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# Modular $A_5$ symmetry for flavour model building

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**ABSTRACT:** In the framework of the modular symmetry approach to lepton flavour, we consider a class of theories where matter superfields transform in representations of the finite modular group  $\Gamma_5 \simeq A_5$ . We explicitly construct a basis for the 11 modular forms of weight 2 and level 5. We show how these forms arrange themselves into two triplets and a quintet of  $A_5$ . We also present multiplets of modular forms of higher weight. Finally, we provide an example of application of our results, constructing two models of neutrino masses and mixing based on the supersymmetric Weinberg operator.

**KEYWORDS:** Beyond Standard Model, Discrete Symmetries, Neutrino Physics, CP violation

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**1 Introduction**

Understanding the origins of flavour remains one of the major problems in particle physics. The power of symmetries in governing laws of particle interactions does not need to be advocated. In this regard, it is rather natural to expect that symmetry(ies) also hold the key to the solution of the flavour problem.

The fact that two out of three neutrino mixing angles are large [1–3] suggests the presence of a new flavour symmetry (at least in the lepton sector) described by a non-Abelian discrete (finite) group (see, e.g., [4–7]). While unifying the three known flavours at high energies, this symmetry may be broken at lower energies to residual symmetries of the charged lepton and neutrino mass terms, which correspond to Abelian subgroups of the original flavour symmetry group. In the bottom-up approach, starting from residual symmetries, one can successfully explain the observed pattern of neutrino mixing and, in addition, predict the value of the Dirac CP violation phase [8–10].<sup>1</sup> However, predicting neutrino

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<sup>1</sup>Predictions for the Dirac CPV phase can be obtained also if the neutrino Majorana mass matrix respects a specific residual symmetry while the mixing originating from the charged lepton sector has a form restricted by additional (GUT, generalised CP) symmetry or phenomenological considerations, see, e.g., [11–13].

masses calls for the construction of specific models, in which the flavour symmetry is typically spontaneously broken by vacuum expectation values (VEVs) of flavons — scalar gauge singlets charged non-trivially under the flavour symmetry group. Usually, a numerous set of these fields is needed. Moreover, one may have to construct rather complicated flavon potentials in order to achieve vacuum alignments leading to viable phenomenology.

A very interesting generalisation of the discrete symmetry approach to lepton flavour has been recently proposed in ref. [14]. In this proposal, modular invariance plays the role of flavour symmetry, and couplings of a theory are modular forms of a certain level  $N$ . In addition, both the couplings and matter supermultiplets are assumed to transform in representations of a finite modular group  $\Gamma_N$ . In the simplest class of such models, the VEV of a complex field  $\tau$  (the modulus) is the only source of flavour symmetry breaking, such that no flavons are needed. Another appealing feature of the proposed framework is that charged lepton and neutrino masses, neutrino mixing and CPV phases are simultaneously determined by the modular symmetry typically in terms of a limited number of constant parameters. This leads to experimentally testable correlations between, e.g., the neutrino mass and mixing observables.

The cornerstone of the new approach is the modular forms of weight 2 and level  $N$ , and their arrangements into multiplets of  $\Gamma_N$ . Modular forms of higher weights can be constructed from these building blocks. Remarkably, for  $N \leq 5$ , the finite modular groups are isomorphic to well-known permutation groups. In ref. [14], the group  $\Gamma_3 \simeq A_4$  has been considered, and the three generating modular forms of weight 2 have been explicitly constructed and shown to furnish a 3-dimensional irreducible representation (irrep) of  $A_4$ . Further, the group  $\Gamma_2 \simeq S_3$  has been considered in [15], and the two forms shaping a doublet of  $S_3$  have been identified. The five generating modular forms in the case of  $N = 4$  have been found to organise themselves into a doublet and a triplet of  $\Gamma_4 \simeq S_4$  in ref. [16], where the first realistic model of lepton masses and mixing without flavons has also been constructed. Very recently, by studying Yukawa couplings in magnetised D-brane models, the authors of ref. [17] have found multiplets of weight 2 modular forms corresponding to a triplet and a sextet of  $\Delta(96)$ , and a triplet of  $\Delta(384)$ . They have also reported an  $S_3$  doublet and an  $S_4$  triplet. Note that  $\Delta(96)$  is isomorphic to a subgroup of  $\Gamma_8$ , while  $\Delta(384)$  is isomorphic to a subgroup of  $\Gamma_{16}$  (see, e.g., [18]).

Lepton flavour models based on  $\Gamma_3 \simeq A_4$  have been studied in more detail in refs. [19, 20], where several viable examples have been presented. In ref. [21], we have constructed in a systematic way models based on  $\Gamma_4 \simeq S_4$ , in which light neutrino masses are generated via the type I seesaw mechanism and where no flavons are introduced. We have shown that models with a relatively small number of free parameters can successfully describe data on the charged lepton masses, neutrino mass-squared differences and mixing angles. Furthermore, we have obtained predictions for the neutrino masses and the Dirac and Majorana CPV phases in the neutrino mixing matrix. In these models, the value of atmospheric mixing angle  $\theta_{23}$  is correlated with i) the Dirac phase  $\delta$ , ii) the sum of neutrino masses, and iii) the effective Majorana mass in neutrinoless double beta decay.

In the present article, for the first time in this context, we consider the finite modular group  $\Gamma_5 \simeq A_5$ . Our main focus is on constructing the 11 generating modular forms

of weight 2 and demonstrating how they can be arranged into multiplets of  $A_5$ , namely two triplets and a quintet. The group  $A_5$  has been investigated in the context of the conventional discrete symmetry approach in refs. [22–27] as well as in combination with so-called generalised CP symmetry in refs. [28–32]. The characteristic phenomenological feature of the models based on the  $A_5$  flavour symmetry is the golden ratio prediction for the solar mixing angle,  $\theta_{12} = \arctan(1/\varphi) \approx 32^\circ$ , with  $\varphi = (1 + \sqrt{5})/2$  being the golden ratio, which is inside the experimentally allowed  $3\sigma$  range [2, 3]. An interesting theoretical feature of  $A_5$  is that it is anomaly-free [5].

The article is organised as follows. In section 2, we first summarise the modular symmetry approach to lepton masses and mixing proposed in ref. [14], and then explicitly construct the two  $A_5$  triplets and the  $A_5$  quintet of modular forms of weight 2. Next, in section 3, we give an example of application of the obtained results constructing a phenomenologically viable model of neutrino masses and mixing based on the Weinberg operator. Finally, in section 4, we draw our conclusions.

## 2 The framework

### 2.1 Modular symmetry and modular-invariant theories

The modular group  $\bar{\Gamma}$  is the group of linear fractional transformations  $\gamma$  acting on the complex variable  $\tau$  belonging to the upper-half complex plane as follows:

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where } a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1, \quad \text{Im}\tau > 0. \quad (2.1)$$

The modular group is isomorphic to the projective special linear group  $\text{PSL}(2, \mathbb{Z})$ , and it is generated by two elements  $S$  and  $T$  satisfying

$$S^2 = (ST)^3 = I, \quad (2.2)$$

$I$  being the identity element of a group. Representing  $S$  and  $T$  as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (2.3)$$

one finds

$$\tau \xrightarrow{S} -\frac{1}{\tau}, \quad \tau \xrightarrow{T} \tau + 1. \quad (2.4)$$

The group  $\text{SL}(2, \mathbb{Z}) = \Gamma(1) \equiv \Gamma$  contains a series of infinite normal subgroups  $\Gamma(N)$ ,  $N = 1, 2, 3, \dots$ :

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \quad (2.5)$$

called the principal congruence subgroups. For  $N = 1$  and 2, we introduce the groups  $\bar{\Gamma}(N) \equiv \Gamma(N)/\{I, -I\}$  (note that  $\bar{\Gamma}(1) \equiv \bar{\Gamma}$ ), and for  $N > 2$ ,  $\bar{\Gamma}(N) \equiv \Gamma(N)$ . For each  $N$ , the associated linear fractional transformations of the form in eq. (2.1) are in a one-to-one

correspondence with the elements of  $\bar{\Gamma}(N)$ . The quotient groups  $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$  are called finite modular groups. For  $N \leq 5$ , these groups are isomorphic to permutation groups widely used to build flavour models (see, e.g., [18]). Namely,  $\Gamma_2 \simeq S_3$ ,  $\Gamma_3 \simeq A_4$ ,  $\Gamma_4 \simeq S_4$  and  $\Gamma_5 \simeq A_5$ .

Modular forms of weight  $k$  and level  $N$  are holomorphic functions  $f(\tau)$  transforming under the action of  $\bar{\Gamma}(N)$  in the following way:

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \bar{\Gamma}(N). \tag{2.6}$$

Here  $k$  is even and non-negative, and  $N$  is natural. Modular forms of weight  $k$  and level  $N$  span a linear space of finite dimension. There exists a basis in this space such that a multiplet of modular forms  $f_i(\tau)$  transforms according to a unitary representation  $\rho$  of the finite group  $\Gamma_N$ :

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau), \quad \gamma \in \bar{\Gamma}. \tag{2.7}$$

In the case of  $N = 2$ , the modular forms of weight 2 span a two-dimensional linear space. In a certain basis the two generating modular forms transform in the 2-dimensional irrep of  $S_3$  [15]. For level  $N = 3$ , weight 2 modular forms arrange themselves in a triplet of  $A_4$  [14]. In the case of  $N = 4$ , the corresponding linear space has dimension 5, and weight 2 modular forms group in a doublet and a triplet of  $S_4$  [16]. For  $N = 5$ , there are 11 modular forms of weight 2. They are organised in two triplets and a quintet of  $A_5$ . In the next subsection, we will explicitly derive them, but before that, let us briefly recall how to construct supersymmetric modular-invariant theories.

In the case of  $\mathcal{N} = 1$  rigid supersymmetry, the matter action  $\mathcal{S}$  reads

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} K(\tau, \bar{\tau}, \chi, \bar{\chi}) + \int d^4x d^2\theta W(\tau, \chi) + \int d^4x d^2\bar{\theta} \bar{W}(\bar{\tau}, \bar{\chi}), \tag{2.8}$$

where  $K$  is the Kähler potential,  $W$  is the superpotential and  $\chi$  denotes a set of chiral supermultiplets contained in the theory apart from the modulus  $\tau$ . The  $\theta$  and  $\bar{\theta}$  denote Grassmann variables. The modular group acts on  $\tau$  and supermultiplets  $\chi_I$  of a sector  $I$  of a theory in a certain way [33, 34]. Assuming, in addition, that the supermultiplets  $\chi_I$  transform according to a representation  $\rho_I$  of  $\Gamma_N$ , we have

$$\begin{cases} \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \\ \chi_I \rightarrow (c\tau + d)^{-k_I} \rho_I(\gamma) \chi_I. \end{cases} \tag{2.9}$$

Note that  $\chi_I$  are not modular forms, and the weight  $(-k_I)$  can be odd and/or negative. Requiring invariance of  $\mathcal{S}$  under eq. (2.9) leads to

$$\begin{cases} W(\tau, \chi) \rightarrow W(\tau, \chi), \\ K(\tau, \bar{\tau}, \chi, \bar{\chi}) \rightarrow K(\tau, \bar{\tau}, \chi, \bar{\chi}) + f_K(\tau, \chi) + \bar{f}_K(\bar{\tau}, \bar{\chi}), \end{cases} \tag{2.10}$$

where the second line represents a Kähler transformation. The superpotential can be expanded in powers of  $\chi_I$ :

$$W(\tau, \chi) = \sum_n \sum_{\{I_1, \dots, I_n\}} (Y_{I_1 \dots I_n}(\tau) \chi_{I_1} \dots \chi_{I_n})_{\mathbf{1}}, \tag{2.11}$$

where  $\mathbf{1}$  stands for an invariant singlet of  $\Gamma_N$ . To ensure invariance of  $W$  under the transformations specified in eq. (2.9), the functions  $Y_{I_1 \dots I_n}(\tau)$  must transform as follows:

$$Y_{I_1 \dots I_n}(\tau) \rightarrow (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 \dots I_n}(\tau), \quad (2.12)$$

where  $\rho_Y$  is a representation of  $\Gamma_N$ , and  $k_Y$  and  $\rho_Y$  are such that

$$k_Y = k_{I_1} + \dots + k_{I_n}, \quad (2.13)$$

$$\rho_Y \otimes \rho_{I_1} \otimes \dots \otimes \rho_{I_n} \supset \mathbf{1}. \quad (2.14)$$

Thereby, the functions  $Y_{I_1 \dots I_n}(\tau)$  form a multiplet of weight  $k_Y$  and level  $N$  modular forms transforming in the representation  $\rho_Y$  of  $\Gamma_N$  (cf. eq. (2.7)).

## 2.2 Generators of modular forms of level $N = 5$

The dimension of the space of modular forms of level  $N = 5$  and lowest nontrivial weight 2 is 11. Expansions for a standard basis  $\{b_1(\tau), \dots, b_{11}(\tau)\}$  of this space of functions are given in appendix A.1. Modular forms of higher weight can be constructed from homogeneous polynomials in these eleven modular forms. The action of the discrete quotient group  $\Gamma_5$  divides the space of lowest weight modular functions into two triplets transforming in irreps  $\mathbf{3}$  and  $\mathbf{3}'$  and a quintet transforming in  $\mathbf{5}$  of  $\Gamma_5 \simeq A_5$  (see also section 4.4 of ref. [35]).

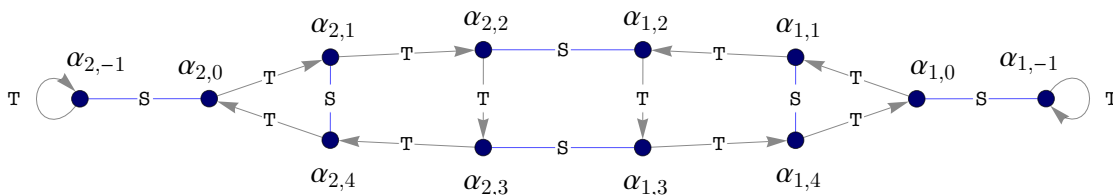
As in the cases of  $\Gamma_3 \simeq A_4$  [14],  $\Gamma_2 \simeq S_3$  [15] and  $\Gamma_4 \simeq S_4$  [16], the lowest weight modular functions correspond to linear combinations of logarithmic derivatives of some “seed” functions  $\alpha_{i,j}(\tau)$ . These functions form a set which is in a certain sense closed under the action of  $A_5$ . As can be inferred from the results in ref. [36], a convenient choice for  $\alpha_{i,j}(\tau)$  is given by the Jacobi theta functions  $\theta_3(z(\tau), t(\tau))$ , and they explicitly read:<sup>2</sup>

$$\begin{aligned} \alpha_{1,-1}(\tau) &\equiv \theta_3\left(\frac{\tau+1}{2}, 5\tau\right), & \alpha_{2,-1}(\tau) &\equiv e^{2\pi i\tau/5} \theta_3\left(\frac{3\tau+1}{2}, 5\tau\right), \\ \alpha_{1,0}(\tau) &\equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau}{5}\right), & \alpha_{2,0}(\tau) &\equiv \theta_3\left(\frac{\tau+7}{10}, \frac{\tau}{5}\right), \\ \alpha_{1,1}(\tau) &\equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+1}{5}\right), & \alpha_{2,1}(\tau) &\equiv \theta_3\left(\frac{\tau+8}{10}, \frac{\tau+1}{5}\right), \\ \alpha_{1,2}(\tau) &\equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+2}{5}\right), & \alpha_{2,2}(\tau) &\equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau+2}{5}\right), \\ \alpha_{1,3}(\tau) &\equiv \theta_3\left(\frac{\tau+2}{10}, \frac{\tau+3}{5}\right), & \alpha_{2,3}(\tau) &\equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+3}{5}\right), \\ \alpha_{1,4}(\tau) &\equiv \theta_3\left(\frac{\tau+3}{10}, \frac{\tau+4}{5}\right), & \alpha_{2,4}(\tau) &\equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+4}{5}\right). \end{aligned} \quad (2.15)$$

Under the action of the generators  $S$  and  $T$  of  $\Gamma_5$  (see appendix B.1), each of these functions is mapped to another, up to (possibly  $\tau$ -dependent) multiplicative factors. A diagram of said map is given in figure 1, and one can check that the actions of  $S^2$ ,  $(ST)^3$

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<sup>2</sup>For properties of these special functions, see, e.g., refs. [37, 38]. In the notations of ref. [37],  $\theta_3 \equiv \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .



**Figure 1.** Graph illustrating the automorphisms of the set of seed functions  $\alpha_{i,j}(\tau)$ , defined in eq. (2.15), under the actions of  $\Gamma_5 \simeq A_5$  generators  $S$  and  $T$ .

and  $T^5$  applied to each element correspond to the identity. Taking logarithmic derivatives, one obtains:

$$\frac{d}{d\tau} \log \alpha_{i,j}(-1/\tau) = \frac{i\pi}{20} \left(1 - \frac{1}{\tau^2}\right) + \frac{1}{2\tau} + \frac{d}{d\tau} \log \alpha_{i,j}^S(\tau), \tag{2.16}$$

$$\frac{d}{d\tau} \log \alpha_{i,j}(\tau + 1) = \frac{d}{d\tau} \log \alpha_{i,j}^T(\tau), \tag{2.17}$$

where  $\alpha_{i,j}^S$  and  $\alpha_{i,j}^T$  are the images of  $\alpha_{i,j}$  under the  $S$  and  $T$  maps of figure 1, respectively.

It then follows that the functions

$$Y(c_{1,-1}, \dots, c_{1,4}; c_{2,-1}, \dots, c_{2,4} | \tau) \equiv \sum_{i,j} c_{i,j} \frac{d}{d\tau} \log \alpha_{i,j}(\tau), \quad \text{with } \sum_{i,j} c_{i,j} = 0, \tag{2.18}$$

span the sought-after 11-dimensional space of lowest weight modular forms of level  $N = 5$ .

