

Modular A_4 invariance and neutrino mixing

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ABSTRACT: We study the phenomenological implications of the modular symmetry $\Gamma(3) \simeq A_4$ of lepton flavors facing recent experimental data of neutrino oscillations. The mass matrices of neutrinos and charged leptons are essentially given by fixing the expectation value of modulus τ , which is the only source of modular invariance breaking. We introduce no flavons in contrast with the conventional flavor models with A_4 symmetry. We classify our neutrino models along with the type I seesaw model, the Weinberg operator model and the Dirac neutrino model. In the normal hierarchy of neutrino masses, the seesaw model is available by taking account of recent experimental data of neutrino oscillations and the cosmological bound of sum of neutrino masses. The predicted $\sin^2 \theta_{23}$ is restricted to be larger than 0.54 and $\delta_{\text{CP}} = \pm(50^\circ - 180^\circ)$. Since the correlation of $\sin^2 \theta_{23}$ and δ_{CP} is sharp, the prediction is testable in the future. It is remarkable that the effective mass m_{ee} of the neutrinoless double beta decay is around 22 meV while the sum of neutrino masses is predicted to be 145 meV. On the other hand, for the inverted hierarchy of neutrino masses, only the Dirac neutrino model is consistent with the experimental data.

KEYWORDS: Discrete Symmetries, Neutrino Physics, Solar and Atmospheric Neutrinos, Compactification and String Models

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Contents

1	Introduction	1
2	Modular symmetry	2
3	Models with modular symmetry	4
3.1	Charged lepton mass matrix	6
3.2	Neutrino mass matrix	6
4	Numerical results	8
4.1	Model I(a): seesaw	9
4.2	Model I(b): seesaw	10
4.3	Model II: Weinberg operator	11
4.4	Model III: Dirac neutrino	11
5	Summary	12
A	Multiplication rule of A_4 group	13
B	Determination of α/γ and β/γ	14
C	Lepton mixing and neutrinoless double beta decay	14

1 Introduction

In spite of the remarkable success of the standard model (SM), the origin of the flavors of quarks and leptons is still unknown. The recent developments of the neutrino oscillation experiments provide us important clues to investigate the flavor physics. Indeed, the neutrino oscillation experiments have determined two neutrino mass squared differences and three neutrino mixing angles precisely. In particular, the recent data of both T2K [1, 2] and NO ν A [3, 4] imply that the atmospheric neutrino mixing angle θ_{23} is expected to be near the maximal angle 45° . The closer the observed θ_{23} is to the maximal mixing, the more likely some flavor symmetry is to exist behind it. In addition to the precise measurements of neutrino mixing angles, T2K and NO ν A strongly indicate CP violation in the neutrino oscillation [2, 4]. Thus, we are in the era to develop the flavor theory of leptons with facing both the flavor mixing angles and CP violating phase.

One of interesting approaches is to impose non-Abelian discrete symmetries for flavors. Many models have been proposed by using S_3 , A_4 , S_4 , A_5 and other groups with larger orders [5–9]. In particular, A_4 flavor models are attractive because A_4 is the minimal group which has a triplet as its irreducible representation and enable us to explain three families of quarks and leptons naturally [10–15]. However, variety of models is so wide that it is difficult to obtain clear clues of flavor symmetry. Indeed, symmetry breakings are required to reproduce realistic mixing angles [16]. The effective Lagrangian of a typical flavor model is given by introducing the gauge singlet scalars which are so-called flavons.

Those vacuum expectation values (VEVs) determine the flavor structure of quarks and leptons. Finally, the breaking sector of flavor symmetry typically produces many unknown parameters.

Superstring theory with certain compactifications can lead to non-Abelian discrete flavor symmetries. For example, heterotic orbifold models lead to D_4 , $\Delta(54)$, etc. [17]. (See also [18, 19].) Similar flavor symmetries are also derived in type II magnetized and intersecting D-brane models [20, 21]. On the other hand, string theory on tori or orbifolds has the modular symmetry which acts non-trivially on flavors of quarks and leptons [22–28]. In this sense, the modular symmetry is a non-Abelian discrete flavor symmetry.

It is interesting that the modular group includes S_3 , A_4 , S_4 , and A_5 as its finite subgroups, $\Gamma(N)$. However, there is a difference between the modular symmetry and the usual flavor symmetry. Yukawa couplings are written as modular forms, functions of the modulus τ , and transform non-trivially under the modular symmetry as well as fields. On the other hand, Yukawa couplings are invariants in the usual flavor symmetries. In this aspect, an attractive ansatz was proposed by taking $\Gamma(3) \simeq A_4$ in ref. [29] where Yukawa couplings are A_4 triplets of modular forms, and both left-handed leptons and right-handed neutrinos are A_4 triplets while right-handed charged leptons are A_4 singlets. Along with this work, $\Gamma(2) \simeq S_3$ [30] and $\Gamma(4) \simeq S_4$ [31] have been discussed as well as the numerical works [32]. These are bottom-up approaches for model building, but not a top-down approach from explicit string models. However, these approaches would make a bridge between neutrino physics and underlying theory such as superstring theory from the viewpoint of flavor symmetries.

In this paper, we present a comprehensive study of $\Gamma(3) \simeq A_4$ numerically by taking account of the recent experimental data of neutrino oscillations. The mass matrices of neutrinos and charged leptons are essentially given by the expectation value of the modulus τ , which is the only source of modular invariance breaking. However, there are freedoms for the assignments of irreducible representations and modular weights to leptons. We study neutrino mass matrices for three classified models: the type I seesaw model, the Weinberg operator model, and the Dirac neutrino model. In order to build models with minimal number of parameters, we introduce no flavons.

The paper is organized as follows. In section 2, we give a brief review on modular symmetry. In section 3, we present the mass matrices for neutrinos and charged leptons in our models. In section 4, we present the numerical results of our models. Section 5 is devoted to a summary. Appendix A shows the relevant multiplication rules of A_4 . In appendix B, we show how to determine the coupling coefficients of the charged lepton sector. Appendix C shows the lepton mixing matrix and the relevant measures which are used in this work for the case of Majorana neutrinos.

2 Modular symmetry

In this section, we give a brief review on the modular symmetry on the torus and its low-energy effective field theory.

The torus compactification is the simplest compactification. For example, the two-dimensional torus T^2 can be constructed as division of \mathbb{R}^2 by a two-dimensional lattice Λ , i.e. $T^2 = \mathbb{R}^2/\Lambda$. Here, we use the complex coordinate on \mathbb{R}^2 with the lattice spanned by two lattice vectors, $\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$; where R is real and τ is a complex modulus parameter. However, there is some ambiguity in choice of the basis vectors. The same lattice can be spanned by the following basis vectors,

$$\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}, \tag{2.1}$$

where a, b, c, d are integer with satisfying $ad - bc = 1$. That is the $SL(2, \mathbb{Z})$ transformation. Under the above transformation, the modulus parameter transforms as

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \tag{2.2}$$

and this modular transformation is generated by S and T ,

$$S : \tau \longrightarrow -\frac{1}{\tau}, \tag{2.3}$$

$$T : \tau \longrightarrow \tau + 1. \tag{2.4}$$

They satisfy the following algebraic relations,

$$S^2 = \mathbb{I}, \quad (ST)^3 = \mathbb{I}. \tag{2.5}$$

If we impose $T^N = \mathbb{I}$ furthermore, we obtain finite subgroups $\Gamma(N)$. $\Gamma(N)$ with $N = 2, 3, 4, 5$ are isomorphic to S_3, A_4, S_4 and A_5 , respectively [33]. Indeed, $\Gamma(N)$ is a quotient of the modular group by the so-called congruence subgroup $\bar{\Gamma}(N)$. Holomorphic functions which transform as

$$f(\tau) \rightarrow (c\tau + d)^k f(\tau), \tag{2.6}$$

under the modular transformation eq. (2.2) are called modular forms of weight k .

