Modified Weibull distribution

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Abstract. The exponential, Rayleigh, linear failure rate and Weibull distributions are the most commonly used distributions for analyzing lifetime data. These distributions have several desirable properties and nice physical interpretations. This paper introduces a new distribution named modified Weibull distribution. This distribution generalizes the following distributions: (1) exponential, (2) Rayleigh, (3) linear failure rate, and (4) Weibull. The properties of the modified Weibull distribution are discussed. The maximum likelihood estimates of its unknown parameters are obtained. A real data set is analyzed and it observed that the present distribution can provide a better fit than some other known distributions.

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1 Introduction

In analyzing lifetime data one often uses the exponential, Rayleigh, linear failure rate or Weibull distribution. These distributions have several desirable properties and nice physical interpretations which enable them to be used frequently, for more details we refer to [2]-[10]. In this paper, we present a new distribution called modified Weibull distribution (MWD). The MWD generalizes all the above mentioned ones.

The linear failure rate distribution with the parameters $\alpha, \beta \geq 0$, such that $\alpha+\beta > 0$, and will be denoted by LFRD(α, β), has the following cumulative distribution function (CDF)

$$F_{LF}(x; \alpha, \beta) = 1 - \exp\left\{-\alpha x - \beta x^2\right\}, \quad x \geq 0. \tag{1.1}$$

It is easily observed that the exponential distribution (ED(α)) and the Rayleigh distribution (RD(β)) can be obtained from LFRD(α, β) by putting $\beta = 0$ and $\alpha = 0$ respectively. Moreover, the probability density function (PDF) of the LFRD(α, β) can be decreasing or unimodal but the failure rate function is either constant or increasing only.

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The Weibull distribution with the parameters $\beta, \gamma > 0$ and will be denoted by $\mathrm{WD}(\beta, \gamma)$, has the following cumulative distribution function (CDF)

$$F_W(x; \beta, \gamma) = 1 - \exp\left\{-\beta x^{\gamma}\right\}, \quad x \geq 0. \tag{1.2}$$

It is easily observed that the exponential distribution ($\mathrm{ED}(\alpha)$) and the Rayleigh distribution ($\mathrm{RD}(\beta)$) can be obtained from $\mathrm{WD}(\beta, \gamma)$ by putting $\gamma = 1$ and $\gamma = 2$ respectively. Moreover, the probability density function (PDF) of the $\mathrm{WD}(\beta, \gamma)$ can be decreasing or unimodal but the failure rate function is either constant or increasing (starting from the origin) or decreasing.

In this paper we introduce a new three-parameter distribution function called as modified Weibull distribution with three parameters α, β, γ and it will be denoted as $\mathrm{MWD}(\alpha, \beta, \gamma)$. It is observed that the $\mathrm{MWD}(\alpha, \beta, \gamma)$ has decreasing or unimodal PDF and it can have increasing (starting from the value of α), decreasing and constant hazard functions. We provide statistical properties of this MWD. The maximum likelihood estimates (MLEs) of the unknown parameters are derived. The asymptotic confidence intervals of the parameters are discussed. A set real data is analyzed and it is observed that the present distribution provides better fit than many existing well known distributions.

The rest of the paper is organized as follows. In section 2 we present $\mathrm{MWD}(\alpha, \beta, \gamma)$ and discuss its properties in Section 3. The MLEs are provided in Section 4. Section 5 gives an application to explain how a real data set can be modeled by $\mathrm{MWD}(\alpha, \beta, \gamma)$. Finally we conclude the paper in Section 6.

2 The MWD

The CDF of the $\mathrm{MWD}(\alpha, \beta, \gamma)$ takes the following form

$$F(x; \alpha, \beta, \gamma) = 1 - e^{-\alpha x - \beta x^{\gamma}}, \ x > 0, \tag{2.1}$$

where $\gamma > 0$, $\alpha, \beta \geq 0$ such that $\alpha + \beta > 0$. Here α is a scale parameter, while β and γ are shape parameters.

The $\mathrm{MWD}(\alpha, \beta, \gamma)$ generalizes the following distributions: (1) $\mathrm{LFRD}(\alpha, \beta)$ by setting $\gamma = 2$, (2) $\mathrm{WD}(\beta, \gamma)$ by setting $\alpha = 0$; (3) $\mathrm{RD}(\beta)$ by setting $\alpha = 0, \gamma = 2$; and (4) $\mathrm{ED}(\alpha)$ by setting $\beta = 0$.
The PDF of $\mathrm{MWD}(\alpha, \beta, \gamma)$ is

$$f(x; \alpha, \beta, \gamma) = \left(\alpha + \beta\gamma x^{\gamma-1}\right) e^{-\alpha x - \beta x^{\gamma}}, \quad x > 0, \tag{2.2}$$

and the hazard function of $\mathrm{MWD}(\alpha, \beta, \gamma)$ is

$$h(x; \alpha, \beta, \gamma) = \alpha + \beta\gamma x^{\gamma-1}, \quad x > 0. \tag{2.3}$$

One can easily verify from (2.3) that: (1) the hazard function is constant when $\gamma = 1$; (2) when $\gamma < 1$, the hazard function is decreasing; and (3) the hazard function will be increasing if $\gamma > 1$.