Under  $S$  and  $T$ , one has the following transformations:

$$\begin{aligned} S: \quad Y(c_{1,-1}, \dots, c_{1,4}; c_{2,-1}, \dots, c_{2,4} | \tau) &\rightarrow Y(c_{1,-1}, \dots, c_{1,4}; c_{2,-1}, \dots, c_{2,4} | -1/\tau) \\ &= Y(c_{1,0}, c_{1,-1}, c_{1,4}, c_{2,2}, c_{2,3}, c_{1,1}; c_{2,0}, c_{2,-1}, c_{2,4}, c_{1,2}, c_{1,3}, c_{2,1} | \tau), \end{aligned} \tag{2.19}$$

$$\begin{aligned} T: \quad Y(c_{1,-1}, \dots, c_{1,4}; c_{2,-1}, \dots, c_{2,4} | \tau) &\rightarrow Y(c_{1,-1}, \dots, c_{1,4}; c_{2,-1}, \dots, c_{2,4} | \tau + 1) \\ &= Y(c_{1,-1}, c_{1,4}, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}; c_{2,-1}, c_{2,4}, c_{2,0}, c_{2,1}, c_{2,2}, c_{2,3} | \tau). \end{aligned} \tag{2.20}$$

Then, as anticipated, the space in question is divided into the following multiplets of  $A_5$ :

$$Y_{\mathbf{5}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{\sqrt{6}} Y(-5, 1, 1, 1, 1, 1; -5, 1, 1, 1, 1, 1 | \tau) \\ Y(0, 1, \zeta^4, \zeta^3, \zeta^2, \zeta; 0, 1, \zeta^4, \zeta^3, \zeta^2, \zeta | \tau) \\ Y(0, 1, \zeta^3, \zeta, \zeta^4, \zeta^2; 0, 1, \zeta^3, \zeta, \zeta^4, \zeta^2 | \tau) \\ Y(0, 1, \zeta^2, \zeta^4, \zeta, \zeta^3; 0, 1, \zeta^2, \zeta^4, \zeta, \zeta^3 | \tau) \\ Y(0, 1, \zeta, \zeta^2, \zeta^3, \zeta^4; 0, 1, \zeta, \zeta^2, \zeta^3, \zeta^4 | \tau) \end{pmatrix}, \tag{2.21}$$

$$Y_{\mathbf{3}}(\tau) = \begin{pmatrix} Y_6(\tau) \\ Y_7(\tau) \\ Y_8(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} Y(-\sqrt{5}, -1, -1, -1, -1, -1; \sqrt{5}, 1, 1, 1, 1, 1 | \tau) \\ Y(0, 1, \zeta^4, \zeta^3, \zeta^2, \zeta; 0, -1, -\zeta^4, -\zeta^3, -\zeta^2, -\zeta | \tau) \\ Y(0, 1, \zeta, \zeta^2, \zeta^3, \zeta^4; 0, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4 | \tau) \end{pmatrix}, \tag{2.22}$$

$$Y_{\mathbf{3}'}(\tau) = \begin{pmatrix} Y_9(\tau) \\ Y_{10}(\tau) \\ Y_{11}(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} Y(\sqrt{5}, -1, -1, -1, -1, -1; -\sqrt{5}, 1, 1, 1, 1, 1 | \tau) \\ Y(0, 1, \zeta^3, \zeta, \zeta^4, \zeta^2; 0, -1, -\zeta^3, -\zeta, -\zeta^4, -\zeta^2 | \tau) \\ Y(0, 1, \zeta^2, \zeta^4, \zeta, \zeta^3; 0, -1, -\zeta^2, -\zeta^4, -\zeta, -\zeta^3 | \tau) \end{pmatrix}, \tag{2.23}$$

where  $\zeta = e^{2\pi i/5}$ . The  $q$ -expansions for these modular forms are given in appendix A.2. In appendix B.1, we specify our basis choice for the representation matrices  $\rho(\gamma)$  of  $A_5$ , and we list the Clebsch-Gordan coefficients for this basis in appendix B.2.

Multiplets transforming in the other representations of  $A_5$  can be obtained from tensor products of the lowest weight multiplets  $Y_5$ ,  $Y_3$  and  $Y_{3'}$ . The missing **1** and **4** representations arise at weight 4. Even though one can form 66 products  $Y_i Y_j$ , the dimension of the space of weight  $k = 4$  (and level 5) forms is  $5k + 1 = 21$ . Therefore, there are 45 constraints between the  $Y_i Y_j$ , which we list in appendix C, and which reduce the 66 potentially independent combinations to 21 truly independent ones. These last combinations arrange themselves into the following multiplets of  $A_5$ :

$$\begin{aligned}
 Y_1^{(4)} &= Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5 \sim \mathbf{1}, \\
 Y_3^{(4)} &= \begin{pmatrix} -2Y_1Y_6 + \sqrt{3}Y_5Y_7 + \sqrt{3}Y_2Y_8 \\ \sqrt{3}Y_2Y_6 + Y_1Y_7 - \sqrt{6}Y_3Y_8 \\ \sqrt{3}Y_5Y_6 - \sqrt{6}Y_4Y_7 + Y_1Y_8 \end{pmatrix} \sim \mathbf{3}, \\
 Y_{3'}^{(4)} &= \begin{pmatrix} \sqrt{3}Y_1Y_6 + Y_5Y_7 + Y_2Y_8 \\ Y_3Y_6 - \sqrt{2}Y_2Y_7 - \sqrt{2}Y_4Y_8 \\ Y_4Y_6 - \sqrt{2}Y_3Y_7 - \sqrt{2}Y_5Y_8 \end{pmatrix} \sim \mathbf{3}', \\
 Y_4^{(4)} &= \begin{pmatrix} 2Y_4^2 + \sqrt{6}Y_1Y_2 - Y_3Y_5 \\ 2Y_2^2 + \sqrt{6}Y_1Y_3 - Y_4Y_5 \\ 2Y_5^2 - Y_2Y_3 + \sqrt{6}Y_1Y_4 \\ 2Y_3^2 - Y_2Y_4 + \sqrt{6}Y_1Y_5 \end{pmatrix} \sim \mathbf{4}, \\
 Y_{5,1}^{(4)} &= \begin{pmatrix} \sqrt{2}Y_1^2 + \sqrt{2}Y_3Y_4 - 2\sqrt{2}Y_2Y_5 \\ \sqrt{3}Y_4^2 - 2\sqrt{2}Y_1Y_2 \\ \sqrt{2}Y_1Y_3 + 2\sqrt{3}Y_4Y_5 \\ 2\sqrt{3}Y_2Y_3 + \sqrt{2}Y_1Y_4 \\ \sqrt{3}Y_3^2 - 2\sqrt{2}Y_1Y_5 \end{pmatrix} \sim \mathbf{5}, \\
 Y_{5,2}^{(4)} &= \begin{pmatrix} \sqrt{3}Y_5Y_7 - \sqrt{3}Y_2Y_8 \\ -Y_2Y_6 - \sqrt{3}Y_1Y_7 - \sqrt{2}Y_3Y_8 \\ -2Y_3Y_6 - \sqrt{2}Y_2Y_7 \\ 2Y_4Y_6 + \sqrt{2}Y_5Y_8 \\ Y_5Y_6 + \sqrt{2}Y_4Y_7 + \sqrt{3}Y_1Y_8 \end{pmatrix} \sim \mathbf{5}.
 \end{aligned} \tag{2.24}$$

Finally, we show in appendix D that relying on the Dedekind eta function for construction of the seed functions (the approach previously employed in the literature) is not enough to fully capture the results reported in eq. (2.15).

### 3 Phenomenology

To illustrate the use of the constructed modular multiplets for model building, we consider a minimal example where the neutrino masses originate from the Weinberg operator. We



assume that the charged lepton mass matrix is diagonal, so it does not contribute to the mixing. We will show later an explicit example with residual symmetry where this possibility is realised. It should be mentioned that here we do not aim to construct a “perfect” minimal model and/or to perform an exhaustive analysis of possible phenomenologically viable models based on the modular  $A_5$  symmetry. This will be done elsewhere. Our goal is only to present a “proof of existence” of such models by constructing a simple model which describes the neutrino data and possibly makes testable predictions.

In the set-up outlined above, the only superpotential term relevant for the mixing is the Weinberg operator:

$$W \supset \frac{g}{\Lambda} (L H_u L H_u Y)_{\mathbf{1}} , \quad (3.1)$$

where  $Y$  is a modular multiplet of weight  $k_Y$ .

We assume that the lepton  $SU(2)_L$  doublets transform as an  $A_5$  triplet ( $\rho_L \sim \mathbf{3}$  or  $\mathbf{3}'$ ) of weight  $-k_L$ , while the Higgs multiplet  $H_u$  is an  $A_5$  singlet ( $\rho_u \sim \mathbf{1}$ ) of zero weight ( $k_u = 0$ ). After the breaking of the modular symmetry, we obtain:

$$\frac{g}{\Lambda} (L H_u L H_u Y)_{\mathbf{1}} \rightarrow c_{ij} (L_i H_u) (L_j H_u) , \quad (3.2)$$

which leads to the Lagrangian term

$$\mathcal{L} \supset -\frac{1}{2} (M_\nu)_{ij} \overline{\nu_{iR}^c} \nu_{jL} + \text{h.c.} , \quad (3.3)$$

written in terms of four-spinors, where  $M_\nu \equiv 2c v_u^2$ , with  $\langle H_u \rangle = (0, v_u)^T$ , and  $\nu_{iR}^c \equiv (\nu_{iL})^c \equiv C \overline{\nu_{iL}^T}$ , with  $C$  being the charge conjugation matrix.

Given the above conditions, one needs to have  $k_Y = 2k_L$  to compensate the overall weight of the Weinberg operator term. Since  $k_Y$  is a non-negative integer, we can systematically explore the possible neutrino mass matrices going from  $k_Y = 0$  to more and more positive integer  $k_Y$ . In the case of  $k_Y = 0$  there are no modular forms in the Weinberg operator and the only possible  $A_5$  singlet is  $(LL)_{\mathbf{1}} = L_1 L_1 + L_2 L_3 + L_3 L_2$  (cf. appendix B.2), which leads to the following neutrino mass matrix:

$$M_\nu = \frac{2v_u^2 g}{\Lambda} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \quad (3.4)$$

The case  $k_Y = 0$  is then excluded, since it leads to degenerate neutrino masses, which are ruled out by the neutrino oscillation data [1]. In the following subsections we consider the cases  $k_Y = 2$  and  $k_Y = 4$  (corresponding to  $k_L = 1$  and  $k_L = 2$ , respectively).

### 3.1 The case of $k_Y = 2$

In this case, the available modular form multiplets are  $Y_{\mathbf{3}}$ ,  $Y_{\mathbf{3}'}$  and  $Y_{\mathbf{5}}$ . Note that  $L^2$  decomposes as  $\mathbf{3}^{(l)} \otimes \mathbf{3}^{(l)} = \mathbf{1} \oplus \mathbf{3}^{(l)} \oplus \mathbf{5}$ , but the  $\mathbf{3}^{(l)}$  component vanishes due to antisymmetry

(see appendix B.2). Therefore the only way to form a singlet is by combining the quintets,  $(L^2 Y_5)_1$ . If  $\rho_L \sim \mathbf{3}$ , one obtains:

$$M_\nu^{\mathbf{3}} = \frac{v_u^2 g}{\Lambda} \begin{pmatrix} 2Y_1 & -\sqrt{3}Y_5 & -\sqrt{3}Y_2 \\ -\sqrt{3}Y_5 & \sqrt{6}Y_4 & -Y_1 \\ -\sqrt{3}Y_2 & -Y_1 & \sqrt{6}Y_3 \end{pmatrix}, \quad (3.5)$$

while if instead  $\rho_L \sim \mathbf{3}'$ , it follows that

$$M_\nu^{\mathbf{3}'} = \frac{v_u^2 g}{\Lambda} \begin{pmatrix} 2Y_1 & -\sqrt{3}Y_4 & -\sqrt{3}Y_3 \\ -\sqrt{3}Y_4 & \sqrt{6}Y_2 & -Y_1 \\ -\sqrt{3}Y_3 & -Y_1 & \sqrt{6}Y_5 \end{pmatrix}. \quad (3.6)$$

The difference between eq. (3.5) and eq. (3.6) resides in the cyclic exchange of  $Y_5$ ,  $Y_4$ ,  $Y_2$  and  $Y_3$  (in this order).

In both cases,  $\langle \tau \rangle$  determines neutrino masses up to an overall mass scale. Furthermore, given our assumption of a diagonal charged lepton mass matrix, after employing the permutation ordering the charged lepton masses,  $\langle \tau \rangle$  additionally determines the mixing parameters. Through numerical search, we find that the agreement with data is optimised by choosing  $\rho_L = \mathbf{3}'$  and  $\langle \tau \rangle = 0.48 + 0.873i$ , giving rise to the following values of observables,<sup>3</sup> for a spectrum with normal ordering:

$$\begin{aligned} r &= 0.03056, & \Delta m_{21}^2 &= 7.427 \cdot 10^{-5} \text{ eV}^2, & \Delta m_{31}^2 &= 2.467 \cdot 10^{-3} \text{ eV}^2, \\ m_1 &= 0.02036 \text{ eV}, & m_2 &= 0.02211 \text{ eV}, & m_3 &= 0.05368 \text{ eV}, & \sum_i m_i &= 0.09616 \text{ eV}, \\ \sin^2 \theta_{12} &= 0.3252, & \sin^2 \theta_{13} &= \mathbf{0.1655}, & \sin^2 \theta_{23} &= 0.4213, \\ \delta/\pi &= 1.498, & \alpha_{21}/\pi &= 1.904, & \alpha_{31}/\pi &= 1.948, \end{aligned} \quad (3.7)$$

given an overall factor  $v_u^2 g/\Lambda \simeq 0.006339 \text{ eV}$ , and assuming the charged lepton sector induces a permutation of the first and third rows of the PMNS mixing matrix. While one obtains a good agreement with data for the mass-squared differences (and hence for the ratio  $r \equiv \Delta m_{21}^2/\Delta m_{31}^2$ ), as well as for the values of  $\sin^2 \theta_{12}$  and of  $\delta$ , the value of  $\sin^2 \theta_{23}$  is slightly outside its  $3\sigma$  range and, more importantly,  $\sin^2 \theta_{13}$  is many standard deviations away from its experimentally allowed range [2, 3]. Nevertheless, it is encouraging to find that the predictions for the mixing angles are in qualitative agreement with the observed pattern, namely,  $\sin^2 \theta_{13} < \sin^2 \theta_{12} < \sin^2 \theta_{23}$ . Note that the indicated value of  $\langle \tau \rangle$  is close to the ‘‘right cusp’’  $\tau_R = 1/2 + i\sqrt{3}/2$ , which preserves a residual  $\mathbb{Z}_3^{TS}$  symmetry (see, e.g., [21]).

### 3.2 The case of $k_Y = 4$

In this case, the available modular form multiplets are those given in eq. (2.24). Again, since  $L^2$  decomposes as  $\mathbf{1} \oplus \mathbf{5}$ , one can form singlets by using  $Y_1^{(4)}$ ,  $Y_{\mathbf{5},1}^{(4)}$  or  $Y_{\mathbf{5},2}^{(4)}$ . All three con-

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<sup>3</sup>Deriving the VEV of  $\tau$  from a potential is out of the scope of our study. Here we treat  $\langle \tau \rangle$  as a free parameter determined by fits to the data.

tributions should enter  $W$  with independent complex coefficients. If  $\rho_L \sim \mathbf{3}$ , one obtains:

$$\begin{aligned}
 M_\nu^{\mathbf{3}} = & \frac{2v_u^2 g_1}{\Lambda} \left[ (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right. \\
 & + \frac{g_2}{g_1} \begin{pmatrix} Y_1^2 + Y_3Y_4 - 2Y_2Y_5 - \frac{3}{2\sqrt{2}}Y_3^2 + \sqrt{3}Y_1Y_5 & -\frac{3}{2\sqrt{2}}Y_4^2 + \sqrt{3}Y_1Y_2 \\ * & 3Y_2Y_3 + \sqrt{\frac{3}{2}}Y_1Y_4 & Y_2Y_5 - \frac{1}{2}(Y_1^2 + Y_3Y_4) \\ * & * & 3Y_4Y_5 + \sqrt{\frac{3}{2}}Y_1Y_3 \end{pmatrix} \\
 & \left. + \frac{g_3}{g_1} \begin{pmatrix} Y_5Y_7 - Y_2Y_8 - \frac{1}{2}Y_5Y_6 - \frac{1}{\sqrt{2}}Y_4Y_7 - \frac{\sqrt{3}}{2}Y_1Y_8 & \frac{1}{2}Y_2Y_6 + \frac{\sqrt{3}}{2}Y_1Y_7 + \frac{1}{\sqrt{2}}Y_3Y_8 \\ * & \sqrt{2}Y_4Y_6 + Y_5Y_8 & \frac{1}{2}(Y_2Y_8 - Y_5Y_7) \\ * & * & -\sqrt{2}Y_3Y_6 - Y_2Y_7 \end{pmatrix} \right], \quad (3.8)
 \end{aligned}$$

where through asterisks we (here and henceforth) omit some entries of symmetric matrices. If instead  $\rho_L \sim \mathbf{3}'$ , it follows that

$$\begin{aligned}
 M_\nu^{\mathbf{3}'} = & \frac{2v_u^2 g_1}{\Lambda} \left[ (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right. \\
 & + \frac{g_2}{g_1} \begin{pmatrix} Y_1^2 + Y_3Y_4 - 2Y_2Y_5 - \frac{3}{\sqrt{2}}Y_2Y_3 - \frac{\sqrt{3}}{2}Y_1Y_4 & -\frac{\sqrt{3}}{2}Y_1Y_3 - \frac{3}{\sqrt{2}}Y_4Y_5 \\ * & \frac{3}{2}Y_4^2 - \sqrt{6}Y_1Y_2 & Y_2Y_5 - \frac{1}{2}(Y_1^2 + Y_3Y_4) \\ * & * & \frac{3}{2}Y_3^2 - \sqrt{6}Y_1Y_5 \end{pmatrix} \\
 & \left. + \frac{g_3}{g_1} \begin{pmatrix} Y_5Y_7 - Y_2Y_8 & -Y_4Y_6 - \frac{1}{\sqrt{2}}Y_5Y_8 & Y_3Y_6 + \frac{1}{\sqrt{2}}Y_2Y_7 \\ * & -\frac{1}{\sqrt{2}}Y_2Y_6 - \sqrt{\frac{3}{2}}Y_1Y_7 - Y_3Y_8 & \frac{1}{2}(Y_2Y_8 - Y_5Y_7) \\ * & * & \frac{1}{\sqrt{2}}Y_5Y_6 + Y_4Y_7 + \sqrt{\frac{3}{2}}Y_1Y_8 \end{pmatrix} \right]. \quad (3.9)
 \end{aligned}$$

