews Nojiri & Odintsov (2006); Sotiriou & Faraoni (2010); Capozziello & De Laurentis (2011); Nojiri & Odintsov (2011); Clifton et al. (2012). Since GR has the same field equations as TEGR, the physical motivation could be the same since the same issues that

GR cannot address, are also true in TEGR. One usually call modified gravity where the Einstein-Hilbert (or TEGR) action is modified/extended. One can modify it by changing the matter/energy (right hand side of the GR's equations) or the parts related to the geometry (left hand side of the GR's equations). Usually, changing the matter/energy is not considered as being part of modified gravity since the geometrical theory is the same but new different kind of sources are considered. This is an alternative way to understand some issues related to cosmology. For example, quintessence models introduce a scalar field to understand the late-time accelerating behaviour of the Universe.

Let us first discuss some issues related to cosmology which is one of the main important subject in this thesis. First of all, GR needs a cosmological constant Λ, which acts as a fluid with a negative pressure $p_\Lambda = -\rho_\Lambda$, to describe the current accelerating behaviour of the Universe. Without evoking a cosmological constant, GR can only produce this scenario by adding some extra scalar fields. As discussed in Sec. 2.5.2.4, the value of the cosmological constant measured by observations is drastically different from that expected from considering quantum and classical aspects related to the vacuum energy. The latter is known as the cosmological constant problem and some researchers believe that there is no way out of this problem without either changing GR, adding extra scalar fields or changing the Standard Model of physics. If one considers modifications of GR, it is possible for some theories, to describe an accelerating late-time behaviour of the expansion of the Universe without evoking any cosmological constant. Further, the new terms coming from the modifications are responsible for this acceleration. This is one of the most important motivations for modified gravity.

At early-times of the Universe, there was a period of an extremely rapid accelerating expansion of the Universe. A cosmological constant Λ cannot be used to describe this epoch since inflation is followed by a radiation era and there is no way to stop inflation (acceleration) with a cosmological constant. This accelerated epoch can be understood by GR with a scalar field called the inflaton. However, GR does not explain the origin of the inflation or its nature. In addition, its predictions have fine-tuning problems in the parameters. Some examples of this are: the initial conditions for inflation and the slow-roll approximations. Modified gravity is able to describe both the inflationary era (early-times) and the current dark energy era in a unified way.

Another interesting problem that cannot be cured with GR, can also be addressed with some modified theories of gravity. As seen in Sec. 2.5.2.3, the value of the energy density of matter (baryonic plus dark matter) is today at the same order of magnitude as the energy density of the dark energy. Why are those values exactly the same order of magnitude today? Is there any physical reason why those quantities should be the same order of magnitude? This problem is known as the *coincidence problem*. Some physicists believe that this is not a problem and indeed is just a coincidence but, a priori, there is no theoretical explanation for this. Some modifications of GR might be able to solve this issue (Nojiri & Odintsov, 2006).

Another issue in cosmology is related to the *cosmological singularities*. In GR, it is not possible to avoid them but for some modified gravity theories, it is possible to construct bounce solutions which avoid some singularities which are related to the initial time of the Universe.

It is well known that in order to understand the observations of the rotation curves of galaxies, one needs to introduce a dark matter component. Since this is just a component which is introduced by hand in GR in order to describe those effects (and others), is there any possibility that one can describe those effects without evoking a fluid like dark matter? Well, the answer is yes. There are some modifications of GR such as $f(\overset{\circ}{R})$ gravity or MOND (Modified Newtonian Dynamics) which mimic some behaviour coming from dark matter. For example, MOND modifies the Newtonian laws to reproduce the flat rotation curves for spiral galaxies and also to achieve the luminosity-rotation velocity relation, known as Tully-Fisher relation (Famaey & McGaugh, 2012). This theory is a empirically motivated theory. It is important to mention that modified gravity has not been so successful to describe dark matter as to describe dark energy.

Another motivation for introducing modified gravity theory is related to the so-called *missing satellite problem*. According to ΛCDM (and then GR), the Milky way should have numerous dwarf galaxies orbiting it. However, only about 30 dwarf galaxies orbiting our galaxy have been observed (Clifton et al., 2012).

Another problems in GR is the fact that it is not *renormalisable* and also it cannot be *quantised*. Some modified theories of gravity have also tried to formulate a renormalisable theory in order to unify it with the other forces.

As a final comment related to motivating GR (or TEGR) is the fact that by studying modified gravity theories, one can learn more about GR (or TEGR). Since modified gravity can be seen as a generalisation of GR (or TEGR), one can learn more properties that could not be seen easily directly by studying GR (or TEGR).

There are many ways of modifying the gravity sector. One can think that at Solar System scales, GR works fine but at large scales such as cosmological ones, GR might be modified. Having this in mind, any viable modification of GR must predict the same observations at Solar System scales as GR. This can be achieved by having an Einstein-Hilbert action and then adding more terms coming from the modifications. Alternately, one can neglect a GR background and work out a modified theory with some screening mechanisms which can ensure the predictions of the theory at Solar System scales. Still, there is not a final modified theory of gravity which can solve all the problems that we know so far.

There are more additional problems related to GR and also to cosmology. For further details, see the reviews Sotiriou & Faraoni (2010); Clifton et al. (2012).

4.2 Lorentz transformations

Since the TEGR action is based on contractions of the torsion tensor (scalar torsion T), then, TEGR is generally covariant under general transformations. However, what happens with Lorentz covariance? In this section, the Lorentz transformations for Teleparallel gravity will be discussed. These transformations are just transformations of the tangent space coordinate x^a. Then, a local Lorentz transformation is given by

$$x^a \to \Lambda^a_b x^b \,, \tag{4.1}$$

where Λ^a_b is the Lorentz matrix which satisfies

$$\eta_{ab} = \eta_{cd}\Lambda^c_a\Lambda^d_b \,, \tag{4.2}$$

where η_{ab} is the Minkowski metric. Special Relativity is consistent with the covariance of the Lorentz transformations, therefore, they are fundamental transformations in physics. GR is invariant under Lorentz transformations (see Sec. 2.2.3), but what happens with TEGR? Further, when one is considering modifications of TEGR, are they also invariant under Lorentz transformations? To study this, let us consider the case where the spin connection is different from zero. By performing a local Lorentz transformations (4.1), one obtains that tetrads and the spin connection transform according to

$$e^a_\mu \to \Lambda^a_c e^c_\mu \,, \quad \overset{\bullet}{w}{}^a{}_{\mu b} \to \Lambda^a_c \overset{\bullet}{w}{}^c{}_{\mu d}(\Lambda^{-1})^d_b - (\Lambda^{-1})^a_c \partial_\mu \Lambda^c_b \,. \tag{4.3}$$

Here, $(\Lambda^{-1})^a_c$ represents the inverse of the Lorentz matrix. Then, one can prove that the torsion tensor with the spin connection different to zero (3.22) is covariant under local Lorentz transformations,

$$T^a{}_{\mu\nu} \to \Lambda^a_b T^b{}_{\mu\nu}\,, \quad \text{when spin connection is non-zero}\,. \tag{4.4}$$

Therefore, the scalar torsion T is also covariant under Lorentz transformations. Since the TEGR action is based on T, then, the TEGR's equations are invariant under Lorentz transformations. However, as mentioned before, for historical and computational reasons, Teleparallel gravity was formulated in the so-called pure tetrad formalism where the spin connection is set to be zero. This is like choosing a gauge (or a frame) from the beginning since this object is related to inertial effects. The field equations do not change when one is considering those frames since the TEGR Lagrangian (3.39) can be written as (Krššák, 2017a)

$$\mathcal{L}_{\text{TEGR}}(e^a_\mu, \overset{\bullet}{w}{}^a{}_{b\mu}) = \mathcal{L}_{\text{TEGR}}(e^a_\mu, 0) + \partial_\mu(e\overset{\bullet}{w}{}^\mu)\,, \tag{4.5}$$

where $\overset{\bullet}{w}{}^\mu = \overset{\bullet}{w}{}^{ab}{}_\nu e^\nu_a e^\mu_b$, and the spin connection appears like a boundary term without affecting the field equations. When one considers the frames where the spin connection vanishes, one obtains that after a Lorentz transformation (4.1), the torsion tensor transforms as

$$T^a{}_{\mu\nu} \to \Lambda^a_b T^b{}_{\mu\nu} + \Lambda^a_b \left(e^c_\nu \partial_\mu \Lambda^b_c - e^c_\mu \partial_\nu \Lambda^b_c\right)\,, \quad \text{when spin connection is zero}\,, \tag{4.6}$$

therefore, it is not covariant under Lorentz transformations. Then, in the pure tetrad formalism, T is not longer covariant under Lorentz transformations. However, according to (4.5), one has that the torsion scalar can be separated as follows

$$T(e^a_\mu, \overset{\bullet}{w}{}^a{}_{b\mu}) = T(e^a_\mu, 0) + \frac{4}{e}\partial_\mu(e\overset{\bullet}{w}{}^\mu)\,. \tag{4.7}$$

Then, from (4.6) and (4.7), it can be shown that even though the torsion tensor is no longer covariant under local Lorentz transformation, the Lagrangian (4.3) is quasi-invariant under local Lorentz transformation (up to a boundary term). The terms which break the Lorentz covariance are the ones in the boundary term. Then, any linear combination in T in the action will lead to a theory which respects the Lorentz covariance. Then, the TEGR action is invariant under local Lorentz covariance in both the pure tetrad formalism and the covariance one. However, how does it happen when one is referring to modifications of Teleparallel gravity where now the Lagrangian is no longer $\mathcal{L}_{\rm TEGR} = eT/2\kappa^2$? In general, those theories will break Lorentz covariance since now the surface term, which is not a local Lorentz invariant quantity, is now no longer a total divergence (Golovnev et al., 2017; Li et al., 2011; Sotiriou et al., 2011). Therefore, there are two alternatives to avoid this issue:

(i) Work within the formalism when the spin connection is different to zero: This is a new formalism, first developed in Kršśák & Saridakis (2016). The spin connection depends on the choice of the observer so that it is not determined uniquely. Moreover, it is only related to inertial effects. In TEGR, this quantity does not affect the field equations since it appears as a boundary term in the action. However, when one is modifying TEGR, it is no longer a boundary term and hence, it contributes. In order to obtain this spin connection, one can use a reference tetrad defined by setting the gravitational constant equal to zero. This methodology is still under development and there is still a debate on how to find the spin connection since there is no equation for this field. It can be understood that this is a field without dynamics so it might be a problematic issue.

(ii) Work within the pure tetrad formalism which breaks the Lorentz covariance but considering the so-called "good tetrads". This will be discussed in Sec. 4.3.2

As mentioned before, both formalisms will give rise to the same field equations.

In this thesis, for practical and historical reasons, the second approach (ii) will be employed for formulating modified Teleparallel theories of gravity. Another reason for choosing this approach is that (i) is currently in progress. For new studies related to this approach, see Krššák & Saridakis (2016); Golovnev et al. (2017); Krššák (2017a,b).

4.3 $f(T)$ gravity

4.3.1 Action and field equations

A well-studied modification of GR is to consider $f(\overset{\circ}{R})$ gravity (De Felice & Tsujikawa (2010); Sotiriou & Faraoni (2010); Capozziello & De Laurentis (2011)) where f is an arbitrary (sufficiently smooth) function of the Ricci scalar

$$\mathcal{S}_{f(\overset{\circ}{R})} = \frac{1}{2\kappa^2}\int f(\overset{\circ}{R})\sqrt{-g}\,d^4x + \mathcal{S}_{\mathrm{m}}\,. \tag{4.8}$$

This action is a straightforward modification of the Einstein-Hilbert action (2.82), where now $\overset{\circ}{R}$ is replaced by an arbitrary function $f(\overset{\circ}{R})$. Then, for the specific case where $f(\overset{\circ}{R}) = \overset{\circ}{R}$, one recovers GR. It should be remembered that the Ricci scalar depends on second derivatives of the metric tensor. Hence variations with respect to the metric will require integration by parts twice which will result in terms of the form $\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu F$ where $F = f'(\overset{\circ}{R})$, making the theory fourth order. Moreover, only the GR case where $f(\overset{\circ}{R}) \propto \overset{\circ}{R}$ yields second order field equations. In cosmology, this theory has interesting applications, for example, the possibility of describing the cosmological history of the Universe without evoking a cosmological constant. For more details about those models, see the reviews Nojiri & Odintsov (2011); Clifton et al. (2012); Nojiri et al. (2017).

In analogy to the above theory, one then can consider a straightforward modification of Teleparallel gravity by changing in the TEGR action (3.39), the scalar torsion T by an arbitrary function $f(T)$. Then, the action of this theory is given by (Bengochea & Ferraro, 2009)

$$\mathcal{S}_{f(T)} = \frac{1}{2\kappa^2}\int f(T)e\, d^4x + \mathcal{S}_{\rm m}\,. \tag{4.9}$$

Since the torsion scalar T only depends on the first derivatives of the tetrads, this theory is a second order theory. Moreover, in the pure tetrad formalism, since T itself is not covariant under local Lorentz transformations (see Sec. 4.2), $f(T)$ gravity is also not locally Lorentz covariant (Li et al., 2011; Maluf, 2013). Hence, there is a trade-off between second order field equations and local Lorentz covariance. Since $f(T)$ does not differ from $f(\overset{\circ}{R})$ by a total derivative term, these theories are no longer equivalent. For a complete discussion about this, see Chap. 5. By taking variations in (4.9) with respect to the tetrads, one obtains

$$\delta\mathcal{S}_{f(T)} = \frac{1}{2\kappa^2}\int \Big[ef_T\delta T + f\delta e + \delta(e\mathcal{L}_{\rm m})\Big]\, d^4x\,, \tag{4.10}$$

where $f_T = df(T)/dT$. The second term in the variations is $\delta e = eE_a^{\lambda}\delta e^a_{\lambda}$ (see (3.45)). The first term in the above equation can be computed by expanding it,

$$ef_T\delta T = ef_T\left(\frac{1}{4}\delta(T^{\mu\nu\lambda}T_{\mu\nu\lambda}) + \frac{1}{2}\delta(T^{\mu\nu\lambda}T_{\nu\mu\lambda}) - \delta(T^{\mu}T_{\mu})\right), \tag{4.11}$$

and then using (3.50)-(3.52) accordingly, and finally ignoring boundary terms, the above term takes the following form

$$ef_T\delta T = \Big[-4e(\partial_\mu f_T)S_a{}^{\mu\lambda} - 4\partial_\mu(eS_a{}^{\mu\lambda})f_T + 4ef_T T^{\sigma}{}_{\mu a}S_{\sigma}{}^{\lambda\mu}\Big]\delta e^a_{\lambda}\,. \tag{4.12}$$

Then, by replacing the above expression, using (3.45) and setting $\delta\mathcal{S}_{f(T)} = 0$, one arrives at the $f(T)$ field equations which are

$$4e\Big[f_{TT}(\partial_\mu T)\Big]S_\nu{}^{\mu\lambda} + 4e_\nu^a\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta_\nu^\lambda = 2\kappa^2 e\mathcal{T}_\nu^\lambda\,, \qquad (4.13)$$

where the energy-momentum tensor $\mathcal{T}_\nu^\lambda$ was defined as (3.55). Clearly, by taking $f(T) = T$, one recovers Teleparallel gravity. To finish this section, let us mention that according to Ferraro & Guzmán (2018), $f(T)$ has three degrees of freedom which means that it has one extra degree of freedom than GR (or TEGR). This extra degree of freedom can be interpreted as scalar one. However, in Bamba et al. (2013a), the authors found that $f(T)$ gravity does not have any further gravitational modes at first linear perturbation level. Therefore, $f(T)$ gravity has two propagating degrees of freedom, exactly as GR. Then, it will not be possible to find any difference between $f(T)$ and GR in future gravitational wave detections.

4.3.2 Good and bad tetrads

This section is devoted to presenting the approach presented in Tamanini & Boehmer (2012) which is a way to work properly when one is considering modifications of Teleparallel gravity. In some sense, the problem that $f(T)$ breaks the Lorentz covariance is somehow alleviated using this approach. It is based on choosing the correct tetrads since different tetrads give rise to different field equations. *Good tetrads* are the special form of tetrad (or frames) where the field equations do not impose any condition on the form of the function f. This definition and formalism can be then extended to any modified Teleparallel theory of gravity since this structure is the same for any modification.

In $f(T)$ gravity, Lorentz transformations change the field equations (not the metric). Let us first study a spherically symmetric case whose line element in spherical coordinates (t, r, θ, ϕ) is

$$ds^2 = A(r,t)^2 dt^2 - B(r,t)^2 dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2)\,. \tag{4.14}$$

As mentioned before, according to $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, different tetrads could give rise to the same metric tensor. The easiest tetrad for the above metric is given by the diagonal one given by

$$e^a_\mu = \mathrm{diag}(A(r,t), B(r,t), r, r\sin\theta)\,. \tag{4.15}$$

Using this diagonal tetrad in the $f(T)$ field equations (4.13), gives us that the theory satisfies the Birkhoff theorem but the Schwartzschild metric is no longer a solution of this theory. Without affecting the metric, one can always change tetrads by a local Lorentz transformation, obtaining Eq. (4.3). Then, let us perform this Lorentz transformation for the diagonal tetrad (4.15). To do this, consider the tetrad (4.15) and perform a general 3-dimensional rotation $\mathcal{R}$ in the tangent space parametrised by three Euler angles α, β, γ so that

$$\Lambda^a_b = \begin{pmatrix} 1 & 0 \\ & \\ 0 & \mathcal{R}(\varphi,\vartheta,\psi) \end{pmatrix}\,, \tag{4.16}$$

where $\mathcal{R}(\varphi,\vartheta,\psi) = R_z(\psi)R_y(\vartheta)R_x(\varphi)$ with R_i being the rotation matrices about the Cartesian coordinate axis with angles φ, ϑ, ψ, respectively. One reduces transformation (4.16) considering the following values for the three Euler angles

$$\alpha = \theta - \frac{\pi}{2}\,, \qquad \beta = \phi\,, \qquad \gamma = \gamma(r)\,, \tag{4.17}$$

where γ is taken to be a general function of both t and r. By doing this, the new rotated tetrad becomes

$$\bar{e}^a_\mu = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B\sin\theta\cos\phi & -r(\cos\theta\cos\phi\sin\gamma+\sin\phi\cos\gamma) & r\sin\theta(\sin\phi\sin\gamma-\cos\theta\cos\phi\cos\gamma) \\ 0 & B\sin\theta\sin\phi & r(\cos\phi\cos\gamma-\cos\theta\sin\phi\sin\gamma) & -r\sin\theta(\cos\theta\sin\phi\cos\gamma+\cos\phi\sin\gamma) \\ 0 & B\cos\theta & r\sin\theta\sin\gamma & r\sin^2\theta\cos\gamma \end{pmatrix}. \tag{4.18}$$

Then, with this new rotated tetrad, the $f(T)$ field equations are reduced to 7 independent equation with two important constraints:

$$f_{TT}T'(r,t)\cos\gamma = 0\,, \quad f_T\dot{B}(r,t) = 0\,, \tag{4.19}$$

where primes denote differentiation with respect to the radial coordinate r and dots with respect to time t. Here, one can see how this approach works. Since the idea is to eliminate all constraints on f, we can assume $f_{TT} \neq 0$ and $f_T \neq 0$, otherwise TEGR is recovered. From the above equation, one directly finds that $B(r,t) = B(r)$. The case $T' = 0$ is also a very specific case where T is a constant, therefore, any solution in GR can be constructed in this case since $f(T)$ will be $f(T_0)$. Then, one good choice is when $\cos\gamma = 0$ which gives us $\gamma = \pi/2$ (or $-\pi/2$). By choosing $\gamma = -\pi/2$ and $B(r,t) = B(r)$, Eqs. (4.19) are satisfied. Then, the rotated tetrad with $\gamma = -\pi/2$ becomes

$$\bar{e}^a_\mu = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & B\sin\theta\sin\phi & r\cos\theta\sin\phi & r\cos\phi\sin\theta \\ 0 & B\cos\theta & -r\sin\theta & 0 \end{pmatrix}, \tag{4.20}$$

which gives us the following torsion scalar

$$T = \frac{2(B-1)\,(A(B-1)-2rA')}{r^2AB^2}\,. \tag{4.21}$$

The rotated tetrad (4.20) is indeed a good tetrad since it does not constrain the value of f nor the value of T (see (4.21)) and contains TEGR (or GR) in the background, i.e., if $f = T$, one recovers the expected results: the Birkhoff theorem is valid and the standard GR spherically symmetric equations are recovered without imposing any condition on T and f. This procedure can also be used for other spacetimes such as FLRW cosmology or different ones.

4.3.3 FLRW Cosmology

One can follow the same methodology described before for FLRW cosmology. Let us discuss the flat case ($k = 0$). Consider the FLRW metric (4.33) in pseudo-spherical coordinates. The FLRW diagonal tetrad can be written as follows

$$e^a_\mu = \text{diag}\Big(1, a(t), a(t)r, a(t)r\sin\theta\Big)\,. \tag{4.22}$$

Using this tetrad, one can notice that the scalar torsion T (see Eq. (3.34)) becomes

$$T = -6H^2 + \frac{2}{r^2a^2}\,, \tag{4.23}$$

where $H = \dot{a}/a$. Here, one notices that this tetrad is a "bad tetrad" since based on the isotropy and homogeneity of the FLRW spacetime, the scalar torsion needs to depend only on time. Moreover, if one uses the diagonal tetrad (4.22), the FLRW cosmological equations for $f(T)$ gravity also depend on the radial coordinate. For example, the first FLRW equation in $f(T)$ gravity for this tetrad reads

$$\left(6H^2 - \frac{1}{r^2a^2}\right) f_T + \frac{f}{2} = \kappa^2\rho \,, \tag{4.24}$$

where a perfect fluid with energy density ρ was considered. Therefore, this tetrad is a bad tetrad and one can follow the procedure employed in the previous section to fix this issue. Let us now perform a rotation in this tetrad, in such a way that $\bar{e}^a_\mu = \Lambda^a_b e^b_\mu$ where Λ^a_b is the rotational matrix (4.16) with the three Euler angles now being

$$\alpha = \theta - \frac{\pi}{2} \,, \qquad \beta = \phi \,, \qquad \gamma = \gamma(r,t) \,, \tag{4.25}$$

where now the function $\gamma = \gamma(t, r)$ depends on time and the radial coordinate. By doing this, one finds the following rotated tetrad

$$\bar{e}^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a\cos\phi\sin\theta & -ra(\cos\gamma\sin\phi + \cos\theta\cos\phi\sin\gamma) & ra\sin\theta(\sin\phi\sin\gamma - \cos\theta\cos\phi\cos\gamma) \\ 0 & a\sin\theta\sin\phi & ra(\cos\phi\cos\gamma - \cos\theta\sin\phi\sin\gamma) & -ra\sin\theta(\cos\theta\cos\gamma\sin\phi + \cos\phi\sin\gamma) \\ 0 & a\cos\theta & ra\sin\theta\sin\gamma & ra\cos\gamma\sin^2\theta \end{pmatrix} . \tag{4.26}$$

It is easy to check that, as expected, the flat FLRW metric (4.33) is invariant under this rotation since the above tetrad also satisfies Eq. (3.6). Using this tetrad, one finds

$$T = \frac{4}{a^2r^2}\left(r\frac{d\gamma}{dr}\cos\gamma + \sin\gamma + 1\right) - 6H^2 \,. \tag{4.27}$$

Since the scalar torsion cannot depend on the radial coordinate, the first term in parenthesis needs to be cancelled, giving us the following equation for γ:

$$r\frac{d\gamma}{dr}\cos\gamma + \sin\gamma + 1 = 0 \,, \tag{4.28}$$

which can be solved and gives us 6 solutions. One can choose one of those solutions to find the correct good tetrad. In this case, the two easiest solutions to work out are: $\gamma = -\pi/2$ or $\gamma = 3\pi/2$. If one uses the first solution ($\gamma = -\pi/2$), one finds that the rotated tetrad (4.26) becomes

$$\bar{e}^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a\cos\phi\sin\theta & ra\cos\theta\cos\phi & -ra\sin\theta\sin\phi \\ 0 & a\sin\theta\sin\phi & ra\cos\theta\sin\phi & ra\sin\theta\cos\phi \\ 0 & a\cos\theta & -ra\sin\theta & 0 \end{pmatrix}. \tag{4.29}$$

In this rotated tetrad, the torsion scalar becomes

$$T = -6H^2\,, \tag{4.30}$$

which does not depend on the radial coordinate. Therefore, this rotated tetrad is a good tetrad. Using the rotated tetrad (4.29) and the $f(T)$ field equations (4.13), the flat FLRW cosmological equations for $f(T)$ become

$$6H^2 f_T + \frac{1}{2}f(T) = \kappa^2\rho\,, \tag{4.31}$$

$$2(3H^2 + \dot{H})f_T + 2H\dot{f}_T + \frac{1}{2}f(T) = -\kappa^2 p\,, \tag{4.32}$$

where $\dot{f}_T = f_{TT}\dot{T}$. Different studies have been conducted to analyse these equations. The most important results in cosmology are the followings:

- Some specific forms of $f(T)$ give an accelerating behaviour of the Universe without introducing any cosmological constant Λ. For some important studies in this direction, see Wu & Yu (2010a,b); Bamba et al. (2011); Dent et al. (2011).

- A phantom divide line becomes plausible. According to observations (Feng et al., 2005), the equation of the state parameter of dark energy crossed the cosmological constant boundary $w_\Lambda = -1$ from above to below (Wu & Yu, 2011).

- It is possible to find bounce solutions which solve the Big Bang singularity issue presented in GR. Some examples of those solutions are described in Cai et al. (2011); Bamba et al. (2013b); de Haro & Amoros (2013); Odintsov et al. (2015).
- According to Nunes (2018), using CMB+BAO+H_0 observations, a small deviation of $f(T)$ gravity with ΛCDM is observed with a better fit in the data. Additionally, it is possible to solve the issue of the tension on measuring the Hubble constant H_0 which appears in GR[1].

For a comprehensive review regarding the physical and cosmological interpretations regarding $f(T)$ cosmology, see Cai et al. (2016).

Alternatively, it is not so difficult to show that in flat FLRW, the metric in Cartesian coordinates

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{4.33}$$

with the diagonal tetrad

$$e^a_\mu = \mathrm{diag}(1, a(t), a(t), a(t)) \tag{4.34}$$

is a good tetrad in the sense of Sec. 4.3.2 and Tamanini & Boehmer (2012) since it does not constrain the function f nor the scalar torsion T. The majority of the cosmological models that will be presented are related to flat FLRW cosmology[2],

[1] The local value of the Hubble constant is slightly different from different observations but matching within the error bars.

[2] For observations constraints $k \approx 0$. See Sec. 2.5.2.3 for more details.

hereafter, the above diagonal tetrad with the flat FLRW metric in Cartesian coordinates will always be chosen since it is a good tetrad. Obviously, tetrad (4.34) gives the same field equations (4.31)-(4.32) than considering the rotated tetrad in spherical coordinates (4.26).

4.4 Scalar-tensor theories

4.4.1 Brans-Dicke and scalar-tensor theories

There are numerous ways to modify both General Relativity and Teleparallel gravity with the aim of being able to understand the late-time acceleration of the Universe without the need for a cosmological constant. One approach is to modify the gravitational sector and one can consider both $f(\overset{\circ}{R})$ theories of gravity and $f(T)$ theories, as discussed in the previous sections. One can also consider a coupling between a scalar field and the gravitational sector. In standard General Relativity, matter fields are only coupled to gravity via the metric $g_{\mu\nu}$ and this ensures the validity of the strong equivalence principle. In the original idea of Brans & Dicke (1961), a non-minimal coupling was introduced to obtain a relativistic theory implementing Mach's principle. Not surprisingly it is known as *Brans-Dicke theory* and it is one of the most studied modifications of Einstein gravity.

The action of Brans-Dicke theory is given by

$$\mathcal{S}_{\rm BD} = \int d^4x\sqrt{-g}\left[\frac{\phi}{2\kappa^2}\overset{\circ}{R} - \frac{\omega_{\rm BD}}{2\phi}\partial\phi^2 + \mathcal{L}_{\rm m}\right] , \tag{4.35}$$

where $\mathcal{L}_{\rm m}$ determines the matter contents, $\kappa^2 = 8\pi G$ and ϕ is the scalar field. The constant $\omega_{\rm BD}$ is called the *Brans-Dicke parameter*. If the latter parameter is positive, then the scalar field is canonical whereas if it is negative it represents a phantom scalar field since it has the other opposite sign in the kinetic term. In later generalisations of this theory a self-interacting potential $V(\phi)$ for the scalar field was introduced in the action, namely

$$\mathcal{S}_{\rm BD} = \int d^4x\sqrt{-g}\left[\frac{\phi}{2\kappa^2}\overset{\circ}{R} - \frac{\omega_{\rm BD}}{2\phi}\partial\phi^2 - V(\phi) + \mathcal{L}_{\rm m}\right] . \tag{4.36}$$

In the presence of matter, Brans-Dicke theory reduces to General Relativity in the limit $\omega_{\rm BD} \to \infty$, and from Solar System experiments, the strong bound $\omega_{\rm BD} \gtrsim 10^4$ can be obtained (see e.g. Bertotti et al. (2003)). One interesting aspect of the action above is that the scalar field ϕ changes the effective Newton's gravitational constant. This implies that now the strength of the gravitational interaction depends on the value of the scalar field, which in turn can depend on the spacetime position.

One can further generalise the Brans-Dicke action by considering any arbitrary coupling function $F(\phi)$ non-minimally coupled with the Ricci curvature. Then, action (4.36) can be generalised to

$$\mathcal{S}_{\rm scalar-tensor} = \int\left[\Big(\frac{1}{\kappa^2} + F(\phi)\Big)\frac{\overset{\circ}{R}}{2} + \frac{1}{2}w(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + \mathcal{L}_{\rm m}\right]\sqrt{-g}\, d^4x\,, \tag{4.37}$$

where now $F(\phi)$ and $w(\phi)$ are coupling functions. The above action gives rise to the so-called *scalar-tensor theories*. For $w(\phi) = 1$ ($w(\phi) = -1$) one obtains a canonical scalar field (phantom scalar field). The standard approach is to consider a coupling between the scalar field and the Ricci scalar, of the form (Chernikov & Tagirov (1968); Callan et al. (1970); Birrell & Davies (1980))

$$F(\phi) = \xi\phi^2 \,. \tag{4.38}$$

Here ξ is a coupling constant. In this notation the effective gravitational constant can be written as

$$G_{\text{eff}} = \frac{G}{1+\kappa^2\xi\phi^2}, \tag{4.39}$$

and one can redefine such a scalar field so that it coincides with the standard Brans-Dicke field. Such a non-minimal coupling has motivations from different contexts. It appears as a result of quantum corrections to the scalar field in curved space-times (Ford, 1987) and it is also required by renormalisation considerations (Callan et al., 1970). It also appears in the context of superstring theories (Maeda, 1986). Such models have attempted to explain the early-time inflationary epoch, however, the simple model with a quadratic scalar potential is now disfavoured by the current Planck data (Martin et al. (2014); Ade et al. (2014a, 2016)).

Minimally coupled quintessence corresponds to taking $F(\phi) = 0$ (or $\xi = 0$) in the Lagrangian (4.37). For a review of quintessence type models, see Copeland et al. (2006). Quintessence alone can give rise to many interesting features from late-time accelerated expansion of the Universe to inflation Ratra & Peebles (1988); Wetterich (1988); Zlatev et al. (1999). However, simple models of scalar field inflation are becoming disfavoured by the latest Planck data. Another issue with a simple quintessence approach is that the effective equation of state must always satisfy $w_{\text{eff}} > -1$ and require a very flat fine tuned potential in order to explain current cosmological observations.

Minimally coupled quintessence is not considered as modified gravity since the action only has the standard Einstein Hilbert action plus some terms related to the scalar field. Then, this theory is in the so-called *Einstein frame*. The frame corresponding to $f(\overset{\circ}{R})$ gravity is usually called *the Jordan frame*. If one considers vacuum case $\mathcal{L}_{\text{m}} = 0$, by making the following conformal transformation,

$$g_{\mu\nu} = e^{-\varphi}\,\widehat{g}_{\mu\nu} \tag{4.40}$$

with $g_{\mu\nu}$ denoting the Einstein frame metric, one obtains that the minimally coupled quintessence action (Einstein frame) is equivalent to the $f(\overset{\circ}{R})$ gravity (Jordan frame). Then, there is a correspondence or equivalence between the Einstein and Jordan frames (Nojiri & Odintsov, 2011). In the case where $\mathcal{L}_{\rm m} \neq 0$, this equivalence still holds but matter in the new frame is now coupled with the conformal factor in a different way, making that particles in the new frame do not follow geodesics. In addition to these, there are also frames in which the scalar field is non-minimally coupled to gravity, and these can be reached by a $f(\overset{\circ}{R})$ gravity by also using a suitably chosen conformal transformation, or directly by the Einstein frame theory by conformally transforming the theory. Generally speaking, one should confront the theoretical predictions of a specific gravitational theory with the observable Universe history supported by the current observational data. In this sense, each of the three mentioned frames, namely the $f(\overset{\circ}{R})$ gravity, and the minimal and non-minimal scalar theories, may give a viable description of the observable Universe history. However, it is not clear that a viable description in one frame gives also a viable and convenient description in the other frame. For instance, it may give a viable but physically inconvenient description. In other words, there appears the question which of these three frames is the most physical one (and in which sense) or, at least, which of these frames gives a convenient description of the Universe history. Eventually, the answer to this question depends very much from the confrontation with the observational data, from the specific choice of the theory and from the observer associated with specific frame. At the same time, the related question is about equivalent results in all three frames and/or about construction of the observable quantities which are invariant under conformal transformations between the three frames. Further, even though mathematically, both frames are equivalent, in Bahamonde et al. (2016), it was shown that it is possible to have various correspondences of finite time cosmological singularities, and in some cases it is possible a singular cosmology in one frame might be non-singular in the other frame. This situation appears since the conformal transformation is singular in those cosmological singularities. Moreover, if acceleration is imposed in one frame, it will not necessarily correspond to an accelerating metric when transformed in another frame (Bahamonde et al., 2017e).

4.4.2 Teleparallel scalar-tensor theories

An alternative approach related to scalar-tensor theories has been to consider a scalar field non-minimally coupled to torsion. The following action for Teleparallel scalar-tensor theories is considered,

$$\mathcal{S}_{\rm TEGR\ ST} = \int \Big[\Big(\frac{1}{\kappa^2} + F(\phi)\Big)\frac{T}{2} + \frac{1}{2}w(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + \mathcal{L}_{\rm m}\Big]\, e\, d^4x\,. \tag{4.41}$$

This gives rise to different dynamics to the case of the non-minimal coupling to the Ricci scalar. When $F(\phi) = 0$, one obtains a minimally-coupled model which is Teleparallel gravity and a kinetic and a potential term, which is equivalent as quintessence models (taking $F(\phi) = 0$ in (4.37)). This action was first introduced in Geng et al. (2011), with the specific case where $w(\phi) = 1$ (canonical scalar field) and, similarly as in (4.38), the coupling function was taken to be

$$F(\phi) = \chi\phi^2\,. \tag{4.42}$$

This theory was labelled as *Teleparallel dark energy*. This theory again has a richer structure than simple standard quintessence behaviour, with both phantom and quintessence type dynamics possible, along with dynamical crossing of the phantom barrier.

The equivalence between General Relativity and Teleparallel gravity breaks down as soon as one non-minimally couples a scalar field and then the field equations result in different dynamics. In Bahamonde & Wright (2015); Zubair et al. (2017), there was introduced a more general action, with the aim of unifying both of the previous considered approaches

$$\mathcal{S}_{\rm TEGR-STE} = \int \Big[\Big(\frac{1}{\kappa^2}+F(\phi)\Big)\frac{T}{2}+\frac{1}{2}G(\phi)B+\frac{1}{2}w(\phi)(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)+\mathcal{L}_{\rm m}\Big]e\,d^4x, \quad (4.43)$$

where now there is a additional coupling function $G(\phi)$ with the boundary term B. Since $\overset{\circ}{R}=-T+B$ (see (3.36)), the above action also contains the case where the Ricci scalar is non-minimally coupled with the scalar field when the functions are chosen to be $F(\phi)=-G(\phi)$. Then, in the latter case, the above action is reduced to (4.37). For the case $G(\phi)=0$, a Teleparallel theory non-minimally coupled with the torsion scalar is recovered and for $F(\phi)=0$, a Teleparallel quintessence with a non-minimal coupling to a boundary term is recovered.

Let us now find the field equations for the theory (4.43). By varying the action (4.43) with respect to the tetrad field, one finds

$$\begin{aligned}\delta\mathcal{S}_{\rm TEGR-STE} \;=\; & \int\Big\{\Big[\frac{1}{2}\Big(\frac{1}{\kappa^2}+F(\phi)\Big)\delta T+\frac{1}{2}G(\phi)\delta B - w(\phi)(\partial_\mu\phi)(\partial^\lambda\phi)E_a^\mu\delta e^a_\lambda\Big]e \\ & +\Big[\frac{1}{2}\Big(\frac{1}{\kappa^2}+F(\phi)\Big)T+\frac{1}{2}G(\phi)B+\frac{1}{2}w(\phi)(\partial_\mu\phi)(\partial^\mu\phi)-V(\phi)\Big]E_a^\lambda\delta e^a_\lambda \\ & +\delta(e\mathcal{L}_{\rm m})\Big\}\,d^4x\,, \qquad (4.44)\end{aligned}$$

where $\delta e = E_a^\lambda\delta e^a_\lambda$ and (3.43) were used. Variations with respect to the scalar torsion $f\delta T$ were found in Sec. 4.3 (see Eq. (4.11)). Hence, one needs to change f_T by $\frac{1}{\kappa^2}+F(\phi)$ in (4.11) in order to obtain the corresponding term which appears in the above equation. The second term in the variations corresponds to find variations with respect to the boundary term B. By doing the corresponding calculations, one obtains

$$G(\phi)\delta B = \Big[2E_a^{\nu}\overset{\circ}{\nabla}{}^{\lambda}\overset{\circ}{\nabla}_{\nu}G(\phi) - 2E_a^{\lambda}\overset{\circ}{\Box}f_B - Bf_BE_a^{\lambda} - 4(\partial_\mu G(\phi))S_a{}^{\mu\lambda}\Big]\delta e^a_\lambda\,. \tag{4.45}$$

For further details about how to compute this term, see Sec. 5.1 where all the important calculations are derived and only one needs to replace f_B by $G(\phi)$ in order to obtain the above result. Hence, by using the above equation and Eq. (4.11), one obtains that the field equations are given by

$$2\Big(\frac{1}{\kappa^2}+F(\phi)\Big)\Big[e^{-1}\partial_\mu(eS_a{}^{\mu\nu}) - E_a^{\lambda}T^{\rho}{}_{\mu\lambda}S_\rho{}^{\nu\mu} - \frac{1}{4}E_a^{\nu}T\Big] - E_a^{\nu}\Big[\frac{1}{2}w(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\Big]$$
$$+w(\phi)E_a^{\mu}\partial^\nu\phi\partial_\mu\phi + 2(\partial_\mu F(\phi)+\partial_\mu G(\phi))E_a^{\rho}S_\rho{}^{\mu\nu} + E_a^{\nu}\overset{\circ}{\Box}G(\phi) - E_a^{\mu}\overset{\circ}{\nabla}{}^{\nu}\overset{\circ}{\nabla}_{\mu}G(\phi) = \mathcal{T}_a^{\nu}, \tag{4.46}$$

where $\overset{\circ}{\Box} = \overset{\circ}{\nabla}_\alpha\overset{\circ}{\nabla}{}^\alpha$, remembering that $\overset{\circ}{\nabla}_\alpha$ is the covariant derivative linked with the Levi-Civita connection and $\mathcal{T}_a^{\nu}$ is the matter energy momentum tensor. We have used units where $\kappa^2 = 1$.