Figure 1 and 2 show, respectively, different patterns of the probability density function and the hazard rate function of $\mathrm{MWD}(\alpha, \beta, \gamma)$ for different parameter values.

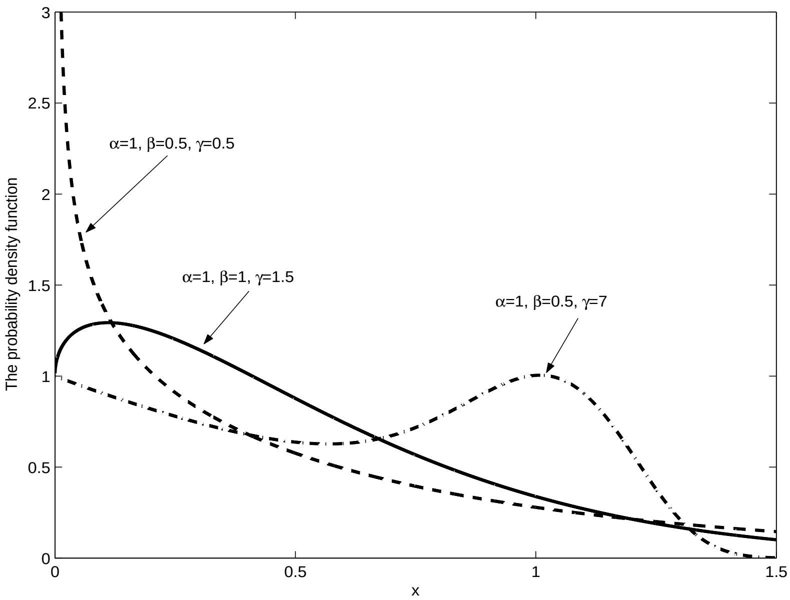

Figure 1: The probability density function

From these Figures, it is immediate that the PDFs can be decreasing or unimodal and the hazard functions can be increasing or decreasing.
INTERPRETATION: It is interesting to observe that the CDF of the MWD(α, β, γ) has a nice physical interpretation. It represents the CDF of the lifetime of a series system. This system consists of two independent units. The lifetime of one unit follows ED(α) and the lifetime of the other follows the WD(β, γ).

3 Statistical properties

In this section we provide some of the basic statistical properties of the MWD(α, β, γ).

3.1 Quantile and median

The quantile x_q of the MWD(α, β, γ) is the real solution of the following equation:

$$\beta {x_q}^{\gamma} + \alpha x_q + \ln(1-q) = 0 \tag{3.1}$$

The above equation has no closed form solution in x_q, so we have to use a numerical technique such as a Newton-Raphson method to get the quantile.
One can use (3.1), to derive the following special cases:

1. The q-th quantile of the LFRD(α, β), by setting $\gamma = 2$, as

$$x_q = \frac{1}{2\beta}\left\{-\alpha + \sqrt{\alpha^2 - 4\beta\,\ln(1-q)}\right\}.$$

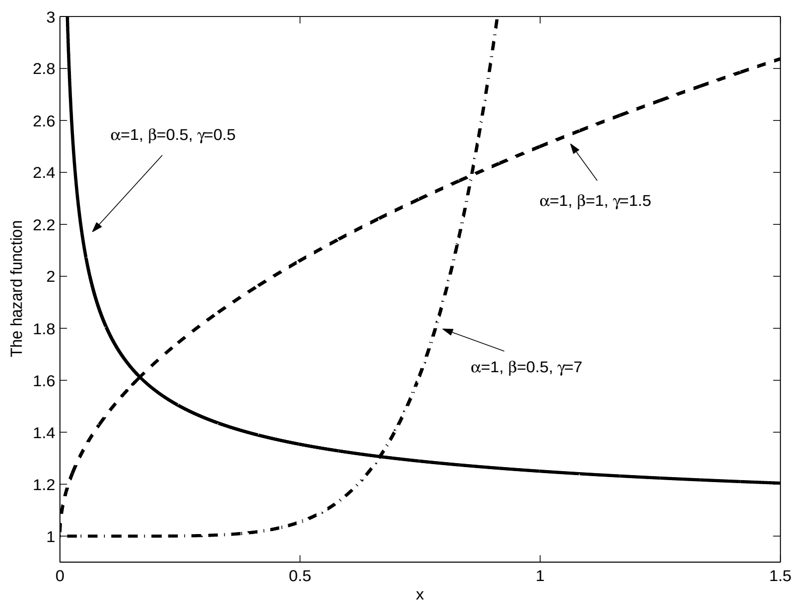

Figure 2: The hazard rate function

2. The q-th quantile of the WD(β, γ), by setting $\alpha = 0$,

$$x_q = \left\{ -\frac{1}{\beta} \ln (1-q) \right\}^{\frac{1}{\gamma}}.$$

3. The q-th quantile of the RD(β), by setting $\gamma = 2$ and $\alpha = 0$,

$$x_q = \sqrt{-\frac{1}{\beta} \ln (1-q)}.$$

4. The q-th quantile of the ED(α), by setting $\beta = 0$,

$$x_q = -\frac{1}{\alpha} \ln (1-q).$$

or by putting $\gamma = 1$.

$$x_q = -\frac{1}{(\alpha+\beta)} \ln (1-q).$$

Put $q = 0.5$ in equation (3.1) one gets the median of MWD(α, β, γ).