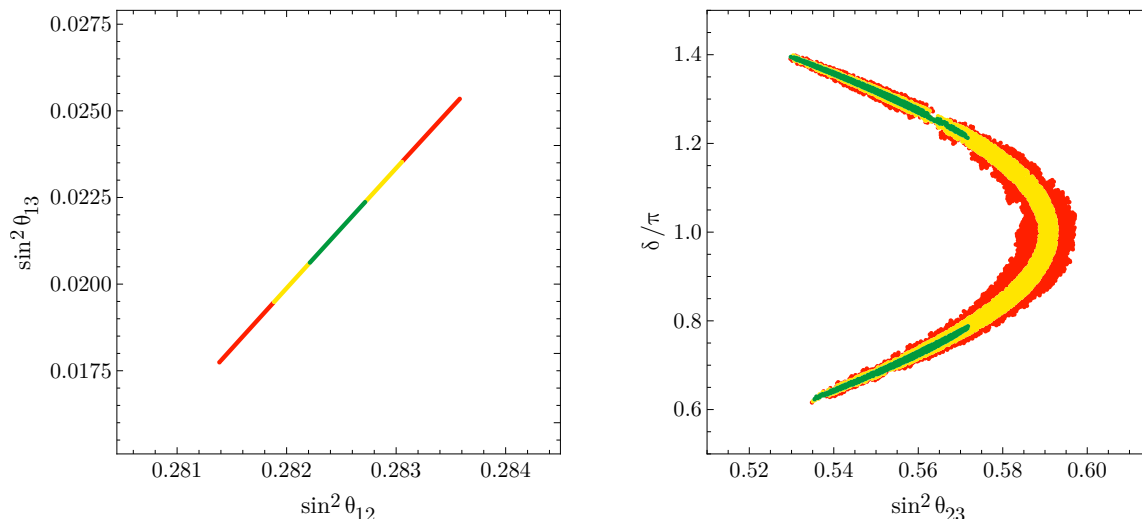
### 3.3 The case of residual symmetries

Let us assume that the modular symmetry is broken to the residual  $\mathbb{Z}_2^S = \{I, S\}$  symmetry of the neutrino mass matrix. This can be achieved by fixing  $\langle \tau \rangle = i$  in the neutrino sector, since this value of  $\langle \tau \rangle$  is invariant under the action of  $S$ ,  $\tau \rightarrow -1/\tau$ .

For the symmetric value of the VEV of the modulus  $\langle \tau \rangle = i$ , the lowest weight modular forms  $Y_i$  take the values:

$$\begin{aligned}
 Y_2 &= \frac{-1 - \sqrt{7-4\varphi}}{\sqrt{6}} Y_1, & Y_3 &= \frac{-1 - \sqrt{18-11\varphi}}{\sqrt{6}} Y_1, & (3.10) \\
 Y_4 &= \frac{-1 + \sqrt{18-11\varphi}}{\sqrt{6}} Y_1, & Y_5 &= \frac{-1 + \sqrt{7-4\varphi}}{\sqrt{6}} Y_1, \\
 Y_6 &= \sqrt{\frac{58-31\varphi}{15}} Y_1, & Y_7 &= \frac{-9+8\varphi+\sqrt{27-4\varphi}}{\sqrt{30}} Y_1, & Y_8 &= \frac{9-8\varphi+\sqrt{27-4\varphi}}{\sqrt{30}} Y_1, \\
 Y_9 &= -\sqrt{\frac{3+4\varphi}{15}} Y_1, & Y_{10} &= \frac{7-4\varphi+\sqrt{2+\varphi}}{\sqrt{30}} Y_1, & Y_{11} &= \frac{-7+4\varphi+\sqrt{2+\varphi}}{\sqrt{30}} Y_1,
 \end{aligned}$$

with  $Y_1(\langle \tau \rangle = i) \simeq 2.594 i$ .



**Figure 2.** Correlations between  $\sin^2 \theta_{12}$  and  $\sin^2 \theta_{13}$  (left) and between  $\sin^2 \theta_{23}$  and  $\delta$  (right) in the model with  $k_Y = 4$  and  $\langle \tau \rangle = i$ , in the vicinity of the viable point of eq. (3.11). The green, yellow and red regions correspond to  $2\sigma$ ,  $3\sigma$  and  $5\sigma$  confidence levels, respectively.

It follows that in both cases  $\rho_L \sim \mathbf{3}, \mathbf{3}'$  for  $k_Y = 4$ , aside from the permutation which orders charged lepton masses, 5 real parameters — namely,  $2v_u^2 g_1/\Lambda$ ,  $\text{Re}(g_2/g_1)$ ,  $\text{Im}(g_2/g_1)$ ,  $\text{Re}(g_3/g_1)$  and  $\text{Im}(g_3/g_1)$  — determine the neutrino masses and mixing. Through numerical search, we find for  $\rho_L \sim \mathbf{3}'$  a point given by:

$$2v_u^2 g_1/\Lambda \simeq 0.005104 \text{ eV}, \quad g_2/g_1 = -0.2205 - 0.1576 i, \quad g_3/g_1 = 0.0246 - 0.0421 i, \quad (3.11)$$

which is consistent with the neutrino oscillation data at  $1.7\sigma$  level,<sup>4</sup> for a spectrum with normal ordering:

$$\begin{aligned} r &= 0.03, & \Delta m_{21}^2 &= 7.399 \cdot 10^{-5} \text{ eV}^2, & \Delta m_{31}^2 &= 2.489 \cdot 10^{-3} \text{ eV}^2, \\ m_1 &= 0.0416 \text{ eV}, & m_2 &= 0.04248 \text{ eV}, & m_3 &= 0.06496 \text{ eV}, \\ \sum_i m_i &= 0.149 \text{ eV}, & |\langle m \rangle| &= 0.04174 \text{ eV}, & & (3.12) \\ \sin^2 \theta_{12} &= 0.2824, & \sin^2 \theta_{13} &= 0.02136, & \sin^2 \theta_{23} &= 0.5504, \\ \delta/\pi &= 1.315, & \alpha_{21}/\pi &= 1.978, & \alpha_{31}/\pi &= 0.9312. \end{aligned}$$

The indicated value of the effective Majorana mass  $|\langle m \rangle|$  which controls the rate of neutrinoless double beta decay may be probed in future experiments aiming to test values down to the  $|\langle m \rangle| \sim 10^{-2} \text{ eV}$  frontier.

In the vicinity of the point described by eq. (3.11), keeping  $\langle \tau \rangle = i$ , we find strong correlations between  $\sin^2 \theta_{12}$  and  $\sin^2 \theta_{13}$ , and between  $\sin^2 \theta_{23}$  and  $\delta$ . These correlations are shown in figure 2.

One possible way to force the charged lepton mass matrix to be diagonal in this set-up is to assume that it originates from a different modulus  $\tau^l$  which develops a VEV  $\langle \tau^l \rangle = i \infty$

<sup>4</sup>The current upper limit on the sum of neutrino masses reported in 2018 by the Planck collaboration depends on the data set used as input and reads [39]:  $\sum_i m_i < 0.120 - 0.160 \text{ eV}$  at 95% C.L.

breaking the modular symmetry to the residual  $\mathbb{Z}$  symmetry generated by  $T$ ,  $\tau \rightarrow \tau + 1$ .<sup>5</sup> The corresponding residual symmetry of the charged lepton mass matrix is  $\mathbb{Z}_5$  generated by the  $T$  generator of  $\Gamma_5$ , which is diagonal for  $\rho_L \sim \mathbf{3}^{(l)}$ . One can show that in the case  $\rho_{E^c} = \rho_L$  the charged lepton Yukawa interaction terms  $(E^c L Y)_1 H_d$  with the multiplets of weight 4 modular forms lead to the following mass matrix (written in the left-right convention):<sup>6</sup>

$$M_e = v_d \alpha_1 \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\alpha_2}{\alpha_1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{\alpha_3}{\alpha_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right], \quad (3.13)$$

where the three matrix terms correspond to contributions from  $Y_{\mathbf{1}}^{(4)}$ ,  $Y_{\mathbf{3}^{(l)}}^{(4)}$  and  $Y_{\mathbf{5},1}^{(4)}$  respectively, and  $\langle H_d \rangle = (v_d, 0)^T$ . The relevant product is diagonal:

$$M_e M_e^\dagger = v_d^2 \alpha_1^2 \text{diag} \left( \left| 1 + 2 \frac{\alpha_3}{\alpha_1} \right|^2, \left| 1 - \frac{\alpha_2}{\alpha_1} - \frac{\alpha_3}{\alpha_1} \right|^2, \left| 1 + \frac{\alpha_2}{\alpha_1} - \frac{\alpha_3}{\alpha_1} \right|^2 \right), \quad (3.14)$$

where  $\alpha_2/\alpha_1$  and  $\alpha_3/\alpha_1$  are complex parameters, and  $v_d \alpha_1$  is the overall mass scale factor. To reproduce the charged lepton masses with these parameters one can choose, e.g.,  $v_d \alpha_1 \simeq 660 \text{ MeV}$ ,  $\alpha_2/\alpha_1 = 1.34$  and  $1 + 2\alpha_3/\alpha_1 = -7.7 \times 10^{-4}$ . It is interesting to note that all three  $\alpha_i$  are of the same order,  $\alpha_1 \sim \alpha_2 \sim \alpha_3$ .

It is also possible to obtain the same matrix without an additional modulus  $\tau^l$ . Instead, suppose that the combination  $E^c L$  has positive modular weight, so that it cannot couple to modular forms. Let us assume that the charged lepton mass matrix originates from Yukawa couplings to three flavons, an  $A_5$  singlet  $\varphi_{\mathbf{1}}$ , an  $A_5$  triplet  $\varphi_{\mathbf{3}^{(l)}}$  and an  $A_5$  quintet  $\varphi_{\mathbf{5}}$  of a negative weight opposite to that of  $E^c L$ , each of which develops a VEV breaking  $A_5$  to  $\mathbb{Z}_5^T$ :

$$\langle \varphi_{\mathbf{1}} \rangle = v_1, \quad \langle \varphi_{\mathbf{3}^{(l)}} \rangle = (v_2, 0, 0) \quad \text{and} \quad \langle \varphi_{\mathbf{5}} \rangle = (v_3, 0, 0, 0, 0). \quad (3.15)$$

In this case the three terms  $(E^c L \varphi_{\mathbf{1}})_1 H_d$ ,  $(E^c L \varphi_{\mathbf{3}^{(l)}})_1 H_d$  and  $(E^c L \varphi_{\mathbf{5}})_1 H_d$  lead to the same mass matrix as in eq. (3.13). This is related to the fact that modular form multiplets of weight 4 take the following values at the symmetric point  $\langle \tau \rangle = i\infty$ :

$$Y_{\mathbf{3}}^{(4)} = \frac{4\pi^2}{\sqrt{15}}(1, 0, 0), \quad Y_{\mathbf{3}'}^{(4)} = -\frac{2\pi^2}{\sqrt{5}}(1, 0, 0), \quad Y_{\mathbf{5},1}^{(4)} = -\frac{2\sqrt{2}\pi^2}{3}(1, 0, 0, 0, 0). \quad (3.16)$$

Hence, they can be thought of as flavon multiplets developing the corresponding VEVs.

The above-described construction of the charged lepton mass matrix has been considered earlier in ref. [24], where the authors provide also an explicit form of the flavon

<sup>5</sup>It is not clear how the two moduli  $\tau$  and  $\tau^l$  can be forced to couple either to neutrinos or charged leptons only, so this possibility is considered on purely phenomenological grounds.

<sup>6</sup>Actually, this is the most general form of the mass matrix for any weight higher than 2, since modular form singlets, triplets and quintets are always present at such weights, and their values at  $\tau = i\infty$  are such that only their first components can be non-zero, cf. eq. (3.16). At weight 2, however, the first term in eq. (3.13) is missing as there is no modular form singlet of weight 2, and it is impossible to recover the charged lepton mass hierarchy in this case.

potential which leads to the VEVs given in eq. (3.15). An auxiliary  $\mathbb{Z}_5$  symmetry is used in ref. [24] to forbid certain terms in the superpotential. This is not needed in our case, since we do not introduce flavons which couple to the neutrino sector, and the  $(E^c L)_1 H_d$  term is forbidden in our construction by a choice of a non-zero modular weight. As a final remark, the parameters  $\alpha_i$  of the charged lepton mass matrix (3.13) have to be fine-tuned in order to reproduce the observed mass hierarchies. To overcome this defect, the authors of ref. [24] propose also an alternative model, which can be adopted directly in our case.

## 4 Summary and conclusions

In the framework of the modular invariance approach to lepton flavour proposed in ref. [14], we have considered a class of theories in which couplings and matter superfields transform in irreps of the finite modular group  $\Gamma_5 \simeq A_5$ . The building blocks needed to construct such theories are modular forms of weight 2 and level 5. We have explicitly constructed the 11 generating modular forms of weight 2, using the Jacobi theta function and its properties, which lead to closure of the set of 12 seed functions (see eq. (2.15)) under the action of  $\Gamma_5$ , as shown in figure 1. Further, we have demonstrated how these 11 modular forms arrange themselves into multiplets of  $A_5$ . Namely, we have found two triplets transforming in the irreps  $\mathbf{3}$  and  $\mathbf{3}'$  of  $A_5$ , and a quintet transforming in the irrep  $\mathbf{5}$  of  $A_5$ . They are given in eqs. (2.21)–(2.23), and their explicit  $q$ -expansions are listed in appendix A.2. From these triplets and quintet we have constructed multiplets of modular forms of weight 4 (see eq. (2.24) and appendix C).

While thorough analysis of modular-invariant theories with the  $\Gamma_5$  symmetry is left for future work, we have presented two examples of application of the obtained results to neutrino masses and mixing. In both of them, we have assumed that neutrino masses are generated via the Weinberg operator, and considered the charged lepton mass matrix to be diagonal. The first model involving the quintet of weight 2 modular forms leads to the neutrino mass matrix containing three real parameters — complex VEV of the modulus  $\langle \tau \rangle$  and a real overall scale. We have found that the value of  $\langle \tau \rangle$  lying very close to the “right cusp”  $\tau_R = 1/2 + i\sqrt{3}/2$  ( $\tau_R$  preserves a residual  $\mathbb{Z}_3^{TS}$  symmetry) leads to a good agreement with neutrino oscillation data except for  $\sin^2 \theta_{13}$ , which falls many standard deviations away from its experimentally allowed region. The second model contains an  $A_5$  singlet and two  $A_5$  quintets of modular forms of weight 4. The neutrino mass matrix in this case depends on five real parameters (three real constants and two phases) apart from  $\langle \tau \rangle$ . Assuming that  $\langle \tau \rangle = i$  — a self-dual point which preserves a residual  $\mathbb{Z}_2^S$  symmetry — we have obtained a viable benchmark point compatible with the data at  $1.7\sigma$  confidence level. In this case the neutrino mass matrix depends on three real parameters and two phases. Varying these free parameters we found strong correlations between the values of  $\sin^2 \theta_{12}$  and  $\sin^2 \theta_{13}$ , and the values of  $\sin^2 \theta_{23}$  and the Dirac CPV phase  $\delta$  (figure 2).

In conclusion, the results obtained in the present study can be used to build in a systematic way modular-invariant flavour models with the  $\Gamma_5 \simeq A_5$  symmetry. In this regard, this article is expected to serve as a useful handbook for future studies.

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## A $q$ -expansions

### A.1 Miller-like basis for the space of lowest weight forms

One can obtain  $q$ -expansions for a basis of the space of lowest weight modular forms for  $\Gamma_5$  from the SageMath algebra system [40]. To obtain the expansions up to (and including)  $\mathcal{O}(q^{10})$  terms, we take as input the code:

```
N=5
k=2
space = ModularForms(GammaH(N^2, [N + 1]), k)
[form.q_expansion(51) for form in space.basis()]
```

We then take  $q \rightarrow q^{1/5}$  in the produced output, and obtain the desired basis:

$$\begin{aligned}
 b_1 &= 1 + 60q^3 - 120q^4 + 240q^5 - 300q^6 + 300q^7 - 180q^9 + 240q^{10} + \dots, \\
 b_2 &= q^{1/5} + 12q^{11/5} + 7q^{16/5} + 8q^{21/5} + 6q^{26/5} + 32q^{31/5} + 7q^{36/5} + 42q^{41/5} + 12q^{46/5} + \dots, \\
 b_3 &= q^{2/5} + 12q^{12/5} - 2q^{17/5} + 12q^{22/5} + 8q^{27/5} + 21q^{32/5} - 6q^{37/5} + 48q^{42/5} - 8q^{47/5} + \dots, \\
 b_4 &= q^{3/5} + 11q^{13/5} - 9q^{18/5} + 21q^{23/5} - q^{28/5} + 12q^{33/5} + 41q^{43/5} - 29q^{48/5} + \dots, \\
 b_5 &= q^{4/5} + 9q^{14/5} - 12q^{19/5} + 29q^{24/5} - 18q^{29/5} + 17q^{34/5} + 8q^{39/5} + 12q^{44/5} - 16q^{49/5} + \dots, \\
 b_6 &= q + 6q^3 - 9q^4 + 27q^5 - 28q^6 + 30q^7 - 11q^9 + 26q^{10} + \dots, \\
 b_7 &= q^{6/5} + 2q^{16/5} + 2q^{21/5} + 3q^{26/5} + 7q^{36/5} + 5q^{46/5} + \dots, \\
 b_8 &= q^{7/5} - q^{12/5} + 3q^{17/5} + 2q^{27/5} + 7q^{37/5} - 6q^{42/5} + 9q^{47/5} + \dots, \\
 b_9 &= q^{8/5} - 2q^{13/5} + 5q^{18/5} - 4q^{23/5} + 4q^{28/5} + 4q^{38/5} - 8q^{43/5} + 16q^{48/5} + \dots, \\
 b_{10} &= q^{9/5} - 3q^{14/5} + 8q^{19/5} - 11q^{24/5} + 12q^{29/5} - 5q^{34/5} + 13q^{49/5} + \dots, \\
 b_{11} &= q^2 - 4q^3 + 12q^4 - 22q^5 + 30q^6 - 24q^7 + 5q^8 + 18q^9 - 21q^{10} + \dots,
 \end{aligned} \tag{A.1}$$

with  $q = e^{2\pi i \tau}$  and where fractional powers  $q^{n/5}$  should be read as  $q^{n/5} = e^{2n\pi i \tau/5}$ .

