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Modular amplitudes and flux-superpotentials on elliptic Calabi-Yau fourfolds

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Abstract: We discuss the period geometry and the topological string amplitudes on elliptically fibered Calabi-Yau fourfolds in toric ambient spaces. In particular, we describe a general procedure to fix integral periods. Using some elementary facts from homological mirror symmetry we then obtain Bridgelands involution and its monodromy action on the integral basis for non-singular elliptically fibered fourfolds. The full monodromy group contains a subgroup that acts as PSL(2,Z) on the Kähler modulus of the fiber and we analyze the consequences of this modularity for the genus zero and genus one amplitudes as well as the associated geometric invariants. We find holomorphic anomaly equations for the amplitudes, reflecting precisely the failure of exact PSL(2,Z) invariance that relates them to quasi-modular forms. Finally we use the integral basis of periods to study the horizontal flux superpotential and the leading order Kähler potential for the moduli fields in F-theory compactifications globally on the complex structure moduli space. For a particular example we verify attractor behaviour at the generic conifold given an aligned choice of flux which we expect to be universal. Furthermore we analyze the superpotential at the orbifold points but find no stable vacua.

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Contents

1 Introduction

At present F-theory compactifications on elliptic Calabi-Yau fourfolds provide the richest class of explicit $N = 1$ effective theories starting from string theory. The reason is that the construction of Calabi-Yau fourfolds as algebraic varieties in a projective ambient space is very simple and toric, or more generally non-abelian gauged linear σ-model descriptions provide immediately trillions of geometries [1].

In fact, geometric classifications of certain compactifications with restricted physical features seem possible even though this has been achieved mostly for elliptic Calabi-Yau threefolds, where it has been argued that there exists only a finite number of topological types in this class [2].

Most of the generic compact toric examples allow for elliptic fibrations and in addition for each of them there is a huge degeneracy of possible flux choices, which together with non-perturbative effects have been argued to solve the moduli stabilization problem by driving the theory to a particular vacuum. Ignoring the details of how this happens for the concrete geometry under consideration it has been shown that by degenerating the fourfold in a controlled way viable phenomenological low energy particle spectra will emerge in four dimensions as was worked out in the F-theory revival starting with the papers of [3–6].

An additional nice feature of F-theory is a largely unified description of gauge- or brane moduli in terms of the complex structure moduli space of the fourfold. Together with mirror symmetry this results in a large variety of geometrical tools that can be used to study the physically relevant structures on these moduli spaces. In this paper we want to improve on these tools following the line of the papers [7–12].

Of particular interest when studying the F-theory effective action associated to a given Calabi-Yau fourfold are the admissible fluxes. There are two different types, namely horizontal and vertical fluxes, and in general both are necessary to construct phenomenologically viable models. While determining a basis of fluxes over $\mathbb{C}$ is relatively straightforward, it has been shown that the fluxes are quantized [13] and finding the proper sublattice — in particular for the horizontal part — is more involved. However, horizontal fluxes on a Calabi-Yau fourfold W can be identified with the charges of topological B-branes on a mirror manifold M. In this work we use the derived category description of the latter and the asymptotic charge formula in terms of the Gamma class [14–16] to determine properly quantized fluxes on W. We provide formulas that allow to write down the integral fluxes — and in many cases an integral basis — in terms of the intersection data on M.

We then restrict to the case of non-singular elliptic Calabi-Yau fourfolds and find explicit expressions for several elements of the monodromy group Γ_M. We show that a generic subgroup of the monodromy generates the $\mathrm{SL}(2, \mathbb{Z})$ action on the Kähler modulus of the fiber. This explains certain modular properties of the topological string amplitudes on M that we also analyze in detail. We find that the genus zero amplitudes in the type II language that determine the Kähler potential and the superpotential are $\mathrm{SL}(2,\mathbb{Z})$ quasi-modular forms, extending results of [17]. We also show that similar features hold for the genus one amplitude, which is conjectured to be related to the gauge kinetic terms. As in the Calabi-Yau threefold case we find that these amplitudes are related via certain holomorphic anomaly equations, from which they can be reconstructed in simple situations [18–22].