3.2 Mode

The mode of MWD(α, β, γ) can be obtained as a solution of the following non-linear equation with respect to x

$$\alpha^2 + \beta^2\gamma^2 x^{2(\gamma-1)} + 2\alpha\beta\gamma x^{\gamma-1} - \beta\gamma(\gamma-1)x^{\gamma-2} = 0. \tag{3.2}$$

As it seems, the equation (3.2) has not an explicit solution in the general case. Therefore, we discuss the following special cases:

(1) $\alpha \neq 0$: there are two cases:

(a) If $\gamma = 1$, the ED($\alpha + \beta$) case , in this case equation (3.2) reduces to $\alpha + \beta = 0$. Since $\alpha + \beta$ must be greater than zero, we get a contradiction which means that there is no mode in this case.

(b) For $\gamma = 2$, the LFRD(α, β) case, in this case (3.2) takes the following form $2\beta - (\alpha + 2\beta x)^2 = 0$. Solving this equation in x, we get the mode as $\text{Mod}(X) = \frac{\sqrt{2\beta}-\alpha}{2\beta}$, such that $\sqrt{2\beta} > \alpha$.

(2) $\alpha = 0$: the WD(β, γ) case. In this case equation (3.2) reduces to

$$\beta\gamma x^{\gamma-2}\{\gamma - 1 - \beta\gamma x^{\gamma}\} = 0.$$

Hence, we have either $x = 0$ or $x = \left(\frac{\gamma-1}{\beta\gamma}\right)^{\frac{1}{\gamma}}$. Since $x > 0$, therefore the mode in this case becomes $\text{Mod}(X) = \left(\frac{\gamma-1}{\beta\gamma}\right)^{\frac{1}{\gamma}}$ such that $\gamma > 1$. It is known that RD(β) can be derived from WD(β, γ) when $\gamma = 2$, therefore the mode of RD(β) becomes $\text{Mod}(X) = \frac{1}{\sqrt{2\beta}}$.

3.3 Moments

The following theorem gives the k^{th} moment of MWD(α, β, γ).

Theorem 3.1. *If X has the MWD(α, β, γ), the k^{th} moment of X, say μ_k, is given as follows:*

$$\mu_k = \begin{cases} \sum_{i=0}^{\infty} \frac{(-\beta)^i}{i!}\left[\frac{\Gamma(i\gamma+k+1)}{\alpha^{i\gamma+k}} + \beta\gamma\frac{\Gamma(k+i\gamma+\gamma)}{\alpha^{i\gamma+\gamma+k}}\right] & \text{for } \alpha, \beta > 0\,, \\ \frac{\Gamma(k/\gamma+1)}{\beta^{k/\gamma}} & \text{for } \alpha = 0, \beta > 0\,, \\ \frac{\Gamma(k+1)}{\alpha^k} & \text{for } \alpha > 0, \beta = 0\,. \end{cases} \tag{3.3}$$

The proof of this theorem is provided in the Appendix.
Based on the results given in theorem (3.1), the measures of skewness and kurtosis of the MWD can be obtained according to the following relations, respectively,

$$\alpha^* = \frac{\mu_3 - 3\mu_1\,\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{\frac{3}{2}}} \tag{3.4}$$

and

$$\beta^* = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\,\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}\,. \tag{3.5}$$

In Figure 3 and 4 we provide α^* and β^* for different values of γ, when $\alpha = 1$ and $\beta = 0.5$. It is observed that α^* and β^* first increase and then start decreasing. Also, α^* takes negative values when γ becomes large.

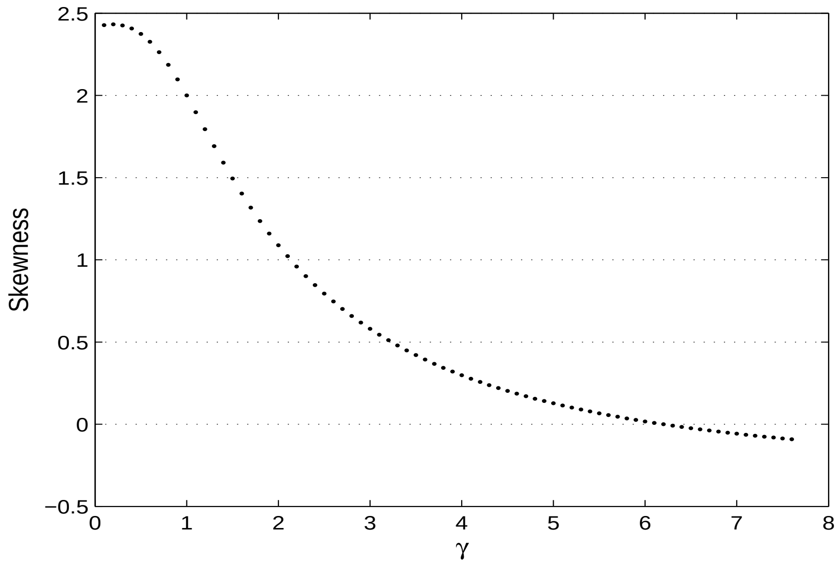

Figure 3: The Skewness measure α^*.