## A.2 Expansions for the lowest weight $A_5$ multiplets

The elements of the quintet  $\mathbf{5}$  of  $A_5$ , given in eq. (2.21), admit the  $q$ -expansions:

$$\begin{aligned}
 -\frac{i}{\pi} \sqrt{\frac{3}{2}} Y_1(\tau) &= 1 + 6q + 18q^2 + 24q^3 + 42q^4 + 6q^5 + \dots = b_1 + 6b_6 + 18b_{11}, \\
 \frac{i}{2\pi} Y_2(\tau) &= q^{1/5} + 12q^{6/5} + 12q^{11/5} + 31q^{16/5} + 32q^{21/5} + \dots = b_2 + 12b_7, \\
 \frac{i}{2\pi} Y_3(\tau) &= 3q^{2/5} + 8q^{7/5} + 28q^{12/5} + 18q^{17/5} + 36q^{22/5} + \dots = 3b_3 + 8b_8, \\
 \frac{i}{2\pi} Y_4(\tau) &= 4q^{3/5} + 15q^{8/5} + 14q^{13/5} + 39q^{18/5} + 24q^{23/5} + \dots = 4b_4 + 15b_9, \\
 \frac{i}{2\pi} Y_5(\tau) &= 7q^{4/5} + 13q^{9/5} + 24q^{14/5} + 20q^{19/5} + 60q^{24/5} + \dots = 7b_5 + 13b_{10},
 \end{aligned}
 \tag{A.2}$$

where, as before and in what follows,  $q = e^{2\pi i \tau}$ .

The elements of the triplet  $\mathbf{3}$ , given in eq. (2.22), admit instead the expansions:

$$\begin{aligned}
 \frac{i}{\pi} \sqrt{\frac{5}{2}} Y_6(\tau) &= -1 + 30q + 20q^2 + 40q^3 + 90q^4 + 130q^5 + \dots = -b_1 + 30b_6 + 20b_{11}, \\
 -\frac{i}{2\sqrt{5}\pi} Y_7(\tau) &= q^{1/5} + 2q^{6/5} + 12q^{11/5} + 11q^{16/5} + 12q^{21/5} + \dots = b_2 + 2b_7, \\
 -\frac{i}{2\sqrt{5}\pi} Y_8(\tau) &= 3q^{4/5} + 7q^{9/5} + 6q^{14/5} + 20q^{19/5} + 10q^{24/5} + \dots = 3b_5 + 7b_{10}.
 \end{aligned}
 \tag{A.3}$$

Finally, the elements of the triplet  $\mathbf{3}'$ , given in eq. (2.23), read:

$$\begin{aligned}
 \frac{i}{\pi} \sqrt{\frac{5}{2}} Y_9(\tau) &= 1 + 20q + 30q^2 + 60q^3 + 60q^4 + 120q^5 + \dots = b_1 + 20b_6 + 30b_{11}, \\
 -\frac{i}{2\sqrt{5}\pi} Y_{10}(\tau) &= q^{2/5} + 6q^{7/5} + 6q^{12/5} + 16q^{17/5} + 12q^{22/5} + \dots = b_3 + 6b_8, \\
 -\frac{i}{2\sqrt{5}\pi} Y_{11}(\tau) &= 2q^{3/5} + 5q^{8/5} + 12q^{13/5} + 7q^{18/5} + 22q^{23/5} + \dots = 2b_4 + 5b_9.
 \end{aligned}
 \tag{A.4}$$

## B $A_5$ group theory

### B.1 Basis

$A_5$  is the group of even permutations of five objects. It contains  $5!/2 = 60$  elements and admits five irreducible representations, namely  $\mathbf{1}$ ,  $\mathbf{3}$ ,  $\mathbf{3}'$ ,  $\mathbf{4}$  and  $\mathbf{5}$  (see, e.g., [5]). It can be generated by two elements  $S$  and  $T$  satisfying

$$S^2 = (ST)^3 = T^5 = I. \tag{B.1}$$



We will employ the group theoretical results of ref. [24], using in particular the following explicit basis for the  $A_5$  generators in different irreps:

$$\mathbf{1} : \quad \rho(S) = 1, \quad \rho(T) = 1, \quad (\text{B.2})$$

$$\mathbf{3} : \quad \rho(S) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\varphi & 1/\varphi \\ -\sqrt{2} & 1/\varphi & -\varphi \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}, \quad (\text{B.3})$$

$$\mathbf{3}' : \quad \rho(S) = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1/\varphi & \varphi \\ \sqrt{2} & \varphi & -1/\varphi \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}, \quad (\text{B.4})$$

$$\mathbf{4} : \quad \rho(S) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1/\varphi & \varphi & -1 \\ 1/\varphi & -1 & 1 & \varphi \\ \varphi & 1 & -1 & 1/\varphi \\ -1 & \varphi & 1/\varphi & 1 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & \zeta^4 \end{pmatrix}, \quad (\text{B.5})$$

$$\mathbf{5} : \quad \rho(S) = \frac{1}{5} \begin{pmatrix} -1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & 1/\varphi^2 & -2\varphi & 2/\varphi & \varphi^2 \\ \sqrt{6} & -2\varphi & \varphi^2 & 1/\varphi^2 & 2/\varphi \\ \sqrt{6} & 2/\varphi & 1/\varphi^2 & \varphi^2 & -2\varphi \\ \sqrt{6} & \varphi^2 & 2/\varphi & -2\varphi & 1/\varphi^2 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^4 \end{pmatrix}, \quad (\text{B.6})$$

where  $\zeta = e^{2\pi i/5}$  and  $\varphi = (1 + \sqrt{5})/2$ .

## B.2 Clebsch-Gordan coefficients

For completeness, we reproduce here the nontrivial Clebsch-Gordan coefficients of ref. [24], given in the above basis. Entries of each multiplet entering the tensor product are denoted by  $\alpha_i$  and  $\beta_i$ .

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \left\{ \begin{array}{l} \mathbf{1} \sim \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\ \mathbf{3} \sim \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix} \\ \mathbf{5} \sim \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ -\sqrt{3}\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 \\ \sqrt{6}\alpha_2\beta_2 \\ \sqrt{6}\alpha_3\beta_3 \\ -\sqrt{3}\alpha_1\beta_3 - \sqrt{3}\alpha_3\beta_1 \end{pmatrix} \end{array} \right. \quad (\text{B.7})$$

$$\begin{aligned}
 \mathbf{3} \otimes \mathbf{3}' &= \mathbf{4} \oplus \mathbf{5} \quad \left\{ \begin{array}{l} \mathbf{4} \sim \begin{pmatrix} \sqrt{2} \alpha_2 \beta_1 + \alpha_3 \beta_2 \\ -\sqrt{2} \alpha_1 \beta_2 - \alpha_3 \beta_3 \\ -\sqrt{2} \alpha_1 \beta_3 - \alpha_2 \beta_2 \\ \sqrt{2} \alpha_3 \beta_1 + \alpha_2 \beta_3 \end{pmatrix} \\ \mathbf{5} \sim \begin{pmatrix} \sqrt{3} \alpha_1 \beta_1 \\ \alpha_2 \beta_1 - \sqrt{2} \alpha_3 \beta_2 \\ \alpha_1 \beta_2 - \sqrt{2} \alpha_3 \beta_3 \\ \alpha_1 \beta_3 - \sqrt{2} \alpha_2 \beta_2 \\ \alpha_3 \beta_1 - \sqrt{2} \alpha_2 \beta_3 \end{pmatrix} \end{array} \right. \\
 \\
 \mathbf{3} \otimes \mathbf{4} &= \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5} \quad \left\{ \begin{array}{l} \mathbf{3}' \sim \begin{pmatrix} -\sqrt{2} \alpha_2 \beta_4 - \sqrt{2} \alpha_3 \beta_1 \\ \sqrt{2} \alpha_1 \beta_2 - \alpha_2 \beta_1 + \alpha_3 \beta_3 \\ \sqrt{2} \alpha_1 \beta_3 + \alpha_2 \beta_2 - \alpha_3 \beta_4 \end{pmatrix} \\ \mathbf{4} \sim \begin{pmatrix} \alpha_1 \beta_1 - \sqrt{2} \alpha_3 \beta_2 \\ -\alpha_1 \beta_2 - \sqrt{2} \alpha_2 \beta_1 \\ \alpha_1 \beta_3 + \sqrt{2} \alpha_3 \beta_4 \\ -\alpha_1 \beta_4 + \sqrt{2} \alpha_2 \beta_3 \end{pmatrix} \\ \mathbf{5} \sim \begin{pmatrix} \sqrt{6} \alpha_2 \beta_4 - \sqrt{6} \alpha_3 \beta_1 \\ 2\sqrt{2} \alpha_1 \beta_1 + 2\alpha_3 \beta_2 \\ -\sqrt{2} \alpha_1 \beta_2 + \alpha_2 \beta_1 + 3\alpha_3 \beta_3 \\ \sqrt{2} \alpha_1 \beta_3 - 3\alpha_2 \beta_2 - \alpha_3 \beta_4 \\ -2\sqrt{2} \alpha_1 \beta_4 - 2\alpha_2 \beta_3 \end{pmatrix} \end{array} \right. \quad (\text{B.8}) \\
 \\
 \mathbf{3} \otimes \mathbf{5} &= \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5} \quad \left\{ \begin{array}{l} \mathbf{3} \sim \begin{pmatrix} -2\alpha_1 \beta_1 + \sqrt{3} \alpha_2 \beta_5 + \sqrt{3} \alpha_3 \beta_2 \\ \sqrt{3} \alpha_1 \beta_2 + \alpha_2 \beta_1 - \sqrt{6} \alpha_3 \beta_3 \\ \sqrt{3} \alpha_1 \beta_5 - \sqrt{6} \alpha_2 \beta_4 + \alpha_3 \beta_1 \end{pmatrix} \\ \mathbf{3}' \sim \begin{pmatrix} \sqrt{3} \alpha_1 \beta_1 + \alpha_2 \beta_5 + \alpha_3 \beta_2 \\ \alpha_1 \beta_3 - \sqrt{2} \alpha_2 \beta_2 - \sqrt{2} \alpha_3 \beta_4 \\ \alpha_1 \beta_4 - \sqrt{2} \alpha_2 \beta_3 - \sqrt{2} \alpha_3 \beta_5 \end{pmatrix} \\ \mathbf{4} \sim \begin{pmatrix} 2\sqrt{2} \alpha_1 \beta_2 - \sqrt{6} \alpha_2 \beta_1 + \alpha_3 \beta_3 \\ -\sqrt{2} \alpha_1 \beta_3 + 2\alpha_2 \beta_2 - 3\alpha_3 \beta_4 \\ \sqrt{2} \alpha_1 \beta_4 + 3\alpha_2 \beta_3 - 2\alpha_3 \beta_5 \\ -2\sqrt{2} \alpha_1 \beta_5 - \alpha_2 \beta_4 + \sqrt{6} \alpha_3 \beta_1 \end{pmatrix} \\ \mathbf{5} \sim \begin{pmatrix} \sqrt{3} \alpha_2 \beta_5 - \sqrt{3} \alpha_3 \beta_2 \\ -\alpha_1 \beta_2 - \sqrt{3} \alpha_2 \beta_1 - \sqrt{2} \alpha_3 \beta_3 \\ -2\alpha_1 \beta_3 - \sqrt{2} \alpha_2 \beta_2 \\ 2\alpha_1 \beta_4 + \sqrt{2} \alpha_3 \beta_5 \\ \alpha_1 \beta_5 + \sqrt{2} \alpha_2 \beta_4 + \sqrt{3} \alpha_3 \beta_1 \end{pmatrix} \end{array} \right.
 \end{aligned}$$

$$\left. \begin{aligned}
 \mathbf{3}' \otimes \mathbf{3}' &= \mathbf{1} \oplus \mathbf{3}' \oplus \mathbf{5} \\
 \\
 \mathbf{3}' \otimes \mathbf{4} &= \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5} \\
 \\
 \mathbf{3}' \otimes \mathbf{5} &= \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}
 \end{aligned} \right\} \begin{cases}
 \mathbf{1} \sim \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\
 \mathbf{3}' \sim \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix} \\
 \mathbf{5} \sim \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ \sqrt{6}\alpha_3\beta_3 \\ -\sqrt{3}\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 \\ -\sqrt{3}\alpha_1\beta_3 - \sqrt{3}\alpha_3\beta_1 \\ \sqrt{6}\alpha_2\beta_2 \end{pmatrix} \\
 \\
 \mathbf{3} \sim \begin{pmatrix} -\sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 \\ \sqrt{2}\alpha_1\beta_1 + \alpha_2\beta_4 - \alpha_3\beta_3 \\ \sqrt{2}\alpha_1\beta_4 - \alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix} \\
 \mathbf{4} \sim \begin{pmatrix} \alpha_1\beta_1 + \sqrt{2}\alpha_3\beta_3 \\ \alpha_1\beta_2 - \sqrt{2}\alpha_3\beta_4 \\ -\alpha_1\beta_3 + \sqrt{2}\alpha_2\beta_1 \\ -\alpha_1\beta_4 - \sqrt{2}\alpha_2\beta_2 \end{pmatrix} \\
 \mathbf{5} \sim \begin{pmatrix} \sqrt{6}\alpha_2\beta_3 - \sqrt{6}\alpha_3\beta_2 \\ \sqrt{2}\alpha_1\beta_1 - 3\alpha_2\beta_4 - \alpha_3\beta_3 \\ 2\sqrt{2}\alpha_1\beta_2 + 2\alpha_3\beta_4 \\ -2\sqrt{2}\alpha_1\beta_3 - 2\alpha_2\beta_1 \\ -\sqrt{2}\alpha_1\beta_4 + \alpha_2\beta_2 + 3\alpha_3\beta_1 \end{pmatrix} \\
 \\
 \mathbf{3} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 + \alpha_2\beta_4 + \alpha_3\beta_3 \\ \alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_5 - \sqrt{2}\alpha_3\beta_4 \\ \alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 \end{pmatrix} \\
 \mathbf{3}' \sim \begin{pmatrix} -2\alpha_1\beta_1 + \sqrt{3}\alpha_2\beta_4 + \sqrt{3}\alpha_3\beta_3 \\ \sqrt{3}\alpha_1\beta_3 + \alpha_2\beta_1 - \sqrt{6}\alpha_3\beta_5 \\ \sqrt{3}\alpha_1\beta_4 - \sqrt{6}\alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix} \\
 \mathbf{4} \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_2 + 3\alpha_2\beta_5 - 2\alpha_3\beta_4 \\ 2\sqrt{2}\alpha_1\beta_3 - \sqrt{6}\alpha_2\beta_1 + \alpha_3\beta_5 \\ -2\sqrt{2}\alpha_1\beta_4 - \alpha_2\beta_2 + \sqrt{6}\alpha_3\beta_1 \\ -\sqrt{2}\alpha_1\beta_5 + 2\alpha_2\beta_3 - 3\alpha_3\beta_2 \end{pmatrix} \\
 \mathbf{5} \sim \begin{pmatrix} \sqrt{3}\alpha_2\beta_4 - \sqrt{3}\alpha_3\beta_3 \\ 2\alpha_1\beta_2 + \sqrt{2}\alpha_3\beta_4 \\ -\alpha_1\beta_3 - \sqrt{3}\alpha_2\beta_1 - \sqrt{2}\alpha_3\beta_5 \\ \alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_2 + \sqrt{3}\alpha_3\beta_1 \\ -2\alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_3 \end{pmatrix}
 \end{cases} \tag{B.9}$$

$$\left. \begin{array}{l}
 \mathbf{4} \otimes \mathbf{4} = \\
 \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \\
 \oplus \mathbf{4} \oplus \mathbf{5}
 \end{array} \right\} \begin{array}{l}
 \mathbf{1} \sim \alpha_1\beta_4 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_4\beta_1 \\
 \mathbf{3} \sim \begin{pmatrix} -\alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_4\beta_1 \\ \sqrt{2}\alpha_2\beta_4 - \sqrt{2}\alpha_4\beta_2 \\ \sqrt{2}\alpha_1\beta_3 - \sqrt{2}\alpha_3\beta_1 \end{pmatrix} \\
 \mathbf{3}' \sim \begin{pmatrix} \alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_4\beta_1 \\ \sqrt{2}\alpha_3\beta_4 - \sqrt{2}\alpha_4\beta_3 \\ \sqrt{2}\alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_1 \end{pmatrix} \\
 \mathbf{4} \sim \begin{pmatrix} \alpha_2\beta_4 + \alpha_3\beta_3 + \alpha_4\beta_2 \\ \alpha_1\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_3 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_4\beta_4 \\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix} \\
 \mathbf{5} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 - \sqrt{3}\alpha_3\beta_2 + \sqrt{3}\alpha_4\beta_1 \\ -\sqrt{2}\alpha_2\beta_4 + 2\sqrt{2}\alpha_3\beta_3 - \sqrt{2}\alpha_4\beta_2 \\ -2\sqrt{2}\alpha_1\beta_1 + \sqrt{2}\alpha_3\beta_4 + \sqrt{2}\alpha_4\beta_3 \\ \sqrt{2}\alpha_1\beta_2 + \sqrt{2}\alpha_2\beta_1 - 2\sqrt{2}\alpha_4\beta_4 \\ -\sqrt{2}\alpha_1\beta_3 + 2\sqrt{2}\alpha_2\beta_2 - \sqrt{2}\alpha_3\beta_1 \end{pmatrix}
 \end{array} \tag{B.10}$$
  