Finally, we study the global structure of the properly quantized horizontal flux superpotential for a particular example. To this end we analytically continue the integral periods to the generic conifold locus, the generic orbifold and the Gepner point. We find that aligned flux stabilizes the theory at the conifold where the scalar potential vanishes. Somewhat surprisingly the complex 8×8 continuation matrix can be expressed analytically up to five real constants.

In the rest of the introduction we describe the principle structures associated to the moduli space of Calabi-Yau fourfolds. This will set our notation and guide the reader in later sections, where we add to this discussion.

Note added: after this article appeared on the arxiv, Georg Oberdieck pointed out that our results in section 4.5 match with his and Aaron Pixton's conjectured holomorphic anomaly equation on Calabi-Yau n-folds appearing in [23, 24]. Moreover, he explained to us the explicit form of the generalized holomorphic anomaly equation for the Gromov-Witten potentials on Calabi-Yau fourfolds, which we include now in appendix B. We performed further non-trivial checks of his conjecture with our data beyond the material that appeared already in appendix A.5.

1.1 Mathematical and physical structures on the moduli space

Let us give a very short account of the complex structure moduli space of Calabi-Yau fourfolds W, its algebraic and differential structures and their physical interpretation.

As far as the differential structure and some aspects of mirror symmetry are concerned this is based on the analysis of [7–9]. The analysis can be viewed as a generalization of the ones that lead to special geometry for Calabi-Yau threefolds [25] and was discussed with emphasis on mirror symmetry in [26].

Calabi-Yau manifolds are equipped with a Kähler $(1,1)$ form ω and a no-where vanishing holomophic $(4,0)$ form Ω with the relation $\omega^4/12 = \Omega\wedge\bar{\Omega}$. The complex structure moduli space $\mathcal{M}$ is unobstructed and of complex dimension $h_{3,1}(W)$. Further key structures are the bilinear intersection form on the horizontal cohomology $\alpha_{pq},\beta_{rs}\in H^4_{\mathrm{hor}}(W) = H^{40}\oplus H^{31}\oplus H^{22}_{\mathrm{hor}}\oplus H^{13}\oplus H^{04}$

$$\langle\alpha_{pq},\beta_{rs}\rangle = \int_W \alpha_{pq}\wedge\beta_{rs} = 0 \qquad \text{unless} \quad p=s \ \ \text{and}\ q=r\,, \tag{1.1}$$

which is even as the dimension is even and transversal with respect to the Hodge type as indicated.

Moreover there is a positive real structure

$$R(\alpha) = i^{p-q}\langle\alpha,\bar{\alpha}\rangle > 0\,, \tag{1.2}$$

where α is a primitive form in $H^{p,q}$ with $p+q=n$. In particular

$$e^{-K(z)} = R(\Omega(z)) \,, \tag{1.3}$$

defines the real Kähler potential K for the Weil-Petersson metric $G_{i\bar{\jmath}} = \partial_j \bar{\partial}_{\bar{\jmath}} K$, which is closely related to kinetic terms of the moduli fields in the $N=1$ 4d effective action. Here $\partial_j = \frac{\partial}{\partial z^i}$ or $\bar{\partial}_{\bar{\jmath}}$ are the derivatives with respect to the generic coordinates z^i on $\mathcal{M}$ and their complex conjugates.