By taking variations in the action (4.43) with respect to the scalar field ϕ, one finds the modified Klein-Gordon equation given by

$$w(\phi)\overset{\circ}{\Box}\phi - \frac{1}{2}w'(\phi)\overset{\circ}{\nabla}_\mu\phi\overset{\circ}{\nabla}{}^\mu\phi + V'(\phi) = \frac{1}{2}\Big(F'(\phi)T + G'(\phi)B\Big)\,. \tag{4.47}$$

Here primes denote differentiation with respect to the scalar field.

For flat FLRW described by the metric (4.33) and the diagonal tetrad (4.34), and considering a canonical scalar field $w(\phi) = 1$, Eqs. (4.46) become

$$3H^2(1+F(\phi)) = \rho_{\rm m} + V(\phi) + \frac{1}{2}\dot{\phi}^2 + 3H\dot{\phi}G'(\phi), \tag{4.48}$$

$$(3H^2+2\dot{H})(1+F(\phi)) = -\Big(p_{\rm m} - V(\phi) + \frac{1}{2}\dot{\phi}^2\Big) - 2H\dot{\phi}F'(\phi) + (G''(\phi)\dot{\phi}^2 + G'(\phi)\ddot{\phi}). \tag{4.49}$$

Here, $H = \dot{a}(t)/a(t)$ is the Hubble parameter and dots and primes denote differentiation with respect to the time coordinate and the argument of the function respectively. It was also considered that the matter is described by a perfect fluid with energy density given by $\rho_{\rm m}$ and pressure $p_{\rm m}$.

The modified Klein-Gordon equation (4.47) becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{1}{2}\Big(F'(\phi)T + G'(\phi)B\Big)\,, \tag{4.50}$$

where in flat FLRW, $T = -6H^2$ and $B = -6(\dot{H} + 3H^2)$. It should be noted that the Klein-Gordon equation can be also obtained directly from the field equations (4.48) and (4.49), so that it is not an extra equation. Thus, this model has two independent equations, three given functions depending on the model ($F(\phi), G(\phi)$ and $V(\phi)$) and four dynamical variables ($\phi, a(t), \rho_{\rm m}$ and $p_{\rm m}$).

One can also rewrite (4.48)-(4.49) in a fluid representation as follows,

$$3H^2 = \frac{1}{1+F(\phi)}\rho_{\rm eff}\,, \tag{4.51}$$

$$2\dot{H} = -\frac{1}{1+F(\phi)}(\rho_{\rm eff} + p_{\rm eff})\,, \tag{4.52}$$

where $\rho_{\rm eff} = \rho_{\rm m} + \rho_\phi$ is the total energy density and $p_{\rm eff} = p_{\rm m} + p_\phi$ is the total pressure. The energy density and the pressure of the scalar field ρ_ϕ and p_ϕ are respectively defined as follows

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 3H\dot{\phi}G'(\phi)\,, \tag{4.53}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) + 2H\dot{\phi}F'(\phi) - (G''(\phi)\dot{\phi}^2 + G'(\phi)\ddot{\phi})\,. \tag{4.54}$$

It can be shown that the conservation equation, i.e. $\overset{\circ}{\nabla}_\mu \mathcal{T}^{\mu\nu} = 0$ is valid, then the standard continuity equations read

$$\dot{\rho}_{\rm eff} + 3H(\rho_{\rm eff} + p_{\rm eff}) = 0\,, \tag{4.55}$$

$$\dot{\rho}_{\rm m} + 3H(\rho_{\rm m} + p_{\rm m}) = 0\,. \tag{4.56}$$

In Bahamonde & Wright (2015), the specific case was studied for the coupling functions

$$F(\phi) = \xi\phi^2\,, \quad G(\phi) = \chi\phi^2\,, \tag{4.57}$$

where ξ and χ are coupling constants. Specifically, using the dynamical system techniques, in the case when there is only a pure coupling to the boundary term ($\xi = 0$), it was found that the system generically evolves to a late-time accelerating attractor solution without requiring any fine tuning of the parameters. A dynamical crossing of the phantom barrier was also shown to be possible. For further details regarding the method of dynamical system, see the review Bahamonde et al. (2017a). In Zubair et al. (2017), the validity of the first and second law of thermodynamics at the apparent horizon was discussed for any coupling $F(\phi)$ and $G(\phi)$. Moreover, the authors also found some analytical cosmological solutions for the Eqs. (4.48)-(4.49).

To finalise this section, let us note that a theory which is minimally coupled with the scalar torsion (replace $F(\phi) = 0$ in (4.43)), which is the analogy of a theory which is minimally coupled with the Ricci scalar (replace $F(\phi) = 0$ in (4.37)) is not equivalent to $f(T)$ gravity. Hence, if one takes $f(T)$ gravity, which can be considered as a Jordan frame, and performs a conformal transformation (4.40), the resulting action does not give a Teleparallel minimally coupled theory. Thus, the dynamics of $f(T)$ gravity is not equivalent to Teleparallel action plus a scalar field since an additional term appears (Yang (2011); Wright (2016)). For more details about conformal transformations in modified Teleparallel theories of gravity, see

Sec. 6.4.

4.4.3 Teleparallel quintom models

For minimally coupled models ($F(\phi) = 0$ in (4.37)) with canonical scalar field, the equation of state must satisfy $w_{\rm de} \geq -1$, while for a phenomenologically acceptable phantom case, (i.e. valid only after the early-time discontinuities in its equation of state have taken place), the phantom scalar field equation of state is constrained to be $w_{\rm de} < -1$. In a minimally coupled model, there is no way to cross this phantom barrier (i.e. the cosmological constant value $w_{\rm de} = -1$) with a single canonical or phantom scalar field. An interesting model which allows for such a crossing to take place was proposed by Feng et al. (2005), which is known as *quintom models*. This scenario of dark energy gives rise to the equation of state larger than -1 in the past and less than -1 today, satisfying current observations. It can be achieved with more general non-canonical scalar fields, but the simplest model is represented by the quintom action made up of two scalar fields, one canonical field ϕ and one phantom field σ (minimally coupled):

$$\mathcal{S}_{\rm quintom} \;=\; \int \Big[\frac{\overset{\circ}{R}}{2\kappa^2} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - V(\phi,\sigma) + \mathcal{L}_{\rm m}\Big]\sqrt{-g}\, d^4x \,, \tag{4.58}$$

where now the energy potential $V(\phi,\sigma)$ depends on both scalar fields. A phantom scalar field has the opposite sign of the kinetic term. It should be noted that phantom scalar fields violate the conservation of probability and also they have unboundedly negative energy density. This lead to the absence of a stable vacuum quantum state. Therefore, phantom scalar fields have Ultra-Violet quantum instabilities (Cline et al., 2004; Copeland et al., 2006). Hence, the model presented above needs to be consider as a toy model to describe the dark energy, and not as a fundamental theory of gravity. For more details about quintom scalar fields, see Cai et al. (2010).

On the other hand, it is also possible to consider Teleparallel quintom models minimally coupled with two scalar fields. Since $\overset{\circ}{R} = -T + B$, a minimally coupled model based on $\overset{\circ}{R}$ (GR plus two scalar fields minimally coupled) will lead to the same theory as a theory based on T (TEGR plus two scalar fields minimally coupled). However, one can generalise the quintom models by considering non-minimally coupling between the Ricci scalar and the scalar fields, namely

$$\begin{aligned}\mathcal{S}_{\rm g-quintom} &= \int \Big[\frac{\overset{\circ}{R}}{2\kappa^2} + \frac{1}{2}\Big(f_1(\phi) + f_2(\sigma)\Big)\overset{\circ}{R} + \frac{1}{2}\xi\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\chi\partial_\mu\sigma\partial^\mu\sigma \\ &\quad -V(\phi,\sigma) + \mathcal{L}_{\rm m}\Big]\sqrt{-g}\,d^4x\,, \qquad (4.59)\end{aligned}$$

where now there are two coupling functions $f_1(\phi)$ and $f_2(\sigma)$ and for being more general, two coupling constants, χ and ξ were introduced in order to have quintom models with $\xi = -\chi = 1$ (one canonical scalar field and one phantom scalar field), two canonical scalar fields ($\xi = \chi = 1$) or two phantom scalar fields ($\xi = \chi = -1$). It should be remarked that in principle one can also introduce coupling functions $w_1(\phi)$ and $w_2(\sigma)$ instead of the coupling constants ξ and χ. For simplicity, coupling constants will be only considered in this section. The above action can be seen as a generalisation of the non-minimally scalar field model presented in (4.37), with now having two scalar fields. This action was first presented in Marciu (2016) with $f_1(\phi) = c_1\phi^2$ and $f_2(\sigma) = c_2\sigma^2$ and $\xi = -\chi = 1$. The author studied its cosmology using numerical techniques finding a late-time accelerating behaviour of the Universe with a possibility of the crossing of the phantom divide line.

In the same spirit as (4.59), recently, there was proposed a generalised Teleparallel action which is a generalisation of (4.44) with two scalar fields, namely (Bahamonde et al., 2018a)

$$\begin{aligned}\mathcal{S}_{\rm TEGR-quintom} &= \int \Big[\frac{T}{2\kappa^2} + \frac{1}{2}\Big(f_1(\phi) + f_2(\sigma)\Big)T + \frac{1}{2}\Big(g_1(\phi) + g_2(\sigma)\Big)B \\ &\quad +\frac{1}{2}\xi\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\chi\partial_\mu\sigma\partial^\mu\sigma - V(\phi,\sigma) + \mathcal{L}_{\rm m}\Big]e\,d^4x\,, \qquad (4.60)\end{aligned}$$

the functions f_1, f_2, g_1 and g_2 are coupling functions which depend on two different scalar field ϕ and σ. The field equations for this model can be easily found by using (4.11) and (4.45). Moreover, the computation is straightforward from the field equations for one scalar field (see Eq. (4.46)). The cosmological equations are then given by

$$
\begin{aligned}
3H^2(1+f_1(\phi)+f_2(\sigma)) &= \rho_{\rm m}+V(\phi,\sigma)+\frac{1}{2}\xi\dot{\phi}^2+\frac{1}{2}\chi\dot{\sigma}^2 \\
&\quad +3H(g_1'(\phi)\dot{\phi}+g_2'(\sigma)\dot{\sigma})\,, \qquad (4.61)\\
(3H^2+2\dot{H})(1+f_1(\phi)+f_2(\sigma)) &= -p_{\rm m}+V(\phi,\sigma)-\frac{1}{2}\xi\dot{\phi}^2-\frac{1}{2}\chi\dot{\sigma}^2 \\
&\quad -2H(\dot{\phi}f_1'(\phi)+\dot{\sigma}f_2'(\sigma))+\ddot{g}_1(\phi)+\ddot{g}_2(\sigma)\,. \qquad (4.62)
\end{aligned}
$$

Here, $\kappa^2=1$ was also assumed. It can be shown that the continuity equations (4.55) and (4.56) are also valid in this theory. The modified FLRW equations (4.61) and (4.62) can be also be rewritten in terms of effective energy and pressure, as follows

$$3H^2 = \rho_{\rm eff}\,, \qquad (4.63)$$

$$3H^2+2\dot{H} = -p_{\rm eff}\,, \qquad (4.64)$$

where $\rho_{\rm eff}=\rho_{\rm m}+\rho_\phi+\rho_\sigma$, $\;p_{\rm eff}=p_{\rm m}+p_\phi+p_\sigma$ were defined and

$$\rho_\phi = -3H^2f_1(\phi)+V_1(\phi)+\frac{1}{2}\xi\dot{\phi}^2+3Hg_1'(\phi)\dot{\phi}\,, \qquad (4.65)$$

$$p_\phi = (3H^2+2\dot{H})f_1(\phi)+\frac{1}{2}\xi\dot{\phi}^2+2H\dot{\phi}f_1'(\phi)-\ddot{g}_1-V_1(\phi)\,, \qquad (4.66)$$

$$\rho_\sigma = -3H^2f_2(\sigma)+V_2(\sigma)+\frac{1}{2}\chi\dot{\sigma}^2+3Hg_2'(\sigma)\dot{\sigma}\,, \qquad (4.67)$$

$$p_\sigma = (3H^2+2\dot{H})f_2(\sigma)+\frac{1}{2}\chi\dot{\sigma}^2+2H\dot{\sigma}f_2'(\sigma)-\ddot{g}_2-V_2(\sigma)\,. \qquad (4.68)$$

One can define the equation of state of the dark energy or scalar fields as the following ratio of the scalar field pressures and energy densities

$$\omega_\phi = \frac{p_\phi}{\rho_\phi}\,, \qquad\qquad \omega_\sigma = \frac{p_\sigma}{\rho_\sigma}\,. \tag{4.69}$$

For the present generalised quintom model in Teleparallel gravity, the dark energy equation of state is:

$$w_{\rm de} = \frac{p_\phi + p_\sigma}{\rho_\phi + \rho_\sigma}. \tag{4.70}$$

One can also define the total or effective equation of state as

$$\omega_{\rm eff} = \frac{p_{\rm eff}}{\rho_{\rm eff}} = \frac{p_{\rm m} + p_\phi + p_\sigma}{\rho_{\rm m} + \rho_\phi + \rho_\sigma}\,, \tag{4.71}$$

and the standard matter energy density as:

$$\Omega_{\rm m} = \frac{\rho_{\rm m}}{3H^2}\,. \tag{4.72}$$

Analogously, one can define the energy density parameter for dark energy or scalar fields as,

$$\Omega_{\rm de} = \Omega_\phi + \Omega_\sigma\,, \quad \Omega_\phi = \frac{\rho_\phi}{3H^2}\,, \quad \Omega_\sigma = \frac{\rho_\sigma}{3H^2}\,, \tag{4.73}$$

such that the relation $\Omega_{\rm m} + \Omega_\phi + \Omega_\sigma = \Omega_{\rm m} + \Omega_{\rm de} = 1$ holds.

In Bahamonde et al. (2018a), the specific case where the coupling functions are

$$f_1(\phi) = c_1\phi^2\,, \quad f_2(\sigma) = c_2\sigma^2\,, \quad g_1(\phi) = c_3\phi^2\,, \quad g_2(\sigma) = c_4\sigma^2\,, \tag{4.74}$$

where c_i $(i = 1, .., 4)$ are constants was studied. Further, the authors considered that the energy potential is an exponential type and also that it can be split into two parts,

$$V(\phi, \sigma) = V_1(\phi) + V_2(\sigma) = V_1 e^{-\lambda_1\phi} + V_2 e^{-\lambda_2\phi}\,, \tag{4.75}$$

where $\lambda_{1,2} > 0$ and V_1 and V_2 are constants. It was also assumed that the fluid is a standard barotropic one with $p_{\rm m} = w\rho_{\rm m}$. The dynamical system of this model was studied in Bahamonde et al. (2018a), finding a 6 dimensional one with 21 critical points (but only 13 ensures that the potentials are positive). One of those points represents a matter dominated era whereas the other ones are dark energy dominated eras. The numerical solutions deduced in the case of scalar torsion and boundary couplings with the scalar fields are depicted in Figs. 4.1-4.5. The study can be divided into two general models: (i) a model where the torsion scalar is non-minimally coupled with the scalar fields ($g_1 = g_2 = 0$); ii) a model where the boundary term is non-minimally coupled with the scalar fields ($f_1 = f_2 = 0$). For each model, four different set of numerical constant were chosen. For the model non-minimally coupled with T, the following four models were studied:

- Model T_1: $c_1 = 0.7, \lambda_1 = \lambda_2 = 0.6, V_1 = V_2 = 1.02, \phi(t_i) = \sigma(t_i) = -0.6,$ $\dot\phi(t_i) = \dot\sigma(t_i) = 0, t_i = 0.008.$
- Model T_2: $c_1 = 0.5, \lambda_1 = \lambda_2 = 0.5, V_1 = V_2 = 1, \phi(t_i) = \sigma(t_i) = 0.5, \dot\phi(t_i) = 0.01, \dot\sigma(t_i) = 0.001, t_i = 0.05.$
- Model T_3: $c_1 = 0.7, \lambda_1 = \lambda_2 = 0.6, V_1 = V_2 = 1.08, \phi(t_i) = \sigma(t_i) = 0.6, \dot\phi(t_i) = 0, \dot\sigma(t_i) = 0, t_i = 0.008.$
- Model T_4: $c_1 = 0.6, \lambda_1 = \lambda_2 = 0.01, V_1 = V_2 = 1.01, \phi(t_i) = \sigma(t_i) = 8, \dot\phi(t_i) = 0.0001, \dot\sigma(t_i) = 0.001, t_i = 0.035.$

On the other hand, for the model non-minimally coupled with the boundary term, the following models were studied:

- Model B_1: $c_3 = 0.5, \lambda_1 = \lambda_2 = 0.7, V_1 = V_2 = 1, \phi(t_i) = \sigma(t_i) = 0.6, \dot\phi(t_i) = 0.001, \dot\sigma(t_i) = 0.000001, t_i = 0.0295.$

- Model B_2: $c_3 = 0.6, \lambda_1 = \lambda_2 = 0.4, V_1 = V_2 = 1.05, \phi(t_i) = \sigma(t_i) = 1.5, \dot{\phi}(t_i) = 0.001, \dot{\sigma}(t_i) = 0.001, t_i = 0.0295$.

- Model B_3: $c_3 = 0.4, \lambda_1 = \lambda_2 = 0.2, V_1 = V_1 = 1.05, \phi(t_i) = \sigma(t_i) = -5, \dot{\phi}(t_i) = 0.00001, \dot{\sigma}(t_i) = 0.00001, t_i = 0.0083$.

- Model B_4: $c_3 = 0.6, \lambda_1 = \lambda_2 = 0.5, V_1 = V_2 = 1.055, \phi(t_i) = \sigma(t_i) = 1.1, \dot{\phi}(t_i) = \dot{\sigma}(t_i) = 0.00001, t_i = 0.0015$.

In the left panel of Fig. 4.1, the evolution of the cosmic scale factor as a function of cosmic time for the four models T_i. One can observe that the dynamics of the Universe in the case of the four scalar torsion models corresponds to an accelerated expansion, very close to a de Sitter expansion at late-times. The present time is at the numerical value of $t_0 = 0.96$, where the cosmic scale factor is approximately equal to unity $a(t_0) \sim 1$, as requested from the numerical method considered. Moreover, one can observe that the values of the coupling coefficient c_1 have a minor influence on the dynamics on the large scale. The Universe in this model is accelerating independently from the values of the coupling parameter c_1.

The case of boundary coupling models, neglecting the scalar torsion couplings are represented in the right panel of Fig. 4.1, where a similar behaviour is observed: the dynamics of the Universe in the case of the four boundary coupling models corresponds to an accelerated expansion, toward a de Sitter stage, independently from the values of the coupling parameter c_3. As previously mentioned, one has a similar behaviour as in the left panel of Fig. 4.1; an evolution of the system toward a de Sitter stage in the distant future.

The influence of the scalar torsion coupling coefficient c_1 in the evolution of the density parameters of the Universe (dark energy density parameter and matter density parameters) are presented in Fig. 4.2. In these figures one can observe that at the present time $t_0 = 0.96$, the density parameters have the values suggested by astrophysical observations, $\Omega_{\mathrm{de}} = 1 - \Omega_{\mathrm{m}} \sim 0.70$. From the numerical evolution, it can be noticed that at the initial time, the Universe faces a matter dominated era, while at the final numerical time, the cosmic picture is dominated by the dark energy quintom fields non-minimally coupled with the scalar torsion. Hence, the present quintom model with a scalar torsion coupling is able to explain the current values of the density parameters in the Universe, in a good agreement with astronomical and astrophysical observations. It is easy to see that the values of the coupling coefficient c_1 have minor effects on the values of the density parameters, since with a proper fine-tuning of the initial conditions, we are able to reproduce the corresponding current values of the density parameters in the Universe. From this, one remarks that the present generalised quintom model in the framework of Teleparallel gravity theory with a scalar torsion coupling is a feasible dark energy model.

Additionally, in Fig. 4.3 is displayed the evolution of the density parameters in the case of boundary coupling models B_i, considering different values of the coupling coefficient c_3, previously discussed. Here, one has a similar behaviour as in the case of a scalar torsion coupling. At the initial time, the Universe faces the matter dominated epoch, while at late-times the dark energy fields dominate the cosmic picture. Hence, boundary coupling models B_i also represent feasible dark energy models, explaining the acceleration of the Universe as well as the current values of the density parameters. The values of the boundary coupling parameters c_3 have a minor influence on the evolution of the density parameters corresponding to the dark energy fluid and matter fluid, respectively, similar to the scalar torsion models T_i. Consequently, in both models with boundary couplings B_i and scalar torsion couplings T_i endowed with decomposable exponential potentials, the Universe is evolving towards a state dominated by dark energy fields over the matter fluid, while the cosmic scale factor is in an accelerated stage.

The time evolution of the dark energy equation of state in the case of scalar torsion coupling models T_i is shown in Fig. 4.4. One can remark that for the scalar torsion coupling models, the dark energy equation of state can exhibit the main specific feature to quintom models; the crossing of the phantom divide line. For scalar torsion coupling models, in the first stage of evolution, the dark energy equation of state presents oscillations around the cosmological constant boundary, while at later times the equation of state evolves asymptotically towards the ΛCDM model, acting almost as the cosmological constant. As a consequence, the generalised quintom model in the Teleparallel gravity theory with a scalar torsion coupling can be in agreement with cosmological observations which have suggested that the cosmological constant boundary might be crossed by the dark energy equation of state. Finally, the evolution of the dark energy equation of state in the case of boundary coupling models B_i, is depicted in Fig. 4.5. As in the previous models, at the initial stage of evolution, the models show the crossing of the phantom divide line, while in the end at the final time, the dark energy quintom model acts asymptotically as a cosmological constant. As a concluding remark, it should be remembered that the present generalised quintom model in the Teleparallel gravity with scalar torsion and boundary couplings represents also a possible dark energy model, an alternative to the ΛCDM model, which can explain the observed crossing of the cosmological constant boundary in the near past of the dark energy equation of state, as suggested by various cosmological observations. Notice that this behaviour can be achieved without evoking any cosmological constant.

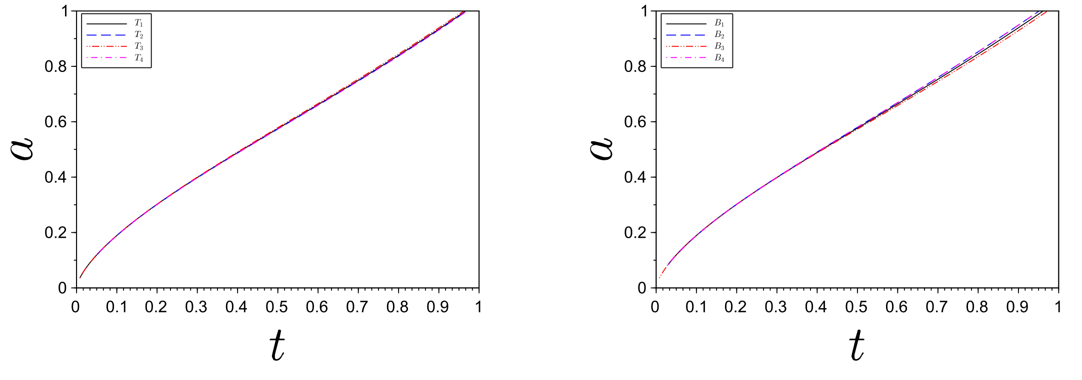


Figure 4.1: The time evolution of the cosmic scale factor for models with scalar torsion coupling(left panel) T_1, T_2, T_3, T_4 and boundary couplings(right panel) B_1, B_2, B_3, B_4

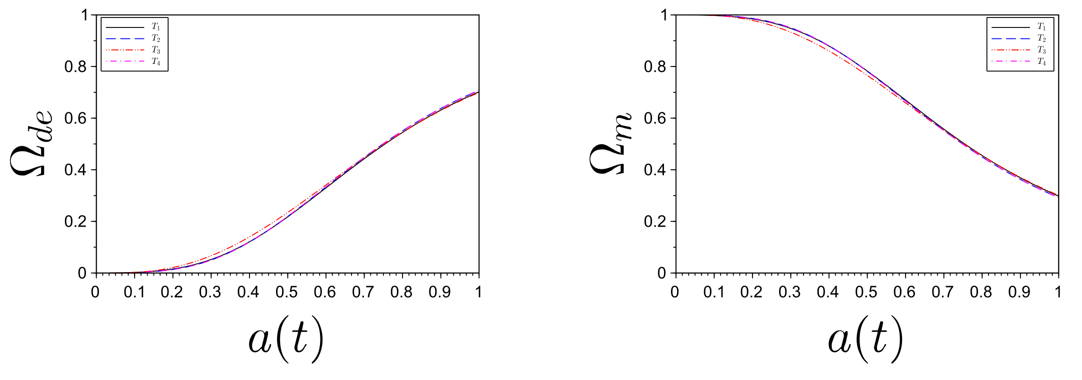


Figure 4.2: The evolution of the quintom energy density and matter energy density for scalar torsion coupling models

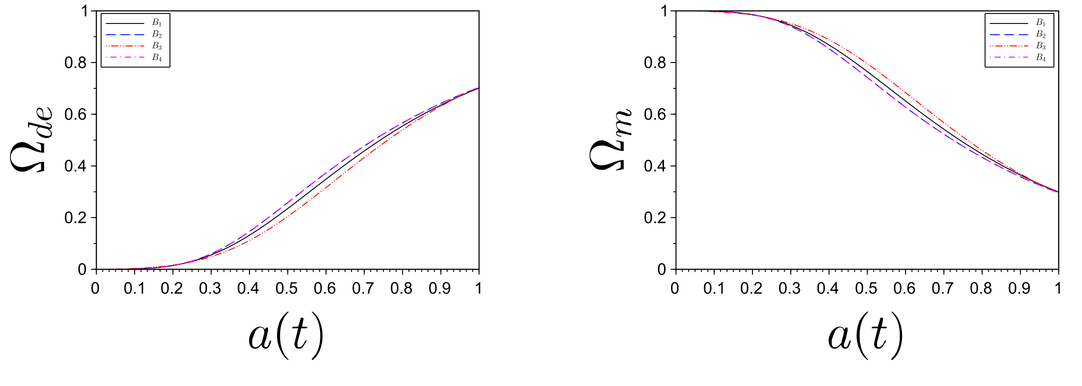


Figure 4.3: The evolution of quintom energy density and matter energy density for boundary coupling models

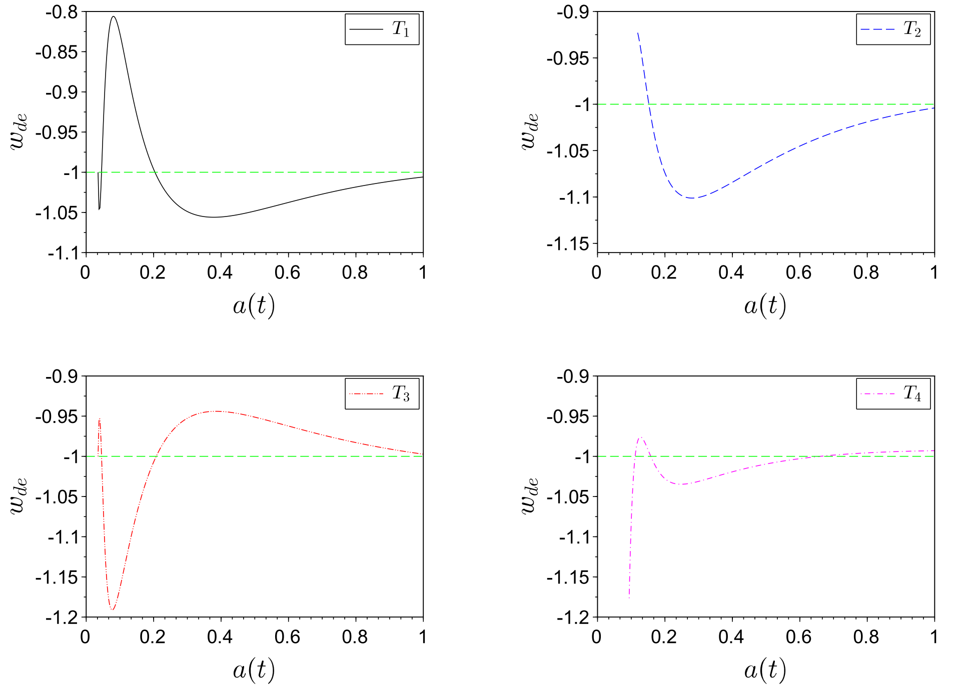


Figure 4.4: The evolution of the dark energy equation of state for boundary coupling models T_1, T_2, T_3, T_4

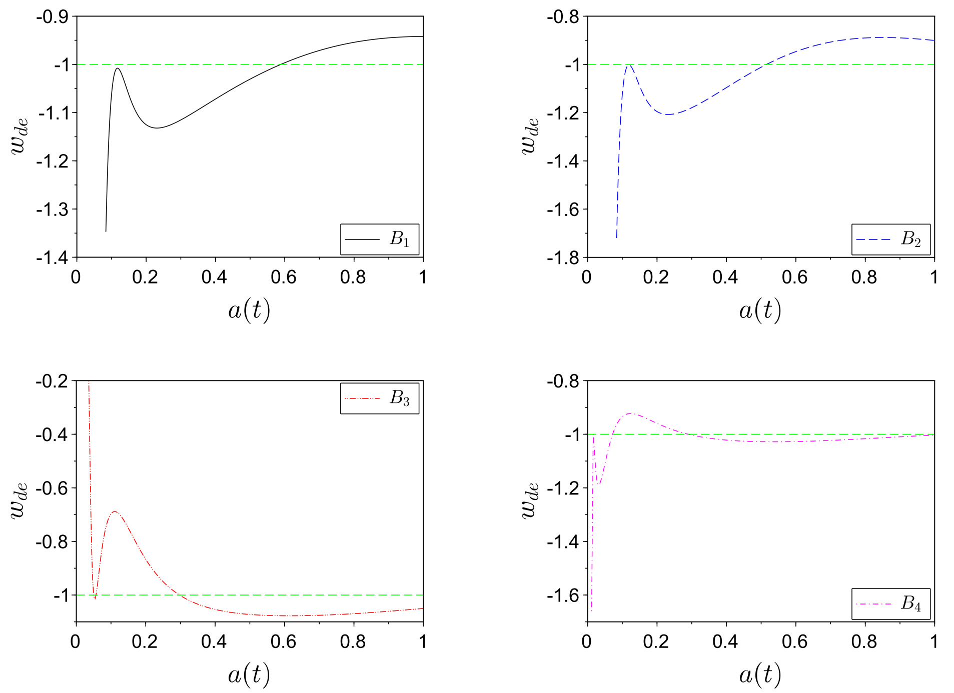


Figure 4.5: The evolution of the dark energy equation of state for boundary coupling models B_1, B_2, B_3, B_4

5

$f(T, B)$ gravity and its extension with non-minimally matter couplings

Chapter Abstract

In this chapter, a modified Teleparallel theory of gravity is presented which is based on an arbitrary function f which depends on the scalar torsion T and a boundary term $B = (2/e)\partial_\mu(eT^\mu)$ which appears via $\overset{\circ}{R} = -T + B$. This theory can become either $f(T)$ gravity or the Teleparallel equivalent of $f(\overset{\circ}{R})$ gravity. Flat FLRW cosmology is discussed in this theory, finding some cosmological solutions using the Noether symmetry approach and also the reconstruction method, which mimics some interesting kinds of universes.

5.1 General equations

As seen in Chap. 4, the first straightforward modification of the Teleparallel equivalent of General Relativity is the so-called $f(T)$ gravity where one generalises the TEGR Lagrangian (see Eq. (3.39)) to an arbitrary function of the scalar torsion. In the standard Teleparallel approach where the spin connection is zero, this theory is not covariant under Lorentz transformations. This theory is very different from its analogous metric counterpart, $f(\overset{\circ}{R})$ gravity. A general framework will be now considered which includes both $f(\overset{\circ}{R})$ gravity and $f(T)$ gravity as special sub-cases. This theory was presented in Bahamonde et al. (2015), and then, this chapter will review the most important results discussed in that paper. Let us emphasize here that in this chapter, the standard approach in Teleparallel theories of gravity will be assumed where the spin connection is zero (pure tetrad formalism). Eq. (3.36) says that the Ricci scalar $\overset{\circ}{R}$ differs by a boundary term B from the scalar torsion T. Therefore, Bahamonde et al. (2015) proposed a new modified Teleparallel theory of gravity given by the following action,

$$\mathcal{S}_{f(T,B)} = \int \left[\frac{1}{2\kappa^2} f(T,B) + \mathcal{L}_{\rm m}\right] e\, d^4x \,, \tag{5.1}$$

where f is a function of both T and $B = (2/e)\partial_\mu(eT^\mu)$ and $\mathcal{L}_{\rm m}$ is a matter Lagrangian. Variations of the action with respect to the tetrad gives

$$\delta\mathcal{S}_{f(T,B)} = \int \left[\frac{1}{2\kappa^2}\Big(f(T,B)\delta e + ef_B(T,B)\delta B + ef_T(T,B)\delta T\Big) + \delta(e\mathcal{L}_{\rm m})\right] d^4x \,, \tag{5.2}$$

where $f_B = \partial f/\partial B$ and $f_T = \partial f/\partial T$. The variations of $F\delta e$ and $eF\delta T$, where F is a function, were derived in Sec. 4.3.1. Using those results, one obtains

$$ef_T(T,B)\delta T = \Big[-4e(\partial_\mu f_T)S_a{}^{\mu\lambda} - 4\partial_\mu(eS_a{}^{\mu\lambda})f_T + 4ef_T T^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu}\Big]\delta e^a_\lambda \,, \tag{5.3}$$

$$f(T,B)\delta e = ef(T,B)E_a^\lambda \delta e^a_\lambda \,. \tag{5.4}$$

Now, let us compute the variations of the new term coming form δB. Performing this variation, one first finds

$$ef_B(T,B)\delta B = -\Big(f_B B + 2(\partial_\mu f_B)T^\mu\Big)\delta e - 2e(\partial_\mu f_B)\delta T^\mu \,, \tag{5.5}$$

where

$$T^\mu = g^{\mu\nu}T^\sigma{}_{\sigma\nu} = g^{\mu\nu}E_a^\sigma\Big(\partial_\sigma e^a_\nu - \partial_\nu e^a_\sigma\Big) \,, \tag{5.6}$$

was used to define the torsion vector. By using (3.49), the last term on the right hand side of (5.5) becomes

$$\begin{aligned} e(\partial_\mu f_B)\delta T^\mu = \Big[&\partial_\nu\Big(E_a^\lambda(eg^{\mu\nu})(\partial_\mu f_B)\Big) - \partial_\nu\Big(E_a^\nu(eg^{\mu\lambda})(\partial_\mu f_B)\Big) \\ &- e(\partial_\mu f_B)\Big(E_a^\mu T^\lambda + g^{\mu\lambda}T_a + T^\lambda{}_a{}^\mu\Big)\Big]\delta e^a_\lambda\,, \end{aligned} \tag{5.7}$$

where boundary terms were neglected. Using $\partial_\lambda e = eg^{\mu\nu}\partial_\lambda g_{\mu\nu}$ and the compatibility equation for the metric $\nabla_\lambda(g^{\mu\nu}) = 0$ one finds

$$\partial_\lambda e = e\overset{\bullet}{\Gamma}_\lambda{}^\rho{}_\rho\,, \quad \partial_\lambda g^{\mu\nu} = -\Big(\overset{\bullet}{\Gamma}_\lambda{}^{\nu\mu} + \overset{\bullet}{\Gamma}_\lambda{}^{\mu\nu}\Big)\,. \tag{5.8}$$

It should be reminded that $\overset{\bullet}{\Gamma}_\lambda{}^{\nu\mu}$ denotes the Weitzenböck connection. Using Eqs. (3.28) and (5.8), the first term of (5.7) can be written in terms of covariant derivatives as

$$\partial_\nu\Big(E_a^\lambda(eg^{\mu\nu})(\partial_\mu f_B)\Big) = eE_a^\lambda\overset{\circ}{\Box}f_B - e(\partial_\mu f_B)\Big(E_a^\lambda\overset{\bullet}{\Gamma}_\nu{}^{\mu\nu} - E_a^\lambda\overset{\bullet}{\Gamma}{}^{\nu\mu}{}_\nu + \overset{\bullet}{\Gamma}{}^{\mu\lambda}{}_a\Big)\,. \tag{5.9}$$

Remind here that $\overset{\circ}{\Box} = \overset{\circ}{\nabla}{}^\mu\overset{\circ}{\nabla}_\mu$ which is computed with the Levi-Civita connection. Using the same idea, the second term of (5.7) becomes

$$\begin{aligned} \partial_\nu\Big(E_a^\nu(eg^{\mu\lambda})(\partial_\mu f_B)\Big) = eE_a^\nu\overset{\circ}{\nabla}{}^\lambda\overset{\circ}{\nabla}_\nu f_B + e(\partial_\mu f_B)\Big(&g^{\mu\lambda}(\overset{\bullet}{\Gamma}_a{}^\nu{}_\nu - \overset{\bullet}{\Gamma}_\nu{}^\nu{}_a) \\ &- \overset{\bullet}{\Gamma}_a{}^{\lambda\mu} - \overset{\bullet}{\Gamma}_a{}^{\mu\lambda} + \overset{\bullet}{\Gamma}{}^{\lambda\mu}{}_a - K^{\lambda\mu}{}_a\Big)\,. \end{aligned} \tag{5.10}$$

By replacing (5.9) and (5.10) into (5.7) one obtains

$$\begin{aligned} e(\partial_\mu f_B)\delta T^\mu = -\Big[&e(\partial_\mu f_B)\Big(E_a^\mu T^\lambda + g^{\mu\lambda}T_a + T^\lambda{}_a{}^\mu + g^{\mu\lambda}(\overset{\bullet}{\Gamma}_a{}^\nu{}_\nu - \overset{\bullet}{\Gamma}_\nu{}^\nu{}_a) - \overset{\bullet}{\Gamma}_a{}^{\lambda\mu} \\ &- \overset{\bullet}{\Gamma}_a{}^{\mu\lambda} + \overset{\bullet}{\Gamma}{}^{\lambda\mu}{}_a - K^{\lambda\mu}{}_a - \overset{\bullet}{\Gamma}_a{}^{\mu\lambda} + \overset{\bullet}{\Gamma}{}^{\mu\lambda}{}_a + E_a^\lambda\overset{\bullet}{\Gamma}_\nu{}^{\mu\nu} - E_a^\lambda\overset{\bullet}{\Gamma}{}^{\nu\mu}{}_\nu\Big) - eE_a^\lambda\overset{\circ}{\Box}f_B \\ &+ eE_a^\nu\overset{\circ}{\nabla}{}^\lambda\overset{\circ}{\nabla}_\nu f_B\Big]\delta e^a_\lambda\,. \end{aligned} \tag{5.11}$$

If one uses the symmetry of the Levi-Civita connection and then uses Eq. (3.28), one can simplify the above equation as follows

$$e(\partial_\mu f_B)\delta T^\mu = -\Big[e(\partial_\mu f_B)\Big(E_a^\mu T^\lambda + \overset{\bullet}{\Gamma}{}^{\lambda\mu}{}_a - \overset{\bullet}{\Gamma}{}_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a\Big) \\ - eE_a^\lambda \overset{\circ}{\Box} f_B + eE_a^\nu \overset{\circ}{\nabla}{}^\lambda \overset{\circ}{\nabla}_\mu f_B\Big]\delta e^a_\lambda\,. \tag{5.12}$$

Now, by replacing this expression into (5.5) and using (3.44) and $2S_a{}^{\lambda\mu} = K_a{}^{\lambda\mu} + E_a^\mu T^\lambda - E_a^\lambda T^\mu$, one finds

$$ef_B(T,B)\delta B = \Big[2eE_a^\nu \overset{\circ}{\nabla}{}^\lambda \overset{\circ}{\nabla}_\nu f_B - 2eE_a^\lambda \overset{\circ}{\Box} f_B - Bef_B E_a^\lambda + 2e(\partial_\mu f_B)\Big(2S_a{}^{\lambda\mu} \\ - K_a{}^{\lambda\mu} + \overset{\bullet}{\Gamma}{}^{\lambda\mu}{}_a - \overset{\bullet}{\Gamma}{}_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a\Big)\Big]\delta e^a_\lambda\,. \tag{5.13}$$

The last four terms on the right hand side are identically zero due to (3.28). Thus, finally, the variations of the boundary term contribution become

$$ef_B(T,B)\delta B = \Big[2eE_a^\nu \overset{\circ}{\nabla}{}^\lambda \overset{\circ}{\nabla}_\nu f_B - 2eE_a^\lambda \overset{\circ}{\Box} f_B - Bef_B E_a^\lambda - 4e(\partial_\mu f_B)S_a{}^{\mu\lambda}\Big]\delta e^a_\lambda\,. \tag{5.14}$$

Hence, the $f(T,B)$ field equations can be obtained by replacing the above expression and the variations of δT and δe (see Eqs. (5.3)-(5.4)) into (5.2) and then using $\delta\mathcal{S}_{f(T,B)} = 0$ giving us

$$2eE_a^\lambda \overset{\circ}{\Box} f_B - 2eE_a^\sigma \overset{\circ}{\nabla}{}^\lambda \overset{\circ}{\nabla}_\sigma f_B + eBf_B E_a^\lambda + 4e\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_a{}^{\mu\lambda} \\ + 4\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu} - efE_a^\lambda = 2\kappa^2 e\mathcal{T}_a^\lambda\,, \tag{5.15}$$

where the energy-momentum tensor was defined as follows

$$\mathcal{T}_a^\lambda = \frac{1}{e}\frac{\delta(e\mathcal{L}_{\rm m})}{\delta e^a_\lambda}\,. \tag{5.16}$$

One can rewrite the above field equation in spacetime indices by contracting with e^a_ν, yielding

$$2e\delta^\lambda_\nu \mathring{\Box} f_B - 2e\mathring{\nabla}^\lambda \mathring{\nabla}_\nu f_B + eBf_B\delta^\lambda_\nu + 4e\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_\nu{}^{\mu\lambda} \\ + 4e^a_\nu \partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta^\lambda_\nu = 2\kappa^2 e\mathcal{T}^\lambda_\nu\,. \quad (5.17)$$

Here, the energy-momentum with only spacetime indices was defined as contracting (5.16) with e^a_ν, namely $\mathcal{T}^\lambda_\nu = e^a_\nu \mathcal{T}^\lambda_a$. In the following, some properties of this theory will be studied, as for example, the limiting cases that one can recover from this approach.