$$\left. \begin{array}{l}
 \mathbf{4} \otimes \mathbf{5} = \\
 \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \\
 \oplus \mathbf{5}_1 \oplus \mathbf{5}_2
 \end{array} \right\} \begin{array}{l}
 \mathbf{3} \sim \begin{pmatrix} 2\sqrt{2}\alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_4 + \sqrt{2}\alpha_3\beta_3 - 2\sqrt{2}\alpha_4\beta_2 \\ -\sqrt{6}\alpha_1\beta_1 + 2\alpha_2\beta_5 + 3\alpha_3\beta_4 - \alpha_4\beta_3 \\ \alpha_1\beta_4 - 3\alpha_2\beta_3 - 2\alpha_3\beta_2 + \sqrt{6}\alpha_4\beta_1 \end{pmatrix} \\
 \mathbf{3}' \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_5 + 2\sqrt{2}\alpha_2\beta_4 - 2\sqrt{2}\alpha_3\beta_3 - \sqrt{2}\alpha_4\beta_2 \\ 3\alpha_1\beta_2 - \sqrt{6}\alpha_2\beta_1 - \alpha_3\beta_5 + 2\alpha_4\beta_4 \\ -2\alpha_1\beta_3 + \alpha_2\beta_2 + \sqrt{6}\alpha_3\beta_1 - 3\alpha_4\beta_5 \end{pmatrix} \\
 \mathbf{4} \sim \begin{pmatrix} \sqrt{3}\alpha_1\beta_1 - \sqrt{2}\alpha_2\beta_5 + \sqrt{2}\alpha_3\beta_4 - 2\sqrt{2}\alpha_4\beta_3 \\ -\sqrt{2}\alpha_1\beta_2 - \sqrt{3}\alpha_2\beta_1 + 2\sqrt{2}\alpha_3\beta_5 + \sqrt{2}\alpha_4\beta_4 \\ \sqrt{2}\alpha_1\beta_3 + 2\sqrt{2}\alpha_2\beta_2 - \sqrt{3}\alpha_3\beta_1 - \sqrt{2}\alpha_4\beta_5 \\ -2\sqrt{2}\alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 + \sqrt{3}\alpha_4\beta_1 \end{pmatrix} \\
 \mathbf{5}_1 \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_5 - \sqrt{2}\alpha_2\beta_4 - \sqrt{2}\alpha_3\beta_3 + \sqrt{2}\alpha_4\beta_2 \\ -\sqrt{2}\alpha_1\beta_1 - \sqrt{3}\alpha_3\beta_4 - \sqrt{3}\alpha_4\beta_3 \\ \sqrt{3}\alpha_1\beta_2 + \sqrt{2}\alpha_2\beta_1 + \sqrt{3}\alpha_3\beta_5 \\ \sqrt{3}\alpha_2\beta_2 + \sqrt{2}\alpha_3\beta_1 + \sqrt{3}\alpha_4\beta_5 \\ -\sqrt{3}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 - \sqrt{2}\alpha_4\beta_1 \end{pmatrix} \\
 \mathbf{5}_2 \sim \begin{pmatrix} 2\alpha_1\beta_5 + 4\alpha_2\beta_4 + 4\alpha_3\beta_3 + 2\alpha_4\beta_2 \\ 4\alpha_1\beta_1 + 2\sqrt{6}\alpha_2\beta_5 \\ -\sqrt{6}\alpha_1\beta_2 + 2\alpha_2\beta_1 - \sqrt{6}\alpha_3\beta_5 + 2\sqrt{6}\alpha_4\beta_4 \\ 2\sqrt{6}\alpha_1\beta_3 - \sqrt{6}\alpha_2\beta_2 + 2\alpha_3\beta_1 - \sqrt{6}\alpha_4\beta_5 \\ 2\sqrt{6}\alpha_3\beta_2 + 4\alpha_4\beta_1 \end{pmatrix}
 \end{array}$$

$$\begin{aligned}
 & \mathbf{5} \otimes \mathbf{5} = \\
 & \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4}_1 \\
 & \oplus \mathbf{4}_2 \oplus \mathbf{5}_1 \oplus \mathbf{5}_2
 \end{aligned}
 \left\{ \begin{array}{l}
 \mathbf{1} \sim \alpha_1\beta_1 + \alpha_2\beta_5 + \alpha_3\beta_4 + \alpha_4\beta_3 + \alpha_5\beta_2 \\
 \mathbf{3} \sim \begin{pmatrix} \alpha_2\beta_5 + 2\alpha_3\beta_4 - 2\alpha_4\beta_3 - \alpha_5\beta_2 \\ -\sqrt{3}\alpha_1\beta_2 + \sqrt{3}\alpha_2\beta_1 + \sqrt{2}\alpha_3\beta_5 - \sqrt{2}\alpha_5\beta_3 \\ \sqrt{3}\alpha_1\beta_5 + \sqrt{2}\alpha_2\beta_4 - \sqrt{2}\alpha_4\beta_2 - \sqrt{3}\alpha_5\beta_1 \end{pmatrix} \\
 \mathbf{3}' \sim \begin{pmatrix} 2\alpha_2\beta_5 - \alpha_3\beta_4 + \alpha_4\beta_3 - 2\alpha_5\beta_2 \\ \sqrt{3}\alpha_1\beta_3 - \sqrt{3}\alpha_3\beta_1 + \sqrt{2}\alpha_4\beta_5 - \sqrt{2}\alpha_5\beta_4 \\ -\sqrt{3}\alpha_1\beta_4 + \sqrt{2}\alpha_2\beta_3 - \sqrt{2}\alpha_3\beta_2 + \sqrt{3}\alpha_4\beta_1 \end{pmatrix} \\
 \mathbf{4}_1 \sim \begin{pmatrix} 3\sqrt{2}\alpha_1\beta_2 + 3\sqrt{2}\alpha_2\beta_1 - \sqrt{3}\alpha_3\beta_5 + 4\sqrt{3}\alpha_4\beta_4 - \sqrt{3}\alpha_5\beta_3 \\ 3\sqrt{2}\alpha_1\beta_3 + 4\sqrt{3}\alpha_2\beta_2 + 3\sqrt{2}\alpha_3\beta_1 - \sqrt{3}\alpha_4\beta_5 - \sqrt{3}\alpha_5\beta_4 \\ 3\sqrt{2}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 - \sqrt{3}\alpha_3\beta_2 + 3\sqrt{2}\alpha_4\beta_1 + 4\sqrt{3}\alpha_5\beta_5 \\ 3\sqrt{2}\alpha_1\beta_5 - \sqrt{3}\alpha_2\beta_4 + 4\sqrt{3}\alpha_3\beta_3 - \sqrt{3}\alpha_4\beta_2 + 3\sqrt{2}\alpha_5\beta_1 \end{pmatrix} \\
 \mathbf{4}_2 \sim \begin{pmatrix} \sqrt{2}\alpha_1\beta_2 - \sqrt{2}\alpha_2\beta_1 + \sqrt{3}\alpha_3\beta_5 - \sqrt{3}\alpha_5\beta_3 \\ -\sqrt{2}\alpha_1\beta_3 + \sqrt{2}\alpha_3\beta_1 + \sqrt{3}\alpha_4\beta_5 - \sqrt{3}\alpha_5\beta_4 \\ -\sqrt{2}\alpha_1\beta_4 - \sqrt{3}\alpha_2\beta_3 + \sqrt{3}\alpha_3\beta_2 + \sqrt{2}\alpha_4\beta_1 \\ \sqrt{2}\alpha_1\beta_5 - \sqrt{3}\alpha_2\beta_4 + \sqrt{3}\alpha_4\beta_2 - \sqrt{2}\alpha_5\beta_1 \end{pmatrix} \\
 \mathbf{5}_1 \sim \begin{pmatrix} 2\alpha_1\beta_1 + \alpha_2\beta_5 - 2\alpha_3\beta_4 - 2\alpha_4\beta_3 + \alpha_5\beta_2 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \sqrt{6}\alpha_3\beta_5 + \sqrt{6}\alpha_5\beta_3 \\ -2\alpha_1\beta_3 + \sqrt{6}\alpha_2\beta_2 - 2\alpha_3\beta_1 \\ -2\alpha_1\beta_4 - 2\alpha_4\beta_1 + \sqrt{6}\alpha_5\beta_5 \\ \alpha_1\beta_5 + \sqrt{6}\alpha_2\beta_4 + \sqrt{6}\alpha_4\beta_2 + \alpha_5\beta_1 \end{pmatrix} \\
 \mathbf{5}_2 \sim \begin{pmatrix} 2\alpha_1\beta_1 - 2\alpha_2\beta_5 + \alpha_3\beta_4 + \alpha_4\beta_3 - 2\alpha_5\beta_2 \\ -2\alpha_1\beta_2 - 2\alpha_2\beta_1 + \sqrt{6}\alpha_4\beta_4 \\ \alpha_1\beta_3 + \alpha_3\beta_1 + \sqrt{6}\alpha_4\beta_5 + \sqrt{6}\alpha_5\beta_4 \\ \alpha_1\beta_4 + \sqrt{6}\alpha_2\beta_3 + \sqrt{6}\alpha_3\beta_2 + \alpha_4\beta_1 \\ -2\alpha_1\beta_5 + \sqrt{6}\alpha_3\beta_3 - 2\alpha_5\beta_1 \end{pmatrix}
 \end{array} \right.
 \tag{B.11}$$

### C Higher weight forms and constraints

Through tensor products of  $Y_5$ ,  $Y_3$  and  $Y_{3'}$ , one can find, at weight 4, the multiplets:

$$\begin{aligned}
 Y_1^{(4)} &= Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5 \sim \mathbf{1}, \\
 Y_1^{(4)'} &= Y_6^2 + 2Y_7Y_8 \sim \mathbf{1}, \\
 Y_1^{(4)''} &= Y_9^2 + 2Y_{10}Y_{11} \sim \mathbf{1},
 \end{aligned}
 \tag{C.1}$$

$$\begin{aligned}
 Y_3^{(4)} &= \begin{pmatrix} -2Y_1Y_6 + \sqrt{3}Y_5Y_7 + \sqrt{3}Y_2Y_8 \\ \sqrt{3}Y_2Y_6 + Y_1Y_7 - \sqrt{6}Y_3Y_8 \\ \sqrt{3}Y_5Y_6 - \sqrt{6}Y_4Y_7 + Y_1Y_8 \end{pmatrix} \sim \mathbf{3}, \\
 Y_3^{(4)'} &= \begin{pmatrix} \sqrt{3}Y_1Y_9 + Y_4Y_{10} + Y_3Y_{11} \\ Y_2Y_9 - \sqrt{2}Y_5Y_{10} - \sqrt{2}Y_4Y_{11} \\ Y_5Y_9 - \sqrt{2}Y_3Y_{10} - \sqrt{2}Y_2Y_{11} \end{pmatrix} \sim \mathbf{3},
 \end{aligned}
 \tag{C.2}$$

$$\begin{aligned}
 Y_{3'}^{(4)} &= \begin{pmatrix} \sqrt{3}Y_1Y_6 + Y_5Y_7 + Y_2Y_8 \\ Y_3Y_6 - \sqrt{2}Y_2Y_7 - \sqrt{2}Y_4Y_8 \\ Y_4Y_6 - \sqrt{2}Y_3Y_7 - \sqrt{2}Y_5Y_8 \end{pmatrix} \sim \mathbf{3}', \\
 Y_{3'}^{(4)'} &= \begin{pmatrix} -2Y_1Y_9 + \sqrt{3}Y_4Y_{10} + \sqrt{3}Y_3Y_{11} \\ \sqrt{3}Y_3Y_9 + Y_1Y_{10} - \sqrt{6}Y_5Y_{11} \\ \sqrt{3}Y_4Y_9 - \sqrt{6}Y_2Y_{10} + Y_1Y_{11} \end{pmatrix} \sim \mathbf{3}',
 \end{aligned}
 \tag{C.3}$$

$$\begin{aligned}
 Y_4^{(4)} &= \begin{pmatrix} 2Y_4^2 + \sqrt{6}Y_1Y_2 - Y_3Y_5 \\ 2Y_2^2 + \sqrt{6}Y_1Y_3 - Y_4Y_5 \\ 2Y_5^2 - Y_2Y_3 + \sqrt{6}Y_1Y_4 \\ 2Y_3^2 - Y_2Y_4 + \sqrt{6}Y_1Y_5 \end{pmatrix} \sim \mathbf{4}, \\
 Y_4^{(4)'} &= \begin{pmatrix} 2\sqrt{2}Y_2Y_6 - \sqrt{6}Y_1Y_7 + Y_3Y_8 \\ -\sqrt{2}Y_3Y_6 + 2Y_2Y_7 - 3Y_4Y_8 \\ \sqrt{2}Y_4Y_6 + 3Y_3Y_7 - 2Y_5Y_8 \\ -2\sqrt{2}Y_5Y_6 - Y_4Y_7 + \sqrt{6}Y_1Y_8 \end{pmatrix} \sim \mathbf{4}, \\
 Y_4^{(4)''} &= \begin{pmatrix} \sqrt{2}Y_7Y_9 + Y_8Y_{10} \\ -\sqrt{2}Y_6Y_{10} - Y_8Y_{11} \\ -Y_7Y_{10} - \sqrt{2}Y_6Y_{11} \\ \sqrt{2}Y_8Y_9 + Y_7Y_{11} \end{pmatrix} \sim \mathbf{4}, \\
 Y_4^{(4)'''} &= \begin{pmatrix} \sqrt{2}Y_2Y_9 + 3Y_5Y_{10} - 2Y_4Y_{11} \\ 2\sqrt{2}Y_3Y_9 - \sqrt{6}Y_1Y_{10} + Y_5Y_{11} \\ -2\sqrt{2}Y_4Y_9 - Y_2Y_{10} + \sqrt{6}Y_1Y_{11} \\ -\sqrt{2}Y_5Y_9 + 2Y_3Y_{10} - 3Y_2Y_{11} \end{pmatrix} \sim \mathbf{4},
 \end{aligned}
 \tag{C.4}$$

$$\begin{aligned}
Y_5^{(4)} &= \begin{pmatrix} \sqrt{2}Y_1^2 + \sqrt{2}Y_3Y_4 - 2\sqrt{2}Y_2Y_5 \\ \sqrt{3}Y_4^2 - 2\sqrt{2}Y_1Y_2 \\ \sqrt{2}Y_1Y_3 + 2\sqrt{3}Y_4Y_5 \\ 2\sqrt{3}Y_2Y_3 + \sqrt{2}Y_1Y_4 \\ \sqrt{3}Y_3^2 - 2\sqrt{2}Y_1Y_5 \end{pmatrix} \sim \mathbf{5}, \\
Y_5^{(4)'} &= \begin{pmatrix} \sqrt{2}Y_1^2 - 2\sqrt{2}Y_3Y_4 + \sqrt{2}Y_2Y_5 \\ \sqrt{2}Y_1Y_2 + 2\sqrt{3}Y_3Y_5 \\ \sqrt{3}Y_2^2 - 2\sqrt{2}Y_1Y_3 \\ \sqrt{3}Y_5^2 - 2\sqrt{2}Y_1Y_4 \\ 2\sqrt{3}Y_2Y_4 + \sqrt{2}Y_1Y_5 \end{pmatrix} \sim \mathbf{5}, \\
Y_5^{(4)''} &= \begin{pmatrix} \sqrt{3}Y_5Y_7 - \sqrt{3}Y_2Y_8 \\ -Y_2Y_6 - \sqrt{3}Y_1Y_7 - \sqrt{2}Y_3Y_8 \\ -2Y_3Y_6 - \sqrt{2}Y_2Y_7 \\ 2Y_4Y_6 + \sqrt{2}Y_5Y_8 \\ Y_5Y_6 + \sqrt{2}Y_4Y_7 + \sqrt{3}Y_1Y_8 \end{pmatrix} \sim \mathbf{5}, \\
Y_5^{(4)'''} &= \begin{pmatrix} \sqrt{2}Y_6^2 - \sqrt{2}Y_7Y_8 \\ -\sqrt{6}Y_6Y_7 \\ \sqrt{3}Y_7^2 \\ \sqrt{3}Y_8^2 \\ -\sqrt{6}Y_6Y_8 \end{pmatrix} \sim \mathbf{5}, \\
Y_5^{(4)''''} &= \begin{pmatrix} \sqrt{3}Y_6Y_9 \\ Y_7Y_9 - \sqrt{2}Y_8Y_{10} \\ Y_6Y_{10} - \sqrt{2}Y_8Y_{11} \\ Y_6Y_{11} - \sqrt{2}Y_7Y_{10} \\ Y_8Y_9 - \sqrt{2}Y_7Y_{11} \end{pmatrix} \sim \mathbf{5}, \\
Y_5^{(4)'''''} &= \begin{pmatrix} \sqrt{3}Y_4Y_{10} - \sqrt{3}Y_3Y_{11} \\ 2Y_2Y_9 + \sqrt{2}Y_4Y_{11} \\ -Y_3Y_9 - \sqrt{3}Y_1Y_{10} - \sqrt{2}Y_5Y_{11} \\ Y_4Y_9 + \sqrt{2}Y_2Y_{10} + \sqrt{3}Y_1Y_{11} \\ -2Y_5Y_9 - \sqrt{2}Y_3Y_{10} \end{pmatrix} \sim \mathbf{5}, \\
Y_5^{(4)''''''} &= \begin{pmatrix} \sqrt{2}Y_9^2 - \sqrt{2}Y_{10}Y_{11} \\ \sqrt{3}Y_{11}^2 \\ -\sqrt{6}Y_9Y_{10} \\ -\sqrt{6}Y_9Y_{11} \\ \sqrt{3}Y_{10}^2 \end{pmatrix} \sim \mathbf{5}.
\end{aligned} \tag{C.5}$$