3.4 The moment generating function

The following theorem gives the moment generating function (mgf) of MWD(α,β,γ)

Theorem 3.2. *If X has the MWD(α,β,γ), then the mgf of X, say $M(t)$ is :*

$$(3.6) \quad M(t) = \begin{cases} \sum_{i=0}^{\infty} \frac{(-\beta)^i}{i!} \left[\frac{\alpha\Gamma(i\gamma+1)}{(\alpha-t)^{i\gamma+1}} + \frac{\beta\gamma\Gamma((i+1)\gamma)}{(\alpha-t)^{(i+1)\gamma}} \right] & \text{for } \alpha,\beta>0,\ \alpha>t, \\ \sum_{i=0}^{\infty} \frac{t^i\,\Gamma(i/\gamma+1)}{\beta^{i/\gamma}} & \text{for } \alpha=0, \beta>0\,, \\ \frac{\alpha}{\alpha-t} & \text{for } \alpha>0, \beta=0,\ \alpha>t. \end{cases}$$

The proof of this theorem is provided in the Appendix.
Note: Theorem 3.1 can be deduced from theorem 3.2 by using the relation between the moments and moment generating function.

4 Parameter estimations

In this section, we derive the maximum likelihood estimates of the unknown parameters α,β and γ of MWD(α,β,γ) based on a complete sample. Let us assume that we have a simple random sample $X_1,\ X_2,\cdots,\ X_n$ from MWD(α,β,γ). The likelihood function of this sample is

$$(4.1) \qquad L = \prod_{i=1}^{n} f(x_i;\alpha,\beta,\gamma)$$

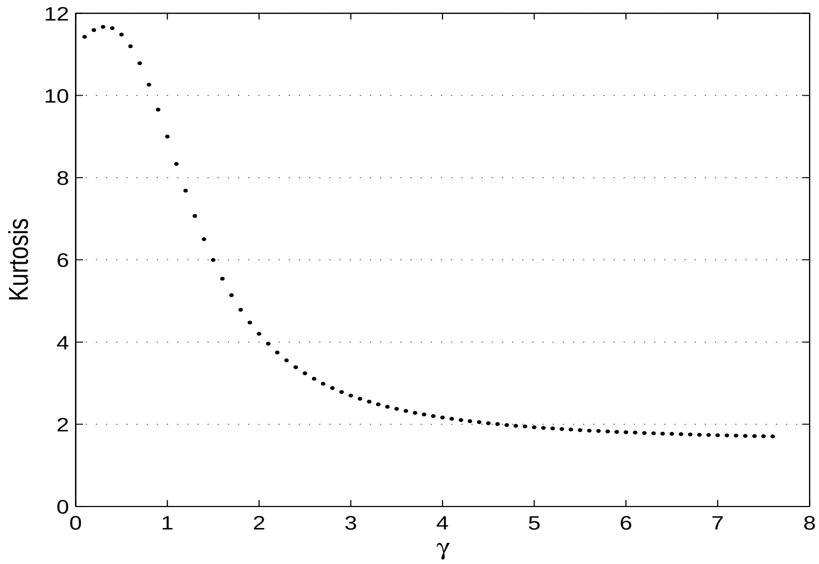

Figure 4: The Kurtosis measure β^*.

Substituting from (2.2) into (4.1), we get

$$L = \prod_{i=1}^{n} \left(\alpha + \beta\gamma x_i^{\gamma-1}\right) e^{-(\alpha x_i + \beta x_i^{\gamma})} \tag{4.2}$$

The log-likelihood function is

$$\mathcal{L} = \sum_{i=1}^{n} \ln(\alpha + \beta\gamma x_i^{\gamma-1}) - \alpha \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^{\gamma}. \tag{4.3}$$

Computing the first partial derivatives of $\mathcal{L}$ and setting the results equal zeros, we get the likelihood equations as in the following form

$$0 = \sum_{i=1}^{n} \frac{1}{\alpha + \beta\gamma x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i\,, \tag{4.4}$$

$$0 = \sum_{i=1}^{n} \frac{\gamma x_i^{\gamma-1}}{\alpha + \beta\gamma x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i^{\gamma}\,, \tag{4.5}$$

$$0 = \sum_{i=1}^{n} \frac{x_i^{\gamma-1}(1 + \gamma \ln(x_i))}{\alpha + \beta\gamma x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i^{\gamma} \ln(x_i)\,. \tag{4.6}$$

Using (4.4) and (4.5), one can derive β as a function of α and γ as in the form

$$\beta = g(\alpha, \gamma) = \frac{n - \alpha \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^{\gamma}}\,. \tag{4.7}$$