Superstring theory on the torus T^2 or orbifold T^2/Z_N has the modular symmetry. Its low-energy effective field theory is described in terms of supergravity theory, and string-derived supergravity theory has also the modular symmetry. Under the modular transformation eq. (2.2), chiral superfields $\phi^{(I)}$ transform as [34],

$$\phi^{(I)} \rightarrow (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)}, \tag{2.7}$$

where $-k_I$ is the so-called modular weight and $\rho^{(I)}(\gamma)$ denotes a unitary representation matrix of $\gamma \in \Gamma(N)$. The kinetic terms of their scalar components are written by

$$\sum_I \frac{|\partial_\mu \phi^{(I)}|^2}{\langle -i\tau + i\bar{\tau} \rangle^{k_I}}, \tag{2.8}$$

which is invariant under the modular transformation. Here, we use the convention that the superfield and its scalar component are denoted by the same letter. Also, the superpotential should be invariant under the modular symmetry. That is, the superpotential

should have vanishing modular weight in global supersymmetric models, while the superpotential in supergravity should be invariant under the modular symmetry up to the Kähler transformation. In the following sections, we study global supersymmetric models, e.g. minimal supersymmetric standard model (MSSM) and its extension with right-handed neutrinos. Thus, the superpotential has vanishing modular weight. However, note that Yukawa couplings as well as higher order couplings depend on modulus, and they can have non-vanishing modular weights. The breaking scale of supersymmetry can be between $\mathcal{O}(1)$ TeV and the compactification scale. The modular symmetry is broken by the vacuum expectation value of τ , i.e. at the compactification scale, which is of order of the Planck scale or slightly lower scale.

The Dedekind eta-function $\eta(\tau)$ is one of famous modular forms, which is written by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{2.9}$$

where $q = e^{2\pi i\tau}$ and $\eta(\tau)^{24}$ is a modular form of weight 12. By use of $\eta(\tau)$ and its derivative, A_4 triplet modular forms (Y_1, Y_2, Y_3) of modular weight 2 are written by [29],

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right), \\ Y_2(\tau) &= \frac{-i}{\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right), \\ Y_3(\tau) &= \frac{-i}{\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2 \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right), \end{aligned} \tag{2.10}$$

where $\omega = e^{2\pi i/3}$. The overall coefficient in eq. (2.10) is one choice and cannot be determined essentially.

3 Models with modular symmetry

Let us consider a modular invariant flavor model with the A_4 symmetry for leptons. At first, we discuss the type I seesaw model where neutrinos are Majorana particles. There are freedoms for the assignments of irreducible representations and modular weights to leptons. We suppose that three left-handed lepton doublets are compiled in a triplet of A_4 . The three right-handed neutrinos are also of a triplet of A_4 . On the other hand, the Higgs doublets are supposed to be singlets of A_4 . The generic assignments of representations and modular weights to the MSSM fields and right-handed neutrino superfields are presented in table 1. In order to build a model with minimal number of parameters, we introduce no flavons.

For the charged leptons, we assign three right-handed charged leptons for three different singlets of A_4 , $(1, 1'', 1')$. Therefore, there are three independent couplings in the superpotential of the charged lepton sector. Those coupling constants can be adjusted to the observed charged lepton masses. Since there are three singlets in the A_4 group, there are six cases for the assignment of three right-handed charged leptons. However, the

	L	e_R, μ_R, τ_R	ν_R	H_u	H_d	Y
SU(2)	2	1	1	2	2	1
A_4	3	1, 1'', 1'	3	1	1	3
$-k_I$	-1 (1)	-1 (-3)	-1	0	0	$k = 2$

Table 1. The charge assignment of SU(2), A_4 , and the modular weight ($-k_I$ for fields and k for coupling Y) in the type I seesaw model. The right-handed charged leptons are assigned three A_4 singlets, respectively. Values of $-k_I$ in the parentheses are alternative assignments of the modular weight.

freedom of these assignments for right-handed charged leptons do not affect the results for lepton mixing angles.

It may be helpful to comment that if the right-handed charged leptons are of a A_4 triplet, we cannot reproduce the well known charged lepton mass hierarchy $1 : \lambda^2 : \lambda^5$, where $\lambda \simeq 0.2$.

The modular invariant mass terms of the leptons are given as the following superpotentials:

$$w_e = \alpha e_R H_d (LY) + \beta \mu_R H_d (LY) + \gamma \tau_R H_d (LY), \quad (3.1)$$

$$w_D = g(\nu_R H_u LY)_1, \quad (3.2)$$

$$w_N = \Lambda(\nu_R \nu_R Y)_1, \quad (3.3)$$

where sums of the modular weights vanish. The parameters α, β, γ, g , and Λ are constant coefficients. The functions $Y_i(\tau)$ are A_4 triplet modular forms and they consist of the modulus parameter τ :

$$Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}, \quad q = e^{2\pi i \tau}, \quad (3.4)$$

where the q -expansion of $Y_i(\tau)$ is used. The $Y_i(\tau)$ satisfy the constraint [29]:

$$Y_2^2 + 2Y_1 Y_3 = 0. \quad (3.5)$$

Since the dimension of the space of modular forms of weight 2 for $\Gamma(3) \simeq A_4$ is 3 (see, e.g. [29, 35]), all Y 's in eqs. (3.1)–(3.3) are the same modular forms.

There is an alternative assignment of the modular weight for the left-handed lepton and the right-handed charged leptons as presented in parentheses of table 1 [32]. For the alternative assignment, the modular invariant superpotential w_D is given with constant parameters without the modular coupling Y as:

$$w_D = g(\nu_R H_u L)_1. \quad (3.6)$$

Next, we discuss the case where neutrino masses originate from the Weinberg operator. We have the unique possibility of the superpotential

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u LLY)_1, \quad (3.7)$$

where both modular weights of L and right-handed charged leptons are -1 as shown in table 1.

There is another possibility for neutrinos, that is, neutrinos are Dirac particles. In this case, the neutrino mass matrix is derived only from w_D in eq. (3.2).

3.1 Charged lepton mass matrix

Let us consider an assignment of A_4 for the right-handed charged leptons as $(e_R, \mu_R, \tau_R) = (1, 1'', 1')$ in table 1. By using the decomposition rule of a A_4 tensor product in appendix A, we obtain the mass matrix of charged leptons as follows:¹

$$M_E = \text{diag}[\alpha, \beta, \gamma] \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}. \quad (3.8)$$

The coefficients α , β , and γ are taken to be real positive by rephasing right-handed charged lepton fields without loss of generality. Those parameters can be written in terms of the modulus parameter τ and the charged lepton masses as seen in appendix B.