Not all of the above multiplets are expected to be independent. Indeed, from the  $q$ -expansions of the  $Y_i(\tau)$  given in appendix A.2 we find 45 constraints between the 66 different  $Y_i(\tau)Y_j(\tau)$  products, namely:

$$\frac{3}{5}(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) = Y_6^2 + 2Y_7Y_8 = Y_9^2 + 2Y_{10}Y_{11}, \quad (C.6)$$

$$\begin{aligned} -2\sqrt{3}Y_1Y_6 + 3Y_5Y_7 + 3Y_2Y_8 &= 2\sqrt{3}Y_1Y_9 + 2Y_4Y_{10} + 2Y_3Y_{11}, \\ 3Y_2Y_6 + \sqrt{3}Y_1Y_7 - 3\sqrt{2}Y_3Y_8 &= 2Y_2Y_9 - 2\sqrt{2}Y_5Y_{10} - 2\sqrt{2}Y_4Y_{11}, \\ 3Y_5Y_6 - 3\sqrt{2}Y_4Y_7 + \sqrt{3}Y_1Y_8 &= 2Y_5Y_9 - 2\sqrt{2}Y_3Y_{10} - 2\sqrt{2}Y_2Y_{11}, \end{aligned} \quad (C.7)$$

$$\begin{aligned} 2\sqrt{3}Y_1Y_6 + 2Y_5Y_7 + 2Y_2Y_8 &= -2\sqrt{3}Y_1Y_9 + 3Y_4Y_{10} + 3Y_3Y_{11}, \\ 2Y_3Y_6 - 2\sqrt{2}Y_2Y_7 - 2\sqrt{2}Y_4Y_8 &= 3Y_3Y_9 + \sqrt{3}Y_1Y_{10} - 3\sqrt{2}Y_5Y_{11}, \\ 2Y_4Y_6 - 2\sqrt{2}Y_3Y_7 - 2\sqrt{2}Y_5Y_8 &= 3Y_4Y_9 - 3\sqrt{2}Y_2Y_{10} + \sqrt{3}Y_1Y_{11}, \end{aligned} \quad (C.8)$$

$$\begin{aligned} \sqrt{6}Y_1Y_2 + 2Y_4^2 - Y_3Y_5 &= (\sqrt{5}/7) \left( 2\sqrt{2}Y_2Y_6 - \sqrt{6}Y_1Y_7 + Y_3Y_8 \right) \\ &= \sqrt{2}Y_7Y_9 + Y_8Y_{10} \\ &= -\sqrt{5} \left( \sqrt{2}Y_2Y_9 + 3Y_5Y_{10} - 2Y_4Y_{11} \right), \\ \sqrt{6}Y_1Y_3 + 2Y_2^2 - Y_4Y_5 &= -(\sqrt{5}/7) \left( \sqrt{2}Y_3Y_6 - 2Y_2Y_7 + 3Y_4Y_8 \right) \\ &= -\sqrt{2}Y_6Y_{10} - Y_8Y_{11} \\ &= -\sqrt{5} \left( 2\sqrt{2}Y_3Y_9 - \sqrt{6}Y_1Y_{10} + Y_5Y_{11} \right), \\ \sqrt{6}Y_1Y_4 + 2Y_5^2 - Y_2Y_3 &= (\sqrt{5}/7) \left( \sqrt{2}Y_4Y_6 + 3Y_3Y_7 - 2Y_5Y_8 \right) \\ &= -\sqrt{2}Y_6Y_{11} - Y_7Y_{10} \\ &= \sqrt{5} \left( 2\sqrt{2}Y_4Y_9 + Y_2Y_{10} - \sqrt{6}Y_1Y_{11} \right), \\ \sqrt{6}Y_1Y_5 + 2Y_3^2 - Y_2Y_4 &= -(\sqrt{5}/7) \left( 2\sqrt{2}Y_5Y_6 - \sqrt{6}Y_1Y_8 + Y_4Y_7 \right) \\ &= \sqrt{2}Y_8Y_9 + Y_7Y_{11} \\ &= \sqrt{5} \left( \sqrt{2}Y_5Y_9 - 2Y_3Y_{10} + 3Y_2Y_{11} \right), \end{aligned} \quad (C.9)$$

$$\begin{aligned} Y_2Y_8 - Y_5Y_7 &= 2Y_4Y_{10} - 2Y_3Y_{11}, \\ Y_2Y_6 + \sqrt{3}Y_1Y_7 + \sqrt{2}Y_3Y_8 &= 4Y_2Y_9 + 2\sqrt{2}Y_4Y_{11}, \\ \sqrt{2}Y_3Y_6 + Y_2Y_7 &= -\sqrt{2}Y_3Y_9 - \sqrt{6}Y_1Y_{10} - 2Y_5Y_{11}, \\ -\sqrt{2}Y_4Y_6 - Y_5Y_8 &= \sqrt{2}Y_4Y_9 + 2Y_2Y_{10} + \sqrt{6}Y_1Y_{11}, \\ Y_5Y_6 + \sqrt{2}Y_4Y_7 + \sqrt{3}Y_1Y_8 &= 4Y_5Y_9 + 2\sqrt{2}Y_3Y_{10}, \end{aligned} \quad (C.10)$$

$$\begin{aligned} 4Y_1^2 + 4Y_3Y_4 - 8Y_2Y_5 - 3\sqrt{5}Y_5Y_7 - 3\sqrt{5}Y_2Y_8 &= 4Y_1^2 - 8Y_3Y_4 + 4Y_2Y_5, \\ 2\sqrt{6}Y_4^2 - 8Y_1Y_2 - \sqrt{15}Y_2Y_6 - 3\sqrt{5}Y_1Y_7 - \sqrt{30}Y_3Y_8 &= 4Y_1Y_2 + 4\sqrt{6}Y_3Y_5, \\ 2\sqrt{2}Y_1Y_3 - \sqrt{30}Y_3Y_6 + 4\sqrt{3}Y_4Y_5 - \sqrt{15}Y_2Y_7 &= 2\sqrt{3}Y_2^2 - 4\sqrt{2}Y_1Y_3, \\ 4\sqrt{3}Y_2Y_3 + 2\sqrt{2}Y_1Y_4 + \sqrt{30}Y_4Y_6 + \sqrt{15}Y_5Y_8 &= 2\sqrt{3}Y_5^2 - 4\sqrt{2}Y_1Y_4, \\ 2\sqrt{6}Y_3^2 - 8Y_1Y_5 + \sqrt{15}Y_5Y_6 + \sqrt{30}Y_4Y_7 - 15Y_1Y_8 &= 4\sqrt{6}Y_2Y_4 + 4Y_1Y_5, \end{aligned} \quad (C.11)$$



$$\begin{aligned}
 & 12Y_1^2 + 12Y_3Y_4 - 24Y_2Y_5 + 21\sqrt{5}Y_5Y_7 - 21\sqrt{5}Y_2Y_8 = 20Y_6^2 - 20Y_7Y_8, \\
 & 6\sqrt{2}Y_4^2 - 8\sqrt{3}Y_1Y_2 - 7\sqrt{5}Y_2Y_6 - 7\sqrt{15}Y_1Y_7 - 7\sqrt{10}Y_3Y_8 = -20Y_6Y_7, \\
 & \quad 2\sqrt{6}Y_1Y_3 - 7\sqrt{10}Y_3Y_6 + 12Y_4Y_5 - 7\sqrt{5}Y_2Y_7 = 10Y_7^2, \\
 & \quad 12Y_2Y_3 + 2\sqrt{6}Y_1Y_4 + 7\sqrt{10}Y_4Y_6 + 7\sqrt{5}Y_5Y_8 = 10Y_8^2, \\
 & 6\sqrt{2}Y_3^2 - 8\sqrt{3}Y_1Y_5 + 7\sqrt{5}Y_5Y_6 + 7\sqrt{10}Y_4Y_7 + 7\sqrt{15}Y_1Y_8 = -20Y_6Y_8,
 \end{aligned} \tag{C.12}$$

$$\begin{aligned}
 & 12Y_1^2 + 12Y_3Y_4 - 24Y_2Y_5 + \sqrt{5}Y_5Y_7 - \sqrt{5}Y_2Y_8 = -20Y_6Y_9, \\
 & 18\sqrt{2}Y_4^2 - 24\sqrt{3}Y_1Y_2 - \sqrt{5}Y_2Y_6 - \sqrt{15}Y_1Y_7 - \sqrt{10}Y_3Y_8 = 20\sqrt{2}Y_8Y_{10} - 20Y_7Y_9, \\
 & \quad 6\sqrt{6}Y_1Y_3 - \sqrt{10}Y_3Y_6 + 36Y_4Y_5 - \sqrt{5}Y_2Y_7 = 20Y_8Y_{11} - 10\sqrt{2}Y_6Y_{10}, \\
 & \quad 36Y_2Y_3 + 6\sqrt{6}Y_1Y_4 + \sqrt{10}Y_4Y_6 + \sqrt{5}Y_5Y_8 = 20Y_7Y_{10} - 10\sqrt{2}Y_6Y_{11}, \\
 & 18\sqrt{2}Y_3^2 - 24\sqrt{3}Y_1Y_5 + \sqrt{5}Y_5Y_6 + \sqrt{10}Y_4Y_7 + \sqrt{15}Y_1Y_8 = 20\sqrt{2}Y_7Y_{11} - 20Y_8Y_9,
 \end{aligned} \tag{C.13}$$

$$\begin{aligned}
 & 6Y_1^2 + 6Y_3Y_4 - 12Y_2Y_5 + 3\sqrt{5}Y_5Y_7 - 3\sqrt{5}Y_2Y_8 = 10Y_9^2 - 10Y_{10}Y_{11}, \\
 & 3\sqrt{2}Y_4^2 - 4\sqrt{3}Y_1Y_2 - \sqrt{5}Y_2Y_6 - \sqrt{15}Y_1Y_7 - \sqrt{10}Y_3Y_8 = 5\sqrt{2}Y_{11}^2, \\
 & \quad \sqrt{6}Y_1Y_3 - \sqrt{10}Y_3Y_6 + 6Y_4Y_5 - \sqrt{5}Y_2Y_7 = -5\sqrt{2}Y_9Y_{10}, \\
 & \quad 6Y_2Y_3 + \sqrt{6}Y_1Y_4 + \sqrt{10}Y_4Y_6 + \sqrt{5}Y_5Y_8 = -5\sqrt{2}Y_9Y_{11}, \\
 & 3\sqrt{2}Y_3^2 - 4\sqrt{3}Y_1Y_5 + \sqrt{5}Y_5Y_6 + \sqrt{10}Y_4Y_7 + \sqrt{15}Y_1Y_8 = 5\sqrt{2}Y_{10}^2.
 \end{aligned} \tag{C.14}$$

The 20 constraints in eqs. (C.6)–(C.9) imply that the primed multiplets  $Y_{\mathbf{r}}^{(4)'\dots'}$  in eqs. (C.1)–(C.4) are proportional among themselves and to the corresponding unprimed ones,  $Y_{\mathbf{r}}^{(4)}$ . Therefore, for  $\mathbf{r} = \mathbf{1}, \mathbf{3}, \mathbf{3}', \mathbf{4}$ , only unprimed multiplets are kept in our discussion. In what concerns the quintets, it follows instead from the 25 constraints in eqs. (C.10)–(C.14) that there are two independent multiplets out of the 7 given in eq. (C.5), which we take to be  $Y_{\mathbf{5},1}^{(4)} \equiv Y_{\mathbf{5}}^{(4)}$  and  $Y_{\mathbf{5},2}^{(4)} \equiv Y_{\mathbf{5}}^{(4)''}$ , cf. eq. (2.24).

We also collect here the linearly independent modular multiplets arising at weights 6, 8 and 10. The linear space of modular forms of weight  $k$  (and level  $N = 5$ , corresponding to  $\Gamma_5 \simeq A_5$ ) has dimension  $5k + 1$ . At weight  $k = 6$ , one finds

$$\begin{aligned}
 Y_{\mathbf{1}}^{(6)} &= 3\sqrt{3}(Y_2Y_3^2 + Y_4^2Y_5) + \sqrt{2}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) \sim \mathbf{1}, \\
 Y_{\mathbf{3},1}^{(6)} &= (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} Y_6 \\ Y_7 \\ Y_8 \end{pmatrix} \sim \mathbf{3}, \\
 Y_{\mathbf{3},2}^{(6)} &= \begin{pmatrix} (Y_5Y_6 - \sqrt{2}Y_4Y_7)Y_7 + (\sqrt{2}Y_3Y_8 - Y_2Y_6)Y_8 \\ (\sqrt{3}Y_1Y_6 - Y_5Y_7)Y_7 - \sqrt{2}Y_3Y_6Y_8 + (Y_6^2 - Y_7Y_8)Y_2 \\ (Y_2Y_8 - \sqrt{3}Y_1Y_6)Y_8 + \sqrt{2}Y_4Y_6Y_7 - (Y_6^2 - Y_7Y_8)Y_5 \end{pmatrix} \sim \mathbf{3}, \\
 Y_{\mathbf{3}',1}^{(6)} &= (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} Y_9 \\ Y_{10} \\ Y_{11} \end{pmatrix} \sim \mathbf{3}',
 \end{aligned}$$

$$\begin{aligned}
 Y_{\mathbf{3}'_2}^{(6)} &= \begin{pmatrix} (Y_4 Y_6 - \sqrt{2} Y_3 Y_7 - \sqrt{2} Y_5 Y_8) Y_{10} - (Y_3 Y_6 - \sqrt{2} Y_2 Y_7 - \sqrt{2} Y_4 Y_8) Y_{11} \\ (Y_3 Y_6 - \sqrt{2} Y_2 Y_7 - \sqrt{2} Y_4 Y_8) Y_9 - (\sqrt{3} Y_1 Y_6 + Y_5 Y_7 + Y_2 Y_8) Y_{10} \\ (\sqrt{3} Y_1 Y_6 + Y_5 Y_7 + Y_2 Y_8) Y_{11} - (Y_4 Y_6 - \sqrt{2} Y_3 Y_7 - \sqrt{2} Y_5 Y_8) Y_9 \end{pmatrix} \sim \mathbf{3}', \\
 Y_{\mathbf{4}_1}^{(6)} &= \begin{pmatrix} \sqrt{2} (\sqrt{6} Y_3 Y_8 - \sqrt{3} Y_2 Y_6 - Y_1 Y_7) Y_9 - (\sqrt{3} Y_5 Y_6 - \sqrt{6} Y_4 Y_7 + Y_1 Y_8) Y_{10} \\ (\sqrt{3} Y_5 Y_6 - \sqrt{6} Y_4 Y_7 + Y_1 Y_8) Y_{11} + \sqrt{2} (\sqrt{3} Y_5 Y_7 - 2 Y_1 Y_6 + \sqrt{3} Y_2 Y_8) Y_{10} \\ (\sqrt{3} Y_2 Y_6 + Y_1 Y_7 - \sqrt{6} Y_3 Y_8) Y_{10} + \sqrt{2} (\sqrt{3} Y_5 Y_7 - 2 Y_1 Y_6 + \sqrt{3} Y_2 Y_8) Y_{11} \\ \sqrt{2} (\sqrt{6} Y_4 Y_7 - \sqrt{3} Y_5 Y_6 - Y_1 Y_8) Y_9 - (\sqrt{3} Y_2 Y_6 + Y_1 Y_7 - \sqrt{6} Y_3 Y_8) Y_{11} \end{pmatrix} \sim \mathbf{4}, \\
 Y_{\mathbf{4}_2}^{(6)} &= \begin{pmatrix} \sqrt{2} (\sqrt{3} Y_1 Y_6 + Y_5 Y_7) Y_7 + (Y_3 Y_6 - \sqrt{2} Y_4 Y_8) Y_8 \\ \sqrt{2} (\sqrt{2} Y_2 Y_7 - Y_3 Y_6) Y_6 + (Y_4 Y_6 + \sqrt{2} Y_3 Y_7 + \sqrt{2} Y_5 Y_8) Y_8 \\ \sqrt{2} (\sqrt{2} Y_5 Y_8 - Y_4 Y_6) Y_6 + (Y_3 Y_6 + \sqrt{2} Y_2 Y_7 + \sqrt{2} Y_4 Y_8) Y_7 \\ \sqrt{2} (\sqrt{3} Y_1 Y_6 + Y_2 Y_8) Y_8 + (Y_4 Y_6 - \sqrt{2} Y_3 Y_7) Y_7 \end{pmatrix} \sim \mathbf{4}, \\
 Y_{\mathbf{5}_1}^{(6)} &= (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} \sim \mathbf{5}, \\
 Y_{\mathbf{5}_2}^{(6)} &= \begin{pmatrix} \sqrt{3} (\sqrt{3} Y_1 Y_6 + Y_5 Y_7 + Y_2 Y_8) Y_6 \\ (Y_5 Y_7 + \sqrt{3} Y_1 Y_6) Y_7 + (3Y_2 Y_7 + 2Y_4 Y_8 - \sqrt{2} Y_3 Y_6) Y_8 \\ (Y_3 Y_6 - \sqrt{2} Y_2 Y_7) Y_6 + 2(Y_5 Y_8 + Y_3 Y_7 - \sqrt{2} Y_4 Y_6) Y_8 \\ (Y_4 Y_6 - \sqrt{2} Y_5 Y_8) Y_6 + 2(Y_2 Y_7 + Y_4 Y_8 - \sqrt{2} Y_3 Y_6) Y_7 \\ (Y_2 Y_8 + \sqrt{3} Y_1 Y_6) Y_8 + (3Y_5 Y_8 + 2Y_3 Y_7 - \sqrt{2} Y_4 Y_6) Y_7 \end{pmatrix} \sim \mathbf{5},
 \end{aligned}$$

corresponding to a total dimension of 31.