5.1.1 Some important special theories

In this section, some interesting theories that can be recovered from $f(T,B)$ gravity will be presented. The first special case that is interesting to mention is a new kind of theory related to the boundary term, choosing

$$f(T,B) = T + F(B)\,, \quad (5.18)$$

where $F(B)$ is a function that depends only on the boundary term. The term T was included above in order to have GR in the background and then this theory can have all the GR solutions in the limit where $F(B) = 0$. This theory is new and in general is a 4th order theory. Cosmological solutions in this theory will be found in forthcoming sections (see Sec. 5.2.1.3). Additionally, in Sec. 5.2.2.4, the reconstruction method for this theory will be employed to mimic some interesting cosmological models.

Another straightforward theory that can be covered from $f(T,B)$ gravity is when one considers

$$f(T,B)=f(T)\,, \tag{5.19}$$

so that $f_B=0$. In this case, one finds the same $f(T)$ field equations given by (4.13). Let us make an important remark about this limit. One verifies immediately that this is the unique form of the function f which will give second order field equations. Recall that linear terms in the boundary term B do not affect the field equations. Therefore, the generic field equations contain terms of the form $\partial_\mu\partial_\nu f_b$ which are always of fourth order and can vanish if and only if f_b is a constant, so that f is linear in the boundary term. Therefore, for a non-linear function f, $f(T)$ gravity is the only possible second order modified theory of gravity constructed out of $\overset{\circ}{R}$, T and B. As mentioned before, the price to pay is the violation of local Lorentz covariance.

Another interesting theory that can be constructed from this theory is found by considering

$$f(T,B)=f(-T+B)=f(\overset{\circ}{R})\,, \tag{5.20}$$

where Eq. (3.36) was used. This is the Teleparallel equivalent of $f(\overset{\circ}{R})$ gravity. Let us now verify that this theory actually has the same field equations as standard $f(\overset{\circ}{R})$ gravity. To do this, let us introduce the standard notation for the derivative of f from $f(\overset{\circ}{R})$ gravity

$$f'(\overset{\circ}{R})=f'(-T+B)=-f_T=f_B\,. \tag{5.21}$$

Inserting this form of function into our general $f(T,B)$ field equation (5.17) leads to the following field equations

$$2e\delta_\nu^\lambda \mathring{\Box} F - 2e\mathring{\nabla}^\lambda \mathring{\nabla}_\nu F + eBF\delta_\nu^\lambda - 4e_\nu^a \partial_\mu (eS_a{}^{\mu\lambda})F + 4eFT^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta_\nu^\lambda = 2\kappa^2 e\mathcal{T}_\nu^\lambda \,. \tag{5.22}$$

This equation is not written in the standard way of $f(\mathring{R})$ gravity since the above equation is written in terms of quantities related to Teleparallel gravity such as the superpotential and the torsion tensor. To verify that this equation is the same as standard $f(\mathring{R})$ gravity, one must change all those terms. One can rewrite the fourth term in (5.22) as

$$4e_\nu^a \partial_\mu (eS_a{}^{\mu\lambda}) = 2\partial_\mu (eK_\nu{}^{\mu\lambda}) - 2\partial_\nu (eT^\lambda) + eB\delta_\nu^\lambda + 4eS_\sigma{}^{\lambda\mu} \overset{\bullet}{\Gamma}_\mu{}^\sigma{}_\nu \,. \tag{5.23}$$

Inserting this back into (5.22) gives

$$\begin{aligned} 2e\delta_\nu^\lambda \mathring{\Box} F - 2e\mathring{\nabla}^\lambda \mathring{\nabla}_\nu F - 2F\partial_\mu (eK_\nu{}^{\mu\lambda}) + 2F\partial_\nu (eT^\lambda) \\ - 4eFS_\sigma{}^{\lambda\mu} \overset{\bullet}{\Gamma}_\nu{}^\sigma{}_\mu - ef\delta_\nu^\lambda = 2\kappa^2 e\mathcal{T}_\nu^\lambda \,. \end{aligned} \tag{5.24}$$

Next, one needs to replace the torsion components with curvature. To do this, one needs to change $\overset{\bullet}{\Gamma}{}^\alpha_{\mu\nu} = \mathring{\Gamma}{}^\alpha_{\mu\nu} + K_\mu{}^\alpha{}_\nu$ and then use Eq. (2.58) to find the following identity

$$\mathring{R}_\nu^\lambda = \frac{1}{e}\Big(\partial_\sigma (eK_\nu{}^{\lambda\sigma}) + \partial_\nu (eT^\lambda)\Big) - 2S_\sigma{}^{\lambda\mu} \overset{\bullet}{\Gamma}_\nu{}^\sigma{}_\mu \,. \tag{5.25}$$

Using this final identity (5.25), it is then easy to see that the field equations reduce to the $f(\mathring{R})$ field equations in standard form

$$F\mathring{R}_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + g_{\mu\nu}\mathring{\Box} F - \mathring{\nabla}_\mu \mathring{\nabla}_\nu F = \kappa^2 \mathcal{T}_{\mu\nu} \,, \tag{5.26}$$

where $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor. Thus it can be concluded that equation (5.22) is the Teleparallel equivalent of $f(\overset{\circ}{R})$ gravity. From here, one can notice how $f(\overset{\circ}{R})$ and $f(T)$ are related. Fig. 5.1 shows how those theories are related. The starting point is a gravitational action based on an arbitrary function $f(T,B)$ which depends on the torsion scalar and a torsion boundary term. If this function is assumed to be independent of the boundary term, one arrives at $f(T)$ gravity which we identified as the unique second order gravitational theory in this approach. Likewise, if the function takes the special $f(-T+B)$, one finds the Teleparallel equivalent of $f(\overset{\circ}{R})$ gravity. Any other form of $f(T,B)$ will result in gravitational theories which are not of second order.

$$\begin{tikzcd}
f(T,B) \arrow[r, "f=f(T)"] \arrow[d, "f=f(-T+B)"'] & f(T) \arrow[d, "f=T"] \\
f(\overset{\circ}{R}) \arrow[r, "f=\overset{\circ}{R}"'] & \text{GR or TEGR}
\end{tikzcd}$$

Figure 5.1: Relationship between $f(T,B)$ and other gravity theories.

5.1.2 Lorentz covariance

One important issue in modified Teleparallel theories of gravity is the loss of the Lorentz covariance. In this section, this property will be studied for $f(T,B)$ gravity. Let us first rewrite the field equations in a covariant way. If one inserts the expression for the Ricci tensor (5.25) into the field equation (5.17), one finds

$$H_{\mu\nu} := -f_T G_{\mu\nu} + g_{\mu\nu}\mathring{\Box} f_B - \mathring{\nabla}_\mu \mathring{\nabla}_\nu f_B + \frac{1}{2}(Bf_B + Tf_T - f)g_{\mu\nu}$$
$$+ 2\Big[(f_{BB} + f_{BT})(\mathring{\nabla}_\lambda B) + (f_{TT} + f_{BT})(\mathring{\nabla}_\lambda T)\Big] S_\nu{}^\lambda{}_\mu = \kappa^2 \mathcal{T}_{\mu\nu}\,, \quad (5.27)$$

where the relations $\mathring{R} = -T + B$ and $\mathring{R}^\lambda_\nu = \mathring{G}^\lambda_\nu + \frac{1}{2}(B - T)\delta^\lambda_\nu$ were used. It is readily seen that if one considers the $f(T)$ limit, then this equation coincides with the covariant form of the $f(T)$ field equations presented in Li et al. (2011), and one notes that this equation is manifestly covariant. However, it is not in general invariant under infinitesimal local Lorentz transformations. Since the Lorentz transformations are symmetric, a necessary condition for the equation to be Lorentz covariant is that the antisymmetric part of the field equations to be identically zero, so the coefficient of $S_\nu{}^\lambda{}_\mu$ must vanish identically, see for example Li et al. (2011). Requiring this gives two conditions

$$f_{BB} + f_{BT} = 0\,, \quad \text{and} \quad f_{TT} + f_{BT} = 0\,, \quad (5.28)$$

which can be satisfied if one chooses

$$f_T + f_B = c_1\,, \quad (5.29)$$

where c_1 is a constant of integration. By introducing $X = T + B$ and $Y = T - B$, one obtains that the above equation becomes $f_X = c_1$ which can be solved yielding

$$f(T,B) = \widetilde{f}(-T + B) + c_1 B = \widetilde{f}(\mathring{R}) + c_1 B\,. \quad (5.30)$$

Since B is a total derivative term, the resulting field equations are unchanged by terms linear in B. Hence, one can set $c_1 = 0$ without loss of generality. As showed before, f of this form simply reduces to the $f(\mathring{R})$ field equations, which are manifestly Lorentz invariant. Hence one can conclude that the above field equations are Lorentz invariant if and only if they are equivalent to $f(\overset{\circ}{R})$ gravity. Therefore, the Teleparallel equivalent of $f(\overset{\circ}{R})$ gravity is the only possible Lorentz invariant theory of gravity constructed out of $\overset{\circ}{R}$, T and B. Conversely to the above, the price to pay is the presence of higher order derivative terms.

5.1.3 Conservation equations

Requiring the matter action to be invariant under both local Lorentz transformations and infinitesimal coordinate transformations gives the condition that $\mathcal{T}_{\mu\nu}$ is symmetric and divergence free

$$\overset{\circ}{\nabla}{}^{\mu}\mathcal{T}_{\mu\nu} = 0\,, \tag{5.31}$$

as shown in Li et al. (2011). Hence one requires the left-hand side of our field equations to also have this property. Let us show that this is indeed the case and that there is no need for this to be imposed as an extra (independent) condition.

For compactness, let us define the vector

$$X_{\lambda} = \Big[(f_{BB} + f_{BT})(\overset{\circ}{\nabla}_{\lambda}B) + (f_{TT} + f_{BT})(\overset{\circ}{\nabla}_{\lambda}T)\Big]. \tag{5.32}$$

Taking the covariant derivative of $H^{\mu\nu}$, one finds after some simplification

$$\overset{\circ}{\nabla}{}^{\mu}H_{\mu\nu} = -\left[\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}Bg_{\mu\nu} + 2\overset{\circ}{\nabla}{}^{\rho}S_{\nu\rho\mu}\right]X^{\mu}. \tag{5.33}$$

Now using

$$\overset{\circ}{R}_{\mu\nu} = -2\overset{\circ}{\nabla}{}^{\rho}S_{\nu\rho\mu} + \frac{1}{2}Bg_{\mu\nu} - 2S^{\rho\sigma}{}_{\mu}K_{\nu\sigma\rho}\,, \tag{5.34}$$

simplifies Eq. 5.33 to

$$\overset{\circ}{\nabla}^{\mu} H_{\mu\nu} = 2S^{\rho\sigma}{}_{\mu} K_{\nu\sigma\rho} X^{\mu}. \tag{5.35}$$

However, one knows that the energy momentum tensor is symmetric, and hence

$$H_{[\mu\nu]} = -S_{[\nu\mu]}{}^{\lambda} X_{\lambda} = 0\,. \tag{5.36}$$

This implies

$$\overset{\circ}{\nabla}^{\mu} H_{\mu\nu} = 2H^{[\rho\sigma]} K_{\nu\rho\sigma} = 0\,, \tag{5.37}$$

which follows from K being antisymmetric in its last two indices. This means that on shell the left-hand side of the field equations are conserved.

5.2 FLRW Cosmology

In this section, flat FLRW cosmology will be studied for $f(T,B)$ gravity. First, the main equations will be derived and then results from Bahamonde et al. (2018b); Bahamonde & Capozziello (2017) will be presented. First, some cosmological analytical solutions will be found using the Noether symmetry approach. These results were presented in Bahamonde & Capozziello (2017). Then, the other sections will be devoted to review Bahamonde et al. (2018b) where the reconstruction method of $f(T,B)$ cosmology will be studied. Let us now introduce the basic equations of a flat FLRW cosmology in $f(T,B)$ gravity. The metric which describes this spacetime in Cartesian coordinates is given by

$$ds^2 \;\; = \;\; dt^2 - a(t)^2(dx^2+dy^2+dz^2)\,, \tag{5.38}$$

where $a(t)$ is the scale factor of the universe. In these coordinates, the tetrad field can be expressed as follows

$$e^a_\mu = \mathrm{diag}\,(1, a(t), a(t), a(t))\ . \tag{5.39}$$

Since $f(T,B)$ is not covariant under Lorentz transformations, one needs to be very careful with the choice of the tetrad. For instance, the unwanted condition $f_{TT} = 0$ appears when one considers a flat diagonal FLRW tetrad in spherical coordinates. The above vierbein is a "good tetrad" as we discussed in Sec. 4.3.3 since it will not constrain our system. By considering a standard perfect fluid as a content of the universe described by an energy-momentum as (2.101) and using the above tetrad, one can find that the field equations (5.17) in flat FLRW become

$$3H^2(3f_B + 2f_T) - 3H\dot{f}_B + 3\dot{H}f_B + \frac{1}{2}f(T,B) = \kappa^2\rho\,, \tag{5.40}$$
$$(3H^2 + \dot{H})(3f_B + 2f_T) + 2H\dot{f}_T - \ddot{f}_B + \frac{1}{2}f(T,B) = -\kappa^2 p\,, \tag{5.41}$$

where dots represent differentiation with respect to the cosmic time and ρ and p are the energy density and pressure of the cosmic fluid. It should be remarked that $\dot{f}_B = f_{BB}\dot{B} + f_{BT}\dot{T}$ and $\dot{f}_T = f_{BT}\dot{B} + f_{TT}\dot{T}$. Clearly, if one chooses that

$$f(T,B) = T = -6H^2\,, \tag{5.42}$$

one recovers the standard flat FLRW equations in GR (see Eqs. (2.102)–(2.103) with $k = 0$).

It can be shown that the scalar torsion and the boundary term in flat FLRW become

$$T = -6\left(\frac{\dot{a}}{a}\right)^2 = -6H^2\,, \tag{5.43}$$

$$B = -6\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2\right] = -6(\dot{H} + 3H^2)\,. \tag{5.44}$$

Therefore, the Ricci scalar is

$$\overset{\circ}{R} = -T + B = -6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] = -6(\dot{H} + 2H^2)\,. \tag{5.45}$$

It should be observed here a very important remark regarding the notation. One needs to be very careful with different metric signature notations. Usually, in Teleparallel theories of gravity, researchers use the metric signature as in this thesis; $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$. However, in standard theories of gravity as in $f(\overset{\circ}{R})$ gravity, authors usually use the other signature notation $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$. The difference of this notation makes that T, B and $\overset{\circ}{R}$ have different signs. If one uses the notation $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$, one obtains that $T = 6H^2$, $B = 6(\dot{H}+3H^2)$ and $\overset{\circ}{R} = 6(\dot{H}+2H^2)$. Since this thesis uses the standard Teleparallel notation, one needs to be careful when one wants to compare our theories with $f(\overset{\circ}{R})$ gravity since usually other papers in that theory use the other signature notation. Hence, to recover $f(\overset{\circ}{R})$ gravity in the standard notation (see for example Sotiriou & Faraoni (2010)), one needs to replace $f(T,B) = f(+T-B) = f(-\overset{\circ}{R})$. Therefore, to recover that case with the standard notation, one must change

$$f_T \to f_R\,, \quad f_B \to -f_R\,, \quad f_{TT} \to f_{RR}\,, \quad f_{BB} \to f_{RR}\,, \quad f_{TB} \to -f_{RR}\,. \tag{5.46}$$

It should be noted that to avoid to write $\circ$ as a sub-index, $f_R = df(\overset{\circ}{R})/d\overset{\circ}{R}$ was defined instead of $f_{\overset{\circ}{R}}$. By doing this, one easily obtains that Eqs. (5.40) and (5.41) become the standard FLRW equations in $f(\overset{\circ}{R})$ gravity reported in Sotiriou & Faraoni (2010), namely

$$-3H^2 f_R + 3H\dot{f}_R - 3\dot{H}f_R + \frac{1}{2}f \;=\; \kappa^2\rho\,, \tag{5.47}$$

$$-3H^2 f_R - \dot{H} f_R + 2H\dot{f}_R + \ddot{f}_R + \frac{1}{2}f \;=\; -\kappa^2 p\,. \tag{5.48}$$

It should be noted here that $\dot{f}_R = f_{RR}\, d\overset{\circ}{R}/dt$ and then $\ddot{f}_R = f_{RRR}\,(d\overset{\circ}{R}/dt)^2 + f_{RR}\, d^2\overset{\circ}{R}/dt^2$. One can also recover $f(T)$ gravity just by assuming that $f(T,B) = f(T)$ in (5.40)–(5.41), giving us the same equations (4.31)–(4.32) studied in Sec. 4.3.3. In the following sections, some properties and consequences for flat FLRW cosmology will be studied for modified $f(T,B)$ gravity.

5.2.1 Noether symmetry approach

In this section, the Noether symmetry approach will be used to find exact cosmological solutions in $f(T,B)$ gravity. This section is a review of the paper by Bahamonde & Capozziello (2017). This technique proved to be very useful for several reasons: *i*) it allows us to fix physically interesting cosmological models related to the conserved quantities (i.e. in particular couplings and potentials) (Capozziello et al., 1996); *ii*) the existence of Noether symmetries allows to reduce dynamics and then to achieve exact solutions (Capozziello et al., 2012); *iii*) symmetries act as a sort of selection rules to obtain viable models in quantum cosmology (Capozziello & Lambiase, 2000). The Noether symmetry approach has been widely used in the literature to find cosmological solutions in modified gravity (see Nojiri & Odintsov (2006); Nojiri & Odintsov (2011); Bamba et al. (2012); Clifton et al. (2012); Nojiri et al. (2017) for recent reviews).

The main idea is to find symmetries in a given model and then to use them to reduce related systems and find exact solutions. As a byproduct, the existence of the symmetries selects the functions inside the models that, in some cases, have a physical meaning. In this sense, the existence of a Noether symmetry is a sort of selection rule. Essentially, the technique consists of deriving constants of motions. Any constant of motion is related to a conserved quantity that allows to reduce the system and then to obtain exact solutions. If the number of constants is equal to the number of degrees of freedom, the system is completely integrable.

In the specific case here, the cosmological equations can be derived both from the field Eqs. (5.40)-(5.41) or deduced by a point-like canonical Lagrangian $\mathcal{L}(a,\dot{a},T,\dot{T},B,\dot{B})$ related to the action (5.2). Here, $\mathbb{Q}\equiv\{a,T,B\}$ is the configuration space from which it is possible to derive $\mathbb{TQ}\equiv\{a,\dot{a},T,\dot{T},B,\dot{B}\}$, the corresponding tangent space on which $\mathcal{L}$ is defined as an application. The variables $a(t)$, $T(t)$ and $B(t)$ are, respectively, the scale factor, the torsion scalar and the boundary term defined in the FLRW metric. The Euler-Lagrange equations are given by

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{a}}=\frac{\partial\mathcal{L}}{\partial a}\,,\qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{T}}=\frac{\partial\mathcal{L}}{\partial T}\,,\qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{B}}=\frac{\partial\mathcal{L}}{\partial B}\,, \tag{5.49}$$

with the energy condition

$$\mathcal{E}_{\mathcal{L}}=\frac{\partial\mathcal{L}}{\partial\dot{a}}\dot{a}+\frac{\partial\mathcal{L}}{\partial\dot{T}}\dot{T}+\frac{\partial\mathcal{L}}{\partial\dot{B}}\dot{B}-\mathcal{L}=0\,. \tag{5.50}$$

As a consequence, the infinite number of degrees of freedom of the original field theory are reduced to a finite number as in mechanical systems.

Let us consider the canonical variables a,T,B in order to derive the $f(T,B)$ action as follows

$$\mathcal{S}_{f(T,B)}=\int\mathcal{L}(a,\dot{a},T,\dot{T},B,\dot{B})dt\,.$$

By using (5.43) and (5.44), one can rewrite the action (5.2) into its point-like representation using the Lagrange multipliers λ_1 and λ_2 as

$$\mathcal{S}_{f(T,B)}=2\pi^2\int dt\left\{f(T,B)a^3-\lambda_1\left[T+6\Big(\frac{\dot{a}}{a}\Big)^2\right]-\lambda_2\left(B+6\left[\frac{\ddot{a}}{a}+2\Big(\frac{\dot{a}}{a}\Big)^2\right]\right)\right\}\,. \tag{5.51}$$

Here $2\pi^2$ is the volume of the unit 3-sphere. By varying this action with respect to

T and B, one finds

$$(a^3 f_T - \lambda_1)\delta T = 0 \ \to \lambda_1 = a^3 f_T \,, \tag{5.52}$$

$$(a^3 f_B - \lambda_2)\delta B = 0 \ \to \lambda_2 = a^3 f_B \,. \tag{5.53}$$

Thus, the action (5.51) becomes

$$\begin{aligned} \mathcal{S}_{f(T,B)} \;&=\; 2\pi^2 \int dt \Big\{ f(T,B)a^3 - a^3 f_T \left(T + 6\Big(\frac{\dot{a}}{a}\Big)^2 \right) \\ &\quad -a^3 f_B \left(B + 6\left[\frac{\ddot{a}}{a} + 2\Big(\frac{\dot{a}}{a}\Big)^2 \right] \right) \Big\} \,, \end{aligned} \tag{5.54}$$

and then the point-like Lagrangian is

$$\mathcal{L}_{f(T,B)} = a^3 \Big[f(T,B) - T f_T - B f_B \Big] - 6a\dot{a}^2 f_T + 6a^2 \dot{a} \Big(f_{BT}\dot{T} + f_{BB}\dot{B} \Big) \,, \tag{5.55}$$

where we have integrated by parts. This Lagrangian is canonical and depends on the three time-dependent fields a, T, and B. If one chooses $f(T,B) = f(T)$, one recovers the Teleparallel $f(T)$ cosmology with the Lagrangian (Basilakos et al., 2013)

$$\mathcal{L}_{f(T)} = a^3 \Big[f(T) - T f_T \Big] - 6a\dot{a}^2 f_T \,. \tag{5.56}$$

In addition, if one chooses $f(T,B) = f(+T - B) = f(-\overset{\circ}{R})$ one obtains the point-like Lagrangian action of $f(\overset{\circ}{R})$ gravity with the standard notation (Capozziello & De Felice, 2008)

$$\mathcal{L}_{f(\overset{\circ}{R})} = a^3 \Big[f(\overset{\circ}{R}) - \overset{\circ}{R} f_R \Big] + 6a\dot{a}^2 f_R + 6a^2 \dot{a}\, d\overset{\circ}{R}/dt f_{RR} \,. \tag{5.57}$$

With these considerations in mind, let us search for cosmological solutions for the above models by the Noether symmetry approach.

In general, a Noether symmetry for a given Lagrangian exists, if the condition

$$L_X\mathcal{L}=0 \quad\to\quad X\mathcal{L}=0\,, \tag{5.58}$$

is satisfied. X is the Noether vector field and L_X is the Lie derivative. It should be observed that (5.58) is not the most general condition for finding Noether symmetries to a specific Lagrangian. In this section, (5.58) will be used but in a forthcoming section (see Sec. 8.4), the most general Noether symmetry condition will be used for another modified Teleparallel theory.

For generalised coordinates q_i, one can construct the Noether vector field X given by

$$X=\alpha^i(q)\frac{\partial}{\partial q^i}+\frac{d\alpha^i(q)}{dt}\frac{\partial}{\partial\dot{q}^i}\,, \tag{5.59}$$

where α^i are functions defined in a given configuration space $\mathbb{Q}$ that assign the Noether vector. In our case, a symmetry generator X in the space $\mathbb{Q}\equiv\{a,T,B\}$ is

$$X=\alpha\partial_a+\beta\partial_T+\gamma\partial_B+\dot{\alpha}\partial_{\dot{a}}+\dot{\beta}\partial_{\dot{T}}+\dot{\gamma}\partial_{\dot{B}}\,, \tag{5.60}$$

where α,β,γ depend on a, T and B. Therefore one has

$$\dot{\alpha}=\Big(\frac{\partial\alpha}{\partial a}\Big)\dot{a}+\Big(\frac{\partial\alpha}{\partial T}\Big)\dot{T}+\Big(\frac{\partial\alpha}{\partial B}\Big)\dot{B}\,, \tag{5.61}$$

$$\dot{\beta}=\Big(\frac{\partial\beta}{\partial a}\Big)\dot{a}+\Big(\frac{\partial\beta}{\partial T}\Big)\dot{T}+\Big(\frac{\partial\beta}{\partial B}\Big)\dot{B}\,, \tag{5.62}$$

$$\dot{\gamma}=\Big(\frac{\partial\gamma}{\partial a}\Big)\dot{a}+\Big(\frac{\partial\gamma}{\partial T}\Big)\dot{T}+\Big(\frac{\partial\gamma}{\partial B}\Big)\dot{B}\,. \tag{5.63}$$

A Noether symmetry exists if at least one of the functions α, β, and γ is different from zero. Their analytic forms can be found by making explicit Eq. (5.58), which corresponds to a set of partial differential equations given by equating to zero the terms in $\dot{a}^2,\dot{a}\dot{T},\dot{a}\dot{B},\dot{T}^2,\dot{B}^2,\dot{B}\dot{T}$ and so on. For a n dimensional configuration space, one has $1+n(n+1)/2$ equations derived from Eq. (5.58). In this case, the configuration space is three dimensional, so one has seven partial differential equations. Explicitly, from (5.58), one finds the following system of partial differential equations

$$f_T\left(2a\frac{\partial\alpha}{\partial a}+\alpha\right)+f_{TB}\left(a\gamma-a^2\frac{\partial\beta}{\partial a}\right)+af_{TT}\beta-a^2f_{BB}\frac{\partial\gamma}{\partial a}=0\,, \quad (5.64)$$

$$f_{TB}\left(a\frac{\partial\alpha}{\partial a}+a\frac{\partial\beta}{\partial T}+2\alpha\right)-2f_T\frac{\partial\alpha}{\partial T}+af_{BB}\frac{\partial\gamma}{\partial T}+a(f_{TTB}\beta+f_{TBB}\gamma)=0\,, \quad (5.65)$$

$$f_{BB}\left(a\frac{\partial\alpha}{\partial a}+a\frac{\partial\gamma}{\partial B}+2\alpha\right)+af_{TB}\frac{\partial\beta}{\partial B}+a(\beta f_{TBB}+\gamma f_{BBB})-2f_T\frac{\partial\alpha}{\partial B}=0\,, \quad (5.66)$$

$$f_{TB}\frac{\partial\alpha}{\partial T}=0\,, \quad (5.67)$$

$$f_{BB}\frac{\partial\alpha}{\partial B}=0\,, \quad (5.68)$$

$$f_{TB}\frac{\partial\alpha}{\partial B}+f_{BB}\frac{\partial\alpha}{\partial T}=0\,, \quad (5.69)$$

$$3\left(f-Bf_B-Tf_T\right)\alpha-a\left(Bf_{TB}+Tf_{TT}\right)\beta-a\left(Bf_{BB}+Tf_{TB}\right)\gamma=0\,, \quad (5.70)$$

where the unknown variables are α, β, γ and the function $f(T,B)$. There are two different strategies to solve it and to find symmetries: (i) one can directly solve the system (5.64)-(5.70) and then find the unknown functions; (ii) one can impose specific forms of $f(T,B)$ and search for the related symmetries. The second approach will be adopted to study $f(T,B)$ cosmology. Hence, some different $f(T,B)$ functions will be imposed to then find out the Noether vector satisfying (5.64)–(5.70). Finally, by using the symmetries, cosmological solutions will be found.

5.2.1.1 $f(T,B)=f(T)$

The first example that will be studied is $f(T)$ gravity. The cases studied in Atazadeh & Darabi (2012); Wei et al. (2012) are straightforwardly obtained. Eqs. (5.67)-(5.69) are identically satisfied since $f_{TB}=f_{BB}=0$. The other equations become

$$f_T\left(2a\frac{\partial\alpha}{\partial a}+\alpha\right)+af_{TT}\beta=0\,, \quad (5.71)$$

$$f_T\frac{\partial\alpha}{\partial T} = 0\,, \tag{5.72}$$
$$f_T\frac{\partial\alpha}{\partial B} = 0\,, \tag{5.73}$$
$$3\,(f - Tf_T)\,\alpha - a\beta T f_{TT} = 0\,. \tag{5.74}$$

By discarding the TEGR case ($f(T) = T$) one has that $f_T \neq 0$ and hence, from Eqs. (5.72) and (5.73), one finds $\alpha = \alpha(a)$. From Eq. (5.74), one finds that

$$\alpha(a) = \frac{af_{TT}T}{3(f - Tf_T)}\beta(a,T,B)\,. \tag{5.75}$$

By replacing this expression in (5.71), one obtains the following differential equation for β

$$\frac{\partial\beta}{\partial a} = -\frac{3f}{2af_T T}\beta(a,T,B)\,. \tag{5.76}$$

To solve this equation, let us assume that β can be separated as $\beta(a,T,B) = \beta_1(a)\beta_2(T)\beta_3(B)$, giving us

$$\frac{2a}{\beta_1}\frac{d\beta_1}{da} = -\frac{3f}{f_T T} = -\frac{3}{C}\,. \tag{5.77}$$

Here, we have used that the l.h.s of the equation only depends on a and the r.h.s only on T, so that C is a constant. Thus, it is easy to solve the above equation yielding

$$f(T) \;=\; f_0 T^C\,, \tag{5.78}$$

where f_0 is an integration constant. Moreover, it is straightforward to find that the Noether symmetry vector becomes

$$X \;=\; -\frac{1}{3}\beta_0 a^{1-\frac{3}{2C}}\partial_a + \frac{\beta_0 T a^{-\frac{3}{2C}}}{C}\partial_T + \gamma\partial_B\,, \tag{5.79}$$

where β_0 is an integration constant. As shown in Atazadeh & Darabi (2012); Wei et al. (2012), using this symmetry, one finds that $f(T)$ gravity admits power-law cosmological solutions of the form of $a(t) \propto t^{-2C/C_3}$, where C_3 is another constant. A more general study of power-law $f(T)$ cosmology is in Basilakos et al. (2013) where the full Noether symmetry condition was used.

5.2.1.2 $f(T,B) = f(-T+B) = f(\overset{\circ}{R})$

One can recover $f(\overset{\circ}{R})$ gravity by assuming $f(T,B) = f(-T+B) = f(\overset{\circ}{R})$. However, as mentioned above, it is better to find the corresponding $f(-\overset{\circ}{R})$ equations to match with the standard notation used in $f(\overset{\circ}{R})$ gravity. Hence, one needs to change the derivatives as (5.46) and then the system of differential equations (5.64)-(5.70) related to the Noether symmetry in $f(\overset{\circ}{R})$ gravity becomes

$$f_R\left(2a\frac{\partial\alpha}{\partial a}+\alpha\right)+af_{RR}\left(\beta+a\frac{\partial\beta}{\partial a}-\gamma-a\frac{\partial\gamma}{\partial a}\right)=0\,, \tag{5.80}$$

$$f_{RR}\left(a\frac{\partial\alpha}{\partial a}+a\frac{\partial\beta}{\partial T}+2\alpha-a\frac{\partial\gamma}{\partial T}\right)+2f_R\frac{\partial\alpha}{\partial T}+af_{RRR}(\beta-\gamma)=0\,, \tag{5.81}$$

$$f_{RR}\left(a\frac{\partial\alpha}{\partial a}+a\frac{\partial\gamma}{\partial B}+2\alpha-a\frac{\partial\beta}{\partial B}\right)+af_{RRR}(\beta-\gamma)-2f_R\frac{\partial\alpha}{\partial B}=0\,, \tag{5.82}$$

$$-f_{RR}\frac{\partial\alpha}{\partial T}=0\,, \tag{5.83}$$

$$f_{RR}\frac{\partial\alpha}{\partial B}=0\,, \tag{5.84}$$

$$f_{RR}\Big(\frac{\partial\alpha}{\partial T}-\frac{\partial\alpha}{\partial B}\Big)=0 \tag{5.85}$$

$$3\alpha\left(f-\overset{\circ}{R}f_R\right)-a\overset{\circ}{R}f_{RR}(\beta-\gamma)=0\,. \tag{5.86}$$

In addition, one requires that $\beta=-\gamma$ to obtain the same generators as in $f(\overset{\circ}{R})$ gravity. In doing this, Eqs. (5.81) and (5.82) are identical due to (5.85) and hence the Noether equations become

$$f_R\left(2a\frac{\partial\alpha}{\partial a}+\alpha\right)+2af_{RR}\left(\beta+a\frac{\partial\beta}{\partial a}\right)=0\,, \tag{5.87}$$

$$f_{RR}\Big(a\frac{\partial\alpha}{\partial a}+2\alpha+2a\frac{\partial\beta}{\partial R}\Big)+2f_R\frac{\partial\alpha}{\partial R}+2a\beta f_{RRR}=0 \tag{5.88}$$

$$f_{RR}\frac{\partial\alpha}{\partial B}=0\,. \tag{5.89}$$

$$3\alpha\left(f-\overset{\circ}{R}f_R\right)-2a\beta\overset{\circ}{R}f_{RR}=0\,. \tag{5.90}$$

It is worth noticing that, in order to recover the same Noether symmetry equations as in Capozziello & De Felice (2008), one requires that $\beta=\frac{1}{2}\widetilde{\beta}$. This issue comes out in the computation of the Lie derivative since the generator and some terms related with the generator of T and B are summed twice. Therefore, by changing $\beta=\frac{1}{2}\widetilde{\beta}$ one finds the same equations as in Capozziello & De Felice (2008), that is

$$f_R\left(2a\frac{\partial\alpha}{\partial a}+\alpha\right)+af_{RR}\left(\widetilde{\beta}+a\frac{\partial\widetilde{\beta}}{\partial a}\right)=0\,, \tag{5.91}$$

$$f_{RR}\Big(a\frac{\partial\alpha}{\partial a}+2\alpha+a\frac{\partial\widetilde{\beta}}{\partial R}\Big)+2f_R\frac{\partial\alpha}{\partial R}+a\widetilde{\beta}f_{RRR}=0 \tag{5.92}$$

$$f_{RR}\frac{\partial\alpha}{\partial R}=0\,, \tag{5.93}$$

$$3\alpha\left(f-\overset{\circ}{R}f_R\right)-a\overset{\circ}{R}f_{RR}\widetilde{\beta}=0\,. \tag{5.94}$$

Since the trivial GR case is not important in this study, $f_{RR}\neq 0$ and, from (5.93), one directly finds that $\alpha=\alpha(a)$. Hence, Eq. (5.92) can be rewritten as

$$\partial_R(\widetilde{\beta}f_{RR}) \;=\; -f_{RR}\Big(\frac{d\alpha}{da}+\frac{2\alpha}{a}\Big)\,, \tag{5.95}$$

and solved yielding

$$\widetilde{\beta}(a,\overset{\circ}{R})=\frac{g(a)}{f_{RR}(\overset{\circ}{R})}-\frac{(a\alpha'(a)+2\alpha(a))\,f_R(\overset{\circ}{R})}{af_{RR}(\overset{\circ}{R})}\,, \tag{5.96}$$

where $g(a)$ is an arbitrary function depending on a. Now if one replaces this solution into (5.91), one obtains

$$f_R(\overset{\circ}{R})\left[\alpha(a) - a\left(a\alpha''(a) + \alpha'(a)\right)\right] + a\left[ag'(a) + g(a)\right] = 0\,, \tag{5.97}$$

which is satisfied only if each bracket is zero. One has

$$\alpha(a) = \frac{(a^2+1)\,\alpha_0}{2a} - \frac{(a^2-1)\,\alpha_1}{2a}\,, \tag{5.98}$$
$$g(a) = \frac{c}{a}\,, \tag{5.99}$$

where c, α_0 and α_1 are integration constants. It is important to mention that this result is more general than that in Capozziello & De Felice (2008) where some terms in $\alpha(a)$ are not present; however the final result does not change since the symmetry vectors are similar. By replacing the above expression into (5.94), one finds

$$\frac{(\alpha_0+\alpha_1)\left(3f(\overset{\circ}{R}) - 2\overset{\circ}{R}f_R(\overset{\circ}{R})\right)}{2a} + \frac{3}{2}a(\alpha_0-\alpha_1)f(\overset{\circ}{R}) - c\overset{\circ}{R} = 0\,, \tag{5.100}$$

which is valid only if $c=0$ and $\alpha_0=\alpha_1$. This gives the result

$$f(\overset{\circ}{R}) = f_0\overset{\circ}{R}^{3/2}\,, \tag{5.101}$$

where f_0 is an integration constant. By using this symmetry it is possible to show that $f(\overset{\circ}{R})$ gravity admits power-law solution of the form

$$a(t) \propto t^{1/2}\,, \quad \text{and} \quad a(t) = a_0[c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0]^{1/2}\,. \tag{5.102}$$

The above analysis for $f(\overset{\circ}{R})$ gravity is not new, so all the details on how to find the above cosmological solutions were not included above. For more details about how those cosmological solutions were found after finding the symmetries, see Capozziello

& De Felice (2008).