3.2 Neutrino mass matrix

Since the tensor product of $3 \otimes 3$ is decomposed into a symmetric triplet and an anti-symmetric triplet as seen in appendix A, the superpotential of the Dirac neutrino mass in eq. (3.2) is expressed with additional two parameters g_1 and g_2 as:

$$\begin{aligned} w_D &= v_u \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix} \otimes \left[g_1 \begin{pmatrix} 2\nu_e Y_1 - \nu_\mu Y_3 - \nu_\tau Y_2 \\ 2\nu_\tau Y_3 - \nu_e Y_2 - \mu Y_1 \\ 2\nu_\mu Y_2 - \nu_\tau Y_1 - \nu_e Y_3 \end{pmatrix} \oplus g_2 \begin{pmatrix} \nu_\mu Y_3 - \nu_\tau Y_2 \\ \nu_e Y_2 - \nu_\mu Y_1 \\ \nu_\tau Y_1 - \nu_e Y_3 \end{pmatrix} \right] \\ &= v_u g_1 [\nu_{R1}(2\nu_e Y_1 - \nu_\mu Y_3 - \nu_\tau Y_2) + \nu_{R2}(2\nu_\mu Y_2 - \nu_\tau Y_1 - \nu_e Y_3) + \nu_{R3}(2\nu_\tau Y_3 - \nu_e Y_2 - \nu_\mu Y_1)] \\ &\quad + v_u g_2 [\nu_{R1}(\nu_\mu Y_3 - \nu_\tau Y_2) + \nu_{R2}(\nu_\tau Y_1 - \nu_e Y_3) + \nu_{R3}(\nu_e Y_2 - \nu_\mu Y_1)]. \end{aligned} \quad (3.9)$$

The Dirac neutrino mass matrix is given as

$$M_D = v_u \begin{pmatrix} 2g_1 Y_1 & (-g_1 + g_2) Y_3 & (-g_1 - g_2) Y_2 \\ (-g_1 - g_2) Y_3 & 2g_1 Y_2 & (-g_1 + g_2) Y_1 \\ (-g_1 + g_2) Y_2 & (-g_1 - g_2) Y_1 & 2g_1 Y_3 \end{pmatrix}_{RL}. \quad (3.10)$$

¹There are six cases to assign A_4 singlets for the right-handed charged leptons as $(e_R, \mu_R, \tau_R) = (1, 1'', 1')$, $(1, 1', 1'')$, $(1', 1, 1'')$, $(1', 1'', 1)$, $(1'', 1', 1)$, $(1'', 1, 1')$. The mass matrices are obtained by permutations of rows each other. Then, the combinations $M_E^\dagger M_E$ are same ones up to re-labeling of parameters α , β , and γ for all cases.

For the alternative case in eq. (3.6), the superpotential of the Dirac neutrino is written as:

$$w_D = v_u g \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix} \otimes \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = v_u g (\nu_{R1}\nu_e + \nu_{R2}\nu_\tau + \nu_{R3}\nu_\mu). \quad (3.11)$$

The Dirac neutrino mass matrix is simply given as

$$M_D = v_u g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{RL}. \quad (3.12)$$

On the other hand, since the Majorana neutrino mass terms are symmetric, the superpotential in eq. (3.3) is expressed simply as

$$\begin{aligned} w_N &= \Lambda \begin{pmatrix} 2\nu_{R1}\nu_{R1} - \nu_{R2}\nu_{R3} - \nu_{R3}\nu_{R2} \\ 2\nu_{R3}\nu_{R3} - \nu_{R1}\nu_{R2} - \nu_{R2}\nu_{R1} \\ 2\nu_{R2}\nu_{R2} - \nu_{R3}\nu_{R1} - \nu_{R1}\nu_{R3} \end{pmatrix} \otimes \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \\ &= \Lambda [(2\nu_{R1}\nu_{R1} - \nu_{R2}\nu_{R3} - \nu_{R3}\nu_{R2})Y_1 + (2\nu_{R3}\nu_{R3} - \nu_{R1}\nu_{R2} - \nu_{R2}\nu_{R1})Y_3 \\ &\quad + (2\nu_{R2}\nu_{R2} - \nu_{R3}\nu_{R1} - \nu_{R1}\nu_{R3})Y_2]. \end{aligned} \quad (3.13)$$

Then, the right-handed Majorana neutrino mass matrix is given as

$$M_N = \Lambda \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}_{RR}. \quad (3.14)$$

Finally, the effective neutrino mass matrix is obtained by the type I seesaw as follows:

$$M_\nu = -M_D^T M_N^{-1} M_D. \quad (3.15)$$

For the case where neutrino masses originate from the Weinberg operator, the superpotential in eq. (3.7) is written as:

$$\begin{aligned} w_\nu &= -\frac{v_u^2}{\Lambda} \begin{pmatrix} 2\nu_e\nu_e - \nu_\mu\nu_\tau - \nu_\tau\nu_\mu \\ 2\nu_\tau\nu_\tau - \nu_e\nu_\mu - \nu_\mu\nu_e \\ 2\nu_\mu\nu_\mu - \nu_\tau\nu_e - \nu_e\nu_\tau \end{pmatrix} \otimes \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \\ &= -\frac{v_u^2}{\Lambda} [(2\nu_e\nu_e - \nu_\mu\nu_\tau - \nu_\tau\nu_\mu)Y_1 + (2\nu_\tau\nu_\tau - \nu_e\nu_\mu - \nu_\mu\nu_e)Y_3 + (2\nu_\mu\nu_\mu - \nu_\tau\nu_e - \nu_e\nu_\tau)Y_2]. \end{aligned} \quad (3.16)$$

The Majorana neutrino mass matrix is given as follows:

$$M_\nu = -\frac{v_u^2}{\Lambda} \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}_{LL}. \quad (3.17)$$

Models	Mass Matrices
I (a): Seesaw	$M_D \sim \begin{pmatrix} 2g_1 Y_1 & (-g_1 + g_2) Y_3 & (-g_1 - g_2) Y_2 \\ (-g_1 - g_2) Y_3 & 2g_1 Y_2 & (-g_1 + g_2) Y_1 \\ (-g_1 + g_2) Y_2 & (-g_1 - g_2) Y_1 & 2g_1 Y_3 \end{pmatrix}, \quad M_N \sim \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$
I (b): Seesaw	$M_D \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_N \sim \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$
II: Weinberg Operator	$M_\nu \sim \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$
III: Dirac Neutrino	$M_\nu \sim \begin{pmatrix} 2g_1 Y_1 & (-g_1 + g_2) Y_3 & (-g_1 - g_2) Y_2 \\ (-g_1 - g_2) Y_3 & 2g_1 Y_2 & (-g_1 + g_2) Y_1 \\ (-g_1 + g_2) Y_2 & (-g_1 - g_2) Y_1 & 2g_1 Y_3 \end{pmatrix}$

Table 2. The classification of the modular invariant mass matrices for neutrino models.

This matrix is the same one as in eq. (3.14) apart from the normalization because both left-handed neutrinos and the right-handed neutrinos are the triplet of A_4 .

For the case where the neutrino is the Dirac particle, we use the mass matrix in eq. (3.10).