Because the intersection (1.1) is even on fourfolds one gets a mixture of algebraic and differential conditions on the periods and if we consider the cohomology over $\mathbb{Z}$ we get lattice structures somewhat similar to that of K3 surfaces. In particular the relations

$$\int_W \Omega \wedge \Omega = 0 \,, \quad \int_W \Omega \wedge \partial_{i_1} \ldots \partial_{i_n} \Omega = 0 \,, \quad \text{for} \quad n \leq 3 \,, \tag{1.4}$$

lead to non-trivial constraints on the periods. In [12] these relations have been used to fix an integral basis for particular one parameter Calabi-Yau fourfolds. Moreover, the authors used the Gamma class formula for the 8-brane charge as a non-trivial check of their results. We verified that the algebraic constraints can be used to fix an integral basis for the mirror of the two-parameter elliptic Calabi-Yau fourfold X_{24} but found that this method quickly becomes unpractical if the number of moduli increases. Our approach is somewhat complementary in that we use the Gamma class formula to fix integral periods and the constraints (1.4) can be used to supplement our technique and as a non-trivial check. In particular, this approach scales well with the number of moduli.

Other immediate data are the 4-point couplings

$$C_{ijkl}(z) = \langle \Omega, \partial_i \partial_j \partial_k \partial_l \Omega(z) \rangle \,. \tag{1.5}$$

By the usual relation of the horizontal and vertical cohomology rings of W to the (chiral,chiral) and (chiral,anti-chiral) rings of the $N=(2,2)$ superconformal theory on the worldsheet — with their $\mathrm{U}(1)_l \times \mathrm{U}(1)_r$ charge bigrading corresponding to the Hodge type grading[1] — and the axioms of the CFT one sees however that these 4-point couplings are not fundamental, but factorize into three-point couplings

$$C_{ijkl}(z) = C^{\alpha}_{ij}(z) \hat{\eta}^{(2)}_{\alpha\beta} C^{\beta}{}_{kl}(z) = C^{\alpha}_{ij}(z) C^{p}_{\alpha k}(z) \hat{\eta}^{(1)}_{pl} \,, \tag{1.6}$$

with the independent associativity condition

$$C^{\alpha}_{ij}(z) \hat{\eta}^{(2)}_{\alpha\beta} C^{\beta}_{kl}(z) = C^{\alpha}_{ik}(z) \hat{\eta}^{(2)}_{\alpha\beta} C^{\beta}_{jl}(z) \,. \tag{1.7}$$

Here the latin indices run over the moduli fields associated to either the complex structure moduli on W whose tangent space is associated to harmonic forms in $H^{3,1}(W)$ (dual to $H^{1,3}(W)$) or Kähler moduli on M whose tangent space is associated to harmonic forms in $H^{1,1}(M)$ (dual to $H^{3,3}(M)$). The greek indices are associated to elements in $H^{2,2}_{\mathrm{hor}}(W)$ and

[1] The exchange of this identification is the essence of mirror symmetry between W and M.

$H^{2,2}_{\rm vert}(M)$, respectively. The $\hat\eta$'s define a constant intersection form with respect to a fixed basis of $H_4^{\rm hor}(W)$ or a suitable K-theory basis extending $H_{*,*}^{vert}(M)$.

More specifically we can identify $\hat\eta^{(2)}$ in a reference complex structure near large radius with the inverse of the pairing on $H^{2,2}_{\rm hor}(W)$ and $\hat\eta^{(1)}$ with the inverse pairing on $H^{3,1}(W)\oplus H^{1,3}(W)$, which by (1.1) is block diagonal. This property is maintained throughout the moduli space due to the charge grading.

The basic idea of mirror symmetry is to calculate these couplings, which are nontrivial sections of tensor bundles over $\mathcal{M}$, from the periods of Ω. The latter can be obtained as the solutions of the Picard-Fuchs differential equations. We denote an integral basis of periods by $\Pi_\kappa(z)=\int_{\Gamma^\kappa}\Omega$, where $\kappa=1,\ldots,\dim H^4_{\rm hor}$ and Γ^κ is a fixed 4-cycle basis in $H_4^{\rm hor}(W,\mathbb{Z})$. This is physically relevant as the flux superpotential

$$W(z)=\int_W G_4\wedge\Omega(z)=n^\kappa\Pi_\kappa(z)\,, \tag{1.8}$$

is given with respect to this basis by (half)[2] integer flux quanta $n^\kappa\in\mathbb{Z}$, quantized due to a Dirac-Zwanziger quantization condition and additional constraints discussed in [13]. The analysis of attractor points and cosmologically suitable minima of the associated scalar potential relies therefore crucially on this basis.