Substituting (4.7) into (4.4) and (4.6), we get the following system of two non-linear equations

$$0 = \sum_{i=1}^{n} \frac{1}{\alpha + g(\alpha,\gamma)\gamma x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i, \tag{4.8}$$

$$0 = \sum_{i=1}^{n} \frac{x_i^{\gamma-1}(1+\gamma\ln(x_i))}{\alpha + g(\alpha,\gamma)\gamma x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i^{\gamma}\ln(x_i) \tag{4.9}$$

To get the MLE of the parameters α and γ, $\hat{\alpha}$ and $\hat{\gamma}$, we have to solve the above system of two non-linear equations with respect to α and γ. Then substituting $\hat{\alpha}$ and $\hat{\gamma}$ instead of α and γ in (4.7) we get the MLE of β. The solution of equations (4.8) and (4.9) is not possible in closed form, so numerical technique is needed to get the MLE.

Asymptotic Confidence bounds. Since the MLE of the unknown parameters α, β, and γ are not obtained in closed forms, then it is not possible to derive the exact distributions of the MLE. In this paragraph, the approximate confidence intervals of the parameters based on the asymptotic distributions of their MLE are derived. For the observed information matrix of α, β, γ, we find the second partial derivatives of $\mathcal{L}$ as

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2} = -I_{11} = \sum_{i=1}^{n} \frac{1}{(\alpha+\beta\gamma x_i^{(\gamma-1)})^2} \tag{4.10}$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta} = -I_{12} = \sum_{i=1}^{n} \frac{\gamma x_i^{\gamma-1}}{(\alpha+\beta\gamma i^{(\gamma-1)})^2} \tag{4.11}$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\gamma} = -I_{13} = \sum_{i=1}^{n} \frac{\beta x_i^{\gamma-1}(1+\gamma\ln(x_i))}{(\alpha+\beta\gamma x_i^{(\gamma-1)})^2} \tag{4.12}$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta^2} = -I_{22} = \sum_{i=1}^{n} \frac{\gamma^2 x_i^{2(\gamma-1)}}{\left(\alpha+\beta\gamma i^{(\gamma-1)}\right)^2} \tag{4.13}$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta\partial\gamma} = -I_{23} = \sum_{i=1}^{n} \frac{\alpha x_i^{\gamma-1}(1+\gamma\ln(x_i))}{(\alpha+\beta\gamma x_i^{(\gamma-1)})^2} + \sum_{i=1}^{n} x_i^{\gamma}\ln(x_i) \tag{4.14}$$

$$\frac{\partial^2\mathcal{L}}{\partial\gamma^2} = -I_{33} = \beta\sum_{i=1}^{n} \frac{x_i^{\gamma-1}(\alpha\gamma\ln^2(x_i)+2\alpha\ln(x_i)-\beta x_i^{\gamma-1})}{(\alpha+\beta\gamma i^{(\gamma-1)})^2} + \beta\sum_{i=1}^{n} x_i^{\gamma}\ln^2(x_i) \tag{4.15}$$

Then the observed information matrix is given by

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix},$$

so that the variance-covariance matrix may be approximated as

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}^{-1}.$$

It is known that the asymptotic distribution of the MLE $\left(\hat{\alpha}, \hat{\beta}, \hat{\gamma}\right)$ is given by, see Miller (1981),

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix} \sim N \left[\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \right]. \tag{4.16}$$

Since $\mathbf{V}$ involves the parameters α, β, γ, we replace the parameters by the corresponding MLE's in order to obtain an estimate of $\mathbf{V}$, which is denoted by

$$\hat{V} = \begin{pmatrix} \hat{I}_{11} & \hat{I}_{12} & \hat{I}_{13} \\ \hat{I}_{12} & \hat{I}_{22} & \hat{I}_{23} \\ \hat{I}_{13} & \hat{I}_{23} & \hat{I}_{33} \end{pmatrix}^{-1}. \tag{4.17}$$

where $\hat{I}_{ij} = I_{ij}$ when $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ replaces (α, β, γ).
By using (4.16), approximate $100(1-\theta)\%$ confidence intervals for α, β, γ are determined, respectively, as

$$\hat{\alpha} \pm z_{\theta/2}\sqrt{\hat{V}_{11}}\,, \quad \hat{\beta} \pm z_{\theta/2}\sqrt{\hat{V}_{22}}\,, \quad \text{and} \quad \hat{\gamma} \pm z_{\theta/2}\sqrt{\hat{V}_{33}}\,, \tag{4.18}$$

where z_θ is the upper $\theta-$th percentile of the standard normal distribution.

5 Application

In this section we provide a data analysis to see how the new model works in practice. The data have been obtained from Aarset [1] and it is provided below. It represents the lifetimes of 50 devices.

.1	.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	18
21	32	36	40	45	46	47	50	55	60	63	63	67	67	67	67	72	75
79	82	82	83	84	84	84	85	85	85	85	85	86	86				

We have used the following different distributions: Rayleigh distribution (RD), exponential distribution (ED), Weibull distribution (WD), linear failure rate distribution (LFRD), and modified Weibull distribution (MWD) to analyze the data. The MLE(s) of the unknown parameter(s) and the corresponding Kolmogorov-Smirnov (K-S) test statistics for different models are given in Table 1.