5.2.1.3 $f(T,B) = T + F(B)$

The case $f(T,B) = T + F(B)$ is a deviation of TEGR up to a function which depends on the boundary term. The Noether conditions (5.64)-(5.70) give

$$2a\frac{\partial\alpha}{\partial a} + \alpha - a^2 F_{BB}\frac{\partial\gamma}{\partial a} = 0\,, \tag{5.103}$$

$$-2\frac{\partial\alpha}{\partial T} + aF_{BB}\frac{\partial\gamma}{\partial T} = 0\,, \tag{5.104}$$

$$F_{BB}\left(a\frac{\partial\alpha}{\partial a} + a\frac{\partial\gamma}{\partial B} + 2\alpha\right) + a\gamma F_{BBB} - 2\frac{\partial\alpha}{\partial B} = 0\,, \tag{5.105}$$

$$F_{BB}\frac{\partial\alpha}{\partial B} = 0\,, \tag{5.106}$$

$$F_{BB}\frac{\partial\alpha}{\partial T} = 0\,, \tag{5.107}$$

$$3\alpha\,(F - BF_B) - aBF_{BB}\gamma = 0\,. \tag{5.108}$$

Discarding the trivial case $F(B) = B$ which gives standard TEGR, from (5.106) and (5.107) one obtains again that $\alpha = \alpha(a)$. Using this condition in (5.104), one finds that $\gamma = \gamma(B,a)$ and the equations become

$$2a\frac{d\alpha}{da} + \alpha - a^2 F_{BB}\frac{\partial\gamma}{\partial a} = 0, \tag{5.109}$$

$$F_{BB}\Big(a\frac{d\alpha}{da} + a\frac{\partial\gamma}{\partial B} + 2\alpha\Big) + a\gamma F_{BBB} = 0, \tag{5.110}$$

$$3\alpha\,(F - BF_B) - aBF_{BB}\gamma = 0\,. \tag{5.111}$$

One can rewrite (5.110) as

$$\partial_B(\gamma F_{BB}) = -F_{BB}\Big(\frac{d\alpha}{da} + 2\frac{\alpha}{a}\Big)\,, \tag{5.112}$$

which can be solved for γ, yielding

$$\gamma = -\Big(\frac{d\alpha}{da} + 2\frac{\alpha}{a}\Big)\frac{F_B}{F_{BB}} + \frac{g(a)}{F_{BB}}\,, \tag{5.113}$$

where $g(a)$ is an arbitrary function of the scale factor. It should be remarked that the latter solution is very similar to the one found in (5.96) for the case $f(T,B) = f(\overset{\circ}{R})$. Therefore, from (5.109) one finds that

$$F_B\Big(2\alpha - 2a\frac{d\alpha}{da} - a^2\frac{d^2\alpha}{da^2}\Big) + \alpha + 2a\frac{d\alpha}{da} + a^2\frac{dg}{da} = 0\,, \tag{5.114}$$

which has the following solution

$$\alpha(a) = c_1 a + \frac{C_2}{a^2}\,,\ \ g(a) = c_3 - \frac{C_2}{a^3} - 3c_1\log a\,. \tag{5.115}$$

Here, c_1, C_2 and c_3 are integration constants. Now, by using (5.111) and (5.113) one finds that

$$a^3(3Bc_1\log(a) - Bc_3 + 3c_1F) + C_2(B + 3F - 3BF_B) = 0\,. \tag{5.116}$$

Since $F = F(B)$, the first term is zero, so that $c_1 = c_3 = 0$, yielding

$$B + 3F - 3BF_B = 0\,, \tag{5.117}$$

which can be solved obtaining

$$F(B) = f_0 B + \frac{1}{3}B\log(B)\,. \tag{5.118}$$

Therefore, one finds the following symmetry solutions

$$X = \frac{C_2}{a^2}\partial_a + \beta\partial_T - \frac{C_2}{a^3 F_{BB}}\partial_B\,, \tag{5.119}$$

$$f(T,B) = T + f_0 B + \frac{1}{3}B\log(B)\,. \tag{5.120}$$

Let us now search for cosmological solutions for this model. Considering (5.119), it is convenient to introduce the following normal coordinates

$$u = \frac{1}{3C_2}a^3\,,\ v = \frac{1}{3C_2}\Big[F_B + \log(a)\Big]\,, \tag{5.121}$$

which transform the Noether vector as

$$X = \partial_u + \beta\partial_T\,. \tag{5.122}$$

Lagrangian (5.55) reads as follows ($\ddot{u} \neq 0$)

$$\mathcal{L} = \frac{2C_2}{\ddot{u}(t)}\left[\ddot{u}(t)^2 + \dddot{u}(t)\dot{u}(t)\right]\,, \tag{5.123}$$

and hence, the Euler-Lagrange equation for $u(t)$ is

$$\ddddot{u}(t) - \frac{\dddot{u}(t)^2}{\ddot{u}(t)} = 0\,. \tag{5.124}$$

Hence, it is easy to find the following solution

$$u(t) = \frac{u_3}{u_1^2}e^{u_1 t} + u_2 t + u_0\,, \tag{5.125}$$

where u_0, u_1, u_2 and u_3 are integration constants. Additionally, since $\mathcal{L} = \mathcal{E} - 2V$, with $\mathcal{E}$ being the Hamiltonian (the energy) of the system and $V(t) = 2C_2 u_3 e^{tu_1}$ can be understood as an energy potential, one finds the following constraint

$$2C_2u_1v_1 \;=\; \mathcal{E}\,. \qquad (5.126)$$

Finally, using (5.121) one can express this cosmological solution in terms of the scale factor as follows,

$$a(t) \;=\; \Big[\frac{3C_2u_3e^{u_1t}}{u_1^2}+3C_2\Big(tu_2+u_0\Big)\Big]^{1/3}. \qquad (5.127)$$

It is easy to see that this solution gives a de-Sitter universe for the specific choice $u_2=u_0=0$.

To conclude this section, let us stress again that the method employed in the Noether symmetry approach is not the most general form. However, using the symmetries found from this approach, some interesting cosmological solutions were found. This method is very useful to study systems with difficult equations as seen in this section. It is important to mention that those symmetries come directly from the first principles of physics (Noether symmetries) and then the solutions found are implicitly part of the symmetries of the equations.

There are other alternative ways to find solutions in this kind of models, like for example the reconstruction method that will be used in the next section.

5.2.2 Reconstruction method in $f(T,B)$ gravity

In this section, the usual reconstruction method will be used to find the specific form of the function $f(T,B)$ which mimics different cosmological models. This section is devoted to review Bahamonde et al. (2018b). Remark that in the latter paper, the authors used the other metric signature notation, therefore, the following solutions would be slightly different from the ones presented there. Hereafter, it will be assumed that the matter pressure is $p_{\rm m}=w\rho_{\rm m}$ where w is the state parameter.

Therefore, by using the matter conservation equation, one finds

$$\rho_{\rm m}(t) \;=\; \rho_0 a(t)^{-3(w+1)}\,. \tag{5.128}$$

5.2.2.1 Power-law Cosmology

It would be interesting to explore the existence of exact power solutions in $f(T,B)$ gravity theory corresponding to different phases of cosmic evolution. Let us consider a model described by a power-law scale factor given by

$$a(t) \;=\; \left(\frac{t}{t_0}\right)^h\,, \tag{5.129}$$

where t_0 is some fiducial time and h is greater than zero. These solutions in General Relativity (GR) help to explain the cosmic history including matter/radiation and dark energy dominated eras. In GR, these solutions provide the scale factor evolution for the standard fluids such as dust ($h = 2/3$) or radiation ($h = 1/2$) dominated eras of the Universe. Also, in GR, $h > 1$ predicts a late-time accelerating Universe. Using (5.43) and (5.44), the scalar torsion and boundary read as follows

$$T \;=\; -\frac{6h^2}{t^2}\,, \tag{5.130}$$

$$B \;=\; -\frac{6h(3h-1)}{t^2}\,. \tag{5.131}$$

Now, for simplicity, let us assume that the function can be written in the following form

$$f(T,B) \;=\; f_1(T) + f_2(B)\,. \tag{5.132}$$

By inverting (5.130) and (5.131), the 00 equation given by (5.40) becomes

$$\frac{1}{2}f_1(T) - Tf_{1,T}(T) - \kappa^2\rho_{\rm m}(t) = K\,, \tag{5.133}$$

$$-2B^2 f_{2,BB}(B) + (1-3h)Bf_{2,B}(B) + (3h-1)f_2(B) = (2-6h)K\,. \tag{5.134}$$

Here, K is a constant for the method of separation of variables and $f_{1,T} = df_1/dT$ and $f_{2,B} = df_2/dB$. One can directly solve the above equations obtaining

$$f_1(T) = \frac{2\kappa^2\rho_0}{1-3h(w+1)}\Big(\frac{t_0}{\sqrt{6h}}\sqrt{-T}\Big)^{3h(w+1)} + C_1\sqrt{-T} + 2K\,, \tag{5.135}$$

$$f_2(B) = -C_2 B^{\frac{1}{2}(1-3h)} - C_3 B - 2K\,. \tag{5.136}$$

Then, one can mimic either radiation dominated era or matter dominated era by choosing $w = 1/3$ and $w = 0$ respectively. Then, the scale factor parameter h can have any value for those epochs. In GR, those parameters are always $h = 1/2$ (radiation era) and $h = 2/3$ (matter era), but in $f(T,B)$ gravity, the scale factor can have different power-law parameters for those eras. It should be noted that this is one specific form of the function which mimics a power-law cosmology. There are other possible functions that also will represent this model. The separation of variables can be done either by choosing that $\rho_{\rm m}$ depends on T or B. The latter comes from the fact that $\rho_{\rm m} = \rho_{\rm m}(t)$ and also $T = T(t)$ an $B = B(t)$. Hence, in principle, the energy density is $\rho(t) = \rho_0(\frac{t}{t_0})^{-3h(w+1)}$ and by using (5.130) and (5.131) one can rewrite the energy density in two ways, namely

$$\rho_{\rm m}(T) = \rho_0\left(6^{h/2}\left(\frac{h}{\sqrt{-T}t_0}\right)^h\right)^{-3(w+1)}\,, \quad \text{or} \tag{5.137}$$

$$\rho_{\rm m}(B) = \rho_0\left(6^{h/2}\left(\frac{\sqrt{h(1-3h)}}{\sqrt{B}t_0}\right)^h\right)^{-3(w+1)}\,. \tag{5.138}$$

Hence, one has a freedom to choose either $\rho_{\rm m} = \rho_{\rm m}(T)$ or $\rho_{\rm m} = \rho_{\rm m}(B)$ in the separation of variables. In the computations done above, $\rho_{\rm m} = \rho_{\rm m}(T)$ was chosen but in principle, another kind of solution for the reconstruction method can be found by choosing $\rho_{\rm m} = \rho_{\rm m}(B)$. A similar approach was done in Sec. 5.1 in de la Cruz-Dombriz & Saez-Gomez (2012) and also in de la Cruz-Dombriz et al. (2017). As stated in those references, in TEGR, the density matter is usually described by T so that without losing any generality, the same approach was used here. Then, for the following sections, $\rho_{\rm m} = \rho_{\rm m}(T)$ instead of $\rho_{\rm m} = \rho_{\rm m}(B)$ will be also used. It should be noted that from Eqs. (5.130) and (5.131), one can also express $T = T(B)$ or $B = B(T)$. In principle, one can try to solve the 00 equation (5.40) just by changing all in terms of T. However, this procedure makes the equation very complicated and it is almost impossible to find an analytical solution for the function f. As pointed out before, this kind of behaviour is something well-known in reconstruction techniques when one is considering two functions in f. See for example de la Cruz-Dombriz & Saez-Gomez (2012); de la Cruz-Dombriz et al. (2017) where the authors also would be able to express either $T = T(T_G)$ or $\overset{\circ}{R} = \overset{\circ}{R}(\overset{\circ}{G})$ in their theories. Here, $\overset{\circ}{G}$ is the Gauss-Bonnet term (see Sec. 7.1 for more details about this term). In those papers, one can also notice the situation described above.

5.2.2.2 de-Sitter reconstruction

If one assumes that the universe is governed by a de-Sitter form, i.e., the scale factor of the universe is an exponential $a(t) \propto e^{H_0 t}$, both the torsion scalar and the boundary term are constants. Explicitly they are given by $T = -6H_0^2$ and $B = -18H_0^2$ respectively. This kind of evolution of the universe is very well known and important since it correctly describes the expansion of the current universe. In GR, for this kind of universe, it is known that the universe must be filled by a dark energy fluid whose state parameter $w = -1$ and hence the energy density is also a constant. To find de-Sitter reconstruction, one must set $H = H_0$. From Eq. (5.40), it can easily be seen that any kind of functions of $f(T,B)$ can admit de-Sitter solution if the following equation is satisfied,

$$H_0^2\left(9f_B(T_0,B_0)+6f_T(T_0,B_0)\right)-\frac{1}{2}f(T_0,B_0)=\kappa^2\rho_0\,. \tag{5.139}$$

For instance, by assuming that the function is separable as $f(T,B)=f_1(B)+f_2(T)$, a possible reconstruction function which describes a de-Sitter universe is given by

$$f(T,B) \;=\; 2\kappa^2\rho_0+f_0 e^{-\frac{B}{18H_0^2}}+\widetilde{f}_0 e^{-\frac{T}{12H_0^2}}\,, \tag{5.140}$$

which of course is a constant function. Here, f_0 and $\widetilde{f_0}$ are integration constants.

5.2.2.3 ΛCDM reconstruction

Here, the reconstruction of the $f(T,B)$ function for a ΛCDM cosmological evolution will be discussed in the absence of any cosmological constant term in the modified Einstein field equations. This model was firstly formulated in Elizalde et al. (2010) in $f(\overset{\circ}{R},\overset{\circ}{G})$ modified Gauss-Bonnet theory of gravity. The cosmological effects of the cosmological constant term in the concordance model is exactly replaced by the modification introduced by $f(T,B)$ function with respect to the usual Einstein-Hilbert Lagrangian.

For simplicity, instead of working with all the variables depending on the cosmic time t, the e-folding parameter defined as $N=\ln(a/a_0)$ will be used. By using $a(t)=a_0/(1+z)$, the e-folding parameter can be also written depending on the redshift function z as $N=-\ln(1+z)=\ln(1/(1+z))$. In terms of this variable, one can express $a(t)$, $H(t)$ and time derivatives as

$$a=a_0e^N\,,\quad H=\frac{\dot{a}}{a}=\frac{dN}{dt}\,,\quad \frac{d}{dt}=H\frac{d}{dN}\,.$$

Therefore, one can rewrite equation (5.40) in terms of N, yielding

$$3H^2(3f_B + 2f_T) - 18H\Big[(H^2H'' + HH'^2 + 6H^2H')f_{BB} + 2H^2H'f_{BT}\Big] \\ + 3HH'f_B + \frac{1}{2}f(T,B) = \kappa^2\rho_{\rm m}(N)\,. \quad (5.141)$$

Here, primes denote differentiation with respect to the e-folding N. Additionally, in term of the e-folding, the scalar torsion and the boundary term are $T = -6H^2$ and $B = -6H(3H + H')$ respectively. Now, for convenience, let us introduce a new variable $g = H^2$ making the above equation become

$$\frac{3}{2}\left(g' + 6g\right) f_B - 18gg'f_{TB} - 9gf_{BB}\left(g'' + 6g'\right) + 6gf_T + \frac{1}{2}f(T,B) = \kappa^2\rho_{\rm m}\,. \quad (5.142)$$

It is easy to find that the torsion scalar and the boundary term written in this variable are $T = -6g$ and $B = -3(g' + 6g)$ respectively. Now, it will be also assumed that the function $f(T, B)$ is separable as Eq. (5.132). Using these assumptions, the above equation becomes

$$\frac{3}{2}\left(g' + 6g\right) f_{1,B}(B) - 9gf_{1,BB}(B)\left(g'' + 6g'\right) + \frac{1}{2}f_1(B) = \\ \kappa^2\rho_{\rm m}(N) - \frac{1}{2}f_2(T) - 6gf_{2,T}(T)\,. \quad (5.143)$$

Let us now reconstruct the ΛCDM model whose function $g = g(N)$ is given by (Elizalde et al., 2010)

$$g = H^2 = H_0^2 + le^{-3N}, \quad l = \frac{\kappa^2\rho_0 a_0^{-3}}{3}\,. \quad (5.144)$$

In this model, the e-folding can be expressed depending on the boundary term and the torsion scalar as follows

$$N=\frac{1}{3}\log\left(-\frac{9l}{B+18H_0^2}\right)=\frac{1}{3}\log\left(-\frac{6l}{6H_0^2+T}\right). \tag{5.145}$$

Therefore, one can rewrite Eq. (5.143) as follows

$$\begin{aligned}
2\left(27BH_0^2+162H_0^4+B^2\right)f_{1,BB}(B)-Bf_{1,B}(B)+f_1(B) &= K\,, & (5.146)\\
\kappa^2\rho_0\left(\frac{6H_0^2+T}{-6la_0^3}\right)^{w+1}-\frac{1}{2}f_2(T)+Tf_{2,T}(T) &= \frac{K}{2}\,, & (5.147)
\end{aligned}$$

where K is a constant since the r.h.s. of (5.143) depends only on T and the l.h.s. only on B. The energy density can be expressed depending on T or B so that, the above equations are one of the possible options to reconstruct a ΛCDM Universe. Thus, by solving the above equations, one finds one way to reconstruct ΛCDM is by taking the following functions,

$$\begin{aligned}
f_1(B) &= 2BC_1+C_2\sqrt{B+9H_0^2}+K\,, & (5.148)\\
f_2(T) &= -K+C_3\sqrt{-T}-\frac{\kappa^2\rho_0}{3}H_0^{2w}(a_0^3l)^{-(w+1)}\Big(6H_0^2\,{}_2F_1\left(-\frac{1}{2},-w;\frac{1}{2};-\frac{T}{6H_0^2}\right)\\
&\quad -T\,{}_2F_1\left(\frac{1}{2},-w;\frac{3}{2};-\frac{T}{6H_0^2}\right)\Big)\,, & (5.149)
\end{aligned}$$

where $H_0\neq 0$ and for the case where $H_0=0$ one finds

$$\begin{aligned}
f_1(B) &= BC_1+C_2\sqrt{-B}+K\,, & (5.150)\\
f_2(T) &= \frac{2^{-w}\kappa^2\rho_0}{2w+1}\left(-\frac{T}{3a_0^3l}\right)^{w+1}+C_3\sqrt{-T}-K\,. & (5.151)
\end{aligned}$$

Here, C_1, C_2 and C_3 are constants and ${}_2F_1$ represents the hypergeometric function of the second kind. The case where $H_0=0$ represents a power-law solution with $h=2/3$. The above solution is consistent with Eqs. (5.135) and (5.136) in that limit. The case $T=-6H_0^2, B=-18H_0^2$ which represents de-Sitter universes can not be recovered directly from the above equations. However, these models can be recovered directly from (5.143) by imposing $g = H_0^2$ (with $l = 0$) which actually gives us the same result obtained in the previous section (see Eq. (5.140)). This issue comes from the fact that one expresses the e-folding depending on B or T, one needs to assume $l \neq 0$. The same issue can be seen in Sec. 2.1 in Elizalde et al. (2010).

5.2.2.4 Reconstruction method in $f(T,B) = T + F(B)$ cosmology

In this section, the specific case where the function takes the form $f(T,B) = T+F(B)$ will be studied, which is similar to models of the form $f(\overset{\circ}{R}) = \overset{\circ}{R} + F(\overset{\circ}{R})$ and $f(T) = T + f(T)$ studied in $f(\overset{\circ}{R})$ and $f(T)$ gravity respectively (Nesseris et al., 2013). This theory is equivalent to consider a Teleparallel background (or GR) plus an additional function which depends on the boundary term which can be also understood as $F(B) = F(T + \overset{\circ}{R})$. It is important to mention that even though the case $f(T,B) = f_1(B) + f_2(T)$ studied in the previous section is more general and in principle could contain the case $f(T) = T + F(B)$, one might obtain a different reconstruction solution. The latter comes from the fact that the case $f(T) = T + F(B)$ is a very specific choice of the function and also that all the functions found before in Secs. 5.2.2.1-5.2.2.3 are one of the possible choices for reconstructing the corresponding models. Moreover, due to the mathematics technique employed before, i.e., the method of separation of variables, if one tries to recover the case $f(T) = T + F(B)$ from the solution, one might not obtain the same answer. As an example, for the power-law case it is not possible to recover $f(T,B) = T + F(B)$ unless one restricts the model with $C_1 = 0$ and $h = \frac{2}{3(w+1)}$ which is only a kind of power-law model (see Eqs. (5.135) and (5.136)). Hence, it is interesting and important to also study if it is possible to reconstruct these cosmological models within this particular theory.

In this model, the 00 field equation (5.40) becomes

$$3H^2+9H^2F_B-3H\dot{F}_B+3\dot{H}F_B+\frac{1}{2}F(B) \;=\; \kappa^2\rho_{\rm m}(t)\,, \tag{5.152}$$

where the energy density is given by (5.128). Equivalently, from (5.141), it is easy to rewrite the above equation in term of the e-folding,

$$3H^2+9H^2F_B-18H\Big[(H^2H''+HH'^2+6H^2H')F_{BB}\Big]+3HH'F_B+\frac{1}{2}F(B)=\kappa^2\rho_{\rm m}(N)\,. \tag{5.153}$$

Let us now perform a reconstruction method for all the same models studied in Secs. 5.2.2.1-5.2.2.3.

For a power-law cosmology described in Sec. 5.2.2.1, Eq. (5.152) can be written as follows,

$$\frac{B\left(h+2BF_{BB}(B)\right)}{2-6h}-\frac{1}{2}BF_B(B)+\frac{F(B)}{2}=\kappa^2\rho_0\left(-\frac{6h(3h-1)}{Bt_0^2}\right)^{-\frac{3}{2}h(w+1)} \tag{5.154}$$

which can be directly solved, yielding the following solution

$$\begin{aligned} F(B) \;&=\; C_1B^{\frac{1-3h}{2}}+B\left(C_2+\frac{2h}{(3h+1)^2}\right)-\frac{hB\log(B)}{3h+1}\\ &\quad-\frac{(3h-1)\kappa^2 2^{p+1}3^p\rho_0\left(\frac{(1-3h)h}{Bt_0^2}\right)^p}{(p+1)(-3h+2p+1)}\,, \end{aligned} \tag{5.155}$$

where C_1 and C_2 are integration constants and $p=-\frac{3}{2}h(w+1)$.

Now, for a de-Sitter reconstruction, the scale factor behaves as $a(t)=a_0e^{H_0t}$, then $B=-18H_0^2$ and hence from (5.152) we directly find that the function takes the following form,

$$F(B) \;=\; C_1e^{-\frac{B}{18H_0^2}}-2\left(3H_0^2-\kappa^2\rho_0\right)\,. \tag{5.156}$$

Here, C_1 is an integration constant. Let us now reconstruct a ΛCDM universe where $g = H_0^2 + le^{-3N}$. In this theory, Eq. (5.143) becomes

$$(B^2 + 27BH_0^2 + 162H_0^4)F_{BB}(B) - \frac{1}{2}BF_B(B) + \frac{F(B)}{2} - \frac{1}{3}\left(B + 9H_0^2\right) = 3^{-2(w+1)}\kappa^2\rho_0\left(a_0\sqrt[3]{-\frac{l}{B+18H_0^2}}\right)^{-3(w+1)}, \quad (5.157)$$

where Eq. (5.145) was used to express all in terms of the boundary term B. The above equation is difficult to solve analytically for all values of w, so that for simplicity, let us assume the cold dust case $w = 0$, which gives us

$$F(B) = \frac{4\kappa^2\rho_0\left(B + 9H_0^2\right)}{9a_0^3 l} + \frac{2B\log\left(B + 18H_0^2\right)\left(3a_0^3 l - \kappa^2\rho_0\right)}{9a_0^3 l} + \frac{8H_0\sqrt{B+9H_0^2}\arctan\left(\frac{\sqrt{B+9H_0^2}}{3H_0}\right)\left(3a_0^3 l - \kappa^2\rho_0\right)}{3a_0^3 l} + C_1\sqrt{B+9H_0^2} + 2BC_2 - \frac{4B}{3} - 18H_0^2\,, \quad (5.158)$$

where C_1 and C_2 are integration constants.

Let us stress here that the final expressions for the function $F(B)$ becomes less complicated than in $f(T,B)$ gravity.

5.3 Generalised non-minimally gravity matter-coupled theory

Other kinds of modified theories of gravity have been considered in the literature. Some interesting ones are theories with non-minimal coupling between matter and gravity. In the standard metric approach, some alternative models have been proposed such as $f(\overset{\circ}{R},\mathcal{T})$ (Harko et al., 2011), where $\mathcal{T}$ is the trace of the energy-momentum tensor or non-minimally coupled theories between the curvature scalar and the matter Lagrangian $f_1(\overset{\circ}{R}) + f_2(\overset{\circ}{R})\mathcal{L}_{\rm m}$ (Bertolami et al., 2007). Further, another more general theory is the so-called $f(\overset{\circ}{R},\mathcal{L}_{\rm m})$ where now an arbitrary function of $\overset{\circ}{R}$ and $\mathcal{L}_{\rm m}$ is considered in the action (Harko & Lobo, 2010). Along the lines of those theories, modified Teleparallel theories of gravity where couplings between matter and the torsion scalar have been also considered. Some important theories are for example: $f(T,\mathcal{T})$ gravity (Harko et al., 2014a) and also non-minimally couplings between the torsion scalar and the matter Lagrangian theory $f_1(T)+f_2(T)\mathcal{L}_{\rm m}$ (Harko et al., 2014b). Along this line, recently, there was presented a new generalised non-minimally gravity-matter coupled theory with the following action (Bahamonde, 2018)

$$\mathcal{S}_{f(T,B,\mathcal{L}_{\rm m})} = \int e f(T,B,\mathcal{L}_{\rm m})\, d^4x\,, \tag{5.159}$$

where the function f depends on the scalar curvature T, the boundary term B and the matter Lagrangian $\mathcal{L}_{\rm m}$. The energy-momentum tensor of matter $\mathcal{T}_a^{\beta}$ is defined as

$$\mathcal{T}_a^{\beta} = \frac{1}{e}\frac{\delta(e\mathcal{L}_{\rm m})}{\delta e^a_{\beta}}\,. \tag{5.160}$$

Let us further assume that the matter Lagrangian depends only on the components of the tetrad (or metric) and not on its derivatives, which according to Harko et al. (2014b) is equivalent as having

$$\frac{\partial\mathcal{L}_{\rm m}}{\partial(\partial_\mu e^a_{\rho})} = 0\,, \tag{5.161}$$

which gives us the following energy-momentum tensor

$$\mathcal{T}_a^{\beta} = \mathcal{L}_{\rm m}E_a^{\beta} + \frac{\partial\mathcal{L}_{\rm m}}{\partial e^a_{\beta}}\,. \tag{5.162}$$

Now, by a variation of action (5.1) with respect to the tetrad, one obtains

$$\begin{aligned}\delta\mathcal{S}_{f(T,B,\mathcal{L}_{\rm m})} &= \int\Big[ef_T\delta T + ef_B\delta B + ef_L\frac{\delta\mathcal{L}_{\rm m}}{\delta e^a_\beta}\delta e^a_\beta + f\delta e\Big]d^4x\,, \qquad (5.163)\\ &= \int e\Big[f_T\delta T + f_B\delta B + f_L\Big(\mathcal{T}_a^{\beta} + \mathcal{L}_{\rm m}E_a^{\beta}\Big)\delta e^a_\beta + fE_a^\beta\delta e^a_\beta\Big]d^4x\,, \qquad (5.164)\end{aligned}$$

where Eq. (5.162) was used and $f_T = \partial f/\partial T, f_B = \partial f/\partial B$ and $f_L = \partial f/\partial\mathcal{L}_{\rm m}$. Variations with respect to the torsion scalar and the boundary term are given by (5.3) and (5.14) respectively. Hence, by setting $\delta\mathcal{S}_{f(T,B,\mathcal{L}_{\rm m})} = 0$, one obtains the $f(T,B,\mathcal{L}_{\rm m})$ field equations given by

$$\begin{aligned}&2E_a^\beta\mathring{\Box}f_B - 2E_a^\sigma\mathring{\nabla}^\beta\mathring{\nabla}_\sigma f_B + Bf_BE_a^\beta + 4\Big[(\partial_\mu f_T) + (\partial_\mu f_B)\Big]S_a{}^{\mu\beta}\\ &\qquad + 4f_T\Big(e^{-1}\partial_\mu(eS_a{}^{\mu\beta}) - T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu}\Big) - fE_a^\beta - f_L\mathcal{L}_{\rm m}E_a^\beta = f_L\mathcal{T}_a^\beta\,.\end{aligned}$$

The above field equations can be also written only in spacetime indices by contracting it by e^a_λ giving us

$$\begin{aligned}&2\delta^\beta_\lambda\mathring{\Box}f_B - 2\mathring{\nabla}^\beta\mathring{\nabla}_\lambda f_B + Bf_B\delta^\beta_\lambda + 4\Big[(\partial_\mu f_T) + (\partial_\mu f_B)\Big]S_\lambda{}^{\mu\beta}\\ &\quad +4f_Te^a_\lambda\Big(e^{-1}\partial_\mu(eS_a{}^{\mu\beta}) - T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu}\Big) - f\delta^\beta_\lambda - f_L\mathcal{L}_{\rm m}\delta^\beta_\lambda \ = f_L\mathcal{T}_\lambda^\beta\,. \qquad (5.165)\end{aligned}$$

From these field equations, one can directly recover Teleparallel gravity by choosing $f(T,\mathcal{L}_{\rm m}) = T/2\kappa^2 + \mathcal{L}_{\rm m}$ which gives us the same action as (3.39). Moreover if one chooses $f(T,\mathcal{L}_{\rm m}) = T/2\kappa^2 + f_1(T) + (1+\lambda f_2(T))\mathcal{L}_{\rm m}$, one recovers the non-minimal torsion-matter coupling extension of $f(T)$ gravity presented in Harko et al. (2014b). Let us now study the conservation equation for this theory. First, $\mathring{R}^\beta_\lambda = \mathring{G}^\beta_\lambda + \frac{1}{2}(B-T)\delta^\beta_\lambda$ will be used, where $\mathring{G}^\beta_\lambda$ is the Einstein tensor. Using this relationship, one can rewrite the field equation (5.165) as follows

$$H_{\lambda\beta} := f_T\overset{\circ}{G}_{\lambda\beta} - \overset{\circ}{\nabla}_\lambda\overset{\circ}{\nabla}_\beta f_B + g_{\lambda\beta}\overset{\circ}{\Box} f_B + \frac{1}{2}\Big(Tf_T + Bf_B - \mathcal{L}_{\rm m}f_L - f\Big)g_{\lambda\beta} \\ +2X_\nu S_\lambda{}^\nu{}_\beta = \frac{1}{2}f_L\mathcal{T}_{\lambda\beta}\,, \quad (5.166)$$

where for simplicity the quantity

$$\begin{aligned} X_\nu &= (f_{BT}+f_{BB}+f_{BT})\overset{\circ}{\nabla}_\nu B + (f_{TT}+f_{TB}+f_{TL})\overset{\circ}{\nabla}_\nu T \\ &\quad +(f_{TL}+f_{BL}+f_{LL})\overset{\circ}{\nabla}_\nu\mathcal{L}_{\rm m} \end{aligned} \quad (5.167)$$

was introduced. By taking the covariant derivative of $H_{\lambda\beta}$ and after some simplifications, one finds that

$$\overset{\circ}{\nabla}{}^\lambda H_{\lambda\beta} = -2S^{\sigma\rho}{}_\lambda K_{\beta\sigma\rho}X^\lambda - \frac{1}{2}g_{\lambda\beta}\overset{\circ}{\nabla}{}^\lambda(\mathcal{L}_{\rm m}f_L) = -\frac{1}{2}g_{\lambda\beta}\overset{\circ}{\nabla}{}^\lambda(\mathcal{L}_{\rm m}f_L)\,, \quad (5.168)$$

where there was used the fact that the energy-momentum tensor is symmetric and hence $S^{\sigma\rho}{}_\lambda K_{\beta\sigma\rho}X^\lambda = 0$. The latter comes from the fact that field equations are symmetric, and hence the energy-momentum tensor is also symmetric. Now, let us find a condition that f needs to satisfy in order to have the standard conservation equation for the energy momentum tensor, i.e., $\overset{\circ}{\nabla}_\mu\mathcal{T}^{\mu\nu} = 0$. By taking the covariant derivative in (5.166) and assuming $\overset{\circ}{\nabla}_\mu\mathcal{T}^{\mu\nu} = 0$, one obtains that the standard conservation equation for the energy-momentum tensor is satisfied if the function f satisfies the following form

$$\Big(\mathcal{T}_{\mu\nu} + g_{\mu\nu}\mathcal{L}_{\rm m}\Big)\overset{\circ}{\nabla}{}^\mu f_L = -2e^a_\mu g_{\beta\nu}\frac{\partial\mathcal{L}_{\rm m}}{\partial e^a_\beta}\overset{\circ}{\nabla}{}^\mu f_L = 0\,, \quad (5.169)$$

which matches with the conservation equation presented in Harko & Lobo (2010). Thus, in general in $f(T,B,\mathcal{L}_{\rm m})$ gravity, the energy-momentum tensor is not covariantly conserved and depending on the metric and the model, the energy-momentum tensor may or may not be conserved.

In Bahamonde (2018), the complete dynamical system for flat FLRW cosmology was presented and for some specific cases of the function, the corresponding cosmological model was studied for a perfect fluid. It should be noted that even though the density matter Lagrangian $\mathcal{L}_{\rm m}$ is not unique for a perfect fluid, according to Bertolami et al. (2008), its choice does not change the energy-momentum tensor and therefore, the field equations would be the same independently of this choice. In general, the theory is very complicated to work since it becomes a 10 dimensional dynamical system. This is somehow expected since the theory is very general and complicated. Using the full dynamical system found for the full theory, different special interesting theories were also studied in that reference.

Fig. 5.2 shows the most important theories that can be constructed from the action (5.159). The graph is divided into three main parts. The left part of the figure represents the scalar-curvature or standard metric theories coupled with the matter Lagrangian. Different interesting cases can be recovered from this branch, such as a generalised $f(\overset{\circ}{R}, \mathcal{L}_{\rm m})$ theory or a non-minimally scalar curvature-matter coupled gravity $T/2\kappa^2 + f_1(B) + f_2(B)\mathcal{L}_{\rm m}$ or just standard $f(\overset{\circ}{R})$ gravity. The entries at the middle of the figure represent all the theories based on the boundary term B and the matter Lagrangian $\mathcal{L}_{\rm m}$. In this branch, new kind of theories are presented based on a general new theory $T/2\kappa^2 + f(B, \mathcal{L}_{\rm m})$, where the term $T/2\kappa^2$ is added in the model to have TEGR (or GR) in the background. The right part of the figure is related to Teleparallel theories constructed by the torsion scalar and the matter Lagrangian. Under these models, a new general theory $f = f(T, \mathcal{L}_{\rm m})$ is highlighted in a box, allowing to have a new kind of theories with new possible couplings between T and $\mathcal{L}_{\rm m}$. As a special case, this theory can also become a non-minimally torsion-matter coupled gravity theory $f = f_1(T)/2\kappa^2 + f_2(T)\mathcal{L}_{\rm m}$, presented previously in Harko & Lobo (2010). Thus, different gravity curvature-matter or torsion-matter coupled theories can be constructed. Some of them have been considered and studied in the past but others were recently presented in Bahamonde (2018). From the figure, one can directly see the connection between modified Teleparallel theories and standard modified theories. The quantity B connects the right and left part of the figure. Hence, the connection between the Teleparallel and standard theories is directly related to this boundary term B. Therefore, one can directly see that the mother of all of those gravity theories coupled with the matter Lagrangian is the one presented in this section, the so-called $f(T,B,\mathcal{L}_{\rm m})$.

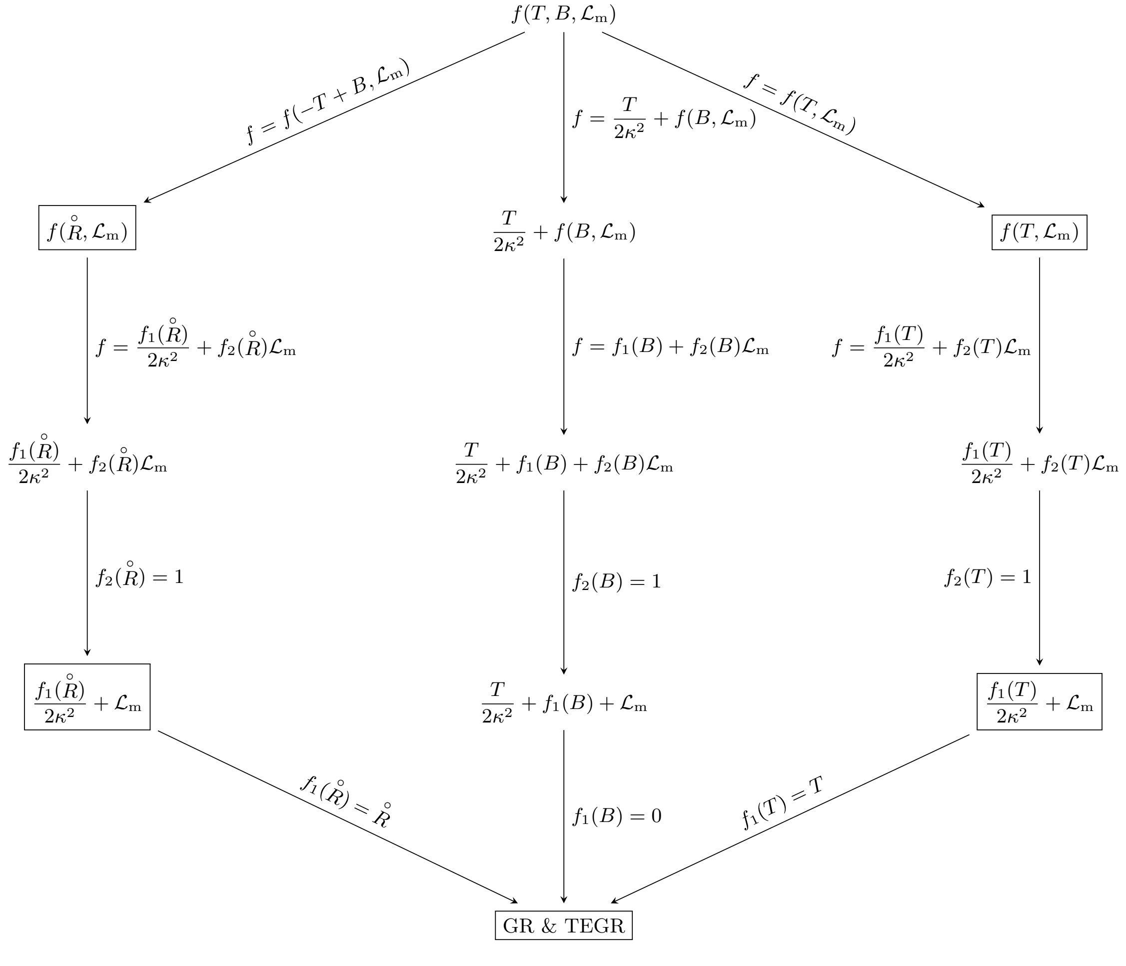


Figure 5.2: Relationship between $f(T,B,\mathcal{L}_{\rm m})$ and other gravity theories.

6

$f(T_{\mathrm{ax}}, T_{\mathrm{ten}}, T_{\mathrm{vec}}, B)$ gravity

Chapter Abstract

This chapter introduces new classes of modified Teleparallel gravity models based on an action constructed to be a function of the irreducible parts of torsion $f(T_{\mathrm{ax}}, T_{\mathrm{ten}}, T_{\mathrm{vec}})$, where $T_{\mathrm{ax}}, T_{\mathrm{ten}}$ and T_{vec} are squares of the axial, tensor and vector components of torsion, respectively. This is the most general (well-motivated) second order Teleparallel theory of gravity that can be constructed from the torsion tensor. Different particular second order theories can be recovered from this theory such as new General Relativity, conformal Teleparallel gravity or $f(T)$ gravity. Additionally, the boundary term B can also be incorporated into the action. By performing a conformal transformation, it is shown that the two unique theories which have an Einstein frame are either the Teleparallel equivalent of General Relativity or $f(-T+B) = f(\overset{\circ}{R})$ gravity, as expected.