Interpreted in the A-model the triple couplings $C^\alpha_{ij}(t)$ in the flat coordinates given by the mirror map $t_k(z)\propto\int_{[C_k]}(\omega+iB)$, where $[C_k]$ is an integral curve class on M and B is the Neveu-Schwarz B-field, encode the quantum cohomology of M. In particular each coefficient of the Fourier expansion $C^\alpha_{ij}(e^{2\pi i t_k})$ counts the contribution of a holomorphic worldsheet instanton in a given topological class. These contributions are directly related to Gromov-Witten invariants at genus zero. Gromov-Witten invariants at genus one can be calculated from the Ray-Singer Torsion, starting with the genus zero data. Both genus zero and genus one worldsheet instanton series give rise to a remarkable integrality structure in terms of additional geometric invariants of embedded curves [27].

An interesting aspect of these generating functions is that they are modular forms of the monodromy group Γ preserving the intersection form in the integral basis. For generic Calabi-Yau fourfolds this aspect is too difficult to appreciate in the sense that not much is known about the corresponding automorphic forms, but for elliptically fibered Calabi-Yau spaces, there is a subgroup of Γ which acts as the modular group on the Kähler modulus τ of the elliptic fiber in M. The precise way this subgroup is embedded in Γ can be inferred using specific auto-equivalences of the derived category of B-branes, as we will see in section 3.2.

It turns out that there is a clash between holomorphicity and modularity in the τ dependence of the triple couplings and the Ray-Singer torsion, which leads for Calabi-Yau threefolds to the holomorphic anomaly equations. We will discuss analogous holomorphic anomaly equations for fourfolds in section 4.

[2] As pointed out in [13] the combination $\left[G_4-\frac{c_2(M)}{2}\right]\in H_4(M,\mathbb{Z})$ has to be integral. However, in the concrete examples discussed below $c_2(M)$ is even.

2 The period geometry of Calabi-Yau fourfolds

In this section we show how to determine integral horizontal fluxes on a Calabi-Yau fourfold W. To this end we interpret the flux lattice as the charge lattice of A-branes on W. This in turn is related via homological mirror symmetry to the charge lattice of B-branes on a mirror manifold M. B-branes on M form the bounded derived category of coherent sheaves $D^b(M)$. Given a brane $\mathcal{E}^\bullet \in D^b(M)$ the asymptotic behaviour of the charge can be calculated using the Γ-class. Moreover, a $\mathbb{C}$-basis of fluxes on W can be obtained as the solution to a set of differential equations, the Picard-Fuchs system. Integral generators are then linear combinations of solutions with the correct asymptotic behaviour.

A similar calculation has been used in [28] to obtain the quantum corrected A-model cohomology ring for certain non-complete intersection Calabi-Yau fourfolds. In some cases the asymptotic behaviour was not sufficient to uniquely determine integral elements. As was pointed out in [28], the Jurkiewicz-Danilov theorem and the Lefschetz hyperplane theorem prevent this behaviour for the induced cohomology on complete intersections in toric ambient spaces. In general algebraic constraints on the periods can be used to supplement the above procedure.

2.1 The structure of $H^4(W, \mathbb{Z})$

The structure of $H^4(W, \mathbb{Z})$ for a Calabi-Yau fourfold is surprisingly subtle and in this paper we will only be interested in finding an integral basis for the period lattice. However, even this notion demands justification.