Table 1. The MLE of the parameter(s) and the associated K-S values.

The model	MLE of the parameter(s)	M_{K-S}
RD(β)	$\hat{\beta} = 6.362 \times 10^{-4}$	0.1058
ED(α)	$\hat{\alpha} = 0.022$	0.0933
WD (α, β)	$\hat{\alpha} = 0.022,\ \hat{\beta} = 0.949$	0.0750
LFRD(α, β)	$\hat{\alpha} = 0.014,\ \hat{\beta} = 2.4 \times 10^{-4}$	0.0923
MWD(α, β, γ)	$\hat{\alpha} = 0.012, \hat{\beta} = 2.159 \times 10^{-8}, \hat{\gamma} = 4.014$	0.0739

To to show that the likelihood equations have a unique solution in the parameters, we plot the profile log-likelihood functions of α, β and γ. Figures 5-7 show these functions.

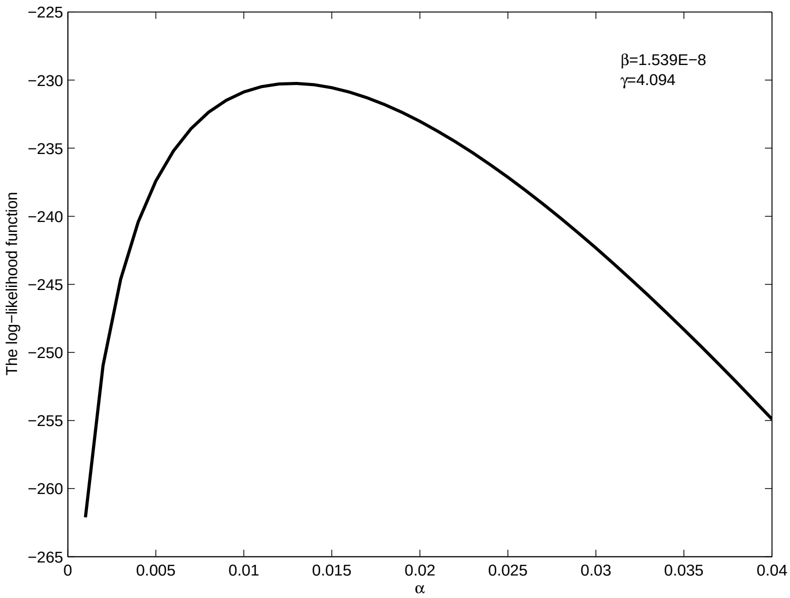

Figure 5: The log-likelihood as a function of α.

Table 2. The MLE, the values of log-likelihood function and p-values.

The model	H_0	$\mathcal{L}$	Λ	d.f.	p-value
RD(β)	$\alpha = 0, \gamma = 2$	-260.053	67.028	2	2.776×10^{-15}
ED(α)	$\beta = 0$	-241.090	21.102	2	4.255×10^{-6}
WD (α, β)	$\alpha = 0$	-241.002	20.926	1	4.773×10^{-6}
LFRD(α, β)	$\gamma = 2$	-238.064	15.05	1	1.047×10^{-4}

Since RD(β), ED(α), WD(α, β) and LFRD(α, β)are special cases of MWD(α, β, γ), we perform the following testing of hypotheses; (i) $H_0 : \alpha = 0, \gamma = 2$, (ii) $H_0 : \beta = 0$, (iii) $\alpha = 0$, (iv) $H_0 : \gamma = 2$. We present the log-likelihood values ($\mathcal{L}$), the values of the likelihood ratio test statistics (Λ) and the corresponding p-values in Table 2. From the p values it is clear that we reject all the hypotheses at any level of significance.

The nonparametric estimate of the survival function and the fitted survival functions are computed and provided in Figure 8. It is clear from Figure 8 that the modified Weibull distribution provides a good fit to the data set.

Substituting the MLE of the unknown parameters in (4.17), we get estimation of the variance covariance matrix as

$$I^{-1} = \begin{bmatrix} 1.546 \times 10^{-5} & 6.412 \times 10^{-11} & -4.859 \times 10^{-4} \\ 6.412 \times 10^{-11} & 1.989 \times 10^{-14} & -1.081 \times 10^{-7} \\ -4.859 \times 10^{-4} & -1.081 \times 10^{-7} & 0.594. \end{bmatrix} \quad (5.1)$$

Therefore, the approximate 95% two sided confidence intervals of the parameters α, β and γ are [0.01, 0.025624], $\left[0, 3.19128 \times 10^{-7}\right]$ and [2.29922, 5.3203] respectively.