It is important to address the transformation needed to put kinetic terms of matter superfields in the canonical form because kinetic terms are given in eq. (2.8). The canonical form is realized by the overall normalization of the lepton mass matrices, which shifts our parameters such as

$$\begin{aligned} \alpha &\rightarrow \alpha' = \alpha(K_L K_{e_R})^{-1/2}, & \beta &\rightarrow \beta' = \beta(K_L K_{\mu_R})^{-1/2}, & \gamma &\rightarrow \gamma' = \gamma(K_L K_{\tau_R})^{-1/2}, \\ g_i &\rightarrow g'_i = g_i(K_L K_{\nu_R})^{-1/2} \quad (i = 1, 2), & \Lambda &\rightarrow \Lambda' = \Lambda K_{\nu_R}^{-1}, \end{aligned} \quad (3.18)$$

where K_ϕ denotes a coefficient of the kinetic term of eq. (2.8). Hereafter, we rewrite α , β , γ , g_i , and Λ for α' , β' , γ' , g'_i , and Λ' in our convention.

Finally, we summarize the classification of mass matrices for neutrino models in table 2.

4 Numerical results

We discuss numerical results for neutrino models in table 2. The lepton mass matrices in the previous section are given by modulus parameter τ . By fixing τ , the modular invariance is broken, and then the lepton mass matrices give the mass eigenvalues and flavor mixing numerically. In order to fix the value of τ , we use the result of NuFIT 3.2 with the 3σ error-bar [36]. We consider both the normal hierarchy (NH) of neutrino masses $m_1 < m_2 < m_3$ and the inverted hierarchy (IH) of neutrino masses $m_3 < m_1 < m_2$, where m_1 , m_2 , and m_3 denote three light neutrino masses. The sum of neutrino masses are restricted by the cosmological observations [37, 38]. Planck 2018 results provide us its cosmological upper

observable	3σ range for NH	3σ range for IH
Δm_{atm}^2	$(2.399\text{--}2.593) \times 10^{-3} \text{ eV}^2$	$(-2.562\text{--}-2.369) \times 10^{-3} \text{ eV}^2$
Δm_{sol}^2	$(6.80\text{--}8.02) \times 10^{-5} \text{ eV}^2$	$(6.80\text{--}8.02) \times 10^{-5} \text{ eV}^2$
$\sin^2 \theta_{23}$	0.418–0.613	0.435–0.616
$\sin^2 \theta_{12}$	0.272–0.346	0.272–0.346
$\sin^2 \theta_{13}$	0.01981–0.02436	0.02006–0.02452

Table 3. The 3σ ranges of neutrino oscillation parameters from NuFIT 3.2 for NH and IH [36].

bound for sum of neutrino masses; 120–160 meV [39] at the 95% C.L. depending on the combined data. We have used the upper bound of 160 meV as a conservative constraint of our models. By inputting the data of $\Delta m_{\text{atm}}^2 \equiv m_3^2 - m_1^2$, $\Delta m_{\text{sol}}^2 \equiv m_2^2 - m_1^2$, and three mixing angles θ_{23} , θ_{12} , and θ_{13} with 3σ error-bar given in table 3, we fix the modulus τ and the other parameters. Then we can predict the CP violating Dirac phases δ_{CP} and Majorana phases α_{31} , α_{21} , which are defined in appendix C.

4.1 Model I(a): seesaw

The coefficients α/γ and β/γ in the charged lepton mass matrix are given only in terms of τ after inputting the observed values m_e/m_τ and m_μ/m_τ as shown in appendix B. Then, we have two free parameters, g_1/g_2 and the modulus τ apart from the overall factors in the neutrino sector. Since these are complex, we set

$$\tau = \text{Re}[\tau] + i \text{Im}[\tau], \quad \frac{g_2}{g_1} = g e^{i\phi_g}. \quad (4.1)$$

The fundamental domain of τ is presented in ref. [29]. In practice, we restrict our parametric search in $\text{Re}[\tau] \in [-1.5, 1.5]$ and $\text{Im}[\tau] > 0.6$. We also take $\phi_g \in [-\pi, \pi]$. These four parameters are fixed by the observed $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$ and three mixing angles θ_{23} , θ_{12} and θ_{13} .

At first, we present the prediction of the Dirac CP violating phase δ_{CP} versus $\sin^2 \theta_{23}$ for NH of neutrino masses in figure 1. It is emphasized that $\sin^2 \theta_{23}$ is restricted to be larger than 0.54, and $\delta_{\text{CP}} = \pm(50^\circ\text{--}180^\circ)$. Since the correlation of $\sin^2 \theta_{23}$ and δ_{CP} is characteristic, this prediction is testable in the future experiments of neutrinos. On the other hand, predicted $\sin^2 \theta_{12}$ and $\sin^2 \theta_{13}$ cover observed full region with 3σ error-bar, and there are no correlations with δ_{CP} .

We also show the predicted Jarlskog invariant J_{CP} [40], characterizing the magnitude of CP violation in neutrino oscillations, versus $\sin^2 \theta_{23}$ for NH of neutrino masses in figure 2. The magnitude of J_{CP} is predicted to be 0–0.035 depending on θ_{23} .

We show the prediction of Majorana phases α_{21} and α_{31} in figure 3. The predicted regions are restricted in $\alpha_{21} = \pm(118^\circ\text{--}138^\circ)$ and $\alpha_{31} = \pm(86^\circ\text{--}130^\circ)$. This result is used in the calculation of neutrinoless double beta decay.

Let us show the prediction of the effective mass m_{ee} which is the measure of the neutrinoless double beta decay as seen in appendix C. The prediction of m_{ee} is presented

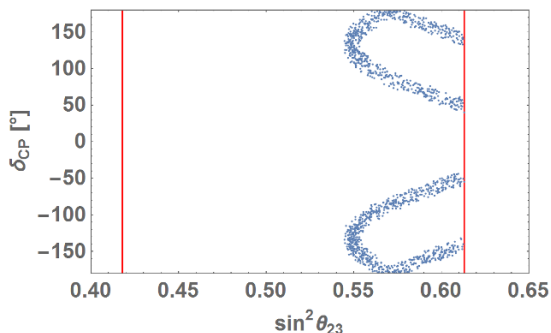


Figure 1. The prediction of δ_{CP} versus $\sin^2 \theta_{23}$ for NH in model I(a). The vertical red lines represent the upper and lower bounds of the experimental data with 3σ .

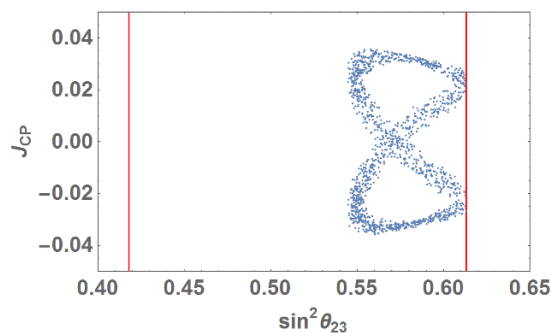


Figure 2. The prediction of J_{CP} versus $\sin^2 \theta_{23}$ for NH in model I(a). The vertical red lines represent the upper and lower bounds of the experimental data with 3σ .

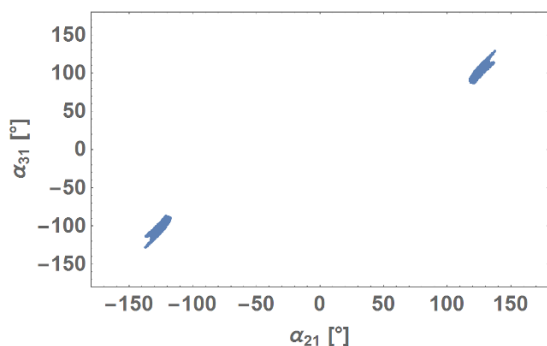


Figure 3. The prediction of Majorana phases α_{21} and α_{31} for NH in model I(a).