At weight  $k = 8$ , one has

$$\begin{aligned}
 Y_{\mathbf{1}}^{(8)} &= (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) Y_6^2 + 2(Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) Y_7 Y_8 \sim \mathbf{1}, \\
 Y_{\mathbf{3}_1}^{(8)} &= \left( 3\sqrt{3} (Y_2 Y_3^2 + Y_4^2 Y_5) + \sqrt{2} Y_1 (Y_1^2 + 3Y_3 Y_4 - 6Y_2 Y_5) \right) \begin{pmatrix} Y_6 \\ Y_7 \\ Y_8 \end{pmatrix} \sim \mathbf{3}, \\
 Y_{\mathbf{3}_2}^{(8)} &= (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) \begin{pmatrix} -2Y_1 Y_6 + \sqrt{3} Y_5 Y_7 + \sqrt{3} Y_2 Y_8 \\ \sqrt{3} Y_2 Y_6 + Y_1 Y_7 - \sqrt{6} Y_3 Y_8 \\ \sqrt{3} Y_5 Y_6 - \sqrt{6} Y_4 Y_7 + Y_1 Y_8 \end{pmatrix} \sim \mathbf{3}, \\
 Y_{\mathbf{3}'_1}^{(8)} &= \left( 3\sqrt{3} (Y_2 Y_3^2 + Y_4^2 Y_5) + \sqrt{2} Y_1 (Y_1^2 + 3Y_3 Y_4 - 6Y_2 Y_5) \right) \begin{pmatrix} Y_9 \\ Y_{10} \\ Y_{11} \end{pmatrix} \sim \mathbf{3}', \\
 Y_{\mathbf{3}'_2}^{(8)} &= (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) \begin{pmatrix} \sqrt{3} Y_1 Y_6 + Y_5 Y_7 + Y_2 Y_8 \\ Y_3 Y_6 - \sqrt{2} Y_2 Y_7 - \sqrt{2} Y_4 Y_8 \\ Y_4 Y_6 - \sqrt{2} Y_3 Y_7 - \sqrt{2} Y_5 Y_8 \end{pmatrix} \sim \mathbf{3}',
 \end{aligned}$$

$$Y_{4,1}^{(8)} = (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} \sqrt{2}Y_7Y_9 + Y_8Y_{10} \\ -\sqrt{2}Y_6Y_{10} - Y_8Y_{11} \\ -Y_7Y_{10} - \sqrt{2}Y_6Y_{11} \\ \sqrt{2}Y_8Y_9 + Y_7Y_{11} \end{pmatrix} \sim 4,$$

$$Y_{4,2}^{(8)} = \begin{pmatrix} \sqrt{3}Y_1(\sqrt{6}Y_1(Y_4^2 + Y_3Y_5) + 4Y_5(Y_2^2 + Y_4Y_5) - 6Y_1^2Y_2 - Y_3^3) \\ \quad + \sqrt{2}(12Y_2^2Y_3^2 - Y_3Y_4(Y_4^2 + Y_3Y_5) + 2Y_2(Y_3Y_5 - 5Y_4^2)Y_5) \\ \sqrt{3}Y_1(\sqrt{6}Y_1(2Y_2^2 + 3Y_4Y_5) - 3Y_4Y_3^2 - 4(2Y_4^2 + Y_3Y_5)Y_2) \\ \quad - \sqrt{2}(3Y_2Y_3^3 + 4Y_2^2Y_3Y_4 - 3Y_4^4 + 4(Y_2^3 - 2Y_3Y_4^2)Y_5 + 10Y_2Y_5^2Y_4) \\ \sqrt{3}Y_1(\sqrt{6}Y_1(2Y_5^2 + 3Y_2Y_3) - 3Y_3Y_4^2 - 4(2Y_3^2 + Y_2Y_4)Y_5) \\ \quad - \sqrt{2}(3Y_4^3Y_5 + 4Y_4Y_5^2Y_3 - 3Y_3^4 + 4Y_2(Y_5^3 - 2Y_4Y_3^2) + 10Y_2^2Y_5Y_3) \\ \sqrt{3}Y_1(\sqrt{6}Y_1(Y_3^2 + Y_2Y_4) + 4Y_2(Y_5^2 + Y_2Y_3) - 6Y_1^2Y_5 - Y_4^3) \\ \quad + \sqrt{2}(12Y_4^2Y_5^2 - Y_3Y_4(Y_3^2 + Y_2Y_4) + 2Y_2(Y_4Y_2 - 5Y_3^2)Y_5) \end{pmatrix} \sim 4,$$

$$Y_{5,1}^{(8)} = (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} \sqrt{2}(Y_1^2 + Y_3Y_4 - 2Y_2Y_5) \\ \sqrt{3}Y_4^2 - 2\sqrt{2}Y_1Y_2 \\ \sqrt{2}(Y_1Y_3 + \sqrt{6}Y_4Y_5) \\ \sqrt{2}(Y_1Y_4 + \sqrt{6}Y_2Y_3) \\ \sqrt{3}Y_3^2 - 2\sqrt{2}Y_1Y_5 \end{pmatrix} \sim 5,$$

$$Y_{5,2}^{(8)} = (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \begin{pmatrix} \sqrt{2}(Y_1^2 + Y_2Y_5 - 2Y_3Y_4) \\ \sqrt{2}(Y_1Y_2 + \sqrt{6}Y_3Y_5) \\ \sqrt{3}Y_2^2 - 2\sqrt{2}Y_1Y_3 \\ \sqrt{3}Y_5^2 - 2\sqrt{2}Y_1Y_4 \\ \sqrt{2}(Y_1Y_5 + \sqrt{6}Y_2Y_4) \end{pmatrix} \sim 5,$$

$$Y_{5,3}^{(8)} = \left( 3\sqrt{3}(Y_2Y_3^2 + Y_4^2Y_5) + \sqrt{2}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) \right) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} \sim 5,$$

$$Y_{5,4}^{(8)} = \begin{pmatrix} Y_2(6\sqrt{6}Y_5Y_1^2 - 3Y_3^2Y_1 - 2\sqrt{6}Y_2^2Y_3 + \sqrt{6}Y_4(Y_4^2 + 2Y_3Y_5) \\ \quad - 2Y_2(2\sqrt{6}Y_5^2 + 3Y_1Y_4)) + Y_5(\sqrt{6}Y_3^3 - 6Y_1Y_5Y_3 - Y_4(2\sqrt{6}Y_5^2 + 3Y_1Y_4)) \\ Y_1(3Y_1(Y_4^2 + 4Y_3Y_5) + \sqrt{6}(2Y_5(Y_2^2 + Y_4Y_5) - 2Y_3^3 - 3Y_2Y_4Y_3)) \\ \quad - Y_2(3\sqrt{6}Y_1^3 + 9Y_2Y_3^2 + Y_5(4Y_3Y_5 - 5Y_4^2)) + 2Y_3Y_4(Y_4^2 + Y_3Y_5) \\ Y_2(\sqrt{6}Y_1(Y_4^2 + 2Y_3Y_5) - 6Y_2Y_1^2 - 2(6Y_3^3 + Y_2Y_4Y_3)) \\ \quad + 2Y_5(5Y_3Y_4^2 + 2Y_2(Y_2^2 + Y_4Y_5)) \\ Y_5(\sqrt{6}Y_1(Y_3^2 + 2Y_2Y_4) - 6Y_5Y_1^2 - 2(6Y_4^3 + Y_3Y_5Y_4)) \\ \quad + 2Y_2(5Y_4Y_3^2 + 2Y_5(Y_5^2 + Y_2Y_3)) \\ Y_1(3Y_1(Y_3^2 + 4Y_2Y_4) + \sqrt{6}(2Y_2(Y_5^2 + Y_2Y_3) - 2Y_4^3 - 3Y_3Y_5Y_4)) \\ \quad - Y_5(3\sqrt{6}Y_1^3 + 9Y_4^2Y_5 + Y_2(4Y_2Y_4 - 5Y_3^2)) + 2Y_3Y_4(Y_3^2 + Y_2Y_4) \end{pmatrix} \sim 5,$$

corresponding to a total dimension of 41.

Finally, at weight  $k = 10$ , one has

$$Y_1^{(10)} = 3\sqrt{3} (Y_2 Y_3^2 + Y_4^2 Y_5) + \sqrt{2} Y_1^3 + 3\sqrt{2} (Y_3 Y_4 - 2Y_2 Y_5) Y_1 \sim \mathbf{1},$$

$$Y_{\mathbf{3},1}^{(10)} = (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5)^2 \begin{pmatrix} Y_6 \\ Y_7 \\ Y_8 \end{pmatrix} \sim \mathbf{3},$$

$$Y_{\mathbf{3},2}^{(10)} = (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) \times \begin{pmatrix} (3Y_1^2 + 2Y_3 Y_4) Y_6 - (\sqrt{2} Y_3^2 + \sqrt{2} Y_2 Y_4 - \sqrt{3} Y_1 Y_5) Y_7 \\ + (\sqrt{3} Y_1 Y_2 - \sqrt{2} (Y_4^2 + Y_3 Y_5)) Y_8 \\ (\sqrt{3} Y_1 Y_2 - \sqrt{2} (Y_4^2 + Y_3 Y_5)) Y_6 + (2Y_3 Y_4 + 3Y_2 Y_5) Y_7 \\ + (Y_2^2 + 4Y_4 Y_5) Y_8 \\ - (\sqrt{2} Y_3^2 + \sqrt{2} Y_2 Y_4 - \sqrt{3} Y_1 Y_5) Y_6 + (Y_5^2 + 4Y_2 Y_3) Y_7 \\ + (2Y_3 Y_4 + 3Y_2 Y_5) Y_8 \end{pmatrix} \sim \mathbf{3},$$

$$Y_{\mathbf{3},3}^{(10)} = \left( 3\sqrt{3} (Y_2 Y_3^2 + Y_4^2 Y_5) + \sqrt{2} Y_1 (Y_1^2 + 3Y_3 Y_4 - 6Y_2 Y_5) \right) \times \begin{pmatrix} \sqrt{3} (Y_5 Y_7 + Y_2 Y_8) - 2Y_1 Y_6 \\ \sqrt{3} Y_2 Y_6 + Y_1 Y_7 - \sqrt{6} Y_3 Y_8 \\ \sqrt{3} Y_5 Y_6 - \sqrt{6} Y_4 Y_7 + Y_1 Y_8 \end{pmatrix} \sim \mathbf{3},$$

$$Y_{\mathbf{3},1}'^{(10)} = (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5)^2 \begin{pmatrix} Y_9 \\ Y_{10} \\ Y_{11} \end{pmatrix} \sim \mathbf{3}',$$

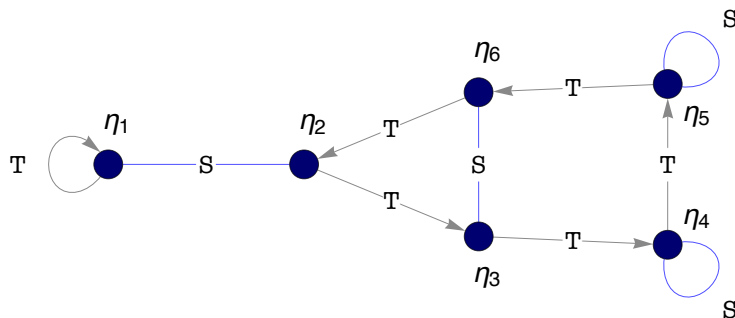
$$Y_{\mathbf{3},2}'^{(10)} = (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) \times \begin{pmatrix} -2(Y_1^2 + Y_3 Y_4 - 2Y_2 Y_5) Y_9 + (3\sqrt{2} Y_2 Y_3 + \sqrt{3} Y_1 Y_4) Y_{10} \\ + (\sqrt{3} Y_1 Y_3 + 3\sqrt{2} Y_4 Y_5) Y_{11} \\ (\sqrt{3} Y_1 Y_3 + 3\sqrt{2} Y_4 Y_5) Y_9 + (Y_1^2 + Y_3 Y_4 - 2Y_2 Y_5) Y_{10} \\ + (2\sqrt{6} Y_1 Y_5 - 3Y_3^2) Y_{11} \\ (3\sqrt{2} Y_2 Y_3 + \sqrt{3} Y_1 Y_4) Y_9 + (2\sqrt{6} Y_1 Y_2 - 3Y_4^2) Y_{10} \\ + (Y_1^2 + Y_3 Y_4 - 2Y_2 Y_5) Y_{11} \end{pmatrix} \sim \mathbf{3}',$$

$$Y_{\mathbf{3},3}'^{(10)} = \left( 3\sqrt{3} (Y_2 Y_3^2 + Y_4^2 Y_5) + \sqrt{2} Y_1 (Y_1^2 + 3Y_3 Y_4 - 6Y_2 Y_5) \right) \times \begin{pmatrix} \sqrt{3} (Y_4 Y_{10} + Y_3 Y_{11}) - 2Y_1 Y_9 \\ \sqrt{3} Y_3 Y_9 + Y_1 Y_{10} - \sqrt{6} Y_5 Y_{11} \\ \sqrt{3} Y_4 Y_9 - \sqrt{6} Y_2 Y_{10} + Y_1 Y_{11} \end{pmatrix} \sim \mathbf{3}',$$

$$Y_{\mathbf{4},1}^{(10)} = (Y_1^2 + 2Y_3 Y_4 + 2Y_2 Y_5) \times \begin{pmatrix} -(Y_3 Y_5 - 2Y_4^2 - \sqrt{6} Y_1 Y_2) Y_6 + (\sqrt{2} (Y_4 Y_5 - 2Y_2^2) - 2\sqrt{3} Y_1 Y_3) Y_8 \\ (Y_4 Y_5 - 2Y_2^2 - \sqrt{6} Y_1 Y_3) Y_6 + (\sqrt{2} (Y_3 Y_5 - 2Y_4^2) - 2\sqrt{3} Y_1 Y_2) Y_7 \\ -(Y_2 Y_3 - 2Y_5^2 - \sqrt{6} Y_1 Y_4) Y_6 - (\sqrt{2} (Y_2 Y_4 - 2Y_3^2) - 2\sqrt{3} Y_1 Y_5) Y_8 \\ (Y_2 Y_4 - 2Y_3^2 - \sqrt{6} Y_1 Y_5) Y_6 - (\sqrt{2} (Y_2 Y_3 - 2Y_5^2) - 2\sqrt{3} Y_1 Y_4) Y_7 \end{pmatrix} \sim \mathbf{4},$$

$$\begin{aligned}
 Y_{4,2}^{(10)} &= (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \\
 &\quad \times \begin{pmatrix} \sqrt{2}(\sqrt{3}Y_1Y_6 + Y_5Y_7)Y_7 + (Y_3Y_6 - \sqrt{2}Y_4Y_8)Y_8 \\ (2Y_2Y_7 + Y_4Y_8 - \sqrt{2}Y_3Y_6)Y_6 + \sqrt{2}(Y_3Y_7 + Y_5Y_8)Y_8 \\ (2Y_5Y_8 + Y_3Y_7 - \sqrt{2}Y_4Y_6)Y_6 + \sqrt{2}(Y_4Y_8 + Y_2Y_7)Y_7 \\ \sqrt{2}(\sqrt{3}Y_1Y_6 + Y_2Y_8)Y_8 + (Y_4Y_6 - \sqrt{2}Y_3Y_7)Y_7 \end{pmatrix} \sim \mathbf{4}, \\
 Y_{4,3}^{(10)} &= \left( 3\sqrt{3}(Y_2Y_3^2 + Y_4^2Y_5) + \sqrt{2}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) \right) \begin{pmatrix} \sqrt{2}Y_7Y_9 + Y_8Y_{10} \\ -\sqrt{2}Y_6Y_{10} - Y_8Y_{11} \\ -\sqrt{2}Y_6Y_{11} - Y_7Y_{10} \\ \sqrt{2}Y_8Y_9 + Y_7Y_{11} \end{pmatrix} \sim \mathbf{4}, \\
 Y_{5,1}^{(10)} &= (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5)^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} \sim \mathbf{5}, \\
 Y_{5,2}^{(10)} &= (Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5) \\
 &\quad \times \begin{pmatrix} 2\sqrt{2}Y_1(Y_1^2 - Y_3Y_4 - 4Y_2Y_5) - 3\sqrt{3}(Y_2Y_3^2 + Y_4^2Y_5) \\ Y_1(\sqrt{3}(Y_4^2 - 2Y_3Y_5) - \sqrt{2}Y_1Y_2) + \sqrt{2}(3Y_3^3 + Y_2Y_4Y_3 - 2Y_5(Y_2^2 - 3Y_4Y_5)) \\ \sqrt{2}(Y_2(3Y_4^2 + 4Y_3Y_5) - 2Y_3^2Y_4) - 4Y_1(\sqrt{3}Y_2^2 + \sqrt{2}Y_1Y_3 + \sqrt{3}Y_4Y_5) \\ \sqrt{2}(Y_5(3Y_3^2 + 4Y_2Y_4) - 2Y_3Y_4^2) - 4Y_1(\sqrt{3}Y_5^2 + \sqrt{2}Y_1Y_4 + \sqrt{3}Y_2Y_3) \\ Y_1(\sqrt{3}(Y_3^2 - 2Y_2Y_4) - \sqrt{2}Y_1Y_5) + \sqrt{2}(3Y_4^3 + Y_3Y_5Y_4 - 2Y_2(Y_5^2 - 3Y_2Y_3)) \end{pmatrix} \sim \mathbf{5}, \\
 Y_{5,3}^{(10)} &= \begin{pmatrix} (2Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) + 3\sqrt{6}(Y_2Y_3^2 + Y_4^2Y_5))(Y_6^2 - Y_7Y_8) \\ - (2\sqrt{3}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) + 9\sqrt{2}(Y_2Y_3^2 + Y_4^2Y_5))Y_6Y_7 \\ (\sqrt{6}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) + 9(Y_2Y_3^2 + Y_4^2Y_5))Y_7^2 \\ (\sqrt{6}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) + 9(Y_2Y_3^2 + Y_4^2Y_5))Y_8^2 \\ - (2\sqrt{3}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) + 9\sqrt{2}(Y_2Y_3^2 + Y_4^2Y_5))Y_6Y_8 \end{pmatrix} \sim \mathbf{5}, \\
 Y_{5,4}^{(10)} &= \left( 3\sqrt{3}(Y_2Y_3^2 + Y_4^2Y_5) + \sqrt{2}Y_1(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5) \right) \\
 &\quad \times \begin{pmatrix} \sqrt{2}(Y_1^2 + Y_3Y_4 - 2Y_2Y_5) \\ \sqrt{3}Y_4^2 - 2\sqrt{2}Y_1Y_2 \\ \sqrt{2}Y_1Y_3 + 2\sqrt{3}Y_4Y_5 \\ \sqrt{2}Y_1Y_4 + 2\sqrt{3}Y_2Y_3 \\ \sqrt{3}Y_3^2 - 2\sqrt{2}Y_1Y_5 \end{pmatrix} \sim \mathbf{5},
 \end{aligned}$$

corresponding to a total dimension of 51. As before, the correct dimensionality of each linear space is guaranteed via an appropriate number of constraints relating products of modular forms.