6.1 Torsion decomposition and other Teleparallel gravity theories

An interesting approach of modifying Teleparallel gravity was considered in the late 1970s (Hayashi & Shirafuji, 1979) called *New General Relativity*. In this model the torsion tensor is decomposed into its three irreducible components, namely

$$T_{\lambda\mu\nu} = \frac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \frac{1}{3}(g_{\lambda\mu}v_\nu - g_{\lambda\nu}v_\mu) + \epsilon_{\lambda\mu\nu\rho}a^\rho\,, \tag{6.1}$$

where

$$v_\mu = T^\lambda{}_{\lambda\mu}\,, \tag{6.2}$$
$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,, \tag{6.3}$$
$$t_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\nu\lambda}v_\mu + g_{\nu\mu}v_\lambda) - \frac{1}{3}g_{\lambda\mu}v_\nu\,, \tag{6.4}$$

are three irreducible parts with respect to the local Lorentz group, known as the vector, axial, and purely tensorial torsions, respectively. One can then square these three pieces to construct the following scalars,

$$T_{\rm ten} = t_{\lambda\mu\nu}t^{\lambda\mu\nu} = \frac{1}{2}\Big(T_{\lambda\mu\nu}T^{\lambda\mu\nu} + T_{\lambda\mu\nu}T^{\mu\lambda\nu}\Big) - \frac{1}{2}T^\lambda{}_{\lambda\mu}T_\nu{}^{\nu\mu}\,, \tag{6.5}$$
$$T_{\rm ax} = a_\mu a^\mu = \frac{1}{18}\Big(T_{\lambda\mu\nu}T^{\lambda\mu\nu} - 2T_{\lambda\mu\nu}T^{\mu\lambda\nu}\Big)\,, \tag{6.6}$$
$$T_{\rm vec} = v_\mu v^\mu = T^\lambda{}_{\lambda\mu}T_\nu{}^{\nu\mu}\,. \tag{6.7}$$

Using these three scalars, one can construct new classes of Teleparallel modified theories. Hayashi & Shirafuji (1979) introduced New General Relativity theory where a linear functional of these squared quantities were considered. This action then reads

$$\mathcal{S}_{\rm NGR} = \int \left[\frac{1}{2\kappa^2}\Big(a_0 + a_1T_{\rm ax} + a_2T_{\rm ten} + a_3T_{\rm vec}\Big) + \mathcal{L}_{\rm m}\right] e\, d^4x \tag{6.8}$$

where the four a_i are arbitrary constants. The number a_0 can be interpreted as the cosmological constant. Clearly, by choosing the following constants

$$a_1 = \frac{3}{2}\,, \quad a_2 = \frac{2}{3}\,, \quad a_3 = -\frac{2}{3}\,, \tag{6.9}$$

one can recover the scalar torsion, yielding

$$T = \frac{3}{2}T_{\rm ax} + \frac{2}{3}T_{\rm ten} - \frac{2}{3}T_{\rm vec}\,. \tag{6.10}$$

Then, for the special case where the constants are chosen as (6.9), one can recover the standard Teleparallel equivalent of General Relativity action. Recently, there has been an increased interest in conformal gravity models that have many attractive features, see for instance Mannheim (2006). As it turns out, it is possible to construct conformal gravity in the Teleparallel framework leading to *conformal Teleparallel gravity* (Maluf & Faria, 2012). This model has second order field equations, which are much simpler than those of the usual Weyl gravity based on the square of the conformal Weyl tensor. The action of this model is taken to be quadratic in the torsion scalar

$$\mathcal{S}_{\rm CTG} = \int \left[\frac{1}{2\kappa^2}\widetilde{T}^2 + \mathcal{L}_{\rm m}\right] e\, d^4x\,, \tag{6.11}$$

where the scalar

$$\widetilde{T} = \frac{3}{2}T_{\rm ax} + \frac{2}{3}T_{\rm ten} \tag{6.12}$$

was introduced. It should be noted that this theory takes the same scalars from the New General Relativity action (6.8), with coefficients $a_1 = 3/2, a_2 = 2/3, a_3 = 0$. Then, this theory is constructed without the vectorial part of torsion and therefore, it not possible to construct TEGR directly from its action. It is interesting to note the absence of the vector torsion in the action. We will return to this observation in Sec 6.4. This model combines some elements of $f(T)$ gravity and New General Relativity. New General Relativity and conformal Teleparallel gravity consider the Lagrangian to be a function of torsion only and do not include its derivatives, as a result of which the equations of motion are always second order. Let us also mention that in Chervova & Vassiliev (2010), a similar action coming from 3-dimensional Cosserat elasticity (see Cosserat et al. (1909)) was studied where only the axial part of torsion was considered. This theory is then conformally invariant under the rescaling of the 3-dimensional metric $g_{\mu\nu}$ by an arbitrary positive scalar function. The authors found if a 3-dimensional space is assumed to be a elastic continuum medium which only experience rotations, in its stationary regime, the equations are equivalent to the Weyl equations (massless Dirac equation). In Cosserat elasticity, torsion is usually called dislocation tensor and it is related to the rotational deformations of the medium (Boehmer et al., 2011). Then, there is an interesting analogy between Cosserat elasticity and Teleparallel gravity.

6.2 New class of modified Teleparallel gravity models

Many Teleparallel models discussed before, except for new General Relativity and Teleparallel conformal gravity, assume the torsion scalar to take the same form as in the Teleparallel equivalent of General Relativity. While this is well-motivated by the fact that the General Relativity limit is easily achievable, these are not the most general models one can consider. The main objective of this chapter is to review Bahamonde et al. (2017b) where there was introduced a new modified Teleparallel theory that would naturally include many Teleparallel models and allow us to analyse their general properties.

Inspired by $f(T)$ gravity and the approach put forward in (6.8), one can then generalise this action to an arbitrary function of the three irreducible torsion pieces.

Hence, let us consider the following action

$$\mathcal{S}_{f(T_{\rm ax},T_{\rm ten},T_{\rm vec})}=\int\Big[\frac{1}{2\kappa^2}f(T_{\rm ax},T_{\rm ten},T_{\rm vec})+\mathcal{L}_{\rm m}\Big]\,e\,d^4x\,, \tag{6.13}$$

which naturally includes all previous models. Since the torsion pieces only contain first partial derivatives of the tetrad, the resulting field equations will be of second order. Variations of this action with respect to tetrad fields yield,

$$\delta\mathcal{S}_{f(T_{\rm ax},T_{\rm ten},T_{\rm vec})}=\int\Big[\frac{1}{2\kappa^2}\Big(f\delta e+ef_{T_{\rm ax}}\delta T_{\rm ax}+ef_{T_{\rm vec}}\delta T_{\rm vec}+ef_{T_{\rm ten}}\delta T_{\rm ten}\Big)+\delta(e\mathcal{L}_{\rm m})\Big]\,d^4x\,, \tag{6.14}$$

where $f_{T_{\rm ax}}=\partial f/\partial T_{\rm vec}$, $f_{T_{\rm vec}}=\partial f/\partial T_{\rm vec}$ and $f_{T_{\rm ten}}=\partial f/\partial T_{\rm ten}$. In the following section, the variations of each piece will be presented and then the field equations of this model. It should be noted that in Bahamonde et al. (2017b), there was presented the case where the spin connection is different to zero, but in this thesis, the field equations in the pure tetrad formalism will be presented.

6.3 Variations and field equations

6.3.1 Variations of $T_{\rm vec}$ and $T_{\rm ax}$

Straightforwardly, variations of the vector torsion with respect to the tetrad yields,

$$ef_{T_{\rm vec}}\delta T_{\rm vec}=ef_{T_{vec}}\delta(v^iv_i)\;=\;2v_if_{T_{vec}}\delta T^{\lambda}{}_{\lambda}{}^{i}=2ev^cf_{T_{\rm vec}}\delta\Big[E_b^{\lambda}E_c^{\nu}T^{b}{}_{\lambda\nu}\Big]\,, \tag{6.15}$$

where v_i was expressed with Latin indices for convenience. If F is any function, it can be proved that by taking of $F\delta T^b{}_{\rho\sigma}$ with respect to tetrads, one obtains

$$F\delta T^b{}_{\rho\sigma} = F\Big[\delta^b_a(\delta^\nu_\rho\delta^\mu_\sigma - \delta^\nu_\sigma\delta^\mu_\rho)\Big]\partial_\nu(\delta e^a_\mu) = -\partial_\nu\Big(F(\delta^\nu_\rho\delta^\mu_\sigma - \delta^\nu_\sigma\delta^\mu_\rho)\Big)\delta e^b_\mu + \text{b.t.} \quad (6.16)$$

Here, the term b.t. represents boundary terms. It should be noted that $\delta^\nu_\rho = e^a_\rho E^\nu_b \delta^b_a$, so that in general the term $\partial_\lambda \delta^\nu_a = \partial_\lambda(E^\mu_a\delta^\nu_\mu) \neq 0$. By using the above expression, $\delta E^\sigma_m = -E^\sigma_n E^\mu_m \delta e^n_\mu$ and neglecting boundary terms, (6.15) becomes

$$ef_{T_{\rm vec}}\delta T_{\rm vec} = -2\Big[ef_{T_{\rm vec}}\Big(v^c T^\beta{}_{ac} + v^\beta v_a\Big) + \partial_\lambda\Big(ef_{\rm T_{vec}}(v^\beta E^\lambda_a - v^\lambda E^\beta_a)\Big)\Big]\delta e^a_\beta\,. \quad (6.17)$$

Now, let us find the variations corresponding to the axial part. To do that, let us rewrite the axial torsion as follows

$$a_i = \frac{1}{6}\epsilon_{ibcd}T^{bcd} = \frac{1}{6}\epsilon_{ib}{}^{cd}E^\rho_c E^\sigma_d T^b{}_{\rho\sigma}\,, \quad (6.18)$$

from where the variations with respect to tetrads follow straightforwardly

$$\begin{aligned} ef_{T_{\rm ax}}\delta(a_i a^i) &= 2ea^i f_{T_{\rm ax}}\delta a_i\,, & (6.19)\\ &= -\frac{2}{3}\left[\epsilon_{ib}{}^{cd}ef_{T_{\rm ax}}a^i E^\beta_c T^b{}_{ad} + \partial_\nu\left(e\epsilon_{ia}{}^{cd}f_{T_{\rm ax}}a^i E^\nu_c E^\beta_d\right)\right]\delta e^a_\beta\,. & (6.20)\end{aligned}$$

6.3.2 Variations of $T_{\rm ten}$

It is useful to rewrite the tensorial torsion as

$$t_{\lambda\mu\nu} = \mathcal{T}_{\lambda\mu\nu} + \mathcal{V}_{\lambda\mu\nu}\,, \quad (6.21)$$

where

$$\mathcal{T}_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu})\,, \qquad \mathcal{V}_{\lambda\mu\nu} = \frac{1}{6}(g_{\nu\lambda}v_\mu + g_{\nu\mu}v_\lambda) - \frac{1}{3}g_{\lambda\mu}v_\nu\,. \tag{6.22}$$

One can then straightforwardly rewrite $T_{\rm ten}$ as

$$T_{\rm ten} = \mathcal{T}_{\lambda\mu\nu}\mathcal{T}^{\lambda\mu\nu} - \frac{1}{2}v_\mu v^\mu\,, \tag{6.23}$$

and then the corresponding variations with respect to the tensorial part yields

$$ef_{T_{\rm ten}}\delta T_{\rm ten} = ef_{T_{\rm ten}}\delta(\mathcal{T}_{\lambda\mu\nu}\mathcal{T}^{\lambda\mu\nu}) - \frac{1}{2}ef_{T_{\rm ten}}\delta(v_\mu v^\mu)\,. \tag{6.24}$$

The first term can be rewritten as

$$\mathcal{T}_{\lambda\mu\nu}\mathcal{T}^{\lambda\mu\nu} = \frac{1}{2}(\eta_{cb}g^{\rho\alpha}g^{\sigma\beta} + E^\alpha_b E^\rho_c g^{\sigma\beta})T^b{}_{\rho\sigma}T^c{}_{\alpha\beta}\,, \tag{6.25}$$

and then after using (6.16) and (6.17), one can compute in a fairly complicated but straightforward way the variations with respect to the tensorial part, yielding

$$\begin{aligned} ef_{T_{\rm ten}}\delta T_{\rm ten} &= -ef_{T_{\rm ten}}\Big(2T^b{}_{a\sigma}T_b{}^{\beta\sigma} + T^\beta{}_{\rho\sigma}T^\rho{}_a{}^\sigma + T^\alpha{}_{\rho a}T^\rho{}_\alpha{}^\beta - T^\beta{}_{ai}v^i - v^\beta v_a\Big)\delta e^a_\beta \\ &\quad -\partial_\nu\Big[ef_{T_{\rm ten}}\left(T^{\beta\nu}{}_a - T^{\nu\beta}{}_a - 2T_a{}^{\beta\nu} - v^\beta E^\nu_a + v^\nu E^\beta_a\right)\Big]\delta e^a_\beta\,. \end{aligned} \tag{6.26}$$

6.3.3 Field equations

We derived the field equations following the pure tetrad approach to Teleparallel theories, where the Teleparallel spin connection is vanishing. The theory is in this case manifestly invariant under general coordinate transformation but non-covariant under local Lorentz transformations. One can work in the particular frame where the spin connection vanishes, which is always possible on account of the pure gauge character of the Teleparallel spin connection. One then naturally loses local Lorentz covariance Li et al. (2011) and must restrict considerations to the case of good tetrads, which need to be calculated following the method of Ferraro & Fiorini (2011a); Tamanini & Boehmer (2012). For sake of deriving the field equations, both methods yield the same result.

By adding up (6.17), (6.20) and (6.26) and using $f\delta e = efE^a_\beta \delta e^\beta_a$, one finally arrives at the field equations of the new classes of Teleparallel theories of gravity, namely

$$2ef_{T_{\rm vec}}\Big(v^c T^\beta{}_{ac} + v^\beta v_a\Big) + 2\partial_\lambda\Big(ef_{T_{\rm vec}}(v^\beta E^\lambda_a - v^\lambda E^\beta_a)\Big) + \frac{2}{3}\epsilon_{ib}{}^{cd} ef_{T_{\rm ax}} a^i E^\beta_c T^b{}_{ad}$$
$$+\frac{2}{3}\partial_\nu\left(e\epsilon_{ia}{}^{cd} f_{T_{\rm ax}} a^i E^\nu_c E^\beta_d\right) + ef_{T_{\rm ten}}\Big(2T^b{}_{a\sigma}T_b{}^{\beta\sigma} + T^\beta{}_{\rho\sigma}T^\rho{}_a{}^\sigma + T^\alpha{}_{\rho a}T^\rho{}_\alpha{}^\beta - T^\beta{}_{ai}v^i - v^\beta v_a\Big)$$
$$+\partial_\nu\Big[ef_{T_{\rm ten}}\left(T^{\beta\nu}{}_a - T^{\nu\beta}{}_a - 2T_a{}^{\beta\nu} - v^\beta E^\nu_a + v^\nu E^\beta_a\right)\Big] = 2\kappa^2 e\mathcal{T}^\beta_a\,. \quad (6.27)$$

6.3.4 Inclusion of parity violating terms and higher-order invariants

Action (6.13) is sufficiently general to include all previously known models of modified Teleparallel models with second order field equations that do not introduce additional fields. For sake of completeness of our approach, let us discuss further viable generalizations to obtain models with second order field equations that are possible in this Teleparallel framework.

One can recall here that three invariants (6.5)–(6.7) are the most general, quadratic, parity preserving, irreducible torsion invariants (Hayashi & Shirafuji, 1979). If one relaxes the requirement of parity preservation, we have two new quadratic parity violating invariants which are (Hayashi & Shirafuji, 1979)

$$P_1 = v^\mu a_\mu\,, \qquad \text{and} \qquad P_2 = \epsilon_{\mu\nu\rho\sigma}t^{\lambda\mu\nu}t_\lambda{}^{\rho\sigma}\,. \quad (6.28)$$

One can then naturally consider a straightforward generalization of the gravity Lagrangian in the following way

$$\mathcal{L} = \frac{e}{2\kappa^2} f(T_{\mathrm{ax}}, T_{\mathrm{ten}}, T_{\mathrm{vec}}, P_1, P_2)\,, \tag{6.29}$$

and derive the corresponding field equations.

The Lagrangian (6.29) is the most general Lagrangian taken as a function of all invariant quadratics in torsion. However, since one considers the Lagrangian to be an arbitrary non-linear function, one can also consider higher order invariants obtainable in this framework. For an illustration, let us consider the two invariant quartic torsion terms

$$S_1 = t^{\lambda\mu\nu} v_\lambda a_\mu v_\nu\,, \qquad S_2 = t^{\lambda\mu\nu} a_\lambda v_\mu a_\nu\,. \tag{6.30}$$

It is obvious that one can construct a large number of such higher-order invariants. It should be remarked that S_1 is a pseudo-scalar while S_2 is a true scalar under spatial inversions. In principle, one can include all of them in the Lagrangian and the resulting field equations will be still of the second order. The derivation of the corresponding field equations is rather straightforward using previous results, but becomes increasingly involved with an increasing number of allowed invariants in the Lagrangian. Therefore, one should exercise caution and consider only well-motivated terms in the Lagrangian. This is the reason why we primarily focus on Lagrangian (6.13), which can be considered to be general enough to include all previous models, allowing the analysis of some of their generic properties, and still have rather manageable field equations.

6.3.5 Inclusion of the boundary term and derivatives of torsion

Another possible extension of the model (6.13) is to include the derivatives of torsion. This results in theories with higher-order field equations, which start to be increasingly complicated when adding further terms. Therefore, one should again exercise caution and consider only those derivative terms that are well-motivated.

One of such well-motivated terms is the so-called boundary term (see Chap. 5), that can be rewritten in terms of the vectorial part of torsion as follows

$$B = \frac{2}{e}\partial_\mu(ev^\mu)\,. \tag{6.31}$$

The boundary term is key in understanding the differences of $f(T)$ and $f(\overset{\circ}{R})$ gravity. For instance, one of the features of $f(T)$ gravity is that the field equations are of second order while the $f(\overset{\circ}{R})$ gravity field equations are of 4th order. It is precisely the boundary term B which contains second derivatives of the tetrads which, after using integration by parts twice, gives the 4th order parts of the field equations seen in $f(\overset{\circ}{R})$ gravity.

One can then include the boundary term and consider the Lagrangian

$$\mathcal{L} = \frac{e}{2\kappa^2}\,f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B)\,. \tag{6.32}$$

As will be seen in the following section, this Lagrangian naturally appears in the analysis of conformal transformations of the model. The corresponding field equations will be given by (6.27) with an addition of terms corresponding to the variation of the boundary term. The new contribution would be the same as adding the terms coming from the variations computed in Eq. (5.14).

6.4 Conformal transformations

6.4.1 Basic equations

It is interesting to study this theory under conformal transformations and the resulting issues of coupling in the Jordan and Einstein frames. The first paper which dealt with conformal transformations in modified Teleparallel theories was Yang (2011). In that paper, the author showed that it is not possible to have an equivalent Einstein frame in $f(T)$ gravity. Thus, for example, it is not possible to constraint $f(T)$ gravity using post-Newtonian parameters from a scalar field equivalent theory. Therefore, it would be interesting to analyse if this characteristic is also valid for new general classes of Teleparallel theories. Let us now consider the conformal transformation properties of the theory given by the action (6.13). Let us introduce the label (index) $A = 1, \ldots, 4$, and then introduce two sets of four auxiliary fields ϕ_A and χ_A. This allows us to rewrite the action as

$$\begin{aligned}\mathcal{S} = \frac{1}{2\kappa^2}\int \Big[f(\phi_A) + \chi_1(T_{\rm ax} - \phi_1) + \chi_2(T_{\rm ten} - \phi_2) + \chi_3(T_{\rm vec} - \phi_3) \\ + \chi_4(B - \phi_4)\Big]\, e\, d^4x\,. \quad (6.33)\end{aligned}$$

Variations with respect to χ_A yield the four equations

$$\phi_1 = T_{\rm ax}\,, \quad \phi_2 = T_{\rm ten}\,, \quad \phi_3 = T_{\rm vec}\,, \quad \phi_4 = B\,. \tag{6.34}$$

Additionally, varying with respect to ϕ_A one arrives at

$$\chi_A = \frac{\partial f(\phi_B)}{\partial \phi_A} := F_A\,. \tag{6.35}$$

Therefore, action (6.33) can be rewritten as

$$\mathcal{S} = \frac{1}{2\kappa^2} \int \Big[\sum_{B=1}^{4} F_B(\phi_A)\phi_B - V(\phi_A) \Big] \, e \, d^4x \,, \tag{6.36}$$

where the energy potential

$$V(\phi_A) = \sum_{B=1}^{4} \phi_B F_B(\phi_A) - f(\phi_A) \tag{6.37}$$

was defined. Next, let us apply a conformal transformation to the metric

$$\widehat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \,, \quad \widehat{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu} \,, \tag{6.38}$$

where Ω is the conformal factor. When conformal transformations are applied at the level of the tetrad, one obtains

$$\widehat{e}^a_\mu = \Omega(x) e^a_\mu \,, \quad \widehat{E}^\mu_a = \Omega^{-1}(x) E^\mu_a \,, \quad \widehat{e} = \Omega^4 e \,. \tag{6.39}$$

Using these transformations one finds that the torsion tensor transforms as

$$\widehat{T}^\rho{}_{\mu\nu} = T^\rho{}_{\mu\nu} + \Omega^{-1}(\delta^\rho_\nu \partial_\mu \Omega - \delta^\rho_\mu \partial_\nu \Omega) \,. \tag{6.40}$$

Hence it is possible to verify that

$$T_{\rm ax} = \Omega^2 \widehat{T}_{\rm ax} \,, \tag{6.41}$$

$$T_{\rm ten} = \Omega^2 \widehat{T}_{\rm ten} \,, \tag{6.42}$$

$$T_{\rm vec} = \Omega^2 \widehat{T}_{\rm vec} + 6\Omega \widehat{v}^\mu \widehat{\partial}_\mu \Omega + 9 \widehat{g}^{\mu\nu} (\widehat{\partial}_\mu \Omega)(\widehat{\partial}_\nu \Omega) \,, \tag{6.43}$$

$$B = \Omega^2 \widehat{B} - 4\Omega \widehat{v}^\mu \widehat{\partial}_\mu \Omega - 18 \widehat{\partial}^\mu \Omega \widehat{\partial}_\mu \Omega + \frac{6}{\widehat{e}} \Omega \widehat{\partial}_\mu (\widehat{e} \widehat{g}^{\mu\nu} \widehat{\partial}_\nu \Omega) \,. \tag{6.44}$$

This shows that the irreducible torsion pieces $T_{\rm ax}$ and $T_{\rm ten}$ transform very simply, they are multiplied by the conformal factor Ω^2.

6.4.2 Minimal and non-minimal couplings

Using the above relationships, action (6.36) takes the following form

$$\mathcal{S} = \frac{1}{2\kappa^2}\int\Big[F_1(\phi_A)\Omega^{-2}\widehat{T}_{\rm ax} + F_2(\phi_A)\Omega^{-2}\widehat{T}_{\rm ten} + F_3(\phi_A)\Big(\Omega^{-2}\widehat{T}_{\rm vec} + 6\Omega^{-3}\widehat{v}^{\mu}\widehat{\partial}_{\mu}\Omega$$
$$+ 9\Omega^{-4}\widehat{g}^{\mu\nu}(\widehat{\partial}_{\mu}\Omega)(\widehat{\partial}_{\nu}\Omega)\Big) + F_4(\phi_A)\Big(\Omega^{-2}\widehat{B} - 4\Omega^{-3}\widehat{v}^{\mu}\widehat{\partial}_{\mu}\Omega - 18\Omega^{-4}\widehat{\partial}^{\mu}\Omega\widehat{\partial}_{\mu}\Omega$$
$$+ \frac{6}{\widehat{e}}\Omega^{-3}\widehat{\partial}_{\mu}(\widehat{e}\widehat{g}^{\mu\nu}\widehat{\partial}_{\nu}\Omega)\Big) - \Omega^{-4}V(\phi_A)\Big]\,\widehat{e}\,d^4x\,. \tag{6.45}$$

From here one can see that if $F_4(\phi_A) = 0$, or in other words, if the function does not depend on the boundary term B, it is not possible to eliminate all the terms related to $\widehat{T}^{\mu}$ in order to obtain a non-minimally coupled theory with T_i or a theory minimally coupled to the torsion scalar (an Einstein frame). Integrating by parts the two terms $\widehat{B}$ and the term $(6\Omega/\widehat{e})\widehat{\partial}_{\mu}(\widehat{e}\,\widehat{g}^{\mu\nu}\widehat{\partial}_{\nu}\Omega)$, one can rewrite the above action as follows

$$\mathcal{S} = \frac{1}{2\kappa^2}\int\Big[F_1(\phi_A)\Omega^{-2}\widehat{T}_{\rm ax} + F_2(\phi_A)\Omega^{-2}\widehat{T}_{\rm ten} + F_3(\phi_A)\Omega^{-2}\widehat{T}_{\rm vec}$$
$$+ 2\Omega^{-2}\widehat{v}^{\mu}\Big(3F_3(\phi_A)\Omega^{-1}\widehat{\partial}_{\mu}\Omega - \partial_{\mu}F_4(\phi_A)\Big) + 9F_3(\phi_A)\Omega^{-4}\widehat{g}^{\mu\nu}(\widehat{\partial}_{\mu}\Omega)(\widehat{\partial}_{\nu}\Omega)$$
$$- 6\Omega^{-3}(\partial^{\mu}\Omega)\partial_{\mu}F_4(\phi_A) - \Omega^{-4}V(\phi_A)\Big]\,\widehat{e}\,d^4x\,. \tag{6.46}$$

Now, let us study the case where one eliminates all the couplings between the scalar field and $\widehat{T}^{\mu}$ (or equivalently $\widehat{B}$). To do that, one must impose the following constraint

$$3F_3(\phi_A)\Omega^{-1}\partial_{\mu}\Omega - \partial_{\mu}F_4(\phi_A) = 0\,. \tag{6.47}$$

It should be remarked that $\partial_\mu = \widehat{\partial}_\mu$ was used in the above equation. By taking derivatives ∂_ν to this equation and then by substituting back into (6.47), one can find the following condition

$$\partial_\nu F_3(\phi_A)\partial_\mu F_4(\phi_A) = \partial_\mu F_3(\phi_A)\partial_\nu F_4(\phi_A)\,, \tag{6.48}$$

which expressed in terms of the initial function $f(\phi_A)$ reads

$$\frac{\partial^2 f(\phi_A)}{\partial\phi_3\partial\phi_C}\frac{\partial^2 f(\phi_A)}{\partial\phi_4\partial\phi_B} = \frac{\partial^2 f(\phi_A)}{\partial\phi_3\partial\phi_B}\frac{\partial^2 f(\phi_A)}{\partial\phi_4\partial\phi_C}\,. \tag{6.49}$$

Here, the chain rule was used to evaluate

$$\partial_\nu F_B(\phi_A) = \partial_\nu\phi_C\frac{\partial^2 f(\phi_A)}{\partial\phi_B\partial\phi_C}\,. \tag{6.50}$$

In general, Eq. (6.49) is a system of sixteen differential equations. However, this reduces to six because the involved second order partial derivatives commute. However, these six equations are not all linearly independent. One can show that, in fact, only three of them are linearly independent, namely

$$f^{(0,0,1,1)}(\phi_A)^2 = f^{(0,0,0,2)}f^{(0,0,2,0)}(\phi_A)\,, \tag{6.51}$$

$$f^{(0,1,1,0)}(\phi_A)f^{(0,0,0,2)}(\phi_A) = f^{(0,0,1,1)}(\phi_A)f^{(0,1,0,1)}(\phi_A)\,, \tag{6.52}$$

$$f^{(1,0,1,0)}(\phi_A)f^{(0,1,0,1)}(\phi_A) = f^{(0,1,1,0)}(\phi_A)f^{(1,0,0,1)}(\phi_A)\,. \tag{6.53}$$

It should be recalled here that $\phi_A = \{T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B\}$ and superscripts denote differentiation (e.g. $f^{(0,0,1,1)} = \partial^2 f/\partial(T_{\rm vec}B)$ or $f^{(0,1,0,1)} = \partial^2 f/\partial(T_{\rm ten}B)$ or $f^{(0,0,0,2)} = \partial^2 f/\partial B^2$). One can directly see that for the special case where $f = f(\frac{3}{2}T_{\rm ax} + \frac{2}{3}T_{\rm ten} - \frac{2}{3}T_{\rm vec}, B) = f(T, B)$, the second and third equations are automatically satisfied and only (6.51) is needed to eliminate all the couplings between the scalar field and $\widehat{T}^\mu$.

This result is consistent with Eq. (91) reported in Wright (2016) where the $f(T, B)$ case was studied.

If one is interested in finding a theory where the scalar field is minimally coupled to the torsion scalar (in the Einstein frame) one must impose

$$\Omega^2 = -\frac{2}{3}F_1(\phi_A) = -\frac{3}{2}F_2(\phi_A) = \frac{3}{2}F_3(\phi_A)\,. \tag{6.54}$$

Additionally, the conditions (6.51)–(6.53) must also hold to eliminate the couplings with $\widehat{T}^\mu$. By solving these equations, we directly find that the Einstein frame is recovered if

$$\begin{aligned} f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B) &= f\Big(-\frac{3}{2}(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B)\Big) = f\Big(-\frac{3}{2}(-T + B)\Big) \\ &= f\Big(-\frac{3}{2}\overset{\circ}{R}\Big) = \widetilde{f}(\overset{\circ}{R})\,, \end{aligned} \tag{6.55}$$

which is $\widetilde{f}(\overset{\circ}{R})$ gravity. As expected, the unique theory with an Einstein frame is either the Teleparallel equivalent of General Relativity or $f(\overset{\circ}{R})$ gravity. From these computations, one can understand better why modified Teleparallel theories of gravity do not have an Einstein frame formulation. It can be noticed that $T_{\rm ax}$ and $T_{\rm ten}$ transform in a simple way under conformal transformations and the problematic term which creates this issue comes from the term $T_{\rm vec}$. This is not possible to see directly if one starts with $f(T)$ gravity. Furthermore, the boundary term B is a derivative of the vectorial part (not the other pieces), so that only theories which contain B might remove the problematic terms coming from the conformal transformations in $T_{\rm vec}$. In principle, one could have speculated that it is possible to remove those new problematic pieces with other kind of theories (not just $f(\overset{\circ}{R})$ gravity), but as was shown here, this is not possible for other theories different than $f(-T + B) = f(\overset{\circ}{R})$ gravity or TEGR gravity. Let us finish this section mentioning that other transformations (such as disformal transformations) have not been studied in full detail in modified Teleparallel theories of gravity. It could be an interesting approach to find if other types of transformations could give rise to an Einstein frame.

6.5 Connection with other theories

In this chapter, a new modified Teleparallel theory of gravity was studied which generalises and includes all of the most important and well-motivated second order field theories that can be constructed from torsion. In this theory, instead of considering a linear combination of the irreducible parts of torsion, a function of them $f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec})$ is proposed in the action. Additionally, the possibility for the function to depend on the boundary term B was also studied, allowing the theory to have an $f(\overset{\circ}{R})$ gravity limiting case. Now, one can recover some well-known theories and this section is devoted to classifying them. Starting with this general theory, Fig. 6.1 shows a classification of the various theories which can be constructed and their relationships. The most relevant theories in the present discussion are highlighted in boxes. Let us begin with $f(T_i, B)$ gravity. This theory is an arbitrary function of the three torsion pieces and the boundary term but could be generalised further, as discussed in Section 6.3.4. It is a large class of theories which contains many of the most studied modified gravity (metric and Teleparallel) models as special cases. As one can see, new General Relativity (NGR), conformal Teleparallel gravity (CTG), $f(T)$ and $f(\overset{\circ}{R})$ gravity and other well-known theories are part of this approach. Two models which have not been studied so far and might be interesting from a theoretical point of view are $f(\widehat{T})$, which corresponds to a modified conformal Teleparallel theory of gravity and $f(T_{\rm NGR})$ which has a clear connection to standard General Relativity.

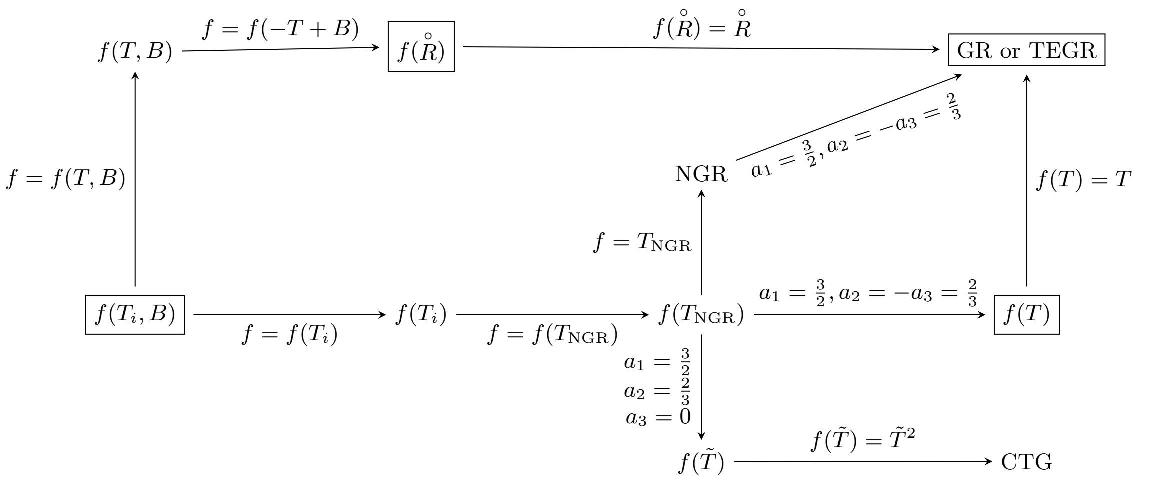


Figure 6.1: Relationship between $f(T_i, B)$ and other gravity theories. In this diagram T is the scalar torsion, $T_i = (T_{\rm ax}, T_{\rm ten}, T_{\rm vec})$, $T_{\rm NGR} = a_1 T_{\rm ax} + a_2 T_{\rm ten} + a_3 T_{\rm vec}$ represents the scalar coming from the new General Relativity theory and $\widetilde{T} = \frac{3}{2} T_{\rm ax} + \frac{2}{3} T_{\rm ten}$ is the scalar coming from the conformal Teleparallel theory. The abbreviations NGR, CTG and TEGR mean new General Relativity, Teleparallel conformal gravity and Teleparallel equivalent of General Relativity, respectively.

7

$f(T, B, T_G, B_G, \mathcal{T})$ gravity

Chapter Abstract

A possible Teleparallel Gauss-Bonnet contribution is being presented with the aim of formulating an extended Teleparallel modified theory of gravity. The possible coupling of gravity with the trace of the energy-momentum tensor is also taken into account. This is motivated by the various different theories formulated in the Teleparallel approach and the metric approach without discussing the exact relationship between them. The connection between different theories is clarified with this formulation. The Teleparallel equivalent of modified Gauss-Bonnet gravity can be recovered from this theory.

7.1 Gauss-Bonnet term

The principal aim of this chapter is to review the work by Bahamonde & Böhmer (2016), where there was presented an extended modified Teleparallel theory of gravity by taking into account the Gauss-Bonnet term and its Teleparallel equivalent. The Gauss-Bonnet scalar is one of the so-called Lovelock scalars Lovelock (1971) which only yields second order field equations in the metric. In more than four dimensions, the study of the Gauss-Bonnet term is quite natural. In four dimensions, on the other hand, the Gauss-Bonnet term can be written as a total derivative and its integral over the manifold is related to the topological Euler number.

The Teleparallel equivalent of the Gauss-Bonnet term was first considered in Kofinas et al. (2014); Kofinas & Saridakis (2014) who studied a theory based on the function $f(T,T_G)$ where T_G is a Teleparallel Gauss-Bonnet term. As is somewhat expected, T_G differs from its Teleparallel equivalent by a divergence term. Hence, as in modified General Relativity, it is possible to formulate modified theories based on the Gauss-Bonnet terms or its Teleparallel equivalent in such a way that both theories are physically distinct. The link between these theories comes from the divergence term which needs to be taken into account when establishing the relationship between the different possible theories.

Teleparallel geometries have been well-understood for many decades. Perhaps more surprising is the fact that the Gauss-Bonnet term was not studied in this context until quite recently (Kofinas & Saridakis, 2014). The Gauss-Bonnet term is a quadratic combination of the Riemann tensor and its contractions given by

$$\overset{\circ}{G} = \overset{\circ}{R}{}^2 - 4\overset{\circ}{R}_{\mu\nu}\overset{\circ}{R}{}^{\mu\nu} + \overset{\circ}{R}_{\mu\nu\kappa\lambda}\overset{\circ}{R}{}^{\mu\nu\kappa\lambda}\,, \tag{7.1}$$

which plays an important role of connecting geometry to topology. It is well known that the addition of the Gauss-Bonnet to the Einstein-Hilbert action does not affect the field equations of General Relativity, provided one works in a four dimensional setting. This fact implies that the topology of the solutions is unconstrained. In more than four dimensions, the addition of the Gauss-Bonnet term affects the resulting gravitational field equations.

Following the procedure outlined in Sec. 3.4, one can again compute the (complete) Gauss-Bonnet term using the connection (2.53) and decompose this result into a Levi-Civita part and an additional part depending on torsion only. Understandably, this process is quite involved. It can be shown, see Kofinas & Saridakis (2014); Gonzalez & Vasquez (2015), that the Gauss-Bonnet term can be expressed in a fashion similar form to (3.36) which simply reads

$$\overset{\circ}{G} = -T_G + B_G\,. \tag{7.2}$$

The Teleparallel Gauss-Bonnet term T_G is given by

$$T_G = (K_a{}^i{}_e K_b{}^{ej} K_c{}^k{}_f K_d{}^{fl} - 2K_a{}^{ij} K_b{}^k{}_e K_c{}^e{}_f K_d{}^{fl} + 2K_a{}^{ij} K_b{}^k{}_e K_f{}^{el} K_d{}^f{}_c + \\ 2K_a{}^{ij} K_b{}^k{}_e K_{c,d}{}^{el})\delta^{abcd}_{ijkl}\,, \tag{7.3}$$

where commas denote differentiation so that $K_{c,d}{}^{el} = \partial_d K_c{}^{el}$, and δ^{abcd}_{ijkl} is the generalised Kronecker delta which in four dimensions is equivalent to $\delta^{abcd}_{ijkl} = \varepsilon^{abcd}\varepsilon_{ijkl}$. This term depends on the contortion tensor and its first partial derivatives, and it is quartic in contortion. This is expected as curvature in general is quadratic in contortion, and the Gauss-Bonnet term is itself quadratic in curvature.

On the other hand, the Gauss-Bonnet boundary term B_G reads

$$B_G = \frac{1}{e}\delta^{abcd}_{ijkl}\partial_a\left[\frac{1}{2}K_b{}^{ij}\overset{\circ}{R}{}^{kl}{}_{cd} + K_b{}^{ij}K_c{}^k{}_f K_d{}^{fl}\right]. \tag{7.4}$$

The above formula was obtained by converting the calculus of forms provided in Kofinas & Saridakis (2014) to a tensorial form in four dimensions. The above expression was first presented in this form in Bahamonde & Böhmer (2016). Equivalently, by using (2.55) with $R^\lambda{}_{\alpha\mu\nu} = \overset{\bullet}{R}{}^\lambda{}_{\alpha\mu\nu} = 0$, one can relate $\overset{\circ}{R}{}^\lambda{}_{\alpha\mu\nu}$ with $K^\alpha{}_{\mu\nu}$ and then the term B_G can be rewritten depending only on the contortion tensor as follows

$$B_G = \frac{1}{e}\delta^{abcd}_{ijkl}\partial_a\left[K_b{}^{ij}\left(K_c{}^{kl}{}_{,d} + K_d{}^m{}_c K_m{}^{kl}\right)\right]. \tag{7.5}$$

It should be noted that it was also used the fact that spacetime indices can be transformed into Latin ones using $\overset{\circ}{R}{}^i{}_{jkl} = E^\alpha_j E^\mu_k E^\nu_l e^i_\lambda \overset{\circ}{R}{}^\lambda{}_{\alpha\mu\nu}$. When discussing the Teleparallel equivalent of General Relativity in the pure tetrad formalism, one touched upon the issue of Lorentz covariance. As before, in the pure tetrad formalism, it is clear that for instance T_G cannot be a Lorentz covariant scalar. To see this, one notes that the final term in the definition (7.3) contains a partial derivative. Therefore this term will contribute second partial derivatives of the local Lorentz transformations which cannot be cancelled by any other term in T_G. Since the Gauss-Bonnet term $\overset{\circ}{G}$ is a Lorentz covariant scalar, these second derivative terms must be cancelled by terms coming from B_G. Consequently the combination $-T_G + B_G$ is the unique Lorentz covariant combination which can be constructed. This fact becomes important when considering modified theories of gravity based on the Teleparallel equivalent of the Gauss-Bonnet scalar.