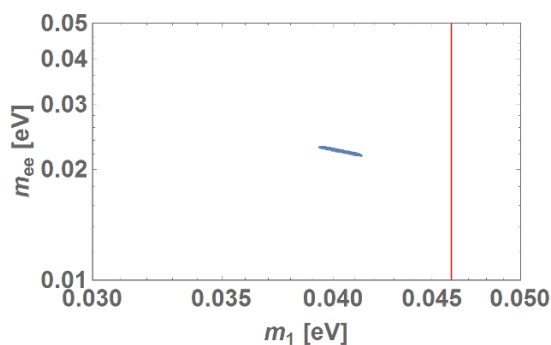


Figure 4. The prediction of m_{ee} versus m_1 for NH in model I(a). The red vertical line denotes the upper-bound of m_1 .

versus m_1 in figure 4. It is remarkable that m_{ee} is around 22 meV while m_1 is 40 meV. The red vertical line in figure 4 denotes the upper bound of m_1 , which is derived from the cosmological bound $\sum m_i < 160$ meV. The obtained value of m_1 indicates near degenerate neutrino mass spectrum, $m_1 \simeq m_2 \simeq 40$ meV and $m_3 \simeq 60$ meV. The prediction of $m_{ee} \simeq 22$ meV is testable in the future experiments of the neutrinoless double beta decay. We predict the rather large sum of neutrino masses as $\sum m_i \simeq 145$ meV, which is required by consistency with the observed value of $\sin^2 \theta_{13}$.

The parameters of our model are determined by the input data of table 3. Numerical values are listed in table 4.

We have also scanned the parameter space for the case of IH of neutrino masses. We have found parameter sets which fit the data of Δm_{sol}^2 , Δm_{atm}^2 and three mixing angles $\sin^2 \theta_{23}$, $\sin^2 \theta_{12}$, and $\sin^2 \theta_{13}$. However, the predicted $\sum m_i$ is around 190–200 meV. Therefore, we also omit to show numerical results.

4.2 Model I(b): seesaw

There is another assignment of the modular weight for the left-handed lepton and the right-handed charged leptons as presented in parentheses of table 1 [32]. Then, the Dirac

Im[τ]	Re[τ]	g	ϕ_g	α/γ	β/γ
0.66–0.73 1.17–1.32	$\pm(0.25\text{--}0.31), \pm(0.46\text{--}0.54),$ $\pm(0.66\text{--}0.75), \pm(1.25\text{--}1.31),$ $\pm(1.46\text{--}1.50)$	1.20–1.22	$\pm(87\text{--}88)^\circ$ $\pm(92\text{--}93)^\circ$	202–203	3286–3306

Table 4. The parameter regions consistent with the experimental data of table 3 for model I(a). Results do not change under the exchange of α/γ and β/γ .

neutrino mass matrix is given by the constant parameter as seen in eq. (3.12). We have scanned the parameter space for both NH and IH of neutrino masses. The parameters to reproduce the observed Δm_{sol}^2 and Δm_{atm}^2 cannot give the large mixing angle of θ_{23} . The predicted value $\sin^2 \theta_{23} \simeq 0.18$ for NH. We also obtain $\sin^2 \theta_{12} \simeq 0.8$ and $\sin^2 \theta_{13} \simeq 0.15$. On the other hand, the predicted value $\sin^2 \theta_{23} \simeq 0$, $\sin^2 \theta_{12} \simeq 0.5$, and $\sin^2 \theta_{13} \simeq 0$ for IH. In conclusion, the model I(b) is inconsistent with the experimental data of table 3.

It may be useful to add the discussion on the model by Criado and Feruglio [32], where the charged lepton mass matrix is different from ours in eq. (3.8), but given by a flavon while the neutrino mass matrix is just same one in model I(b). We have reproduced the numerical results of ref. [32], in which the three mixing angles and masses are consistent with the experimental data and the cosmological bound, respectively, for NH of neutrino masses. The predicted CP violating phase is $\delta_{\text{CP}} \simeq \pm 100^\circ$.

4.3 Model II: Weinberg operator

In this case, the modulus τ is the only parameter in the neutrino mass matrix apart from the overall factors. We can find the parameter space to be consistent with the observed $\sin^2 \theta_{12}$ as well as Δm_{sol}^2 and Δm_{atm}^2 for both NH and IH. However, the predicted $\sin^2 \theta_{23}$ is around 0.8 and $\sin^2 \theta_{13}$ is very large as 0.45 for NH. On the other hand, for IH, the predicted $\sin^2 \theta_{23}$ is rather small as 0.35 and $\sin^2 \theta_{13}$ is around 0.04, which is larger than 1.6 times of the observed value. Thus, the neutrino mass matrix by the Weinberg operator do not lead to the realistic flavor mixing.

4.4 Model III: Dirac neutrino

There is still a possibility of the neutrino being the Dirac particle. Then, the neutrino mass matrix is different from the Majorana one as shown in table 2 although parameters are τ and g likewise in the case of the seesaw model I(a).

We have found the parameter space to be consistent with both observed $\sin^2 \theta_{23}$ and $\sin^2 \theta_{12}$ as well as Δm_{sol}^2 and Δm_{atm}^2 for NH. However, the predicted $\sin^2 \theta_{13}$ is much smaller than the observed value of $\mathcal{O}(10^{-3})$.

On the other hand, the $\sin^2 \theta_{13}$ is completely consistent with the observed value for IH of neutrino masses. We present the prediction of the Dirac CP violating phase δ_{CP} versus $\sin^2 \theta_{23}$ for IH in figure 5. The predicted δ_{CP} is still allowed in $[-\pi, \pi]$ depending on the magnitude of $\sin^2 \theta_{23}$. Since there are no correlations among $\sin^2 \theta_{12}$, $\sin^2 \theta_{13}$, and δ_{CP} , we omit figures of $\sin^2 \theta_{12}$ and $\sin^2 \theta_{13}$.

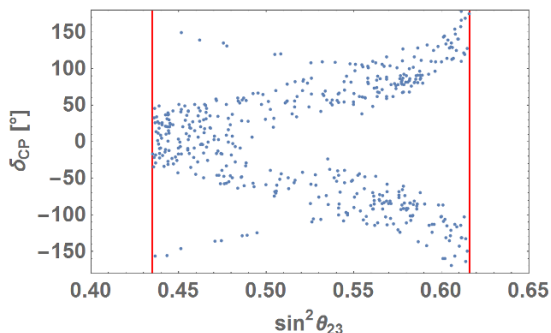


Figure 5. The prediction of δ_{CP} versus $\sin^2 \theta_{23}$ for IH in model III. The vertical red lines represent the upper and lower bounds of the experimental data with 3σ .