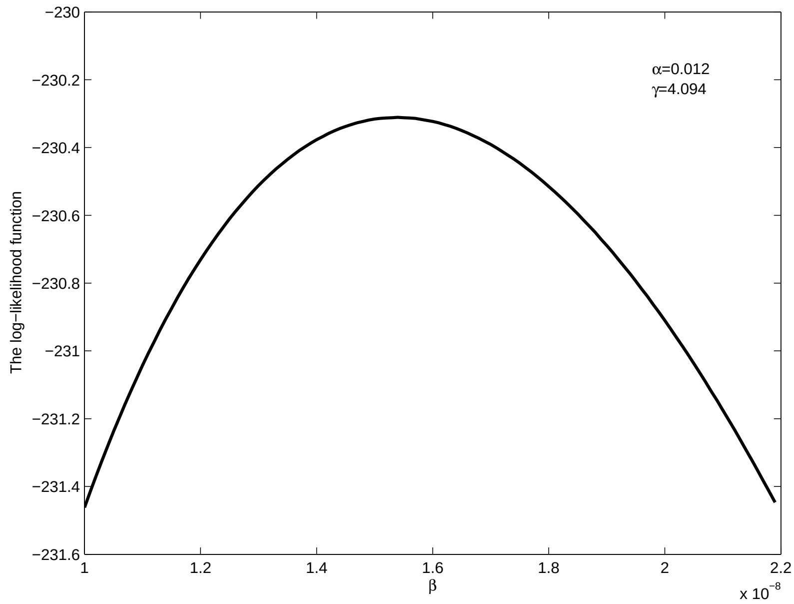

Figure 6: The log-likelihood as a function of β.

6 Conclusions

In this paper we introduced a three-parameter modified Weibull distribution (MWD) and studied its different properties. It is observed that the proposed MWD has several desirable properties and several existing well known distributions can be obtained as special cases of this distribution. It is observed that the MWD can have constant, increasing and decreasing hazard rate functions which are desirable for data analysis purposes. Both point and asymptotic confidence interval estimates of the parameters are derived using the maximum likelihood method. Application on set of real data showed that the MWD can be used rather than other known distribution. To study the properties of the MLEs of the parameters extensive simulations are required. Also, the Bayes procedure can be used to derive the point interval estimates of the parameters. More work is needed in this direction.

Appendix

The Proof of Theorem 3.1. Starting with

$$\mu_k = \int_0^\infty x^k f(x;\alpha,\beta,\gamma)dx$$

then substituting from (2.2) into the above relation we have

$$\mu_k = \int_0^\infty x^k(\alpha + \beta\gamma x^{\gamma-1})\, e^{-\alpha x - \beta x^\gamma} dx \tag{6.1}$$

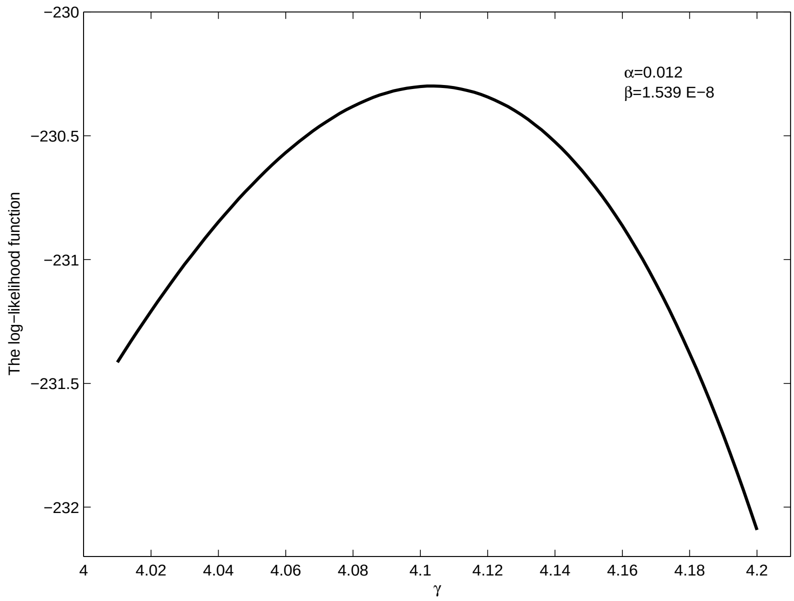

Figure 7: The log-likelihood as a function of γ.

Now, there are three cases. Let us start with the first general case, namely, when $\alpha > 0$ and $\beta > 0$. Using the following expansion of $e^{-\beta x^{\gamma}}$ given by

$$e^{-\beta x^{\gamma}} = \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i x^{i\gamma}}{i!} \tag{6.2}$$

equation (6.1) takes the following form

$$\mu_k = \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i}{i!} \int_0^{\infty} x^k (\alpha + \beta\gamma x^{\gamma-1}) x^{i\gamma} e^{-\alpha x} dx$$

which can be rewritten as

$$\begin{aligned} \mu_k &= \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i}{i!} \left[\int_0^{\infty} \alpha x^{k+i\gamma} e^{-\alpha x} dx + \int_0^{\infty} \beta\gamma x^{k+(i+1)\gamma-1} e^{-\alpha x} dx \right] \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i}{i!} \left[\frac{\Gamma(k + i\gamma + 1)}{\alpha^{k+i\gamma}} + \frac{\beta\gamma\Gamma(k + (i+1)\gamma)}{\alpha^{k+(i+1)\gamma}} \right] \end{aligned}$$