**Figure 3.** Graph illustrating the automorphisms of the set of seed functions  $\eta_i(\tau)$ , defined in eq. (D.5), under the actions of  $\Gamma_5 \simeq A_5$  generators  $S$  and  $T$ .

### D Correspondence with the Dedekind eta function

For the groups  $\Gamma_2 \simeq S_3$ ,  $\Gamma_3 \simeq A_4$  and  $\Gamma_4 \simeq S_4$ , the seed functions, from which the modular forms of weight 2 are constructed, are given by the Dedekind eta function

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}. \tag{D.1}$$

Namely, for  $\Gamma_2$ , the set of interest contains three seed functions  $\tilde{\eta}_i$  [15]:

$$\{\tilde{\eta}_i\} = \left\{ \eta(2\tau), \eta\left(\frac{\tau}{2}\right), \eta\left(\frac{\tau+1}{2}\right) \right\}. \tag{D.2}$$

In the case of  $\Gamma_3$ , the corresponding set of four seed functions  $\tilde{\eta}_i$  reads [14]:

$$\{\tilde{\eta}_i\} = \left\{ \eta(3\tau), \eta\left(\frac{\tau}{3}\right), \eta\left(\frac{\tau+1}{3}\right), \eta\left(\frac{\tau+2}{3}\right) \right\}, \tag{D.3}$$

while for  $\Gamma_4$ , the desired set contains six seed functions  $\hat{\eta}_i$  [16]:

$$\{\hat{\eta}_i\} = \left\{ \eta(4\tau), \eta\left(\frac{\tau}{4}\right), \eta\left(\frac{\tau+1}{4}\right), \eta\left(\frac{\tau+2}{4}\right), \eta\left(\frac{\tau+3}{4}\right), \eta\left(\tau + \frac{1}{2}\right) \right\}. \tag{D.4}$$

It is interesting to continue this chain to the case of  $\Gamma_5 \simeq A_5$  considering the following set of six eta functions:

$$\{\eta_i\} = \left\{ \eta(5\tau), \eta\left(\frac{\tau}{5}\right), \eta\left(\frac{\tau+1}{5}\right), \eta\left(\frac{\tau+2}{5}\right), \eta\left(\frac{\tau+3}{5}\right), \eta\left(\frac{\tau+4}{5}\right) \right\}. \tag{D.5}$$

First, we notice that this set is closed under the action of the modular group (and under the action of  $\Gamma_5$ ), i.e., upon acting with  $S$  or  $T$  each of the  $\eta_i$  functions is mapped to another function from the set (up to (sometimes  $\tau$ -dependent) multiplicative factor). A graph of this mapping is shown in figure 3. Taking logarithmic derivatives, we find:

$$\frac{d}{d\tau} \log \eta_i(-1/\tau) = \frac{1}{2\tau} + \frac{d}{d\tau} \log \eta_i^S(\tau), \tag{D.6}$$

$$\frac{d}{d\tau} \log \eta_i(\tau+1) = \frac{d}{d\tau} \log \eta_i^T(\tau), \tag{D.7}$$

where  $\eta_i^S$  and  $\eta_i^T$  are the images of  $\eta_i$  under the  $S$  and  $T$  maps of figure 3, respectively.

Now we construct a sum

$$X(a_1, \dots, a_6|\tau) \equiv \sum_{i=1}^6 a_i \frac{d}{d\tau} \log \eta_i(\tau), \quad \text{with} \quad \sum_i a_i = 0. \quad (\text{D.8})$$

The functions  $X$  transform as weight 2 modular forms. Under the action of  $S$  and  $T$  we find

$$S : X(a_1, \dots, a_6|\tau) \rightarrow X(a_1, a_2, a_3, a_4, a_5, a_6|-1/\tau) = \tau^2 X(a_2, a_1, a_6, a_4, a_5, a_3|\tau), \quad (\text{D.9})$$

$$T : X(a_1, \dots, a_6|\tau) \rightarrow X(a_1, a_2, a_3, a_4, a_5, a_6|\tau + 1) = X(a_1, a_6, a_2, a_3, a_4, a_5|\tau). \quad (\text{D.10})$$

We search for 5 independent modular forms  $X_i(\tau)$  transforming in  $\mathbf{5}$  of  $A_5$  according to eq. (2.12), which for  $\gamma = S$  and  $T$  reads

$$X_{\mathbf{5}}(-1/\tau) = \tau^2 \rho(S) X_{\mathbf{5}}(\tau) \quad \text{and} \quad X_{\mathbf{5}}(\tau + 1) = \rho(T) X_{\mathbf{5}}(\tau), \quad (\text{D.11})$$

where  $X_{\mathbf{5}} = (X_1, X_2, X_3, X_4, X_5)^T$ , and the matrices  $\rho(S)$  and  $\rho(T)$  are given by eq. (B.6). The transformation  $T$  fixes  $a_i$  up to coefficients  $c_i$ , namely,

$$X_1(\tau) = c_1 X(-5, 1, 1, 1, 1, 1|\tau), \quad (\text{D.12})$$

$$X_2(\tau) = c_2 X(0, \zeta^4, \zeta^3, \zeta^2, \zeta, 1|\tau), \quad (\text{D.13})$$

$$X_3(\tau) = c_3 X(0, \zeta^3, \zeta, \zeta^4, \zeta^2, 1|\tau), \quad (\text{D.14})$$

$$X_4(\tau) = c_4 X(0, \zeta^2, \zeta^4, \zeta, \zeta^3, 1|\tau), \quad (\text{D.15})$$

$$X_5(\tau) = c_5 X(0, \zeta, \zeta^2, \zeta^3, \zeta^4, 1|\tau), \quad (\text{D.16})$$

where  $\zeta = e^{2\pi i/5}$ . The transformation  $S$  fixes the coefficients  $c_i$  up to an overall factor  $c$ :

$$c_1 = c, \quad (\text{D.17})$$

$$c_2 = -c\sqrt{3}\sqrt{-\varphi + i\sqrt{\sqrt{5}/\varphi}}, \quad c_3 = c\sqrt{3}\sqrt{1/\varphi - i\sqrt{\sqrt{5}/\varphi}}, \quad (\text{D.18})$$

$$c_4 = c\sqrt{3}\sqrt{1/\varphi + i\sqrt{\sqrt{5}/\varphi}}, \quad c_5 = -c\sqrt{3}\sqrt{-\varphi - i\sqrt{\sqrt{5}/\varphi}}, \quad (\text{D.19})$$

with  $\varphi = (1 + \sqrt{5})/2$ . Choosing for convenience  $c = -1/\sqrt{6}$ , which leads to the following simplification:

$$c_1 = -\frac{1}{\sqrt{6}}, \quad c_2 = \zeta, \quad c_3 = \zeta^2, \quad c_4 = \zeta^3, \quad c_5 = \zeta^4, \quad (\text{D.20})$$

we finally obtain

$$X_{\mathbf{5}}(\tau) = \begin{pmatrix} X_1(\tau) \\ X_2(\tau) \\ X_3(\tau) \\ X_4(\tau) \\ X_5(\tau) \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{\sqrt{6}}X(-5, 1, 1, 1, 1, 1|\tau) \\ X(0, 1, \zeta^4, \zeta^3, \zeta^2, \zeta|\tau) \\ X(0, 1, \zeta^3, \zeta, \zeta^4, \zeta^2|\tau) \\ X(0, 1, \zeta^2, \zeta^4, \zeta, \zeta^3|\tau) \\ X(0, 1, \zeta, \zeta^2, \zeta^3, \zeta^4|\tau) \end{pmatrix}, \quad (\text{D.21})$$

cf. eq. (2.21). Thus, starting from the set of eta functions in eq. (D.5), one can construct the  $A_5$  quintet of weight 2 modular forms. It is interesting to notice that the graph in figure 3 represents a “half” of that in figure 1, which in addition to the quintet allows for construction of the two  $A_5$  triplets.

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## References

- [1] PARTICLE DATA GROUP collaboration, *Review of Particle Physics*, *Phys. Rev. D* **98** (2018) 030001 [[INSPIRE](#)].
- [2] F. Capozzi, E. Lisi, A. Marrone and A. Palazzo, *Current unknowns in the three neutrino framework*, *Prog. Part. Nucl. Phys.* **102** (2018) 48 [[arXiv:1804.09678](#)] [[INSPIRE](#)].
- [3] I. Esteban, M.C. Gonzalez-Garcia, A. Hernandez-Cabezudo, M. Maltoni and T. Schwetz, *Global analysis of three-flavour neutrino oscillations: synergies and tensions in the determination of  $\theta_{23}$ ,  $\delta_{CP}$  and the mass ordering*, *JHEP* **01** (2019) 106 [[arXiv:1811.05487](#)] [[INSPIRE](#)].
- [4] G. Altarelli and F. Feruglio, *Discrete Flavor Symmetries and Models of Neutrino Mixing*, *Rev. Mod. Phys.* **82** (2010) 2701 [[arXiv:1002.0211](#)] [[INSPIRE](#)].
- [5] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, *Non-Abelian Discrete Symmetries in Particle Physics*, *Prog. Theor. Phys. Suppl.* **183** (2010) 1 [[arXiv:1003.3552](#)] [[INSPIRE](#)].
- [6] S.F. King and C. Luhn, *Neutrino Mass and Mixing with Discrete Symmetry*, *Rept. Prog. Phys.* **76** (2013) 056201 [[arXiv:1301.1340](#)] [[INSPIRE](#)].
- [7] S.T. Petcov, *Discrete Flavour Symmetries, Neutrino Mixing and Leptonic CP-violation*, *Eur. Phys. J. C* **78** (2018) 709 [[arXiv:1711.10806](#)] [[INSPIRE](#)].
- [8] R.M. Fonseca and W. Grimus, *Classification of lepton mixing matrices from finite residual symmetries*, *JHEP* **09** (2014) 033 [[arXiv:1405.3678](#)] [[INSPIRE](#)].
- [9] I. Girardi, S.T. Petcov, A.J. Stuart and A.V. Titov, *Leptonic Dirac CP-violation Predictions from Residual Discrete Symmetries*, *Nucl. Phys. B* **902** (2016) 1 [[arXiv:1509.02502](#)] [[INSPIRE](#)].
- [10] S.T. Petcov and A.V. Titov, *Assessing the Viability of  $A_4$ ,  $S_4$  and  $A_5$  Flavour Symmetries for Description of Neutrino Mixing*, *Phys. Rev. D* **97** (2018) 115045 [[arXiv:1804.00182](#)] [[INSPIRE](#)].
- [11] I. Girardi, A. Meroni, S.T. Petcov and M. Spinrath, *Generalised geometrical CP-violation in a  $T'$  lepton flavour model*, *JHEP* **02** (2014) 050 [[arXiv:1312.1966](#)] [[INSPIRE](#)].
- [12] S.T. Petcov, *Predicting the values of the leptonic CP-violation phases in theories with discrete flavour symmetries*, *Nucl. Phys. B* **892** (2015) 400 [[arXiv:1405.6006](#)] [[INSPIRE](#)].
- [13] I. Girardi, S.T. Petcov and A.V. Titov, *Predictions for the Leptonic Dirac CP-violation Phase: a Systematic Phenomenological Analysis*, *Eur. Phys. J. C* **75** (2015) 345 [[arXiv:1504.00658](#)] [[INSPIRE](#)].
- [14] F. Feruglio, *Are neutrino masses modular forms?*, in *From My Vast Repertoire . . . : Guido Altarelli's Legacy*, A. Levy, S. Forte and G. Ridolfi eds., World Scientific (2018), pp. 227–266 [[DOI:10.1142/9789813238053\\_0012](#)] [[arXiv:1706.08749](#)] [[INSPIRE](#)].
- [15] T. Kobayashi, K. Tanaka and T.H. Tatsuishi, *Neutrino mixing from finite modular groups*, *Phys. Rev. D* **98** (2018) 016004 [[arXiv:1803.10391](#)] [[INSPIRE](#)].



- [16] J.T. Penedo and S.T. Petcov, *Lepton Masses and Mixing from Modular  $S_4$  Symmetry*, *Nucl. Phys. B* **939** (2019) 292 [[arXiv:1806.11040](#)] [[INSPIRE](#)].
- [17] T. Kobayashi and S. Tamba, *Modular forms of finite modular subgroups from magnetized D-brane models*, *Phys. Rev. D* **99** (2019) 046001 [[arXiv:1811.11384](#)] [[INSPIRE](#)].
- [18] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, *Finite Modular Groups and Lepton Mixing*, *Nucl. Phys. B* **858** (2012) 437 [[arXiv:1112.1340](#)] [[INSPIRE](#)].
- [19] J.C. Criado and F. Feruglio, *Modular Invariance Faces Precision Neutrino Data*, *SciPost Phys.* **5** (2018) 042 [[arXiv:1807.01125](#)] [[INSPIRE](#)].
- [20] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, *Modular  $A_4$  invariance and neutrino mixing*, *JHEP* **11** (2018) 196 [[arXiv:1808.03012](#)] [[INSPIRE](#)].
- [21] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Modular  $S_4$  models of lepton masses and mixing*, *JHEP* **04** (2019) 005 [[arXiv:1811.04933](#)] [[INSPIRE](#)].
- [22] L.L. Everett and A.J. Stuart, *Icosahedral ( $A_5$ ) Family Symmetry and the Golden Ratio Prediction for Solar Neutrino Mixing*, *Phys. Rev. D* **79** (2009) 085005 [[arXiv:0812.1057](#)] [[INSPIRE](#)].
- [23] F. Feruglio and A. Paris, *The Golden Ratio Prediction for the Solar Angle from a Natural Model with  $A_5$  Flavour Symmetry*, *JHEP* **03** (2011) 101 [[arXiv:1101.0393](#)] [[INSPIRE](#)].
- [24] G.-J. Ding, L.L. Everett and A.J. Stuart, *Golden Ratio Neutrino Mixing and  $A_5$  Flavor Symmetry*, *Nucl. Phys. B* **857** (2012) 219 [[arXiv:1110.1688](#)] [[INSPIRE](#)].
- [25] I.K. Cooper, S.F. King and A.J. Stuart, *A Golden  $A_5$  Model of Leptons with a Minimal NLO Correction*, *Nucl. Phys. B* **875** (2013) 650 [[arXiv:1212.1066](#)] [[INSPIRE](#)].
- [26] I. de Medeiros Varzielas and L. Lavoura, *Golden ratio lepton mixing and nonzero reactor angle with  $A_5$* , *J. Phys. G* **41** (2014) 055005 [[arXiv:1312.0215](#)] [[INSPIRE](#)].
- [27] J. Gehrlein, J.P. Oppermann, D. Schäfer and M. Spinrath, *An  $SU(5) \times A_5$  golden ratio flavour model*, *Nucl. Phys. B* **890** (2014) 539 [[arXiv:1410.2057](#)] [[INSPIRE](#)].
- [28] C.-C. Li and G.-J. Ding, *Lepton Mixing in  $A_5$  Family Symmetry and Generalized CP*, *JHEP* **05** (2015) 100 [[arXiv:1503.03711](#)] [[INSPIRE](#)].
- [29] A. Di Iura, C. Hagedorn and D. Meloni, *Lepton mixing from the interplay of the alternating group  $A_5$  and CP*, *JHEP* **08** (2015) 037 [[arXiv:1503.04140](#)] [[INSPIRE](#)].
- [30] P. Ballett, S. Pascoli and J. Turner, *Mixing angle and phase correlations from  $A_5$  with generalized CP and their prospects for discovery*, *Phys. Rev. D* **92** (2015) 093008 [[arXiv:1503.07543](#)] [[INSPIRE](#)].
- [31] J. Turner, *Predictions for leptonic mixing angle correlations and nontrivial Dirac CP-violation from  $A_5$  with generalized CP symmetry*, *Phys. Rev. D* **92** (2015) 116007 [[arXiv:1507.06224](#)] [[INSPIRE](#)].
- [32] A. Di Iura, M.L. López-Ibáñez and D. Meloni, *Neutrino masses and lepton mixing from  $A_5 \times CP$* , [arXiv:1811.09662](#) [[INSPIRE](#)].
- [33] S. Ferrara, D. Lüst, A.D. Shapere and S. Theisen, *Modular Invariance in Supersymmetric Field Theories*, *Phys. Lett. B* **225** (1989) 363 [[INSPIRE](#)].
- [34] S. Ferrara, D. Lüst and S. Theisen, *Target Space Modular Invariance and Low-Energy Couplings in Orbifold Compactifications*, *Phys. Lett. B* **233** (1989) 147 [[INSPIRE](#)].

- [35] C. Franc and G. Mason, *Hypergeometric series, modular linear differential equations and vector-valued modular forms*, *Ramanujan J.* **41** (2016) 233 [[arXiv:1503.05519](#)].
- [36] T. Mano, *Differential relations for modular forms of level five*, *J. Math. Kyoto Univ.* **42** (2002) 41.
- [37] H.M. Farkas and I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group*, Graduate Studies in Mathematics, vol. 37, American Mathematical Society (2001).
- [38] S. Kharchev and A. Zabrodin, *Theta vocabulary I*, *J. Geom. Phys.* **94** (2015) 19 [[arXiv:1502.04603](#)].
- [39] PLANCK collaboration, *Planck 2018 results. VI. Cosmological parameters*, [arXiv:1807.06209](#) [[INSPIRE](#)].
- [40] The Sage Developers, *SageMath, the Sage Mathematics Software System*, version 8.4 (2018).