In the following some simple examples of the Gauss-Bonnet term and its Teleparallel equivalent will be presented.

7.1.1 Example: FLRW spacetime with diagonal tetrad

Let us begin with the FLRW metric

$$ds^2 = dt^2 - a(t)^2 \frac{(dx^2 + dy^2 + dz^2)}{(1 + \frac{k}{4}r^2)^2} \tag{7.6}$$

and the diagonal tetrad given by

$$\begin{aligned} e^a_\mu \quad = \quad & \mathrm{diag}\Big(1, \frac{a(t)}{1 + (k/4)(x^2 + y^2 + z^2)}, \frac{a(t)}{1 + (k/4)(x^2 + y^2 + z^2)}, \\ & \frac{a(t)}{1 + (k/4)(x^2 + y^2 + z^2)}\Big), \end{aligned} \tag{7.7}$$

using spatial Cartesian coordinates. It is straightforward to verify that

$$\overset{\circ}{G} = -24\frac{k}{a^2}\frac{\ddot{a}}{a} - 24\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2}\,, \tag{7.8}$$

$$T_G = 24\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2} - 2k\frac{k}{a^2}\frac{\ddot{a}}{a}(x^2+y^2+z^2)\,, \tag{7.9}$$

$$B_G = -24\frac{k}{a^2}\frac{\ddot{a}}{a} - 2k\frac{k}{a^2}\frac{\ddot{a}}{a}(x^2+y^2+z^2)\,. \tag{7.10}$$

These three quantities display some of the key properties important in this context. Firstly, one notes that T_G and B_G depend on the Euclidean distance from the origin while the Gauss-Bonnet term $\overset{\circ}{G}$ is independent of the Cartesian coordinates. The unique linear combination $-T_G + B_G$ is independent of position. Secondly, in case of a spatially flat universe, these terms are absent and the term B_G identically vanishes. The terms depending on the spatial coordinates can be changed by working with a different tetrad, or in other words, these terms are affected by local Lorentz transformations. In the context of extended or modified Teleparallel theories of gravity, tetrads (7.7) should be avoided. The construction of a suitable static and spherically symmetric tetrad in $f(T)$ gravity, for instance, is rather involved, see Ferraro & Fiorini (2011b). In general the choice of a suitable parallelisation is a subtle and non-trivial issue (Ferraro & Fiorini, 2015). Finding a tetrad for which T_G and B_G are both independent of the spatial coordinates is a rather involved task, however, following the approach in Secs. 4.3.2 and 4.3.3, it will be shown that a tetrad with this property can be constructed. Before doing so, we discuss another example with different symmetry properties.

7.1.2 Example: static spherically symmetric spacetime – isotropic coordinates

In this example we consider static and spherically symmetric spacetimes and work with isotropic coordinates (t, x, y, z) to avoid coordinate issues with the tetrads. Let us choose

$$e^a_\mu = \mathrm{diag}\Big(A(r), B(r), B(r), B(r)\Big)\,, \tag{7.11}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean distance from the origin.

The metric then takes the isotropic form

$$ds^2 = A(r)^2 dt^2 - B(r)^2(dx^2 + dy^2 + dz^2)\,. \tag{7.12}$$

The first three quantities of interest $\overset{\circ}{R}$, T and B are given by

$$\overset{\circ}{R} = -\frac{1}{B^2}\Big(\frac{4}{r}\frac{A'}{A} + \frac{8}{r}\frac{B'}{B} + 2\frac{A'}{A}\frac{B'}{B} - 2\frac{B'^2}{B^2} + 2\frac{A''}{A} + 4\frac{B''}{B}\Big)\,, \tag{7.13}$$

$$T = -\frac{1}{B^2}\Big(4\frac{A'}{A}\frac{B'}{B} + 2\frac{B'^2}{B^2}\Big)\,, \tag{7.14}$$

$$B = -\frac{1}{B^2}\Big(\frac{4}{r}\frac{A'}{A} + \frac{8}{r}\frac{B'}{B} + 6\frac{A'}{A}\frac{B'}{B} + 2\frac{A''}{A} + 4\frac{B''}{B}\Big)\,. \tag{7.15}$$

Here, primes denote derivation with respect to the radial coordinate r and dots with respect to time t. A direct calculation verifies that indeed $\overset{\circ}{R} = -T + B$ as expected. Next, the explicit forms of $\overset{\circ}{G}$, T_G and B_G are given by

$$\begin{aligned}\overset{\circ}{G} \;&=\; -\frac{8}{B^4}\Big(\frac{2}{r^2}\frac{A'}{A}\frac{B'}{B} - \frac{2}{r}\frac{A'}{A}\frac{B'^2}{B^2} - 3\frac{A'}{A}\frac{B'^3}{B^3} + \frac{2}{r}\frac{A''}{A}\frac{B'}{B} + \frac{A''}{A}\frac{B'^2}{B^2} \\ &\quad + \frac{2}{r}\frac{A'}{A}\frac{B''}{B} + 2\frac{A'}{A}\frac{B'B''}{B^2}\Big)\,,\end{aligned} \tag{7.16}$$

$$T_G \;=\; \frac{8}{B^4}\Big(\frac{2}{r}\frac{A'}{A}\frac{B'^2}{B^2} - 3\frac{A'}{A}\frac{B'^3}{B^3} + \frac{A''}{A}\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'B''}{B^2}\Big)\,, \tag{7.17}$$

$$B_G \;=\; -\frac{8}{B^4}\Big(\frac{2}{r^2}\frac{A'}{A}\frac{B'}{B} - \frac{4}{r}\frac{A'}{A}\frac{B'^2}{B^2} + \frac{2}{r}\frac{A''}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A}\frac{B''}{B}\Big)\,, \tag{7.18}$$

which indeed satisfies the required identity $\overset{\circ}{G} = -T_G + B_G$. It should be observed that the expressions are considerably more complicated than in the previous case. The tetrads used in (7.7) and (7.11) serve as simple examples which are useful to compute the relevant quantities.

7.1.3 Example: FLRW spacetime – good tetrad

Let us next consider FLRW metric in spherical coordinates given by

$$ds^2 = dt^2 - a(t)^2\left[\frac{1}{1-kr^2}dr^2 + d\Omega^2\right], \tag{7.19}$$

where $a(t)$ is the scale factor of the universe and $k = \{0, \pm 1\}$ is the spatial curvature which corresponds to flat, close and open cosmologies, respectively. This section will generalise the calculations shown in Sec. 4.3.3 where only the flat case was considered.

The simplest tetrad field which yields the above metric is the diagonal one given by

$$e^a_\mu = \mathrm{diag}\Big(1, \frac{a(t)}{\sqrt{1-kr^2}}, a(t)r, a(t)r\sin\theta\Big). \tag{7.20}$$

However, when this tetrad is used in $f(T)$ gravity it implies an off-diagonal field equation which is highly restrictive, namely the condition $f_{TT} = 0$. Such a theory is equivalent to General Relativity and hence not a modification. In order to avoid this issue, one can follow the procedure outlined in Ferraro & Fiorini (2011a); Tamanini & Boehmer (2012) (described also in Sec. 4.3.2) which allows for the construction of tetrads which result in more favourable field equations. As it was done in Sec. 4.3.2, let us perform a 3-dimensional rotation which gives us a new rotated tetrad (7.20) given by

$$\bar{e}^a_\mu = \Lambda^a_b e^b_\mu\,. \tag{7.21}$$

Then, the rotated tetrad for FLRW in spherical coordinates for any spatial curvature takes the following form

$$e^a_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a\cos\phi\sin\theta}{\sqrt{1-kr^2}} & -ra(\cos\gamma\sin\phi+\cos\theta\cos\phi\sin\gamma) & ra\sin\theta(\sin\phi\sin\gamma-\cos\theta\cos\phi\cos\gamma) \\ 0 & \frac{a\sin\theta\sin\phi}{\sqrt{1-kr^2}} & ra(\cos\phi\cos\gamma-\cos\theta\sin\phi\sin\gamma) & -ra\sin\theta(\cos\theta\cos\gamma\sin\phi+\cos\phi\sin\gamma) \\ 0 & \frac{a\cos\theta}{\sqrt{1-kr^2}} & ra\sin\theta\sin\gamma & ra\cos\gamma\sin^2\theta \end{pmatrix}. \tag{7.22}$$

Next, let us focus on the non-flat case $k \neq 0$, since the Gauss-Bonnet boundary term $B_G = 0$ and hence directly $\overset{\circ}{G} = -T_G$ when $k = 0$. By using the rotated tetrad (7.22), the torsion scalar T and the boundary term B becomes

$$T = \frac{4}{a^2}\Big(\frac{\sqrt{1-kr^2}}{r^2}\Big[r\gamma'\cos\gamma+\sin\gamma\Big]+\frac{1}{r^2}\Big)-6\frac{\dot{a}^2}{a^2}-2\frac{k}{a^2}\,, \tag{7.23}$$

$$B = \frac{4}{a^2}\Big(\frac{\sqrt{1-kr^2}}{r^2}\Big[r\gamma'\cos\gamma+\sin\gamma\Big]+\frac{1}{r^2}\Big)-6\frac{\ddot{a}}{a}-12\frac{\dot{a}^2}{a^2}-8\frac{k}{a^2}\,. \tag{7.24}$$

In order to have T and B position independent, one must choose our function γ to satisfy

$$\sqrt{1-kr^2}\Big[r\gamma'\cos\gamma+\sin\gamma\Big]+1 = 0\,. \tag{7.25}$$

For an open universe $k = -1$ gives us the following function

$$\gamma(r) = -\arcsin\left[\mathrm{arcsinh}(r)/r\right]\,, \tag{7.26}$$

where the constant of integration was set to zero. Using this choice of γ ensures that the first terms in (7.23) and (7.24) disappear thereby making T and B time dependent only. Therefore, the rotated tetrad (7.21) with $k = -1$ and the function γ given by (7.26) is a "good tetrad" in the sense of Tamanini & Boehmer (2012).

Independently of the choice of tetrad, one always obtains the usual Ricci scalar

$$\overset{\circ}{R} = -T + B = 6 - \frac{\ddot{a}}{a} - 6\frac{\dot{a}^2}{a^2} - 6\frac{k}{a^2}\,. \tag{7.27}$$

Moreover, by using this rotated tetrad one finds that the Gauss-Bonnet terms are also independent of r, one can verify that

$$T_G = 8\frac{\ddot{a}}{a}\Big(3H^2 - \frac{1}{a^2}\Big)\,, \tag{7.28}$$

$$B_G = 16\frac{\ddot{a}}{a^3}\,, \tag{7.29}$$

and hence the Gauss-Bonnet term in an open universe becomes

$$\overset{\circ}{G} = -T_G + B_G = -24\frac{\ddot{a}}{a}\Big(H^2 - \frac{1}{a^2}\Big)\,. \tag{7.30}$$

On the other hand, for the closed FLRW universe ($k = +1$), one finds that the function γ has to be of the form

$$\gamma(r) = -\operatorname{arcsinh}\left(\sqrt{1+r^2}\right)\,. \tag{7.31}$$

This yields

$$T_G = 24\frac{\ddot{a}}{a}\Big(H^2 - \frac{1}{a^2}\Big)\,, \tag{7.32}$$

$$B_G = -48\frac{\ddot{a}}{a^3}\,, \tag{7.33}$$

with the Gauss-Bonnet term given by

$$\overset{\circ}{G} = -T_G + B_G = -24\frac{\ddot{a}}{a}\Big(H^2 + \frac{1}{a^2}\Big)\,. \tag{7.34}$$

7.2 Modified Teleparallel theory: Gauss-Bonnet and trace extension

We are now ready to discuss the general framework of modified theories of gravity and their Teleparallel counterparts. In principle, the following approach could be applied to any metric theory of gravity whose action is based on objects derived from the Riemann curvature tensor. Any such theory can in principle be re-written using the torsion tensor thereby allowing for a Teleparallel representation of that same theory.

7.2.1 Action and variations

Let us now consider the framework which includes the Teleparallel Gauss-Bonnet and the classical Gauss-Bonnet modified theories of gravity. Inspired by the above discussion, one can define the action

$$\mathcal{S}_{f(T,B,T_G,B_G)} = \int \left[\frac{1}{2\kappa^2} f(T, B, T_G, B_G) + \mathcal{L}_{\rm m}\right] e\, d^4x\,, \tag{7.35}$$

where f is a smooth function of the scalar torsion T, the boundary term B, the Gauss-Bonnet scalar torsion T_G and the boundary Gauss-Bonnet term B_G.

Variations of the action (7.35) with respect to the tetrad gives

$$\begin{aligned}\delta\mathcal{S}_{f(T,B,T_G,B_G)} \;\; = \;\; & \int \Big[\frac{1}{2\kappa^2}\Big(f\delta e + ef_B\delta B + ef_T\delta T + ef_{T_G}\delta T_G + ef_{B_G}\delta B_G\Big) \\ & +\delta(e\mathcal{L}_{\rm m})\Big]\, d^4x\,. \end{aligned} \tag{7.36}$$

The first three terms above, $f\delta e$, $ef_B\delta B$ and $ef_T\delta T$, are the same terms derived before in Eqs. (5.3), (5.4) and (5.14) respectively. In the following sections, the remaining two variations which depend on T_G and B_G will be presented.

7.2.1.1 Variation of B_G

The Gauss-Bonnet boundary term is given by

$$B_G = \frac{1}{e}\epsilon_{ijkl}\epsilon^{bcda}\partial_a\Big[\frac{1}{2}K_b{}^{ij}\overset{\circ}{R}{}^{kl}{}_{cd} + K_b{}^{ij}K_c{}^{k}{}_{f}K_d{}^{fl}\Big]\,, \tag{7.37}$$

or equivalently

$$B_G = \frac{1}{e}\partial_\mu(eE_a^\mu B_G^a)\,, \tag{7.38}$$

where the vector B_G^a

$$B_G^a = \epsilon_{ijkl}\epsilon^{bcda}\Big(\frac{1}{2}K_b{}^{ij}\overset{\circ}{R}{}^{kl}{}_{cd} + K_b{}^{ij}K_c{}^{k}{}_{f}K_d{}^{fl}\Big) \tag{7.39}$$

was introduced. Using the relationship between the contortion tensor and the Riemann tensor, this means Eq. (2.58), and recalling that in four dimensions $\epsilon_{ijkl}\epsilon^{bcd\mu} = \delta^{bcd\mu}_{ijkl}$, the above term (7.39) can be rewritten as

$$B_G^a = \delta^{abcd}_{ijkl}K_b{}^{ij}\Big((K_c{}^{kl})_{,d} + K_d{}^{m}{}_{c}K_m{}^{kl}\Big)\,. \tag{7.40}$$

Considering variations of the function $f(T,B,T_G,B_G)$ with respect to the tetrad fields, the contribution of B_G becomes

$$ef_{B_G}\delta B_G = \Big[e\partial_\mu(f_{B_G})(E_a^\mu B_G^\beta - E_a^\beta B_G^\mu) - ef_{B_G}B_G E_a^\beta\Big]\delta e_\beta^a - eE_a^\mu\partial_\mu(f_{B_G})\delta B_G^a\,, \tag{7.41}$$

where $f_{B_G} = \partial f(T,B,T_G,B_G)/\partial B_G$, boundary terms were neglected and, $\delta E_m^\sigma = -E_n^\sigma E_m^\mu \delta e_\mu^n$ and $\delta e = eE_a^\beta\delta e_\beta^a$ were used. The final term in the above equation reads

$$eE_a^{\mu}\partial_\mu(f_{B_G})\delta B_G^a = P^b{}_{ij}\delta K_b{}^{ij} - \delta^{mbcd}_{ijkl}eE_d^{\beta}\partial_m(f_{B_G})K_b{}^{ij}(\partial_a K_c{}^{kl})\delta e_\beta^a\,, \tag{7.42}$$

where again boundary terms were neglected and for simplicity the following tensor was introduced

$$\begin{aligned} P^b{}_{ij} &= eE_m^{\mu}\partial_\mu(f_{B_G})\Big\{\Big((K_c{}^{kl})_{,d} + K_d{}^p{}_c K_p{}^{kl}\Big)\delta^{mbcd}_{ijkl} + \eta_{pj}\delta^{mdpb}_{qckl}K_d{}^{qc}K_i{}^{kl} \\ &\quad +\delta^{mpcd}_{klij}K_p{}^{kl}K_d{}^b{}_c\Big\} - \delta^{acbd}_{klij}\partial_\sigma\Big(eE_d^{\sigma}E_a^{\mu}\partial_\mu(f_{B_G})K_c{}^{kl}\Big)\,. \end{aligned} \tag{7.43}$$

It should be noted that from (7.42), one notices that the term $\delta K_b{}^{ij}$ needs to be expressed as a variation with respect to the tetrad δe_β^a. Therefore, one can firstly compute how an arbitrary tensor $D^b{}_{ij}\delta K_b{}^{ij}$ changes its form in this context. This formula will be useful for computing $P^b{}_{ij}\delta K_b{}^{ij}$ and is also needed when computing the variations of T_G in the next section.

Recall the contortion and torsion tensors, respectively

$$K_b{}^{ij} = \frac{1}{2}\Big(T^i{}_b{}^j - T^j{}_b{}^i + T_b{}^{ij}\Big)\,, \tag{7.44}$$

$$T^i{}_{bh} = E_b^{\mu}E_h^{\nu}\Big(\partial_\mu e_\nu^i - \partial_\nu e_\mu^i\Big)\,. \tag{7.45}$$

Beginning with (7.44), one obtains that

$$D^b{}_{ij}\delta K_b{}^{ij} = D^b{}_{[ij]}\delta K_b{}^{[ij]} = \frac{1}{2}\Big(D^b{}_i{}^h - D^h{}_i{}^b + D_i{}^{[bh]}\Big)\delta T^i{}_{bh} = \frac{1}{2}C_i{}^{bh}\delta T^i{}_{bh}\,, \tag{7.46}$$

where for simplicity the tensor

$$C_i{}^{bh} = D^b{}_i{}^h - D^h{}_i{}^b + D_i{}^{[bh]} = -C_i{}^{hb}\,, \tag{7.47}$$

was introduced. This tensor needs to be skew-symmetric in its last two indices since $\delta T^i{}_{bh}$ is skew-symmetric in this pair.

Next, by using (7.45) and neglecting boundary terms one finds

$$D^b{}_{ij}\delta K_b{}^{ij} = \Big[\partial_\mu\Big(C_a{}^{bh}E^\mu_h E^\beta_b\Big) + T^i{}_{ab}E^\beta_h C_i{}^{bh}\Big]\delta e^a_\beta\,. \tag{7.48}$$

Equivalently, by using (7.47), one can find explicitly that for any specific tensor $D_i{}^{bh}$, the transformation from $D^b{}_{ij}\delta K_b{}^{ij}$ to terms with δe^a_β will be

$$D^b{}_{ij}\delta K_b{}^{ij} = \Big[\partial_\mu\Big((D^b{}_a{}^h - D^h{}_a{}^b + D_a{}^{[bh]})E^\mu_h E^\beta_b\Big) + T^i{}_{ab}E^\beta_h(D^b{}_i{}^h - D^h{}_i{}^b + D_i{}^{[bh]})\Big]\delta e^a_\beta\,. \tag{7.49}$$

Now, if one changes $D^b{}_{ij} \to P^b{}_{ij}$, one obtains the useful equation

$$P^b{}_{ij}\delta K_b{}^{ij} = \Big[\partial_\mu\Big((P^b{}_a{}^h - P^h{}_a{}^b + P_a{}^{[bh]})E^\mu_h E^\beta_b\Big) + T^i{}_{ab}E^\beta_h(P^b{}_i{}^h - P^h{}_i{}^b + P_i{}^{[bh]})\Big]\delta e^a_\beta\,. \tag{7.50}$$

Finally, if one replaces (7.50) in (7.42) and then replaces that expression in (7.41), one finds the variations of the Gauss-Bonnet boundary term with respect to the tetrad. This is given by

$$\begin{aligned} e f_{B_G}\delta B_G \;&=\; -\Big[\partial_\mu\Big((P^b{}_a{}^h - P^h{}_a{}^b + P_a{}^{[bh]})E^\mu_h E^\beta_b\Big) + T^i{}_{ab}E^\beta_h(P^b{}_i{}^h - P^h{}_i{}^b + P_i{}^{[bh]}) \\ &\quad -\delta^{mbcd}_{ijkl}eE^\beta_d\partial_m(f_{B_G})K_b{}^{ij}K_c{}^{kl}{}_{,a} + e\partial_\mu(f_{B_G})(E^\beta_a B^\mu_G - E^\mu_a B^\beta_G) \\ &\quad +ef_{B_G}B_G E^\beta_a\Big]\delta e^a_\beta\,, \end{aligned} \tag{7.51}$$

where $P^b{}_{ij}$ is explicitly given by Eq. (7.43).

7.2.1.2 Variation of T_G

For simplicity, T_G can be split into four parts as follows

$$\begin{aligned}
T_G &= ({K_a}^i{}_e {K_b}^{ej} {K_c}^k{}_f {K_d}^{fl} - 2{K_a}^{ij} {K_b}^k{}_e {K_c}^e{}_f {K_d}^{fl} + 2{K_a}^{ij} {K_b}^k{}_e {K_f}^{el} {K_d}^f{}_c \\
&\quad +2{K_a}^{ij} {K_b}^k{}_e {K_c}^{el}{}_{,d})\delta^{abcd}_{ijkl}\,, \\
&= T_{G1} + T_{G2} + T_{G3} + T_{G4}\,, \qquad (7.52)
\end{aligned}$$

where T_{G1}, T_{G2}, T_{G3} and T_{G4} are the first, second, third and fourth term of the right-hand sides, respectively. Variations of the $T_{G(i)}$, $i=1,2,3,4$ contributions with respect to the tetrad can be expressed as

$$ef_{T_G}\delta T_G = ef_{T_G}(\delta T_{G1} + \delta T_{G2} + \delta T_{G3} + \delta T_{G4})\,. \qquad (7.53)$$

Here, f_{T_G} stands for the partial derivative of $f(T,B,T_G,T_B)$ with respect to T_G. The first, second and third term can be computed without difficulty, yielding

$$\begin{aligned}
\delta T_{G1} &= \Big[{K_{bj}}^e {K_c}^k{}_f {K_d}^{fl}\delta^{abcd}_{iekl} + {K_b}^e{}_i {K_c}^k{}_f {K_d}^{fl}\delta^{bacd}_{ejkl} + {K_c}^k{}_e {K_b}^{ef} {K_{dj}}^l\delta^{cbad}_{kfil} \\
&\quad + {K_d}^f{}_e {K_b}^{el} {K_c}^k{}_i\delta^{dbca}_{flkj}\Big]\delta {K_a}^{ij}\,, \qquad (7.54) \\
\delta T_{G2} &= -2\Big[{K_b}^k{}_e {K_c}^e{}_f {K_d}^{fl}\delta^{abcd}_{ijkl} + {K_b}^{ke} K_{cjf} {K_d}^{fl}\delta^{bacd}_{keil} + {K_c}^{ef} {K_b}^k{}_i {K_{dj}}^l\delta^{cbad}_{efkl} \\
&\quad + {K_d}^{fl} {K_b}^k{}_e {K_c}^e{}_i\delta^{dbca}_{flkj}\Big]\delta {K_a}^{ij}\,, \qquad (7.55) \\
\delta T_{G3} &= 2\Big[{K_b}^k{}_e {K_f}^{el} {K_d}^f{}_c\delta^{abcd}_{ijkl} + {K_b}^{ke} {K_{fj}}^l {K_d}^f{}_c\delta^{bacd}_{keil} + {K_f}^{el} {K_b}^k{}_i {K_d}^a{}_c\delta^{fbcd}_{elkj} \\
&\quad + {K_d}^{fm} {K_b}^k{}_e {K_i}^{el}\eta_{jc}\delta^{dbca}_{fmkl}\Big]\delta {K_a}^{ij}\,. \qquad (7.56)
\end{aligned}$$

For the final term $ef_{T_G}\delta T_{G4}$, one needs to be careful since one needs to integrate by parts and hence needs to change ∂_d to $\partial_d = E^\mu_d\partial_\mu$. Therefore, one needs to compute the following term

$$ef_{T_G}\delta T_{G4} = 2ef_{T_G}\delta\Big[K_a{}^{ij}K_b{}^k{}_eK_c{}^{el}{}_{,d}\Big]\delta^{abcd}_{ijkl} = 2ef_{TG}\delta\Big[E^\mu_d K_a{}^{ij}K_b{}^k{}_e\partial_\mu(K_c{}^{el})\Big]\delta^{abcd}_{ijkl}\,. \tag{7.57}$$

By ignoring boundary terms, these terms become

$$\begin{aligned} ef_{TG}\delta T_{G4} \;&=\; 2\Big[ef_{T_G}\Big(K_b{}^k{}_eK_c{}^{el}{}_{,d}\delta^{abcd}_{ijkl} + K_b{}^{ke}K_{cj}{}^l{}_{,d}\delta^{bacd}_{keil}\Big) \\ &\quad -\delta^{cbad}_{elkj}\partial_\mu\Big(eE^\mu_d f_{T_G}K_c{}^{el}K_b{}^k{}_i\Big)\Big]\delta K_a{}^{ij} \\ &\quad -2ef_{T_G}\delta^{mbcd}_{ijkl}E^\beta_d E^\mu_a K_m{}^{ij}K_b{}^k{}_e\partial_\mu(K_c{}^{el})\delta e^a_\beta\,. \end{aligned} \tag{7.58}$$

Now, by adding (7.54)-(7.56) and (7.58) one finds

$$\begin{aligned} ef_{T_G}\delta T_G \;&=\; \Big[ef_{T_G}X^a{}_{ij} - 2\delta^{cbad}_{elkj}\partial_\mu\Big(eE^\mu_d f_{T_G}K_c{}^{el}K_b{}^k{}_i\Big)\Big]\delta K_a{}^{ij} \\ &\quad -2ef_{T_G}\delta^{mbcd}_{ijkl}E^\beta_d K_m{}^{ij}K_b{}^k{}_eK_c{}^{el}{}_{,a}\delta e^a_\beta\,, \end{aligned} \tag{7.59}$$

where the following tensor

$$\begin{aligned} X^a{}_{ij} \;&=\; K_{bj}{}^eK_c{}^k{}_fK_d{}^{fl}\delta^{abcd}_{iekl} + K_b{}^e{}_iK_c{}^k{}_fK_d{}^{fl}\delta^{bacd}_{ejkl} + K_c{}^k{}_eK_b{}^{ef}K_{dj}{}^l\delta^{cbad}_{kfil} \\ &\quad +K_d{}^f{}_eK_b{}^{el}K_c{}^k{}_i\delta^{dbca}_{flkj} - 2K_b{}^k{}_eK_c{}^e{}_fK_d{}^{fl}\delta^{abcd}_{ijkl} - 2K_b{}^{ke}K_{cjf}K_d{}^{fl}\delta^{bacd}_{keil} \\ &\quad -2K_c{}^{ef}K_b{}^k{}_iK_{dj}{}^l\delta^{cbad}_{efkl} - 2K_d{}^{fl}K_b{}^k{}_eK_c{}^e{}_i\delta^{dbca}_{flkj} + 2K_b{}^k{}_eK_f{}^{el}K_d{}^f{}_c\delta^{abcd}_{ijkl} \\ &\quad +2K_b{}^{ke}K_{fj}{}^lK_d{}^f{}_c\delta^{bacd}_{keil} + 2K_f{}^{el}K_b{}^k{}_iK_d{}^a{}_c\delta^{fbcd}_{elkj} + 2K_d{}^{fc}K_b{}^k{}_eK_i{}^{el}\eta_{mj}\delta^{dbma}_{fckl} \\ &\quad +2K_b{}^k{}_eK_c{}^{el}{}_{,d}\delta^{abcd}_{ijkl} + 2K_b{}^{ke}K_{cj}{}^l{}_{,d}\delta^{bacd}_{keil}\,, \end{aligned} \tag{7.60}$$

was introduced. It can be shown easily that this long expression is also equivalent to

$$X^a{}_{ij} = \frac{\partial T_G}{\partial K_a{}^{ij}} = \frac{\partial T_{G1}}{\partial K_a{}^{ij}} + \frac{\partial T_{G2}}{\partial K_a{}^{ij}} + \frac{\partial T_{G3}}{\partial K_a{}^{ij}} + 2\delta^{fbcd}_{mnkl}K_c{}^{el}{}_{,d}\frac{\partial}{\partial K_a{}^{ij}}\Big[K_f{}^{mn}K_b{}^k{}_e\Big]\,. \tag{7.61}$$

Next, for simplicity the tensor

$$Y^b{}_{ij} = ef_{T_G}X^b{}_{ij} - 2\delta^{cabd}_{elkj}\partial_\mu\Big(ef_{TG}E^\mu_d K_c{}^{el}K_a{}^k{}_i\Big)\,, \tag{7.62}$$

will be introduced to rewrite Eq. (7.59) as

$$ef_{T_G}\delta T_G = Y^b{}_{ij}\delta K_b{}^{ij} - 2ef_{T_G}\delta^{mbcd}_{ijkl}E^\beta_d K_m{}^{ij}K_b{}^k{}_e K_c{}^{el}{}_{,a}\delta e^a_\beta\,. \tag{7.63}$$

Finally, by using equation (7.49), one can change $\delta K_a{}^{ij}$ to δe^a_β by changing $D^a{}_{ij}$ to $Y^a{}_{ij}$. Doing that, one finally finds that the variations with respect to the T_G part is

$$\begin{aligned} ef_{T_G}\delta T_G \;\; = \;\; & \Big[\partial_\mu\Big((Y^b{}_a{}^h - Y^h{}_a{}^b + Y_a{}^{[bh]})E^\mu_h E^\beta_b\Big) + T^i{}_{ab}E^\beta_h(Y^b{}_i{}^h - Y^h{}_i{}^b + Y_i^{[bh]}) \\ & -2ef_{T_G}\delta^{mbcd}_{ijkl}E^\beta_d K_m{}^{ij}K_b{}^k{}_e K_c{}^{el}{}_{,a}\Big]\delta e^a_\beta\,, \end{aligned} \tag{7.64}$$

where $Y^b{}_{ij}$ is explicitly given by Eq. (7.62).

7.2.2 Equations of motion and FLRW example

Let us now find the equation of motion of the full model with T_G and B_G contribution. By replacing (5.4), (5.3), (5.14), (7.51) and (7.64) into (7.36) and then by setting $\delta\mathcal{S}_{f(T,B,T_G,B_G)} = 0$, one finds

$$\begin{aligned} &\partial_\mu\Big((P^b{}_a{}^h - P^h{}_a{}^b + P_a{}^{[bh]})E^\mu_h E^\beta_b\Big) + T^i{}_{ab}E^\beta_h(P^b{}_i{}^h - P^h{}_i{}^b + P_i^{[bh]}) \\ &-\delta^{mbcd}_{ijkl}eE^\beta_d\partial_m(f_{B_G})K_b{}^{ij}K_c{}^{kl}{}_{,a} + e\partial_\mu(f_{B_G})(E^\beta_a B^\mu_G - E^\mu_a B^\beta_G) + ef_{B_G}B_G E^\beta_a \\ &-\partial_\mu\Big((Y^b{}_a{}^h - Y^h{}_a{}^b + Y_a{}^{[bh]})E^\mu_h E^\beta_b\Big) - T^i{}_{ab}E^\beta_h(Y^b{}_i{}^h - Y^h{}_i{}^b + Y_i^{[bh]}) \\ &+2ef_{T_G}\delta^{mbcd}_{ijkl}E^\beta_d K_m{}^{ij}K_b{}^k{}_e K_c{}^{el}{}_{,a} + 2eE^\beta_a\mathring{\Box}f_B - 2eE^\sigma_a\mathring{\nabla}^\beta\mathring{\nabla}_\sigma f_B + eBf_B E^\beta_a \\ &+4e\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_a{}^{\mu\beta} + 4\partial_\mu(eS_a{}^{\mu\beta})f_T - 4ef_T T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu} - efE^\beta_a = 2\kappa^2 e\mathcal{T}^\beta_a\,. \end{aligned} \tag{7.65}$$

The field equations are very complicated, however, when considering a homogeneous and isotropic spacetime, they simplify considerably and can be presented in closed form. Comparison of these equations with previous results serves as a good consistency check of these calculations.

Let us now consider flat FLRW, where the field equations for the $f(T,B,T_G,B_G)$ theory are given by

$$3H^2\left(3f_B+2f_T-4H^2f_{TG}\right)+3\dot{H}\left(f_B-4H^2f_{TG}\right)-3H\dot{f}_B+12H^3\dot{f}_{TG}+\frac{f}{2}=\kappa^2\rho\,,$$
$$H^2\left(9f_B+6f_T+4\ddot{f}_{TG}\right)+\dot{H}\left(3f_B+2f_T-12H^2f_{TG}+8H\dot{f}_{TG}\right)+2H\dot{f}_T$$
$$+8H^3\dot{f}_{TG}-12H^4f_{TG}-\ddot{f}_B+\frac{f}{2}=-\kappa^2p\,.$$

Here, for simplicity, it was considered that the matter is a standard perfect fluid with energy density ρ and isotropic pressure p. One should make explicit that $\dot{f}_B=f_{BB}\dot{B}+f_{BT}\dot{T}+f_{BT_G}\dot{T}_G+f_{BB_G}\dot{B}_G$ using the chain rule, so that dot denotes differentiation with respect to cosmic time. Remark that in flat FLRW, $B_G=0$ and then, it does not appear in the equations. As was discussed in Sec. 5.2, to recover modified Gauss-Bonnet gravity theory with the standard metric signature notation ($\eta_{ab}=\text{diag}(-+++)$), one needs to change $T\to -T$, $B\to -B$ and $T_G\to -T_G$. In other words, since this thesis is based in the standard Teleparallel signature notation $\eta_{ab}=\text{diag}(+---)$, one needs to change $f(T,B,T_G,B_G)=f(+T-B,+T_G-B_G)=f(-\overset{\circ}{R},-\overset{\circ}{G})$ to recover the corresponding equations in standard signature notation in the modified Gauss-Bonnet gravity theory. Then, to recover the standard notation, derivatives change as follows

$$f_T\to f_R\,,\quad f_B\to -f_R\,,\quad f_{TG}\to f_G\,,\quad f_{BG}\to -f_G\,,\tag{7.66}$$

where $f_R = \partial f/\partial \overset{\circ}{R}$ and $f_G = \partial f/\partial \overset{\circ}{G}$. Doing this, gives us

$$-3(H^2+\dot{H})\big(f_R + 4H^2 f_G\big) + 3H\dot{f}_R + 12H^3\dot{f}_G + \frac{f}{2} = \kappa^2\rho\,, \tag{7.67}$$

$$\begin{aligned}-H^2(3f_R - 4\ddot{f}_G) - \dot{H}(f_R + 12H^2 f_G - 8H\dot{f}_G) + 2H\dot{f}_R + 8H^3\dot{f}_G\\ -12H^4 f_G + \ddot{f}_R + \frac{f}{2} = -\kappa^2 p\,.\end{aligned} \tag{7.68}$$

These equations match those reported in Cognola et al. (2006); Kofinas et al. (2014) which serves as a good consistency check of our field equations in the Teleparallel formulation.

Let us now consider a close FLRW spacetime ($k=+1$) where the rotated tetrad is given by Eq. (7.22) with γ equal to (7.31). In this case, the modified FLRW equations become

$$\begin{aligned}3H^2(3f_B + 2f_T - 4H^2 f_{TG}) + \dot{H}(3f_B - 12H^2 f_{TG}) - 3H\dot{f}_B\\ +12H^3\dot{f}_{TG} + \frac{12}{a^2}\Big[(H^2+\dot{H})(2f_{BG} + f_{TG}) - H(2\dot{f}_{BG} + \dot{f}_{TG})\Big] + \frac{f}{2} = \kappa^2\rho\,,\end{aligned} \tag{7.69}$$

$$\begin{aligned}H^2(9f_B + 6f_T + 4\ddot{f}_{TG}) + \dot{H}(3f_B + 2f_T - 12H^2 f_{TG} + 8H\dot{f}_{TG})\\ +\frac{2}{a^2}\Big[6(H^2+\dot{H})(2f_{BG} + f_{TG}) - f_T - 4\ddot{f}_{BG} - 2\ddot{f}_{TG}\Big]\\ +2H\dot{f}_T + 8H^3\dot{f}_{TG} - 12H^4 f_{TG} - \ddot{f}_B + \frac{f}{2} = -\kappa^2 p\,.\end{aligned} \tag{7.70}$$

In the close FLRW case, $B_G = -48\ddot{a}/a^3 \neq 0$ so that this term now contributes to the equations. By setting $f(T,B,T_G,B_G) = f(+T-B, +T_G - B_G) = f(-\overset{\circ}{R}, -\overset{\circ}{G})$, one then recovers the Gauss-Bonnet equations $f(\overset{\circ}{R}, \overset{\circ}{G})$ with the standard metric signature notation in a close universe ($k=+1$) which are given by

$$\begin{aligned}-3(H^2+\dot{H})(f_R + 4H^2 f_G) + 3H\dot{f}_R + 12H^3\dot{f}_G\\ -\frac{12}{a^2}\Big[(H^2+\dot{H})f_G - H\dot{f}_G\Big] + \frac{f}{2} = \kappa^2\rho\,,\end{aligned} \tag{7.71}$$

$$-H^2(3f_R-4\ddot{f}_G)-\dot{H}(f_R+12H^2f_G-8H\dot{f}_G)+2H\dot{f}_R+8H^3\dot{f}_G$$
$$-12H^4f_G+\ddot{f}_R-\frac{2}{a^2}\Big[6(H^2+\dot{H})f_G+f_R-2\ddot{f}_G\Big]+\frac{f}{2}=-\kappa^2 p\,. \tag{7.72}$$

Finally let us consider the Einstein static universe where all dynamical variables are assumed to be constants which give

$$f=2\kappa^2\,\rho_0\,,\qquad f-f_R\frac{4}{a_0^2}=-2\kappa^2\,p_0\,. \tag{7.73}$$

For the choice $f(\overset{\circ}{R},\overset{\circ}{G})=\overset{\circ}{R}+\kappa^2\,g(\overset{\circ}{G})$, one notes that $\overset{\circ}{R}=6/a_0^2$ and $\overset{\circ}{G}=0$ in this case. The field equations reduce simply to

$$\frac{6}{a_0^2}+\kappa^2\,g(0)=2\kappa^2\,\rho_0\,,\qquad\Leftrightarrow\qquad\frac{3}{a_0^2}=\kappa^2\,\rho_0-\frac{\kappa^2}{2}g(0)\,, \tag{7.74}$$
$$\frac{6}{a_0^2}+\kappa^2\,g(0)-\frac{4}{a_0^2}=-2\kappa^2\,p_0\,,\qquad\Leftrightarrow\qquad-\frac{1}{a_0^2}=\kappa^2\,p_0+\frac{\kappa^2}{2}g(0)\,, \tag{7.75}$$

which are exactly the $k=+1$ equations reported in Boehmer & Lobo (2009), providing a second consistency check. It should be noted that the rotated tetrad used for this calculation proves computationally very challenging.