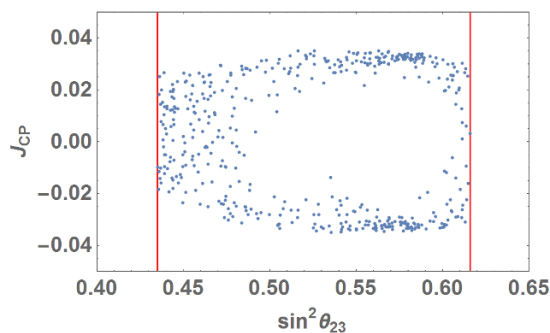


Figure 6. The prediction of J_{CP} versus $\sin^2 \theta_{23}$ for IH in model III. The vertical red lines represent the upper and lower bounds of the experimental data with 3σ .

$\text{Im}[\tau]$	$\text{Re}[\tau]$	g	ϕ_g	α/γ	β/γ
0.90–1.12	$\pm(0.01\text{--}0.07)$ $\pm(0.94\text{--}1.10)$	1.43–2.12	$\pm(76\text{--}104)^\circ$	59–88	857–1302

Table 5. The parameter regions consistent with the experimental data of table 3 for model III. Results do not change under the exchange of α/γ and β/γ .

We also show the predicted Jarlskog invariant J_{CP} versus $\sin^2 \theta_{23}$ for IH of neutrino masses in figure 6. The magnitude of J_{CP} is predicted to be 0–0.035.

The $\sum m_i$ is required in 102–150 meV to be consistent with the observed value of $\sin^2 \theta_{13}$.

We summarize numerical values of parameters in table 5. In the Dirac neutrino model, the neutrinoless double beta decay is forbidden.

5 Summary

We study the phenomenological implications of the modular symmetry $\Gamma(3) \simeq A_4$ facing recent experimental data of neutrino oscillations. The mass matrices of neutrinos and charged leptons are essentially given by fixing the expectation value of the modulus τ , which is the only source of modular invariance breaking. We introduce no flavons in contrast with conventional flavor models with the A_4 symmetry. We classify the neutrino models along with type I seesaw (model I(a) and I(b)), Weinberg operator (model II), and Dirac neutrino (model III). For the charged lepton mass matrix, three right-handed charged leptons e_R , μ_R , and τ_R are assigned to three different singlets 1, $1''$, and $1'$ of A_4 , respectively.

For NH of neutrino masses, we have found that the seesaw model I(a) is available facing recent experimental data of NuFIT 3.2 [36] and the cosmological bound of the sum of neutrino masses [39]. The predicted $\sin^2 \theta_{23}$ is restricted to be larger than 0.54 and $\delta_{\text{CP}} = \pm(50^\circ\text{--}180^\circ)$. The sharp correlation between $\sin^2 \theta_{23}$ and δ_{CP} is testable in the

future experiments of the neutrino oscillations. It is remarkable that m_{ee} is around 22 meV while the sum of neutrino masses is 145 meV.

For IH of neutrino masses, the Dirac neutrino model III is completely consistent with the experimental data of NuFIT 3.2 and the cosmological bound of the sum of neutrino masses. The predicted δ_{CP} is still allowed in $[-\pi, \pi]$ depending on the magnitude of $\sin^2 \theta_{23}$. The $\sum m_i = 102\text{--}150$ meV is required by consistency with the observed value of $\sin^2 \theta_{13}$.

The seesaw model I(b) and the Weinberg operator model II cannot reproduce the observed mixing angles after inputting the data of Δm_{sol}^2 and Δm_{atm}^2 for both NH and IH.

It is helpful to comment on the effects of the supersymmetry (SUSY) breaking and the radiative corrections because we have discussed our model in the limit of exact SUSY. The SUSY breaking effect can be neglected if the separation between the SUSY breaking scale and the SUSY breaking mediator scale is sufficiently large [32]. In our numerical results, the corrections by the renormalization are very small as far as we take the relatively small value of $\tan \beta$.

We have focused on the models with Yukawa couplings and masses, which correspond to modular forms of lower weights. Such models have strong constraints. Similarly, we can construct models with modular forms of higher weights. Such models would have a variety in model building.

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A Multiplication rule of A_4 group

We use the multiplication rule of the A_4 triplet as follows:

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} &= (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}} \oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_{\mathbf{1}'} \\ &\oplus (a_2 b_2 + a_1 b_3 + a_3 b_1)_{\mathbf{1}''} \\ &\oplus \frac{1}{3} \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2} \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}} \quad , \\ \mathbf{1} \otimes \mathbf{1} &= \mathbf{1}, \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'', \quad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}', \quad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}. \quad (\text{A.1}) \end{aligned}$$

More details are shown in the review [6, 7].

B Determination of α/γ and β/γ

The coefficients α , β , and γ in eq. (3.8) are taken to be real positive without loss of generality. We show these parameters are described in terms of the modular parameter τ and the charged lepton masses. We rewrite the mass matrix of eq. (3.8) as

$$M_E^{(1)} = \gamma Y_3 \text{diag}[\hat{\alpha}, \hat{\beta}, 1] \begin{pmatrix} \hat{Y}_1 & \hat{Y}_2 & 1 \\ 1 & \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_2 & 1 & \hat{Y}_1 \end{pmatrix}_{RL}, \quad (\text{B.1})$$

where $\hat{\alpha} \equiv \alpha/\gamma$, $\hat{\beta} \equiv \beta/\gamma$, $\hat{Y}_1 \equiv Y_1/Y_3$, and $\hat{Y}_2 \equiv Y_2/Y_3$. We use the relation $Y_2^2 + 2Y_1Y_3 = 0$ to eliminate Y_1 in the equation. Then, we obtain the following three equations:

$$\text{Tr}[M_E^{(1)\dagger} M_E^{(1)}] = \sum_{i=e}^{\tau} m_i^2 = \frac{|\gamma Y_3|^2}{4} (1 + \hat{\alpha}^2 + \hat{\beta}^2) C_1, \quad (\text{B.2})$$

$$\text{Det}[M_E^{(1)\dagger} M_E^{(1)}] = \prod_{i=e}^{\tau} m_i^2 = \frac{|\gamma Y_3|^6}{64} \hat{\alpha}^2 \hat{\beta}^2 C_2, \quad (\text{B.3})$$

$$\frac{\text{Tr}[M_E^{(1)\dagger} M_E^{(1)}]^2 - \text{Tr}[(M_E^{(1)\dagger} M_E^{(1)})^2]}{2} = \chi = \frac{|\gamma Y_3|^4}{16} (\hat{\alpha}^2 + \hat{\alpha}^2 \hat{\beta}^2 + \hat{\beta}^2) C_3, \quad (\text{B.4})$$

where $\chi \equiv m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2$. The coefficients C_1 , C_2 , and C_3 depend only on $\hat{Y}_2 \equiv Y e^{i\phi_Y}$, where Y is real positive and ϕ_Y is a phase parameter,

$$\begin{aligned} C_1 &= (2 + Y^2)^2, \\ C_2 &= 64 + 400Y^6 + Y^{12} - 40Y^3(Y^6 - 8) \cos(3\phi_Y) - 16Y^6 \cos(6\phi_Y), \\ C_3 &= 16 + 16Y^2 + 36Y^4 + 4Y^6 + Y^8 - 8Y^3(Y^2 - 2) \cos(3\phi_Y). \end{aligned} \quad (\text{B.5})$$