In the second case, we assume that $\alpha = 0$ and $\beta > 0$. In this case relation (6.1) reduces to

$$\mu_k = \int_0^{\infty} \beta\gamma x^{k+\gamma-1} e^{-\beta x^{\gamma}} dx$$

Setting $\beta x^{\gamma} = u$, then

$$\mu_k = \int_0^{\infty} \left(\frac{u}{\beta} \right)^{\frac{k}{\gamma}} e^{-u} du = \frac{\Gamma(\frac{k}{\gamma} + 1)}{\beta^{\frac{k}{\gamma}}}$$

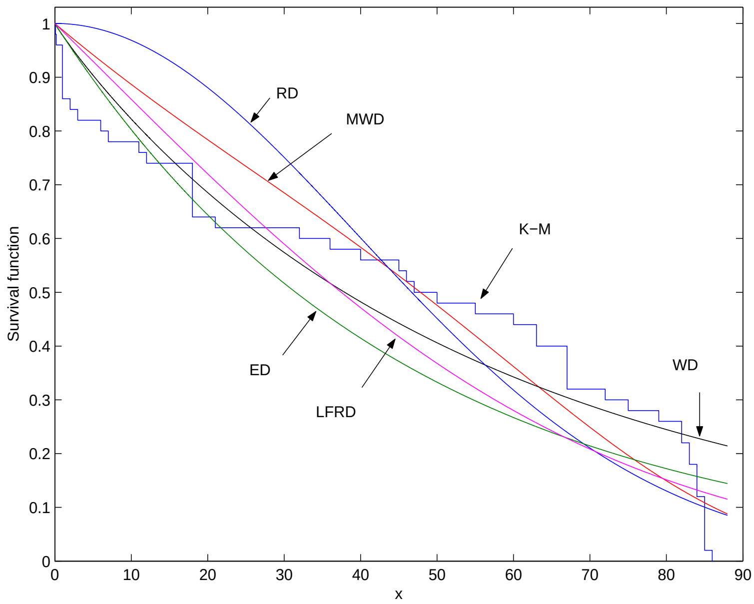

Figure 8: The empirical and fitted survival function.

Finally, assume that $\beta = 0$ and $\alpha > 0$, then

$$\mu_k = \int_0^\infty \alpha x^k e^{-\alpha x} dx = \frac{\Gamma(k+1)}{\alpha^k}$$

which completes the proof.

The Proof of Theorem 3.2. Starting with

$$M(t) = E\left[e^{tX}\right] = \int_0^\infty e^{tx} f(x; \alpha, \beta, \gamma) dx$$

then substituting (2.2) into the above relation we have

$$M(t) = \int_0^\infty \left(\alpha + \beta\gamma x^{\gamma-1}\right) e^{tx-\alpha x-\beta x^\gamma} dx \tag{6.3}$$

Now, there are three cases. Let us start with the case when $\alpha > 0$ and $\beta > 0$. Using the relation (6.2), equation (6.3) takes the following form

$$M(t) = \sum_{i=0}^\infty \frac{(-1)^i \beta^i}{i!} \int_0^\infty (\alpha + \beta\gamma x^{\gamma-1}) x^{i\gamma} e^{-(\alpha-t)x} dx$$

which can be rewritten as

$$\begin{aligned} M(t) &= \sum_{i=0}^\infty \frac{(-1)^i \beta^i}{i!} \left[\int_0^\infty \alpha x^{i\gamma} e^{-(\alpha-t)x} dx + \int_0^\infty \beta\gamma x^{(i+1)\gamma-1} e^{-(\alpha-t)x} dx\right] \\ &= \sum_{i=0}^\infty \frac{(-1)^i \beta^i}{i!} \left[\frac{\alpha\Gamma(i\gamma+1)}{(\alpha-t)^{1+i\gamma}} + \frac{\beta\gamma\Gamma((i+1)\gamma)}{(\alpha-t)^{(i+1)\gamma}}\right], \ \alpha > t. \end{aligned}$$

Now, assume that $\alpha = 0$ and $\beta > 0$. In this case relation (6.3) reduces to

$$M(t) = \int_0^\infty \beta\gamma x^{\gamma-1} e^{tx-\beta x^\gamma} dx$$

Using the expansion $e^{tx} = \sum_{i=0}^\infty \frac{(xt)^i}{i!}$, and the transformation $u = \beta x^\gamma$, we get

$$M(t) = \sum_{i=0}^\infty \frac{(t)^i}{i!} \int_0^\infty \left(\frac{u}{\beta}\right)^{\frac{i}{\gamma}} e^{-u} du = \sum_{i=0}^\infty \frac{(t)^i\,\Gamma(1+i/\gamma)}{i!\,\beta^{i/\gamma}}.$$

Finally, assume that $\beta = 0$ and $\alpha > 0$, then

$$M(t) = \int_0^\infty \alpha\, e^{-(\alpha-t)x} dx = \frac{\alpha}{\alpha-t}\,,\ \alpha > t. \qquad \square$$

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