7.2.3 Energy-momentum trace extension and connection with other theories

Let us now consider the above framework and include the trace of the energy-momentum tensor to the action (7.35). This gives the extended action

$$\mathcal{S}_{f(T,B,T_G,B_G,\mathcal{T})}=\int\left[\frac{1}{2\kappa^2}f(T,B,T_G,B_G,\mathcal{T})+\mathcal{L}_{\rm m}\right]e\,d^4x\,, \tag{7.76}$$

where additionally f is a function of the trace of the energy-momentum tensor $\mathcal{T} = E_a^{\beta}\mathcal{T}^a_{\beta}$. As before $\mathcal{L}_{\rm m}$ denotes an arbitrary matter Lagrangian density. One can define the energy-momentum tensor as

$$\mathcal{T}^a_{\mu} = \frac{1}{e}\frac{\delta(e\mathcal{L}_{\rm m})}{\delta E_a^{\mu}}\,, \tag{7.77}$$

and assume that the matter Lagrangian only depends explicitly on the tetrads and its derivatives and does not depend on the connection independently. The energy-momentum tensor is then given by

$$\mathcal{T}^a_{\mu} = -2e^a_{\mu}\mathcal{L}_{\rm m} - 2\Big(\frac{\partial\mathcal{L}_{\rm m}}{\partial E_a^{\mu}}\Big)\,. \tag{7.78}$$

Variations of the action (7.76) with respect to the tetrad gives one additional term, namely

$$ef_{\mathcal{T}}\delta\mathcal{T} = ef_{\mathcal{T}}(4\Omega_a^{\beta} + \mathcal{T}_a^{\beta})\delta e^a_{\beta}\,, \tag{7.79}$$

$$\Omega_a^{\beta} = \frac{1}{4}e^b_{\alpha}\Big(\frac{\delta\mathcal{T}_b^{\alpha}}{\delta e^a_{\beta}}\Big) = \mathcal{T}_a^{\beta} + \frac{3}{2}E_a^{\beta}\mathcal{L}_{\rm m} - \frac{1}{2}e^b_{\alpha}\Big(\frac{\partial^2\mathcal{L}_{\rm m}}{\partial e^a_{\beta}\partial e^b_{\alpha}}\Big)\,. \tag{7.80}$$

This completes the statement of the field equations.

Let us finish this chapter with a discussion of the relationship between the various modified theories of gravity which are governed by the function $f(T,B,T_G,B_G,\mathcal{T})$. In the pure tetrad formalism, in general these are fourth order theories which violate local Lorentz covariance. Therefore, these theories are quite different from General Relativity in many ways. However, for a particular choice of this function, one is able to recover General Relativity or its Teleparallel equivalent. Therein lies the power of this approach, namely one can recover the two equivalent formulations of General Relativity using a single unified approach. This in particular clarifies the roles of the total derivative terms present in our framework. As was shown in Chap. 5, by considering the function $f(T, B)$ one can formulate the Teleparallel equivalent of $f(\mathring{R})$ gravity and identify those parts of the field equations which are part of $f(T)$ gravity, the second order part of the equations which is not locally Lorentz invariant. In analogy to this, one can also make the relationships between various modified theories of gravity clear which are based on the Gauss-Bonnet term and also the trace of the energy-momentum tensor.

The top left corner of Fig. 7.1 refers to $f(T, B, T_G, B_G)$ gravity, the most general theory one can formulate based on the four variables. One can think of the top entries as the Teleparallel row and the bottom as the metric row. The arrows indicate the specific choices that have to be made in order to move from one theory to the other. If one now includes the trace of the energy-momentum tensor to this approach, things get slightly more complicated as the number of possible theories increases quite dramatically. Fig. 7.2 visualises the entire set of possible theories. Now, the left half of the figure corresponds to the metric approach while the right half corresponds to the Teleparallel framework. The four main theories of the previous discussions are highlighted by boxes. Many of these theories were considered in isolation in the past and their relationship with other similarly looking theories was only made implicitly. One should also point out that the representation of these theories is only one of the many possibilities and moreover, Fig. 7.2 is incomplete. There are many more

$$\begin{tikzcd}
f(T, B, T_G, B_G) \arrow[r, "f = f(T, T_G)"] \arrow[d, "f = f(-T + B, -T_G + B_G)"] & f(T, T_G) \arrow[r, "f = f(T)"] & f(T) \arrow[d, "f = T"'] \\
f(\mathring{R}, \mathring{G}) \arrow[r, "f = f(-T + B)"'] & f(\mathring{R}) \arrow[r, "f = \mathring{R}"'] & \text{GR or TEGR}
\end{tikzcd}$$

Figure 7.1: Relationship between $f(T, B, T_G, B_G)$ and other gravity theories.

theories one could potentially construct which we have not mentioned so far. The diagram was constructed having in mind those theories which have been studied in the past.

In constructing the diagram, one also made the interesting observation that the theory based on the function $f(\mathring{R}, T)$ should be viewed as a special case of the Teleparallel gravity theory $f(T, B)$. To see this, simply recall the principal identity $\mathring{R} = -T + B$ which shows that the special choice $f(-T + B, T)$ is the Teleparallel equivalent of $f(\mathring{R}, T)$ theory and also that the Teleparallel framework should be viewed as the slightly more natural choice for this theory. Additionally, the theory $f(\mathring{R}, T)$ is based on two different connections which is somehow a problematic approach.

A short list of the most important theories is written in Table 7.1 with accompanying references for the interested reader, mainly focusing on the primary sources or reviews where such theories were considered. Of the many possible theories

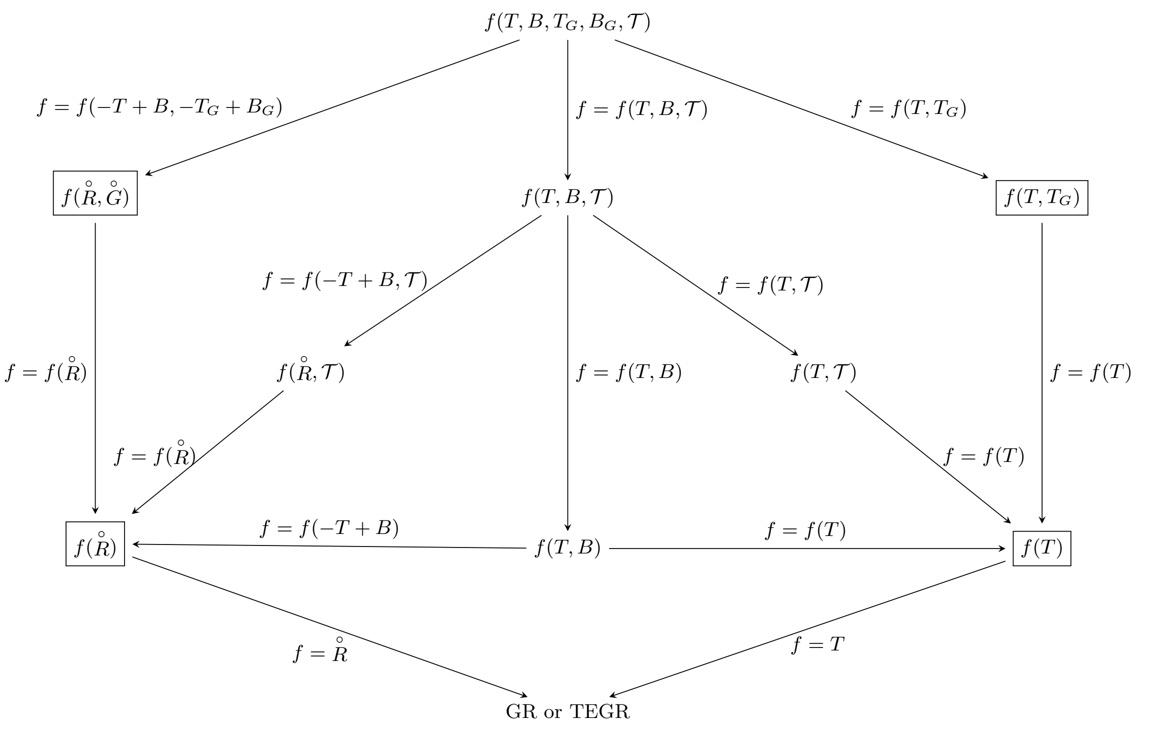


Figure 7.2: Relationship between $f(T, B, T_G, B_G, \mathcal{T})$ and other gravity theories.

Theory	Some key references
$f(\overset{\circ}{R})$	Reviews Sotiriou & Faraoni (2010) & De Felice & Tsujikawa (2010)
$f(T)$	Ferraro & Fiorini (2007), and review by Cai et al. (2016)
$f(T, B)$	Bahamonde et al. (2015)
$f(\overset{\circ}{R}, T)$	Myrzakulov (2012)
$f(\overset{\circ}{R}, \overset{\circ}{G})$	Nojiri et al. (2005)
$f(T, T_G)$	Kofinas & Saridakis (2014) & Kofinas et al. (2014)
$f(\overset{\circ}{R}, \mathcal{T})$	Harko et al. (2011)
$f(T, \mathcal{T})$	Harko et al. (2014a)

Table 7.1: Short list of previously studied theories covered by the function $f(T, B, T_G, B_G, \mathcal{T})$.

one could potentially construct from $f(T, B, T_G, B_G, \mathcal{T})$, one identified some which might be of interest for future studies. Clearly, there are many theories which do not have a general relativistic counter part like $f(B, T_G, B_B, \mathcal{T})$ since no theory in this class can reduce to General Relativity. However, it is always possible to consider such a theory in addition to General Relativity by considering for instance a theory based on $T + f(B, T_G, B_G, \mathcal{T})$. For a function linear in its arguments this yields the Teleparallel equivalent of General Relativity.

It is also useful to make explicit the limitations of the current approach. In essence, one is dealing with modified theories of gravity which are based on scalars derived from tensorial quantities of interest, for instance, the Ricci scalar or the trace of the energy-momentum tensor. However, theories containing the square of the Ricci tensor or theories containing the term $\overset{\circ}{R}_{\mu\nu}\mathcal{T}^{\mu\nu}$ are not currently covered. In principle, it is straightforward though to extend our formalism to such theories. In case of the quantity $\overset{\circ}{R}_{\mu\nu}\mathcal{T}^{\mu\nu}$, one would have to recall Eq. (2.58) with $R_{\mu\nu} = \overset{\bullet}{R}_{\mu\nu} = 0$, so that, $\overset{\circ}{R}_{\mu\nu}$ can be expressed in the Teleparallel setting, something that also has not been done yet. Likewise, one could also address quadratic gravity models (Deser & Tekin, 2003) which contain squares of the Riemann tensor and use Eq. (2.55). Theories depending on higher order derivative terms also require a separate treatment (Otalora & Saridakis, 2016).

The current approach is entirely based on the torsion scalar T which is motivated by its close relation to the Ricci tensor. However, in principle one could follow the approach done in Chap. 6 and decompose the torsion tensor into its three irreducible pieces and construct their respective scalars. This would allow us to study a larger class of models based on those three scalars and the boundary term.

For many years now, an ever increasing number of modifications of General Relativity have been considered. This chapter focused only on theories where the gravitational field can either be modelled using the metric of the tetrad. Hence, all types of metric affine theories where the metric and the torsion tensor are treated as two independent dynamical variables were excluded. It would be almost impossible to present a visualisation that also encompasses all those theories. Even this would represent only a fraction of what is referred to as modified gravity. It would still exclude higher dimensional models, Einstein-Aether models, Hořava-Lifshitz theory and many others. It is also interesting to note that $f(\overset{\circ}{R})$ gravity for instance can be formulated as a theory based on a non-minimally coupled scalar field. Hence, many of the theories in Fig. 7.2 might also have various other representations which in turn might be connected in different manners.

8 Teleparallel non-local theories

Chapter Abstract

Even though it is not possible to distinguish General Relativity from Teleparallel gravity using classical experiments, it could be possible to discriminate between them by quantum gravitational effects. These effects have motivated the introduction of non-local deformations of General Relativity, and similar effects are also expected to occur in Teleparallel gravity. This chapter is devoted to studying non-local deformations of Teleparallel gravity along with cosmological solutions. Along this track, future experiments probing non-local effects could be used to test whether General Relativity or Teleparallel gravity give the most consistent picture of gravitational interaction. Additionally, a further generalised non-local theory of gravity is introduced which, in specific limits, can become either the curvature non-local or Teleparallel non-local theory. Using the Noether symmetry approach, it is found that the coupling functions coming from the non-local terms are constrained to be either exponential or linear in form, which in some non-local theories are needed in order to achieve a renormalisable theory.

8.1 Teleparallel non-local gravity

GR tells us that gravitational interaction is described by the curvature of torsion-less spacetimes. On the other hand, TEGR describes gravity by the torsion of spacetime, so that the curvature picture is not necessary. Even though these two approaches are fundamentally different, they produce the same classical field equations. Thus, both theories predict the same dynamics for classical gravitational systems, and so classical gravitational experiments cannot be used to test them. In other words, they are equivalent at classical level.

However, because these theories are conceptually different, they are expected to produce different quantum effects. An important remark is in order at this point. One can deal with TEGR only at the classical level because it produces the same classical field equations as GR. Considering quantum effects and non-locality, it is improper to speak of the equivalence of the two theories since they could be fundamentally different.

Even though there does not exist a fully developed quantum theory of gravity, there are various proposals for quantum gravity, and a universal prediction from almost all of these approaches seems to be the existence of an intrinsic extended structure in the geometry of spacetime (Das & Vagenas, 2008), and such an extended structure would be related to an effective non-local behaviour for spacetime (Modesto, 2016; Modesto & Shapiro, 2016). For example, in perturbative string theory, it is not possible to measure spacetime below string length scale, as the string is the smallest available probe. As it is not possible to define point-like local structures, string theory produces an effective non-local behaviour (Eliezer & Woodard (1989); Calcagni & Modesto (2014); Calcagni et al. (2016). Similarly, there is an intrinsic minimal area in loop-quantum gravity (Major & Seifert, 2002), and this extended structure is expected to produce a non-local behaviour. It can be argued, from black hole physics, that any theory of quantum gravity should present intrinsic extended structures of the order of the Planck length, and it would not be possible to probe the spacetime below this scale. In fact, the energy needed to probe the spacetime below this scale is more than the energy needed to form a mini black hole in that region of spacetime (Maggiore, 1993).

Thus, quantum gravitational effects produce effective extended structures in spacetime that would lead to non-locality (Das & Vagenas, 2008). Hence, it can be argued that the first order corrections from quantum gravity will produce non-local deformations of GR (Elizalde & Odintsov, 1995; Modesto et al., 2011), and this will, in turn, produce a non-locality in cosmology. The effect of non-local deformations in cosmology could be a straightforward explanation for cosmic acceleration (see for example Jhingan et al. (2008); Deffayet & Woodard (2009)).

Furthermore, the non-locality induced by GR deformations could be used to describe the transition from radiation to the matter dominated era if it is consistently constrained with the observations.

As non-locality is produced by first order quantum gravitational effects, it is expected that they would also occur in Teleparallel gravity. Unlike the standard local classical dynamics, the behaviour of such non-local effects could be very different in Teleparallel gravity and GR, and they can be used to experimentally discriminate between these two theories. Therefore, it is interesting to study the non-local deformation of both GR and Teleparallel gravity. Even though the non-local deformation of GR has been extensively studied, the non-local deformation of Teleparallel gravity has recently been introduced in Bahamonde et al. (2017c,d). Thus, this chapter is devoted to review the latter references related to non-local Teleparallel gravity.

At present, the non-local Teleparallel gravity satisfies all the existing cosmological experimental constraints, and can explain phenomena that are explained using non-local deformations of GR. However, as the non-local Teleparallel gravity is fundamentally different from non-local deformation of GR, future experiments can be used to verify which of these theories is the correct theory of gravity. Thus, the action for General Relativity $\mathcal{S}_{\rm GR}$, can be corrected by a non-local terms $\mathcal{S}_{\rm GRNL}$ due to quantum corrections, and so the quantum corrected non-local GR can be written as (Deser & Woodard, 2007)

$$\mathcal{S}_1 = \mathcal{S}_{\mathrm{GR}} + \mathcal{S}_{\mathrm{GRNL}} \,. \tag{8.1}$$

Similarly, the standard classical action of TEGR, $\mathcal{S}_{\mathrm{TEGR}}$, can be corrected by a non-local term due to quantum corrections $\mathcal{S}_{\mathrm{TEGRNL}}$, and so the quantum corrected non-local Teleparallel gravity can be written as (Bahamonde et al., 2017d)

$$\mathcal{S}_2 = \mathcal{S}_{\mathrm{TEGR}} + \mathcal{S}_{\mathrm{TEGRNL}} \,. \tag{8.2}$$

It is not possible to experimentally differentiate between $\mathcal{S}_{\mathrm{GR}}$ and $\mathcal{S}_{\mathrm{TEGR}}$, but the quantum corrections to these theories $\mathcal{S}_{\mathrm{GRNL}}$ and $\mathcal{S}_{\mathrm{TEGRNL}}$ are very different. Thus, it is experimentally possible to discriminate between $\mathcal{S}_1$ and $\mathcal{S}_2$. It may be noted that as in the non-local GR case, the non-local correction to Teleparallel gravity is motivated by quantum gravitational effects, and it is not arbitrary added to the original action.

It may be noted that non-local Teleparallel formalism could be a better approach to study quantum gravitational effects. This is due to the fact that TEGR does not require the equivalence principle to be formulated (see Sec. 3.5), and it has been argued that quantum effects can cause the violation of the equivalence principle (Seveso & Paris, 2017). Furthermore, a violation of the equivalence principle can be related to a violation of the Lorentz symmetry (Wang et al., 2016), and the Lorentz symmetry is also expected to be broken at the UV scale in various approaches to quantum gravity, such as discrete spacetime ('t Hooft, 1996), spacetime foam (Amelino-Camelia et al., 1998), among others. In our approach, the pure tetrad formalism will be used. In this formalism, the torsion tensor does not transform covariantly under local Lorentz transformations. Hence, the torsion scalar also is not covariant under local Lorentz transformations. In standard Teleparallel gravity where just a linear combination of the scalar torsion is considered in the action $\mathcal{S}_{\rm TEGR}$, the theory becomes quasi-local Lorentz invariant, or invariant up to a boundary term. However, when one is considering modifications of Teleparallel theories of gravity, such as $f(T)$ gravity or in our case non-local Teleparallel gravity, the theory is no longer local Lorentz invariant. The loss of this covariance in Teleparallel theories might be an interesting behaviour on quantum scales.

8.1.1 Action and field equations

In this section, a non-local deformation of Teleparallel gravity will be presented along with the field equations of the model. Adopting the formalism developed for non-local deformations of GR (Deser & Woodard, 2007), one can write a non-local deformation for Teleparallel gravity as

$$\mathcal{S}_{\rm Teleparallel-NL} = \frac{1}{2\kappa^2}\int d^4x\, e(x)\, T(x)\Big[f(\mathcal{G}[T](x))+1\Big] + \int d^4x\, e(x)\,\mathcal{L}_{\rm m} \tag{8.3}$$

$$= \mathcal{S}_{\rm TEGR} + \frac{1}{2\kappa^2}\int d^4x\, e(x)\, T(x) f\Big(\mathring{\Box}^{-1}T)(x)\Big) + \mathcal{S}_{\rm m}\,, \tag{8.4}$$

where f is an arbitrary function which depends on the retarded Green function evaluated at the torsion scalar[1], $\mathcal{L}_{\rm m}$ is any matter Lagrangian, $\mathring{\Box} \equiv \partial_\rho(eg^{\sigma\rho}\partial_\sigma)/e$ is the scalar-wave operator, and $\mathcal{G}[f](x)$ is a non-local operator which can be written in terms of the Green function $G(x,x')$ as

$$\mathcal{G}[f](x) = (\mathring{\Box}^{-1}f)(x) = \int d^4x'\, e(x')f(x')G(x,x')\,. \tag{8.5}$$

Furthermore, like the non-local corrections to the GR, these non-local corrections to the Teleparallel gravity are also motivated from quantum gravitational effects. It can be noted that, as for non-local GR, the Green function is evaluated at the Ricci scalar $\mathring{R}$, in non-local Teleparallel gravity, the Green function is evaluated at the

[1] Quantum effects such as the Planck constant have been absorbed in the definition of this function

torsion scalar T.

It is worth noticing that unlike GR which produces the same equations of motion as the TEGR, the non-local deformation of GR is different from the non-local deformation of Teleparallel gravity. The latter comes from the fact that $\overset{\circ}{R} = -T + B$, where B is the boundary term so that $\mathcal{S}_{\rm GR}$ (which is constructed by $\overset{\circ}{R}$) and $\mathcal{S}_{\rm TEGR}$ (which is constructed by T) produces the same field equations. However, the non-local terms $\sqrt{-g}\overset{\circ}{R}f_1(\overset{\circ}{\Box}{}^{-1}\overset{\circ}{R})$ and $eTf_2(\overset{\circ}{\Box}{}^{-1}T)$ coming from the non-local actions $\mathcal{S}_{\rm GRNL}$ and $\mathcal{S}_{\rm TEGRNL}$, will produce different field equations even for the case where $f_1 = \overset{\circ}{\Box}{}^{-1}\overset{\circ}{R}$ and $f_2 = \overset{\circ}{\Box}{}^{-1}T$. This happens since the boundary term B, which is the difference between T and $\overset{\circ}{R}$, produces a contribution in the variational process in non-local terms. This fact is in the same spirit as was discussed in the other chapters (see for example Chap. 5).

Now by a variation with respect to the tetrad, one obtains

$$\begin{aligned}\delta\mathcal{S}_{\rm Teleparallel-NL} &= \delta\mathcal{S}_{\rm TEGR} + \frac{1}{2\kappa^2}\int\Big[Tf(\mathcal{G}[T])\delta e + ef(\mathcal{G}[T])\delta T + e\,T\delta f(\mathcal{G}[T])\Big]d^4x \\ &\quad + \int d^4x\delta(e\mathcal{L}_{\rm m})\,. \qquad (8.6)\end{aligned}$$

The first two terms in the integrand are easily computed since they are similar to the terms in $f(T)$ gravity, giving us

$$\begin{aligned} ef(\mathcal{G}[T])\delta T &= -4\Big[e(\partial_\mu f(\mathcal{G}[T]))S_a{}^{\mu\beta} + \partial_\mu(eS_a{}^{\mu\beta})f(\mathcal{G}[T]) \\ &\quad - ef(\mathcal{G}[T])T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu}\Big]\delta e^a_\beta\,, \qquad (8.7)\\ Tf(\mathcal{G}[T])\delta e &= eTf(\mathcal{G}[T])E_a^\beta\delta e^a_\beta\,. \qquad (8.8)\end{aligned}$$

The last term $e\,T\delta f(\mathcal{G}[T])$ in (8.6) is new and therefore, it will be computed here. This term can be written as

$$eT\delta f(\mathcal{G}[T]) = eT\delta f\Big(\mathring{\Box}^{-1}T\Big) = eTf'(\mathcal{G})\Big(\mathring{\Box}^{-1}\delta T - \mathring{\Box}^{-1}(\delta\mathring{\Box})\,\mathring{\Box}^{-1}T\Big) \quad (8.9)$$

$$= -e\mathring{\Box}^{-1}(Tf'(\mathcal{G}))\delta\Big(\frac{\partial_\mu(eg^{\mu\nu}\partial_\nu)}{e}\Big)\mathring{\Box}^{-1}T + e\mathring{\Box}^{-1}(Tf'(\mathcal{G}))\delta T\,. \quad (8.10)$$

Let us now work out the second term on the right-hand side since the variation of $F(e)\delta T$ is well-known for any function $F(e)$ which depends on the tetrad. Now, if one expands the first term, one obtains

$$-e\mathring{\Box}^{-1}(Tf'(\mathcal{G}))\delta\Big(\frac{\partial_\mu(eg^{\mu\nu}\partial_\nu)}{e}\Big)\mathring{\Box}^{-1}T = (\mathring{\Box}^{-1}Tf'(\mathcal{G}))T\delta e - \mathring{\Box}^{-1}(Tf'(\mathcal{G}))\partial_\mu(\delta(eg^{\mu\nu}\partial_\nu))\mathring{\Box}^{-1}T\,, \quad (8.11)$$

$$= T\mathcal{G}[Tf'(\mathcal{G})]\delta e + \partial_\mu(\mathcal{G}[Tf'(\mathcal{G})])(\partial_\nu T)\Big(g^{\mu\nu}\delta e + e\delta g^{\mu\nu}\Big)\,, \quad (8.12)$$

where $\mathring{\Box}\times\mathring{\Box}^{-1}T = T$ was used and boundary terms were neglected[2]. Now, if one takes into account that $\delta e = eE_a^\beta e^a_\beta$ and $\delta g^{\sigma\rho} = -(g^{\sigma\beta}E_a^\rho + g^{\rho\beta}E_a^\sigma)\delta e^a_\beta$, one can expand the above term yielding

$$-e\mathring{\Box}^{-1}(Tf'(\mathcal{G}))\delta\Big(\frac{\partial_\mu(eg^{\mu\nu}\partial_\nu)}{e}\Big)\mathring{\Box}^{-1}T = e\Big[\partial_\mu(\mathcal{G}[Tf'(\mathcal{G})])(\partial_\nu T)\Big(g^{\mu\nu}E_a^\beta - 2g^{\beta(\mu}E_a^{\nu)}\Big) + T\mathcal{G}[Tf'(\mathcal{G})]E_a^\beta\Big]\delta e^a_\beta\,. \quad (8.13)$$

Therefore, variations of the non-local term is

$$eT\delta f(\mathcal{G}[T]) = e\Big[T\mathcal{G}[Tf'(\mathcal{G})]E_a^\beta + \partial_\mu(\mathcal{G}[Tf'(\mathcal{G})])(\partial_\nu T)\Big(g^{\mu\nu}E_a^\beta - 2g^{\beta(\mu}E_a^{\nu)}\Big)\Big]\delta e^a_\beta + e\,\mathcal{G}[Tf'(\mathcal{G})]\delta T\,. \quad (8.14)$$

[2] $\times$ here denotes the function composition so that $\mathring{\Box}\times\mathring{\Box}^{-1}f = f$.

Therefore, by adding (8.7), (8.8) and (8.14) and using the TEGR field equations, one can find the non-local Teleparallel gravity field equations that can be written as

$$
\begin{aligned}
&4\Big[S_a{}^{\mu\beta}\partial_\mu + \frac{1}{e}\partial_\mu(eS_a{}^{\mu\beta}) - T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu} - T\,E_a^\beta\Big]\Big[f(\mathcal{G}[T]) + \mathcal{G}[Tf'(\mathcal{G})]\Big] \\
&\quad - \partial_\rho\Big(\mathcal{G}[Tf'(\mathcal{G})]\Big)(\partial_\sigma T)\Big(g^{\sigma\rho}E_a^\beta - 2g^{\beta(\rho}E_a^{\sigma)}\Big) - \frac{4}{e}\partial_\mu(eS_a{}^{\mu\beta}) + 4T^\sigma{}_{\mu a}S_\sigma{}^{\beta\mu} \\
&\qquad + TE_a^\beta = 2\kappa^2\mathcal{T}_a^\beta\,. \qquad (8.15)
\end{aligned}
$$

In the next section, Teleparallel non-local cosmology will be introduced and studied.

8.2 Teleparallel non-local cosmology

Since the non-local field equations are difficult to use, it is easier to first rewrite the non-local action (8.4) in terms of two scalar fields ϕ and θ. This approach is in the same spirit as was done in Nojiri & Odintsov (2008). Doing this, one has

$$
\mathcal{S}_{\rm Teleparallel-NL} = \frac{1}{2\kappa^2}\int d^4x\, e\Big[T(f(\phi)+1) - \partial_\mu\theta\partial^\mu\phi - \theta T\Big] + \mathcal{S}_{\rm m}\,. \qquad (8.16)
$$

Now by varying the above action with respect to ϕ and θ one obtains $\phi = \mathring{\Box}^{-1}T$ and $\mathring{\Box}\theta = -f'(\phi)T$, respectively. By varying this non-local action with respect to the tetrads, one obtains

$$
\begin{aligned}
&2(1-f(\phi)+\theta)\left[e^{-1}\partial_\mu(eS_a{}^{\mu\beta}) - E_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\beta\mu} - \frac{1}{4}E_a^\beta T\right] - 2\partial_\mu(\theta - f(\phi))E_a^\rho S_\rho{}^{\mu\beta} \\
&\qquad - \frac{1}{2}\Big[(\partial^\lambda\theta)(\partial_\lambda\phi)E_a^\beta - (\partial^\beta\theta)(\partial_a\phi) - (\partial_a\theta)(\partial^\beta\phi)\Big] = \kappa^2\mathcal{T}_a^\beta\,. \qquad (8.17)
\end{aligned}
$$

These equations are straightforward to obtain (see Sec. 4.4.2 where a similar action was used). It should be noted that by introducing these scalar fields, one is localising the nonlocal field equations given by (8.15). Let us study non-local cosmology from the field equations (8.17). Let us assume a flat FLRW cosmology with the tetrad in

Euclidean coordinates $e^a_\beta = \mathrm{diag}(1, a(t), a(t), a(t))$, and write the FLRW metric as $ds^2 = dt^2 - a(t)^2(dx^2+dy^2+dz^2)$ for a spatially flat spacetime. Thus, the cosmological field equations can be written as

$$3H^2(1+\theta - f(\phi)) = \frac{1}{2}\dot{\theta}\dot{\phi} + \kappa(\rho_m + \rho_\Lambda)\,, \tag{8.18}$$

$$(1+\theta - f(\phi))(3H^2 + 2\dot{H}) = -\frac{1}{2}\dot{\theta}\dot{\phi} + 2H(\dot{\theta} - \dot{f}(\phi)) - \kappa(p_m + p_\Lambda)\,, \tag{8.19}$$

where dots represent differentiation with respect to the cosmic time and it was assumed that the matter is described by the energy density of standard matter ρ_{m} and an energy density related to a cosmological constant ρ_Λ. The equations for the scalar fields can be written as

$$-6H^2 f'(\phi) + 3H\dot{\theta} + \ddot{\theta} = 0\,, \tag{8.20}$$

$$3H\dot{\phi} + 6H^2 + \ddot{\phi} = 0\,. \tag{8.21}$$

These equations describe a non-local model of Teleparallel cosmology. One can take into account constraints on them from recent cosmological data. If one assumes $f(\phi) = Ae^{n\phi}$, where A and n are constants, the dynamics of the model given by the system (8.18)-(8.21) can be tested. In order to constrain the free parameters of the model, SNe Ia (type Ia supernovae) + BAO (baryon acoustic oscillations) + CC (cosmic chronometers) + H_0 were used in Bahamonde et al. (2017d). At 1σ and 2σ confidence levels (CL) from the joint analysis SNIa + BAO + CC + H_0, it was found the following constraints : $A = -0.009713^{+0.017}_{-0.021}$, $n = 0.02086^{+0.0013}_{-0.0208}$, $h = 0.7127^{+0.013}_{-0.015}$ km/s/Mpc, $\Omega_\Lambda = 0.7018^{+0.018}_{-0.02}$, and $\Omega_{m0} = 0.2981^{+0.02}_{-0.018}$, with $\chi^2_{\mathrm{min}} = 707.4$. For further details about the data used, see Bahamonde et al. (2017d). It should be noted that the constraints are close to the ΛCDM model, without any evidence for non-local effects in the present analysis, which here are characterised by the

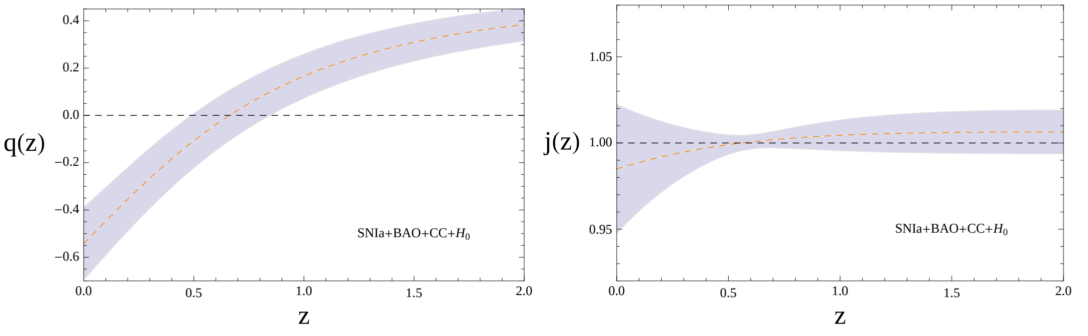


Figure 8.1: Reconstruction of the $q(z)$ (deceleration parameter) and $j(z)$ (jerk parameter) from SNIa + BAO + CC + H_0 data set at 1σ CL.

parameters A and n. Let us define two important cosmological parameters:

$$q(z) = -\frac{\ddot{a}a}{\dot{a}^2}, \tag{8.22}$$

$$j(z) = \frac{\dddot{a}}{aH^3}. \tag{8.23}$$

Here, $q(z)$ and $j(z)$ are known as the *deceleration parameter* and the *jerk parameter* respectively. The jerk parameter is a dimensionless parameter which measures the third derivative of the scale factor with respect to the cosmic time whereas the deceleration parameter measures the cosmic acceleration of the expansion of the universe. They are usually expressed in terms of the redshift z. A positive $j(z)$ with a negative deceleration parameter $q(z)$, gives a deceleration to acceleration transition. In order to investigate kinematic effects, Fig. 8.1 shows the deceleration $q(z)$ and jerk $j(z)$ parameters as a function of the redshift. The standard error propagation is used using the best fit values from SNIa + BAO + CC + H_0 in the reconstruction (grey region) of both parameters. On the left panel, $q(z)$ is shown, where the transition from decelerated to accelerated phase occurs at $z \sim 0.6$, with $q_0 = -0.54 \pm 0.15$. As expected, $q \to 1/2$ for high redshift. The right panel shows the jerk parameter $j(z)$ obtained from the joint analysis, the dotted black line ($j = 1$) represents the ΛCDM model. In general, small deviations can be noted when non-local effects are introduced, but such effects are close to the dynamics of the ΛCDM model.

The free parameters of the non-local Teleparallel cosmology are strongly constrained by present cosmological data. Furthermore, since non-local GR and non-local Teleparallel gravity are fundamentally different, it is possible that future cosmological data can be used to test which of these two proposals is the correct theory of gravity. As these theories are fundamentally different, experiments can be performed to distinguish each other. Here some possible experimental tests will be proposed that could be pursued in the near future to know which is the correct theory of gravity.

The first experiment that could be performed is based on the violation of the equivalence principle, as this could only occur in non-local Teleparallel gravity. The accuracy of the weak equivalence principle has been measured from the acceleration of Beryllium and Titanium test bodies using a rotating torsion balance (Schlamminger et al., 2008). It has been found that for acceleration a, the accuracy is of the order $\Delta a/a \sim 1.8 \cdot 10^{-13}$. The accuracy was increased to $\Delta a/a \sim 2 \cdot 10^{-15}$ using The MICROSCOPE satellite Touboul et al. (2017) and in the following years, the sensitivity will be increased even more. It is possible to use more accurate future experiments to observe a violation of the weak equivalence principle. As such, a violation would only occur in non-local Teleparallel gravity and it could be used as an experimental test to know which of these theories is the correct theory of nature.

One can also test these theories by performing experiments using photon time delay and gravitational red shifts measured by high energy gamma rays. Both these non-local effects would produce different photon time delays that have been observed by measuring the round trip time of a bounced radar beam off the surface of Venus (Shapiro et al., 1971). This kind of experiments, performed with more precision, can be compared with effects produced by the non-local deformation of both theories, and any discrepancy between results could be used to discriminate between them. Similarly, gravitational red shift could be used to distinguish between the two theories. The gravitational red shift derived by gamma rays of energy 14.4×10^{-6}GeV has been measured in the Pound-Snider experiment (Pound & Snider, 1965), and it is possible to perform similar experiments with higher energy gamma rays with present day technology. Since non-local Teleparallel gravity and non-local GR predict different gravitational red sifting, such difference could be compared with these more accurate experiments.

8.3 Generalised non-local gravity and cosmology

Let us now present a generalisation of (8.4) and standard non-local gravity described by the scalar curvature (Deser & Woodard, 2007), which for simplicity, one can label it as a generalised non-local Teleparallel gravity (GNTG). Its action is given by

$$\begin{aligned} \mathcal{S}_{\rm GNTG} = \mathcal{S}_{\rm TEGR} + \frac{1}{2\kappa^2} \int d^4x\, e(x)\, (\xi T(x) + \chi B(x)) f\Big((\mathring{\Box}^{-1} T)(x), (\mathring{\Box}^{-1} B)(x) \Big) \\ + \int d^4x\, e\, \mathcal{L}_{\rm m}\,. \end{aligned} \tag{8.24}$$

Here, T is the torsion scalar, B is a boundary term and $f(\mathring{\Box}^{-1}T, \mathring{\Box}^{-1}B)$ is now an arbitrary function of the non-local torsion and the non-local boundary term. The Greek letters ξ and χ denote coupling constants. It is easily seen, that by choosing $\xi = -\chi = -1$ one obtains the standard Ricci scalar introduced in Deser & Woodard (2007). From (8.5), one can directly see that the following relation is also true

$$\mathring{\Box}^{-1} \mathring{R} = -\mathring{\Box}^{-1} T + \mathring{\Box}^{-1} B\,, \tag{8.25}$$

and thus, if $f(\mathring{\Box}^{-1}T, \mathring{\Box}^{-1}B) = f(-\mathring{\Box}^{-1}T + \mathring{\Box}^{-1}B)$, the action takes the well known form $\mathring{R} f(\mathring{\Box}^{-1}\mathring{R})$ given by the action described in Deser & Woodard (2007). Moreover, non-local Teleparallel gravity given by the action (8.4) is recovered if $\chi = 0$ and

$f(\mathring{\Box}^{-1}T, \mathring{\Box}^{-1}B) = f(\mathring{\Box}^{-1}T)$.

Since the field equations for the GNTG theory are very cumbersome, the action (8.24) will be rewritten in a more suitable way using scalar fields, according to Nojiri & Odintsov (2008). Specifically, the action can be rewritten introducing four scalar fields $\phi, \psi, \theta, \zeta$ as follows

$$\begin{aligned}\mathcal{S}_{\rm GNTG} &= \mathcal{S}_{\rm TEGR} + \frac{1}{2\kappa^2}\int d^4x\, e\,\Big[\,(\xi T + \chi B)f(\phi,\varphi) + \theta(\mathring{\Box}\phi - T) + \zeta(\mathring{\Box}\varphi - B)\Big] \\ &\quad + \int d^4x\, e\,\mathcal{L}_{\rm m}\,, \\ &= \mathcal{S}_{\rm TEGR} + \frac{1}{2\kappa^2}\int d^4x\, e\,[\,(\xi T + \chi B)f(\phi,\varphi) - \partial_\mu\theta\partial^\mu\phi - \theta T - \partial_\mu\zeta\partial^\mu\varphi - \zeta B] \\ &\quad + \int d^4x\, e\,\mathcal{L}_{\rm m}\,. \end{aligned} \tag{8.26}$$

By varying this action with respect to θ and ζ one obtains $\phi = \mathring{\Box}^{-1}T$ and $\varphi = \mathring{\Box}^{-1}B$ respectively. In addition, by varying this action with respect to ϕ and φ one obtains

$$\mathring{\Box}\theta = (\xi T + \chi B)\frac{\partial f(\phi,\varphi)}{\partial\phi}\,, \tag{8.27}$$

$$\mathring{\Box}\zeta = (\xi T + \chi B)\frac{\partial f(\phi,\varphi)}{\partial\varphi}\,. \tag{8.28}$$

In the scalar representation it is not straightforward how curvature or Teleparallel non-local gravity can be recovered. Let us explicitly recover these theories under scalar formalism. For example, by setting $\xi = -1 = -\chi$, $f(\phi,\varphi) = f(-\phi+\varphi)$, and $\theta = -\zeta$ one obtains standard non-local curvature gravity, namely

$$\mathcal{S}_{\rm standard-NL} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\Big[\,\mathring{R} + \mathring{R}f(\psi) - \partial_\mu\zeta\partial^\mu\psi - \zeta\mathring{R}\Big] + \mathcal{S}_{\rm m}\,, \tag{8.29}$$

$$= \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\,\Big[\,\mathring{R} + \mathring{R}f(\mathring{\Box}^{-1}\mathring{R})\Big] + \mathcal{S}_{\rm m}\,, \tag{8.30}$$

where $\psi = -\phi + \varphi$. On the other hand, the non-local TEGR is recovered if in the action (8.26) one chooses $\xi = 1$, $\chi = 0$, $f(\phi,\varphi) = f(\phi)$ and $\zeta = 0$. One then obtains

$$\begin{aligned}
\mathcal{S}_{\rm Teleparallel-NL} &= \frac{1}{2\kappa^2}\int d^4x\, e\,[T\,(f(\phi)+1) - \partial_\mu\theta\partial^\mu\phi - \theta T] + \mathcal{S}_{\rm m}\,, && (8.31)\\
&= \frac{1}{2\kappa^2}\int d^4x\, e\left[T\left(f(\mathring{\Box}^{-1}T)+1\right)\right] + \mathcal{S}_{\rm m}\,, && (8.32)
\end{aligned}$$

which is equivalent as the action considered before (see Eq. (8.4)). A more general class of theories, like $T + (\xi T + \chi B)f(\mathring{\Box}^{-1}T)$ or $T + (\xi T + \chi B)f(\mathring{\Box}^{-1}B)$ can be obtained by setting $f(\phi,\varphi) = f(\phi)$ and $f(\phi,\varphi) = f(\varphi)$ respectively. Obviously, in these cases, one can change the values of ξ and χ to obtain other couplings like

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\, e\left[T + Bf(\mathring{\Box}^{-1}T)\right] + \mathcal{S}_{\rm m}\,, \quad (8.33)$$

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\, e\left[T + Tf(\mathring{\Box}^{-1}B)\right] + \mathcal{S}_{\rm m}\,, \quad (8.34)$$

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\, e\left[T + Bf(\mathring{\Box}^{-1}B)\right] + \mathcal{S}_{\rm m}\,. \quad (8.35)$$

Fig. 8.2 is a comprehensive diagram representing all the theories that can be recovered from the action (8.26). Here, we have not considered unnatural couplings like $\mathring{R}f(\mathring{\Box}^{-1}T)$ or $Tf(\mathring{\Box}^{-1}\mathring{R})$ because $\mathring{R}$ and T, B are quantities defined in different connections, so mixed terms like $\mathring{R}f(\mathring{\Box}^{-1}T)$ are not natural terms to consider. The above half part of the figure represents different non-local Teleparallel theories and the below part of it, the standard curvature counterpart. As can be easily seen, only TEGR and GR dynamically coincide while this is not the case for other theories defined by T, $\mathring{R}$ and B. From a fundamental point of view, this fact is extremely relevant because the various representations of gravity can have different dynamical contents. For example, it is well known that $f(T)$ gravity gives second order field equations while $f(\mathring{R})$ gravity, in metric representation, is fourth order. These facts are strictly related to the dynamical roles of torsion and curvature and their discrimination at fundamental level could constitute an important insight to really understand the nature of gravitational field (see Cai et al. (2016) for a detailed discussion).