These values are determined if the value of modulus τ is fixed. Then, we obtain the general equations which describe $\hat{\alpha}$ and $\hat{\beta}$ as functions of charged lepton masses and τ :

$$\frac{(1+s)(s+t)}{t} = \frac{(\sum m_i^2/C_1)(\chi/C_3)}{\prod m_i^2/C_2}, \quad \frac{(1+s)^2}{s+t} = \frac{(\sum m_i^2/C_1)^2}{\chi/C_3}, \quad (\text{B.6})$$

where we redefine the parameters $\hat{\alpha}^2 + \hat{\beta}^2 = s$ and $\hat{\alpha}^2 \hat{\beta}^2 = t$. They are related as follows,

$$\hat{\alpha}^2 = \frac{s \pm \sqrt{s^2 - 4t}}{2}, \quad \hat{\beta}^2 = \frac{s \mp \sqrt{s^2 - 4t}}{2}. \quad (\text{B.7})$$

C Lepton mixing and neutrinoless double beta decay

Supposing neutrinos to be Majorana particles, the PMNS matrix U_{PMNS} [41, 42] is parametrized in terms of the three mixing angles θ_{ij} ($i, j = 1, 2, 3$; $i < j$), one CP violating Dirac phase δ_{CP} , and two Majorana phases α_{21} , α_{31} as follows:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\text{CP}}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}, \quad (\text{C.1})$$

where c_{ij} and s_{ij} denote $\cos \theta_{ij}$ and $\sin \theta_{ij}$, respectively.

The rephasing invariant CP violating measure, the Jarlskog invariant [40], is defined by the PMNS matrix elements $U_{\alpha i}$. It is written in terms of the mixing angles and the CP violating Dirac phase as:

$$J_{\text{CP}} = \text{Im} [U_{e1}U_{\mu 2}U_{e2}^*U_{\mu 1}^*] = s_{23}c_{23}s_{12}c_{12}s_{13}c_{13}^2 \sin \delta_{\text{CP}}. \quad (\text{C.2})$$

There are also other invariants I_1 and I_2 associated with Majorana phases [43–47],

$$I_1 = \text{Im} [U_{e1}^*U_{e2}] = c_{12}c_{12}c_{13}^2 \sin \left(\frac{\alpha_{21}}{2} \right), \quad I_2 = \text{Im} [U_{e1}^*U_{e3}] = c_{12}s_{13}c_{13} \sin \left(\frac{\alpha_{31}}{2} - \delta_{\text{CP}} \right). \quad (\text{C.3})$$

We calculate δ_{CP} , α_{21} , and α_{31} with these relations.

In terms of these parametrization, the effective mass for the $0\nu\beta\beta$ decay is given as follows:

$$m_{ee} = \left| m_1c_{12}^2c_{13}^2 + m_2s_{12}^2c_{13}^2e^{i\alpha_{21}} + m_3s_{13}^2e^{i(\alpha_{31}-2\delta_{\text{CP}})} \right|. \quad (\text{C.4})$$

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References

- [1] T2K collaboration, K. Abe et al., *Measurement of neutrino and antineutrino oscillations by the T2K experiment including a new additional sample of ν_e interactions at the far detector*, *Phys. Rev. D* **96** (2017) 092006 [*Erratum ibid.* **D 98** (2018) 019902] [[arXiv:1707.01048](https://arxiv.org/abs/1707.01048)] [[INSPIRE](https://inspirehep.net/literature/1707010)].
- [2] W. Morgan, *T2K Status, Results, and Plans*, talk given at the *XXVIII International Conference on Neutrino Physics and Astrophysics*, Heidelberg, Germany, 4–9 June 2018, <https://doi.org/10.5281/zenodo.1286751>.
- [3] NOvA collaboration, P. Adamson et al., *Constraints on Oscillation Parameters from ν_e Appearance and ν_μ Disappearance in NOvA*, *Phys. Rev. Lett.* **118** (2017) 231801 [[arXiv:1703.03328](https://arxiv.org/abs/1703.03328)] [[INSPIRE](https://inspirehep.net/literature/1703033)].
- [4] M. Sanchez, *NOvA Results and Prospects?*, talk given at the *XXVIII International Conference on Neutrino Physics and Astrophysics*, Heidelberg, Germany, 4–9 June 2018, <https://doi.org/10.5281/zenodo.1286757>.
- [5] G. Altarelli and F. Feruglio, *Discrete Flavor Symmetries and Models of Neutrino Mixing*, *Rev. Mod. Phys.* **82** (2010) 2701 [[arXiv:1002.0211](https://arxiv.org/abs/1002.0211)] [[INSPIRE](https://inspirehep.net/literature/85481)].
- [6] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, *Non-Abelian Discrete Symmetries in Particle Physics*, *Prog. Theor. Phys. Suppl.* **183** (2010) 1 [[arXiv:1003.3552](https://arxiv.org/abs/1003.3552)] [[INSPIRE](https://inspirehep.net/literature/85481)].
- [7] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu and M. Tanimoto, *An introduction to non-Abelian discrete symmetries for particle physicists*, *Springer, Lect. Notes Phys.* **858** (2012) 1 [[INSPIRE](https://inspirehep.net/literature/1003352)].
- [8] S.F. King and C. Luhn, *Neutrino Mass and Mixing with Discrete Symmetry*, *Rept. Prog. Phys.* **76** (2013) 056201 [[arXiv:1301.1340](https://arxiv.org/abs/1301.1340)] [[INSPIRE](https://inspirehep.net/literature/1100000)].

- [9] S.F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, *Neutrino Mass and Mixing: from Theory to Experiment*, *New J. Phys.* **16** (2014) 045018 [[arXiv:1402.4271](#)] [[INSPIRE](#)].
- [10] E. Ma and G. Rajasekaran, *Softly broken A_4 symmetry for nearly degenerate neutrino masses*, *Phys. Rev. D* **64** (2001) 113012 [[hep-ph/0106291](#)] [[INSPIRE](#)].
- [11] K.S. Babu, E. Ma and J.W.F. Valle, *Underlying A_4 symmetry for the neutrino mass matrix and the quark mixing matrix*, *Phys. Lett. B* **552** (2003) 207 [[hep-ph/0206292](#)] [[INSPIRE](#)].
- [12] G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions*, *Nucl. Phys. B* **720** (2005) 64 [[hep-ph/0504165](#)] [[INSPIRE](#)].
- [13] G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing, A_4 and the modular symmetry*, *Nucl. Phys. B* **741** (2006) 215 [[hep-ph/0512103](#)] [[INSPIRE](#)].
- [14] Y. Shimizu, M. Tanimoto and A. Watanabe, *Breaking Tri-bimaximal Mixing and Large θ_{13}* , *Prog. Theor. Phys.* **126** (2011) 81 [[arXiv:1105.2929](#)] [[INSPIRE](#)].
- [15] S.K. Kang, Y. Shimizu, K. Takagi, S. Takahashi and M. Tanimoto, *Revisiting A_4 model for leptons in light of NuFIT 3.2*, *Prog. Theor. Exp. Phys.* **2018** (2018) 083B01 [[arXiv:1804.10468](#)] [[INSPIRE](#)].
- [16] S.T. Petcov and A.V. Titov, *Assessing the Viability of A_4 , S_4 and A_5 Flavour Symmetries for Description of Neutrino Mixing*, *Phys. Rev. D* **97** (2018) 115045 [[arXiv:1804.00182](#)] [[INSPIRE](#)].
- [17] T. Kobayashi, H.P. Nilles, F. Ploger, S. Raby and M. Ratz, *Stringy origin of non-Abelian discrete flavor symmetries*, *Nucl. Phys. B* **768** (2007) 135 [[hep-ph/0611020](#)] [[INSPIRE](#)].
- [18] T. Kobayashi, S. Raby and R.-J. Zhang, *Searching for realistic 4d string models with a Pati-Salam symmetry: Orbifold grand unified theories from heterotic string compactification on a \mathbb{Z}_6 orbifold*, *Nucl. Phys. B* **704** (2005) 3 [[hep-ph/0409098](#)] [[INSPIRE](#)].
- [19] P. Ko, T. Kobayashi, J.-h. Park and S. Raby, *String-derived D_4 flavor symmetry and phenomenological implications*, *Phys. Rev. D* **76** (2007) 035005 [Erratum *ibid.* **D 76** (2007) 059901] [[arXiv:0704.2807](#)] [[INSPIRE](#)].
- [20] H. Abe, K.-S. Choi, T. Kobayashi and H. Ohki, *Non-Abelian Discrete Flavor Symmetries from Magnetized/Intersecting Brane Models*, *Nucl. Phys. B* **820** (2009) 317 [[arXiv:0904.2631](#)] [[INSPIRE](#)].
- [21] M. Berasaluce-Gonzalez, P.G. Camara, F. Marchesano, D. Regalado and A.M. Uranga, *Non-Abelian discrete gauge symmetries in 4d string models*, *JHEP* **09** (2012) 059 [[arXiv:1206.2383](#)] [[INSPIRE](#)].
- [22] J. Lauer, J. Mas and H.P. Nilles, *Duality and the Role of Nonperturbative Effects on the World Sheet*, *Phys. Lett. B* **226** (1989) 251 [[INSPIRE](#)].
- [23] J. Lauer, J. Mas and H.P. Nilles, *Twisted sector representations of discrete background symmetries for two-dimensional orbifolds*, *Nucl. Phys. B* **351** (1991) 353 [[INSPIRE](#)].
- [24] W. Lerche, D. Lüüst and N.P. Warner, *Duality Symmetries in $N = 2$ Landau-Ginzburg Models*, *Phys. Lett. B* **231** (1989) 417 [[INSPIRE](#)].
- [25] S. Ferrara, D. Lüüst and S. Theisen, *Target Space Modular Invariance and Low-Energy Couplings in Orbifold Compactifications*, *Phys. Lett. B* **233** (1989) 147 [[INSPIRE](#)].

- [26] D. Cremades, L.E. Ibáñez and F. Marchesano, *Computing Yukawa couplings from magnetized extra dimensions*, *JHEP* **05** (2004) 079 [[hep-th/0404229](#)] [[INSPIRE](#)].
- [27] T. Kobayashi and S. Nagamoto, *Zero-modes on orbifolds: magnetized orbifold models by modular transformation*, *Phys. Rev. D* **96** (2017) 096011 [[arXiv:1709.09784](#)] [[INSPIRE](#)].
- [28] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba and T.H. Tatsuishi, *Modular symmetry and non-Abelian discrete flavor symmetries in string compactification*, *Phys. Rev. D* **97** (2018) 116002 [[arXiv:1804.06644](#)] [[INSPIRE](#)].
- [29] F. Feruglio, *Are neutrino masses modular forms?*, [arXiv:1706.08749](#) [[INSPIRE](#)].
- [30] T. Kobayashi, K. Tanaka and T.H. Tatsuishi, *Neutrino mixing from finite modular groups*, *Phys. Rev. D* **98** (2018) 016004 [[arXiv:1803.10391](#)] [[INSPIRE](#)].
- [31] J.T. Penedo and S.T. Petcov, *Lepton Masses and Mixing from Modular S_4 Symmetry*, [arXiv:1806.11040](#) [[INSPIRE](#)].
- [32] J.C. Criado and F. Feruglio, *Modular Invariance Faces Precision Neutrino Data*, *SciPost Phys.* **5** (2018) 042 [[arXiv:1807.01125](#)] [[INSPIRE](#)].
- [33] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, *Finite Modular Groups and Lepton Mixing*, *Nucl. Phys. B* **858** (2012) 437 [[arXiv:1112.1340](#)] [[INSPIRE](#)].
- [34] S. Ferrara, D. Lüst, A.D. Shapere and S. Theisen, *Modular Invariance in Supersymmetric Field Theories*, *Phys. Lett. B* **225** (1989) 363 [[INSPIRE](#)].
- [35] R.C. Gunning, *Lectures on Modular Forms*, Princeton University Press, Princeton NJ U.S.A. (1962).
- [36] NuFIT 3.2, (2018) <http://www.nu-fit.org/>.
- [37] E. Giusarma, M. Gerbino, O. Mena, S. Vagnozzi, S. Ho and K. Freese, *Improvement of cosmological neutrino mass bounds*, *Phys. Rev. D* **94** (2016) 083522 [[arXiv:1605.04320](#)] [[INSPIRE](#)].
- [38] S. Vagnozzi et al., *Unveiling ν secrets with cosmological data: neutrino masses and mass hierarchy*, *Phys. Rev. D* **96** (2017) 123503 [[arXiv:1701.08172](#)] [[INSPIRE](#)].
- [39] PLANCK collaboration, N. Aghanim et al., *Planck 2018 results. VI. Cosmological parameters*, [arXiv:1807.06209](#) [[INSPIRE](#)].
- [40] C. Jarlskog, *Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP-violation*, *Phys. Rev. Lett.* **55** (1985) 1039 [[INSPIRE](#)].
- [41] Z. Maki, M. Nakagawa and S. Sakata, *Remarks on the unified model of elementary particles*, *Prog. Theor. Phys.* **28** (1962) 870 [[INSPIRE](#)].
- [42] B. Pontecorvo, *Neutrino Experiments and the Problem of Conservation of Leptonic Charge*, *Sov. Phys. JETP* **26** (1968) 984 [*Zh. Eksp. Teor. Fiz.* **53** (1967) 1717] [[INSPIRE](#)].
- [43] S.M. Bilenky, S. Pascoli and S.T. Petcov, *Majorana neutrinos, neutrino mass spectrum, CP-violation and neutrinoless double beta decay. 1. The Three neutrino mixing case*, *Phys. Rev. D* **64** (2001) 053010 [[hep-ph/0102265](#)] [[INSPIRE](#)].
- [44] J.F. Nieves and P.B. Pal, *Minimal Rephasing Invariant CP Violating Parameters With Dirac and Majorana Fermions*, *Phys. Rev. D* **36** (1987) 315 [[INSPIRE](#)].

- [45] J.F. Nieves and P.B. Pal, *Rephasing invariant CP-violating parameters with Majorana neutrinos*, *Phys. Rev. D* **64** (2001) 076005 [[hep-ph/0105305](#)] [[INSPIRE](#)].
- [46] J.A. Aguilar-Saavedra and G.C. Branco, *Unitarity triangles and geometrical description of CP-violation with Majorana neutrinos*, *Phys. Rev. D* **62** (2000) 096009 [[hep-ph/0007025](#)] [[INSPIRE](#)].
- [47] I. Girardi, S.T. Petcov and A.V. Titov, *Predictions for the Majorana CP-violation Phases in the Neutrino Mixing Matrix and Neutrinoless Double Beta Decay*, *Nucl. Phys. B* **911** (2016) 754 [[arXiv:1605.04172](#)] [[INSPIRE](#)].