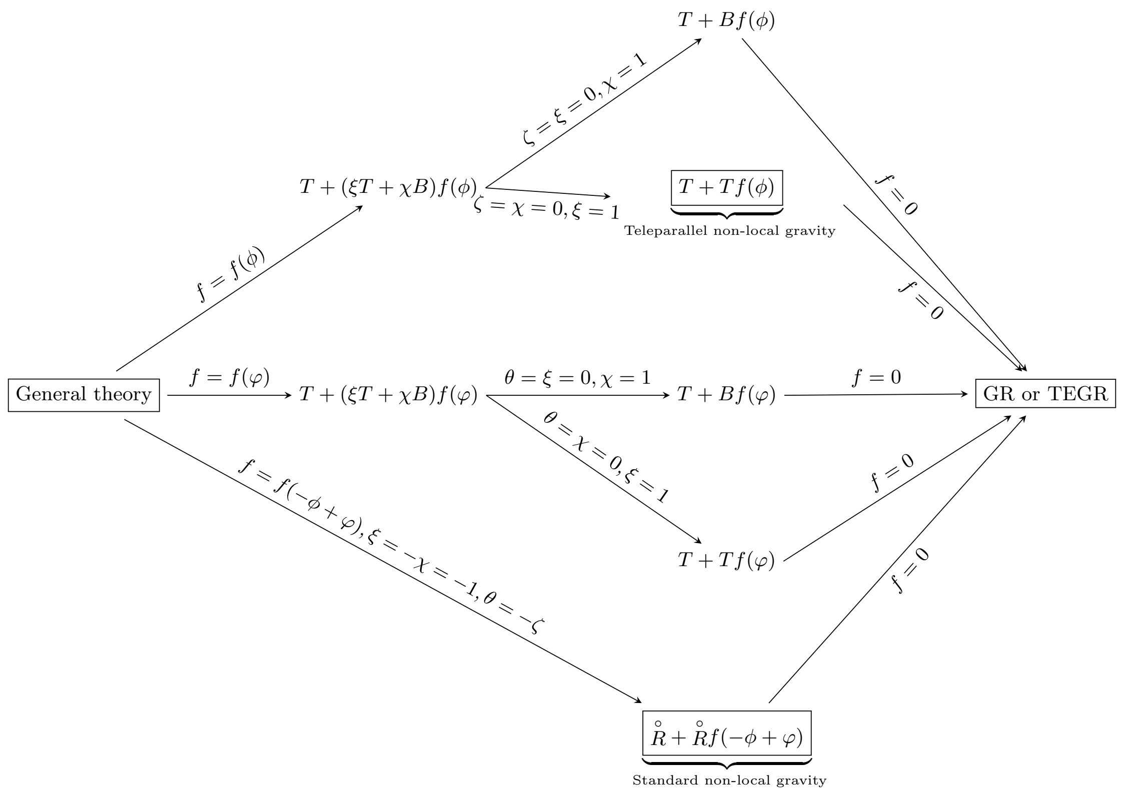


Figure 8.2: Relationship between GNLG and other gravity theories. The diagram shows how to recover the different theories of gravity starting from the scalar-field representation of the general theory. It should be noted that $\phi = \mathring{\Box}^{-1}T$ and $\varphi = \mathring{\Box}^{-1}B$ so that $-\phi + \varphi = \mathring{\Box}^{-1}\mathring{R}$. Clearly, the curvature and torsion representations "converge" only for the linear theories in $\mathring{R}$, the GR, and in T, the TEGR.

By varying the generalised non-local action (8.26) with respect to the tetrads, one obtains the following field equations

$$
\begin{aligned}
&2(1 - \xi(f(\phi,\varphi) - \theta)) \left[e^{-1}\partial_\mu (e S_a{}^{\mu\beta}) - E_a^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\beta\mu} - \frac{1}{4}E_a^\beta T \right] - \\
&- \frac{1}{2}\Big[(\partial^\lambda\theta)(\partial_\lambda\phi)E_a^\beta - (\partial^\beta\theta)(\partial_a\phi) - (\partial_a\theta)(\partial^\beta\phi)\Big] - \frac{1}{2}\Big[(\partial^\lambda\zeta)(\partial_\lambda\varphi)E_a^\beta - (\partial^\beta\zeta)(\partial_a\varphi) \\
&\qquad - (\partial_a\zeta)(\partial^\beta\varphi)\Big] + 2\,\partial_\mu\Big[f(\phi,\varphi)(\xi+\chi) - \theta - \zeta\Big] E_a^\rho S_\rho{}^{\mu\beta} + \\
&\qquad\qquad \Big(E_a^\beta \mathring{\Box} - E_a^\mu \mathring{\nabla}^\beta \mathring{\nabla}_\mu\Big)(\zeta - \chi f(\phi,\varphi)) = \kappa^2 \mathcal{T}_a^\beta \,. \quad (8.36)
\end{aligned}
$$

Let us now take into account the tetrad $e^a_\beta = \mathrm{diag}(1, a(t), a(t), a(t))$, which reproduces the flat FLRW metric $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$. For this geometry, the modified FLRW equations are

$$
\begin{aligned}
3H^2(\theta - \xi f + 1) &= \frac{1}{2}\dot{\zeta}\dot{\varphi} + \frac{1}{2}\dot{\theta}\dot{\phi} + 3H(\dot{\zeta} - \chi\dot{f}) + \kappa^2\rho_m\,, \qquad (8.37)\\
(2\dot{H} + 3H^2)(\theta - \xi f + 1) &= -\frac{1}{2}\dot{\zeta}\dot{\varphi} - \frac{1}{2}\dot{\theta}\dot{\phi} - \dot{f}(2H(\xi + 2\chi) + \chi)\\
&\quad +2H(2\dot{\zeta} + \dot{\theta}) + \ddot{\zeta} - \kappa^2 p_{\mathrm{m}}\,, \qquad (8.38)
\end{aligned}
$$

where ρ_m and p_m are the energy density and the pressure of the cosmic fluid respectively and dots denote differentiation with respect to the cosmic time. The equations for the scalar fields can be written as

$$
\begin{aligned}
6H^2 + 3H\dot{\phi} + \ddot{\phi} &= 0\,, \qquad (8.39)\\
6(\dot{H} + 3H^2) + 3H\dot{\varphi} + \ddot{\varphi} &= 0\,, \qquad (8.40)\\
-6H^2\left(\xi f_\varphi + 3\chi f_\varphi\right) - 6\dot{H}\chi f_\varphi + 3H\dot{\zeta} + \ddot{\zeta} &= 0\,, \qquad (8.41)\\
-6H^2\left(\xi f_\phi + 3\chi f_\phi\right) - 6\dot{H}\chi f_\phi + 3H\dot{\theta} + \ddot{\theta} &= 0\,, \qquad (8.42)
\end{aligned}
$$

where the sub-indices represent the partial derivative $f_\phi = \partial f/\partial\phi$ and $f_\varphi = \partial f/\partial\varphi$. In the following section, the Noether symmetry approach will be used to seek for conserved quantities.

8.4 The Noether symmetry approach

Let us use the Noether symmetry approach in order to find symmetries and cosmological solutions for the generalised action (8.26). For simplicity, hereafter the vacuum case will be assumed, i.e., $\rho_m = p_m = 0$. It can be shown that the torsion scalar and the boundary term in a flat FLRW are given by

$$T = -6H^2\,, \quad B = -18H^2 - 6\dot{H}\,, \tag{8.43}$$

so that the action (8.26) takes the following form

$$\begin{aligned} \mathcal{S}_{\rm GNLT} &= 2\pi^2 \int a^3 dt \Big\{ -6\frac{\dot{a}^2}{a^2}(\xi f(\phi,\varphi) - \theta - 1) - 6\Big(2\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\Big)(\chi f(\phi,\varphi) - \zeta) \\ &\quad -\dot{\theta}\dot{\phi} - \dot{\zeta}\dot{\varphi}\Big\}\,. \end{aligned} \tag{8.44}$$

Considering the procedure in Capozziello et al. (1996), one finds that the point-like Lagrangian is given by

$$\mathcal{L} = 6a\dot{a}^2(\theta + 1 - \xi f(\phi,\varphi)) + 6a^2\dot{a}(\chi \dot{f}(\phi,\varphi) - \dot{\zeta}) - a^3\dot{\theta}\dot{\phi} - a^3\dot{\zeta}\dot{\varphi}\,. \tag{8.45}$$

Let us now apply a more general formalism than was done in Sec. 5.2.1. The generator of infinitesimal transformations is given by (Paliathanasis, 2014)

$$X = \lambda(t, x^\mu)\partial_t + \eta^i(t, x^\mu)\partial_i\,, \tag{8.46}$$

where $x^\mu = (a, \theta, \phi, \varphi, \zeta)$ and the vector η^i is

$$\eta^i(t, x^\mu) = \Big(\eta^a, \eta^\theta, \eta^\phi, \eta^\varphi, \eta^\zeta\Big)\,. \tag{8.47}$$

In general, each function depends on t and x^μ. If there exists a function $h = h(t, x^\mu)$ such that

$$X^{[1]}\mathcal{L} + \mathcal{L}\frac{d\lambda}{dt} = \frac{dh}{dt}\,, \tag{8.48}$$

where $\mathcal{L} = \mathcal{L}(t, x^\mu, \dot{x}^\mu)$ is the Lagrangian of a system and $X^{[1]}$ is the first prolongation of the vector X (Paliathanasis, 2014), then the Euler-Lagrange equations remain invariant under these transformations. The generator is a Noether symmetry of the system described by $\mathcal{L}$ and the relative integral of motion is given by

$$I = \lambda \left(\dot{x}^\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \mathcal{L} \right) - \eta^i \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + h \,. \tag{8.49}$$

The set of generalised coordinates $x^\mu = \{t, a, \theta, \phi, \varphi, \zeta\}$ gives rise to the configuration space $\mathcal{Q} \equiv \{x^\mu \,, \mu = 1, ..., 6\}$ and the tangent space $\mathcal{TQ} \equiv \{x^\mu, \dot{x}^\mu\}$ of the Lagrangian $\mathcal{L} = \mathcal{L}(t, x^\mu, \dot{x}^\mu)$. Clearly, the procedure can be applied to many different models starting from Fig. 8.2.

From Eq. (8.48), considering the general point-like Lagrangian (8.45), the full Noether conditions give rise to 43 differential equations. However, there are 24 differential equations that can be easily solved giving us the functions

$$\lambda = \lambda(t) \,, \quad h = h(a, \theta, \phi, \psi, \zeta) \,, \quad \eta_\phi = \eta_\phi(a, \phi, \psi, \zeta, t) \,. \tag{8.50}$$

The remaining 19 differential equations are the followings:

$$6h_{,\theta} + a^3 \eta_{\phi,t} = 0 \,, \tag{8.51}$$

$$6h_{,\psi} + a^2 \left(a\eta_{\zeta,t} - 6\chi f_{,\psi} \eta_{a,t} \right) = 0 \,, \tag{8.52}$$

$$h_{,\phi} + a^2 \left(a\eta_{\theta,t} - 6\chi f_{,\phi} \eta_{a,t} \right) = 0 \,, \tag{8.53}$$

$$h_{,\zeta} + a^2 \left(6\eta_{a,t} + a\eta_{\psi,t} \right) = 0 \,, \tag{8.54}$$

$$a \left(\eta_{\phi,\psi} + \eta_{\zeta,\theta} \right) - 6\chi f_{,\psi} \eta_{a,\theta} = 0 \,, \tag{8.55}$$

$$a \left(\eta_{\phi,\zeta} + \eta_{\psi,\theta} \right) + 6\eta_{a,\theta} = 0 \,, \tag{8.56}$$

$$6\chi f_{,\phi} \eta_{a,\phi} - a\eta_{\theta,\phi} = 0 \,, \tag{8.57}$$

$$6\chi f_{,\psi} \eta_{a,\psi} - a\eta_{\zeta,\psi} = 0 \,, \tag{8.58}$$

$$6\eta_{a,\zeta} + a\eta_{\psi,\zeta} = 0 \,, \tag{8.59}$$

$$h_{,a} - 6a^2 \left(\chi f_{,\phi} \eta_{\phi,t} + \chi f_{,\psi} \eta_{\psi,t} - \eta_{\zeta,t} \right) + 12a(\xi f - \theta - 1)\eta_{a,t} = 0 \,, \tag{8.60}$$

$$6a\left(\chi f_{,\psi}\eta_{\psi,\theta}-\eta_{\zeta,\theta}\right)+12(\theta+1-\xi f)\eta_{a,\theta}-a^{2}\eta_{\phi,a}=0\,, \tag{8.61}$$

$$6\chi(f_{,\phi}\eta_{a,\psi}+f_{,\psi}\eta_{a,\phi})-a\left(\eta_{\theta,\psi}+\eta_{\zeta,\phi}\right)=0\,, \tag{8.62}$$

$$6(\chi f_{,\phi}\eta_{a,\zeta}-\eta_{a,\phi})-a\left(\eta_{\psi,\phi}+\eta_{\theta,\zeta}\right)=0\,, \tag{8.63}$$

$$6\chi f_{,\phi}\eta_{a,\theta}-a\left(\eta_{\phi,\phi}+\eta_{\theta,\theta}-\lambda_{,t}\right)-3\eta_{a}=0\,, \tag{8.64}$$

$$6\chi f_{,\psi}\eta_{a,\zeta}-6\eta_{a,\psi}-3\eta_{a}-a\left(\eta_{\psi,\psi}+\eta_{\zeta,\zeta}-\lambda_{,t}\right)=0\,, \tag{8.65}$$

$$\begin{aligned}&6a\left(\chi f_{,\phi}\eta_{\phi,\phi}+\chi f_{,\phi\phi}\eta_{\phi}+\chi f_{,\phi}\eta_{a,a}+\chi f_{,\psi}\eta_{\psi,\phi}+\chi f_{,\phi\psi}\eta_{\psi}-\chi\lambda_{,t}f_{,\phi}-\eta_{\zeta,\phi}\right)\\&+12\chi f_{,\phi}\eta_{a}+12(\theta+1-\xi f)\eta_{a,\phi}-a^{2}\eta_{\theta,a}=0\,,\end{aligned} \tag{8.66}$$

$$\begin{aligned}&6a\left(\chi f_{,\phi}\eta_{\phi,\zeta}+\chi f_{,\psi}\eta_{\psi,\zeta}-\eta_{a,a}-\eta_{\zeta,\zeta}+\lambda_{t}\right)+12(\theta+1-\xi f)\eta_{a,\zeta}\\&-12\eta_{a}-a^{2}\eta_{\psi,a}=0\,,\end{aligned} \tag{8.67}$$

$$\begin{aligned}&6a\left(\chi f_{,\phi}\eta_{\phi,\psi}+\chi f_{,\phi\psi}\eta_{\phi}+\chi f_{,\psi}\left(\eta_{a,a}+\eta_{\psi,\psi}-\lambda_{t}\right)+\chi f_{,\psi\psi}\eta_{\psi}-\eta_{\zeta,\psi}\right)\\&+12\chi f_{,\psi}\eta_{a}+12(\theta+1-\xi f)\eta_{a,\psi}-a^{2}\eta_{\zeta,a}=0\,,\end{aligned} \tag{8.68}$$

$$\begin{aligned}&a\Big(\chi a f_{,\phi}\eta_{\phi,a}-\xi f_{,\phi}\eta_{\phi}+\chi a f_{,\psi}\eta_{\psi,a}-\xi f_{,\psi}\eta_{\psi}-2\xi f\eta_{a,a}+\xi\lambda_{,t}f+2\theta\eta_{a,a}+2\eta_{a,a}\\&+\eta_{\theta}-a\eta_{\zeta,a}-(\theta+1)\lambda_{t}\Big)+(\theta+1-\xi f)\eta_{a}=0\,.\end{aligned} \tag{8.69}$$

Here commas denote partial derivatives. Clearly, being a system of non-linear partial differential equations, the solution is not unique. This means that several Noether symmetries can be selected according to different functions. In the next subsections, Noether symmetries will be found in specific non-local Lagrangians. The first subsection will be focused on the Teleparallel non-local case where $f=Tf(\overset{\circ}{\square}^{-1}T)$ and the second subsection will be focused on the standard non-local curvature case where $\overset{\circ}{R}f(\overset{\circ}{\square}^{-1}\overset{\circ}{R})$.

8.4.1 Noether symmetries in Teleparallel non-local gravity with coupling $Tf(\mathring{\Box}^{-1}T)$

8.4.1.1 Finding Noether symmetries

Let us first study the case where one recovers the Teleparallel non-local case studied in Sec. 8.1.1. In this case, the torsion scalar T is coupled to a non-local function evaluated at the torsion scalar, that is $f(\mathring{\Box}^{-1}T) = f(\phi)$. For Noether symmetries, one needs to consider,

$$f(\phi,\varphi) = f(\phi)\,, \quad \chi = 0\,, \quad \xi = 1 \quad \text{and} \quad \zeta = 0\,, \tag{8.70}$$

in the general action (8.26) and thus the point-like Lagrangian (8.45) becomes

$$\mathcal{L} = 6a\left(-f(\phi) + \theta + 1\right)\dot{a}^2 - a^3\dot{\theta}\dot{\phi}\,. \tag{8.71}$$

From Eqs. (8.50)-(8.69), it can be immediately seen that the dependence on the coordinates of the Noether vector components is

$$\begin{aligned}
\lambda(a,\theta,\phi,t) &= \lambda(t)\,, &(8.72)\\
\eta_a(a,\theta,\phi,t) &= \eta_a(a,\theta,\phi,t)\,, &(8.73)\\
\eta_\phi(a,\theta,\phi,t) &= \eta_\phi(a,\phi,t)\,, &(8.74)\\
\eta_\theta(a,\theta,\phi,t) &= \eta_\theta(a,\theta,t)\,, &(8.75)\\
h(a,\theta,\phi,t) &= h(a,\theta,\phi)\,. &(8.76)
\end{aligned}$$

It should be remarked that there is no need to impose any assumption to find the symmetries. Even though the system is given by 19 differential equations, one can solve it in full generality. This computation is straightforward but lengthy so it will not be explicitly showed here. By solving Eqs. (8.51)-(8.69) for f, one obtains

$$c_1 f'(\phi) - c_2 f(\phi) + c_2 - c_3 = 0\,, \tag{8.77}$$

where c_1, c_2 and c_3 are constants. There are two non trivial solutions ($f \neq$ constant) to (8.77) depending on the value of c_2, i.e.

$$f(\phi) = \begin{cases} c_7 e^{\frac{c_2\phi}{c_1}} - \frac{c_3}{c_2} + 1\,, & c_2 \neq 0\,, \\ \\ c_7 + \frac{c_3}{c_1}\phi\,, & c_2 = 0\,, \end{cases} \tag{8.78}$$

where c_7 is another integration constant. From (8.32), one can notice that for having a TEGR (or GR) background we must have that $c_3 = c_2$ in the exponential form and $c_7 = 0$ in the linear form. The Noether vector has the following form

$$X = (c_4 + c_5 t)\partial_t - \frac{1}{3}(c_2 - c_4)a\partial_a + (c_3 + c_2\theta)\partial_\theta + c_1\partial_\phi\,, \tag{8.79}$$

and the integral of motion is

$$\begin{aligned} I \;=\; & \left[4a^2\left(c_2 - c_4\right)\dot{a} + 6a\dot{a}^2\left(c_4 t + c_5\right)\right]\left(-f(\phi) + \theta + 1\right) + c_6 \\ & a^3 c_1\dot{\theta} + a^3 c_2(\theta + 1)\dot{\phi} - a^3\left(c_4 t + c_5\right)\dot{\theta}\dot{\phi}\,. \end{aligned} \tag{8.80}$$

8.4.1.2 Cosmological solutions

In the previous subsection the form of the function f was found to be constrained as an exponential or a linear form of the non-local term (8.78). It can be shown that for the linear form, there are no power-law or de-Sitter solution. Here, we will find solutions for the exponential form of the coupling function.

As pointed out before, it is physically convenient to choose $c_2 = c_3$ in order to have a GR (or TEGR) background. Hence, in this section, this condition will be assumed for the constants. For the exponential form of the function $f(\phi)$ given by (8.78), the point-like Lagrangian (8.71) takes now the form

$$\mathcal{L} = -6a\dot{a}^2\left(c_7 e^{\frac{c_3\phi}{c_1}} - \theta - 1\right) - a^3\dot{\theta}\dot{\phi}\,, \tag{8.81}$$

so that the Euler-Lagrange equations are given by

$$c_1\left(4\dot{H}(c_7 e^{\frac{c_2\phi}{c_1}} - \theta - 1) - \dot{\theta}\dot{\phi}\right) + H\left(4c_2c_7\dot{\phi}e^{\frac{c_2\phi}{c_1}} - 4c_1\dot{\theta}\right) + 6c_1H^2\left(c_7 e^{\frac{c_2\phi}{c_1}} - \theta - 1\right) = 0\,, \tag{8.82}$$

$$6H^2 + 3H\dot{\phi} + \ddot{\phi} = 0\,, \tag{8.83}$$

$$-\frac{6c_2c_7}{c_1}H^2 e^{\frac{c_2\phi}{c_1}} + \ddot{\theta} + 3H\dot{\theta} = 0\,, \tag{8.84}$$

$$6H^2\left(-c_7 e^{\frac{c_2\phi}{c_1}} + \theta + 1\right) - \dot{\theta}\dot{\phi} + 6\theta H^2 = 0\,, \tag{8.85}$$

for a, θ, ϕ and the energy equation, respectively. If one considers de-Sitter solution for the scale factor,

$$a(t) = e^{H_0 t} \Rightarrow H(t) = H_0\,,$$

one immediately finds from (8.83) that

$$\phi(t) = -2H_0 t - \frac{\phi_1 e^{-3H_0 t}}{3H_0} + \phi_2\,. \tag{8.86}$$

For the sake of simplicity, $\phi_1 = \phi_2 = 0$ will be considered otherwise Eq. (8.84) cannot be integrated easily. By this assumption, one can directly finds that

$$\theta(t) = e^{-3H_0 t}\left(-c_7(3H_0 t + 1) - \frac{\theta_1}{3H_0}\right) + \theta_2\,, \tag{8.87}$$

where θ_1 and θ_2 are integration constants and the branch $c_1 = 2c_2/3$ was needed to choose, otherwise Eq. (8.82) cannot be satisfied. Hence, from (8.82) one directly see that $\theta_2 = -1$, giving us the following cosmological solution,

$$a(t) = e^{H_0 t}\,, \quad \phi(t) = -2H_0 t\,, \quad \theta(t) = e^{-3H_0 t}\left(-c_7(3H_0 t + 1) - \frac{\theta_1}{3H_0}\right) - 1\,, \quad (8.88)$$

and

$$f(\phi) = c_7 e^{-3H_0 t}\,. \quad (8.89)$$

If one considers that the scale factor behaves as a power-law $a(t) = a_0 t^p$, where p is a constant, from (8.83) one directly finds that

$$\phi(t) = \frac{6p^2 \log(t - 3pt)}{1-3p} + \frac{\phi_1}{1-3p} t^{1-3p} + \phi_0\,, \quad (8.90)$$

where ϕ_1 and ϕ_0 are integration constants that for simplicity, will be assumed that are zero, otherwise (8.84) cannot be integrated directly (as was done before). By doing this, one obtains

$$\theta(t) = \frac{c_1 t^{1-3p}}{1-3p} + c_2 + \frac{c_7(3p-1)(c_1 - 3c_1 p)}{c_1(1-3p)^2 - 6c_2 p^2}(t - 3pt)^{\frac{6c_2 p^2}{c_1 - 3c_1 p}}\,, \quad (8.91)$$

where θ_0 and θ_1 are integration constants and also $c_1 \neq \frac{6c_2 p^2}{(3p-1)^2}$ and $p \neq 1/3$ were assumed since there are not solutions for these other two branches. By replacing this solution into (8.82) one obtains that $c_2 = \frac{c_1(2-9p+9p^2)}{6p^2}$ and $\theta_1 = -1$ yielding the following solution

$$\phi(t) = \frac{6p^2 \log(t - 3pt)}{1-3p}\,, \quad \theta(t) = c_7(1-3p)^{3-3p} t^{2-3p} + \frac{\theta_0 t^{1-3p}}{1-3p} - 1\,, \quad (8.92)$$

$$a(t) = a_0 t^p\,, \quad f(\phi) = c_7 e^{\frac{\left(9p^2 - 9p + 2\right)\phi}{6p^2}}\,. \quad (8.93)$$

As seen, the Noether symmetry approach constrained the function f to be either an exponential or a linear one. Then, it was found that the exponential case admits both de-Sitter and power-law solutions. Thus, the Noether symmetry approach provides a good tool to find cosmological solutions.

8.4.2 Noether symmetries in curvature non-local gravity with coupling $\overset{\circ}{R} f(\overset{\circ}{\Box}^{-1}\overset{\circ}{R})$

Let us find now Noether symmetries for the case where curvature non-local gravity is considered. Then, the coupling $\overset{\circ}{R} f(\overset{\circ}{\Box}^{-1}\overset{\circ}{R})$ is present in the action. To recover this case, one must set

$$f(\phi,\varphi)=f(-\phi+\varphi)=f(\psi)\,,\quad \chi=1\,,\quad \xi=-1\,,\quad \theta=-\zeta\,. \tag{8.94}$$

In this way, the point-like Lagrangian (8.45) reads as follows

$$\mathcal{L}=6a\dot{a}^2(f(\psi)+\theta+1)+6a^2\dot{a}(f'(\psi)\dot{\psi}+\dot{\theta})+a^3\dot{\theta}\dot{\psi}\,. \tag{8.95}$$

and Noether condition equation (8.48), gives a system of 18 differential equations. Also this is a special case of that presented in (8.51)-(8.69). The result is

$$\lambda(a,\theta,\psi,t)=\lambda(t)\quad \text{and}\quad h(a,\theta,\psi,t)=h(a,\theta,\psi)\,, \tag{8.96}$$

and the system reduces to 9 equations. However, the full system is still difficult to solve without any assumption. A simple assumption is choosing $h(a,\theta,\psi)=\text{constant}$. The last two equations of Noether condition for $f(\psi)$ are

$$2c_2f'(\psi)+c_1f(\psi)+c_1-c_3 \;=\; 0\,, \tag{8.97}$$

$$2c_2f''(\psi)+c_1f'(\psi) \;=\; 0\,. \tag{8.98}$$

and the Noether vector results to be

$$X = (c_5 + c_4 t)\partial_t + \frac{1}{3}a(c_4 - c_1)\partial_a + (c_3 + c_1\theta)\partial_\theta - 2c_2\partial_\psi\,. \tag{8.99}$$

Eqs. (8.97) and (8.98) are easily solved and the form of f is

$$f(\psi) = \begin{cases} -1 + \frac{c_3}{c_1} + c_6 e^{-\frac{c_1}{c_2}\psi} & c_1 \neq 0\,, \\ \\ c_6 + \frac{c_3}{2c_2}\psi\,. & c_1 = 0 \end{cases}\,. \tag{8.100}$$

Again, the form of the function is either exponential or linear in $\psi = \mathring{\Box}^{-1}\mathring{R}$. This result is very interesting since, without further assumptions than $h = const.$, the symmetries give the same kind of couplings for both Teleparallel and curvature non-local theories. These two couplings can be particularly relevant to get a renormalisable theory of gravity. As discussed in Modesto & Rachwal (2014, 2017), the form of the coupling is extremely important to achieve a regular theory. In particular, the exponential coupling plays an important role in calculations. Here, the symmetry itself is imposing this kind of coupling. In other words, it is not put by hand but is related to a fundamental principle, i.e. the existence of the Noether symmetry. Even though this method has only been used here to find exact solutions, it might be possible to identify the conserved Noether quantities, which appear from first principles for each model, to a physical real conserved quantity. However, this has not been done before due to the increase number of new conserved charges which appears in modified gravity. Following a similar approach as in Sec. 8.4.1.2, it is possible to find that curvature non-local gravity admits both de-Sitter and power-law solutions for the exponential and the linear f. For more details about those solutions, see Bahamonde et al. (2017c).

Let us stress again one of the most important results of what was found using the Noether symmetry approach. It was proved that, in most physically interesting cases, the only forms of the distortion function selected by the Noether symmetries, are the exponential and the linear ones. According to the literature (Jhingan et al., 2008; Nojiri & Odintsov, 2008), this is an important result, because, up to now, these kinds of couplings were chosen by hand in order to find cosmological solutions while, in our case, they come out from a first principle. In addition, there is a specific class of exponential non-local gravity models which are renormalisable (see Modesto & Rachwal (2014, 2017)). This means that the Noether symmetries dictate the form of the action and choose exponential form for the distortion function. As discussed in Bahamonde & Capozziello (2017) (see Sec. 5.2.1), the existence of Noether symmetries is a selection criterion for physically motivated models.

9 Concluding remarks

This thesis dealt with a not so well explored fundamental theory of gravity based on a manifold with torsion but with vanishing curvature tensor. This theory is called Teleparallel gravity (TEGR) and its field equations are equivalent to General Relativity (GR). Perhaps, its lack of study has been related mostly from a historical point of view since Einstein developed GR before TEGR. Since both theories have the same equations, they can be seen as fundamental theories of gravity. The physical interpretation of them is quite different. GR understands gravity as a deformation of the spacetime and TEGR returns the point of view of thinking that gravity is a force mediated by the torsion tensor. Then, even though they are equivalent on field equations, their physical interpretations are very different.

Since it is possible to consider extensions or modifications of GR in order to explore or solve different theoretical/observational issues, it is also possible to do it considering Teleparallel gravity as the fundamental theory of gravity. In this thesis, a plethora of different Teleparallel modified theories of gravity have been presented with emphasis on constructing theories that can be physically well motivated and also that can be somehow related to their counterpart related to modifications of General Relativity. It is not possible to classically distinguish between GR to TEGR since they have the same field equations but their generalisations are different. A possible future way to distinguish between which theory of gravity describes better our Universe is by studying modifications of TEGR (or GR) or also considering quantum approaches such as non-local effects. One powerful way to discriminate between modified theories could be related to their validity to describe the current cosmological observations or possible new gravitational wave detections.

One of the aims of this thesis was to try to use modified Teleparallel theories of gravity to tackle some cosmological problems such as the dark energy problem. It was found that similar to modifications of GR, Teleparallel modified theories are good candidates to explain all the cosmological history of the Universe; starting from a Big Bang, facing an inflation era, then passing from a radiation to a matter dominated era and finally entering a dark energy dominated era. Many Teleparallel models have been presented and they have been classified and compared with the standard modified models. In term of cosmology, both approaches tend to be very successful.

Let us here now try to discuss some advantages when one is working with modifications of TEGR. All Teleparallel theories of gravity are based on the scalar torsion T which contains only first derivatives of tetrads. On the other hand, GR is based on the Ricci scalar $\overset{\circ}{R}$ which depends on second derivatives of the metric tensor. Therefore, some Teleparallel theories are mathematically simpler than other modified theories coming from GR. For example, $f(T)$ gravity is only a second order theory whereas $f(\overset{\circ}{R})$ is a fourth order theory. Then, for example, further generalisations containing non-minimally couplings with scalar fields (see Sec. 4.4.2) or matter (see Sec. 5.3) are mathematically simpler in modified Teleparallel gravity than modified GR. It should be noted that in those sections, a further generalisation was taken into account considering couplings with the boundary term B which makes the equations more difficult, but, in principle those theories coming from Teleparallel gravity only based on T are mathematically simpler than the ones based with GR. Another interesting advantage on working within Teleparallel gravity is the possibility of reconstructing all the models coming from GR just by adding the corresponding boundary terms B

or B_G if Gauss-Bonnet terms are considered. Then, one can have a broader family of theories of gravity when one is considering Teleparallel gravity. It should be remembered again that the relationship $\overset{\circ}{R} = -T + B$ is a fundamental relationship which is derived from the Ricci theorem and then assuming that the spacetime is globally flat. In modified GR, this is not possible since torsion tensor is identically zero and therefore Ricci scalar cannot be related to the torsion scalar. Moreover, from the Ricci theorem, one only obtain that the full curvature Ricci scalar is $R = \overset{\circ}{R}$, where $\overset{\circ}{R}$ is the quantity computed with the Levi-Civita connection. Therefore, one cannot extend modifications of GR in such a way that one can reconstruct/relate modifications of Teleparallel gravity such as for example $f(T)$ gravity or $f(T, \mathcal{L}_{\rm m})$ gravity. In easy words, if one assumes GR as the fundamental theory, one cannot find for example, theories such as $f(T)$ gravity or Teleparallel dark energy, but if one assumes TEGR as the fundamental theory, one can find an equivalent Teleparallel version of other modifications of GR such as $f(\overset{\circ}{R})$ gravity or $f(\overset{\circ}{R}, \mathcal{L}_{\rm m})$ gravity. This is of course an advantage since all the models coming from modifications of GR can also be reconstructed using the boundary terms. This feature was seen in all the sections related to modified Teleparallel gravity where, it was always possible to construct the counterpart theory coming from the GR point of view.

Another interesting advantage of TEGR and (its modifications) are that the theory is based on the gauge theory of the translations. It is well known that gravity cannot be integrated to the other three forces to create a Unified Theory of Everything. However, all the remaining three forces are also based on gauge theories, so that, in principle, TEGR could be a good candidate in order to construct modifications or generalisations in order to unify gravity with the other forces. GR on the other hand cannot be fully written as a gauge theory, therefore, it is from its basis written in a different language than the other forces. This of course is just a conjecture since there does not exist a final quantum gravity theory but since

TEGR is based on the same gauge structure as the other forces, it might be a very good candidate for constructing quantum gravity theories. One effect that is very common in quantum gravity is the existence of an intrinsic extended structure in the geometry of spacetime, and such an extended structure would be related to an effective non-local behaviour for spacetime. Then, as was seen in Chap. 8, non-local effects are very natural if they are considered from Teleparallel gravity. This kind of approaches seem very interesting and natural to follow for future work.

GR is based on the validity of the equivalence principle (see Sec. 2.2.1), therefore, if this principle is violated at some scales, then GR will be ruled out at those scales. On the other hand, another advantage about TEGR is that it is consistent with universality but it is not needed to construct the theory. One can further consider theories with a generalised Teleparallel force as presented in Sec. 3.5 where universality is no longer true and the theory is still consistent. This is a very interesting approach that can be considered for future work in order to see their consequences in modified gravity.

Only future experiments and observations will tell us about whether modifications of GR or TEGR could be the most adequate way to understand gravity. We are facing a very promising era in physics where much accurate data will be available with new instruments. Gravitational waves opened a new window to discriminate or rule out modified theories of gravity and in the following years, it will become a more powerful tool for physicists. TEGR is now viewed by some people as an exotic gravitational theory but this is only historical. Future experiments could help us on constraining their modifications and possibly, in the future, TEGR will be considered as a more natural theory than GR.

A
Conventions

This thesis works with the standard metric signature notation in Teleparallel gravity which is $\eta_{ab} = \text{diag}(+ - - -)$. This can be a little confusing since mostly all the literature based on GR is written in the other metric signature notation. A list of symbols is presented in Tab. A.1. It should be noted that torsion tensor and its contractions are denoted without a bullet above from Chapter 3 to avoid writing them in all the quantities. In Chapter 2, torsion tensor refers to any connection. Then, other objects constructed with the torsion tensor are also denoted without a bullet. For example: superpotential $S^{\alpha}{}_{\mu\nu}$, contortion tensor $K_{\mu}{}^{\lambda}{}_{\nu}$ or boundary term B are denoted without a bullet above. This means that only covariant derivative and connections can have a bullet above, meaning that they are computed with respect to the Weitzenböck connection.

Mathematical symbol	**Convention**
$(+---)$	Metric signature notation
Greek indices (e.g. $\alpha, \beta, \gamma, ..$) running from $0, 1, 2, 3$	Space-time indices
Latin indices (e.g. $a, b, c, ..$) running from $0, 1, 2, 3$	Tangent space indices
e^a_μ	Tetrad
E^μ_a	Inverse of the tetrad
$e = \det(e^a_\mu)$	Determinant of the tetrad
$c = 1$ and $\kappa^2 = 8\pi G$	Coupling gravitational constant in natural units
Ricci curvature $R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}$	Riemann curvature contracted first and third index
Quantities without symbols above (e.g. $R_\alpha{}^\mu{}_{\gamma\lambda}, \Gamma^\alpha_{\mu\nu}, \nabla_\mu, R$)	Quantities computed for any general connection
Quantities with $\circ$ above (e.g. $\overset{\circ}{R}_\alpha{}^\mu{}_{\gamma\lambda}, \overset{\circ}{R}, \overset{\circ}{\Gamma}{}^\alpha_{\mu\nu}, \overset{\circ}{\nabla}_\mu$)	Quantities computed with the Levi-Civita connection
Quantities with $\bullet$ above (e.g. $\overset{\bullet}{R}_\alpha{}^\mu{}_{\gamma\lambda}, \overset{\bullet}{R}, \overset{\bullet}{\Gamma}{}^\alpha_{\mu\nu}, \overset{\bullet}{\nabla}_\mu$)	Quantities computed with the Weitzenböck connection
Torsion tensor $T^\lambda{}_{\mu\nu}$, torsion vector $T_\mu \equiv T^\lambda{}_{\lambda\mu}$ and torsion scalar T	Quantities computed with the Weitzenböck connection (excepted in Chap. 2 where they refer to any connection)
$\overset{\circ}{\Box} \equiv \overset{\circ}{\nabla}_\mu \overset{\circ}{\nabla}{}^\mu$	D'alembertian computed with the Levi-Civita connection
$\mathcal{T}_{\mu\nu}$	Energy-momentum tensor

Table A.1: List of symbols and their conventions. Exceptions of some convention rules are written explicitly.

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