We first discuss the structure of $H^4(W, \mathbb{C})$. By the definition of a Calabi-Yau manifold, $H^{4,0}(W, \mathbb{C})$ is generated by a unique, holomorphic 4-form that we call Ω. Then $H^{3,1}(W, \mathbb{C})$ is generated by first-order derivatives $\partial_{z_i}\Omega$ — modulo a part in $H^{4,0}(W, \mathbb{C})$ — where z_i are complex structure coordinates. Due to the existence of the harmonic $(4, 0)$ form, $H^{3,1}(W, \mathbb{C})$ can be identified with the first order deformations of the complex structures and by the Tian-Todorov theorem the latter are unobstructed. $H^{1,3}(W, \mathbb{C})$ and $H^{0,4}(W, \mathbb{C})$ are obtained from these spaces by complex conjugation.

The interesting part is thus $H^{2,2}(W, \mathbb{C})$. By Lefschetz decomposition the cohomology splits into

$$H^{2,2}(W, \mathbb{C}) = H^{2,2}_{\text{prim}}(W, \mathbb{C}) \oplus H^{2,2}_V(W, \mathbb{C})\,. \tag{2.1}$$

Here the subgroup of primitive classes is given by

$$H^{2,2}_{\text{prim}}(W, \mathbb{C}) = \{\alpha \in H^{2,2}(W, \mathbb{C}) \,|\, \omega \wedge \alpha = 0\}\,, \tag{2.2}$$

where ω is the Kähler form. On the other hand the so-called *primary vertical cohomology* is generated by the $\mathrm{SL}(2, \mathbb{Z})$ Lefschetz action from the primitive classes in $H^{1,1}(W, \mathbb{C})$, i.e.

$$H^{2,2}_V(W, \mathbb{C}) = \{\omega \wedge \beta \,|\, \beta \in H^{1,1}(W, \mathbb{C})\,,\, \omega^3 \wedge \beta = 0\}\,. \tag{2.3}$$

We now denote the subspace of cohomology generated by derivatives $\partial_{z_{i_1}} \cdots \partial_{z_{i_n}}\Omega$ of the holomorphic 4-form as the *primary horizontal cohomology* $H^4_H(W, \mathbb{C})$. Since the Kähler class is independent of the complex structure, it follows from

$$\omega \wedge \Omega = 0\,, \tag{2.4}$$

that $H^{2,2}_H(W,\mathbb{C}) = H^4_H(W,\mathbb{C}) \cap H^{2,2}(W,\mathbb{C})$ lies inside $H^{2,2}_{\text{prim}}(W,\mathbb{C})$. However, as was shown in [29], there can be additional primitive classes in $H^{2,2}_{\text{prim}}(W,\mathbb{C}) \backslash H^{2,2}_H(W,\mathbb{C})$. The structure is thus

$$H^{2,2}(W,\mathbb{C}) = H^{2,2}_H(W,\mathbb{C}) \oplus H^{2,2}_{RM}(W,\mathbb{C}) \oplus H^{2,2}_V(W,\mathbb{C})\,, \tag{2.5}$$

where $H^{2,2}_{RM}(W,\mathbb{C})$ is the subgroup of primitive classes that are neither horizontal nor vertical.

The naive expectation that mirror symmetry maps vertical into horizontal classes and vice versa while the remaining component maps into itself can not hold. It would lead to a contradiction when applied to the geometry studied in [28], where additional "vertical" cycles appear in the quantum deformed A-model intersections. A true statement about the relation under mirror symmetry would therefore require a more refined notion of verticality. This subtlety is avoided when phrasing the problem in terms of branes and homological mirror symmetry.

2.2 Fixing an integral basis

A 4-cycle Σ dual to an element in $H^4_H(W,\mathbb{C}) \cap H^4(W,\mathbb{Z})$ is calibrated symplectically, i.e.

$$\mathrm{Re}\, e^{i\theta}\Omega\Big|_\Sigma = 0\,, \tag{2.6}$$

and the Kähler class restricts to zero $\omega|_\Sigma = 0$. In other words, Σ is a special lagrangian cycle that can be wrapped by a topological A-brane L. The central charge of this brane is then given by the period

$$Z_A(L) = \int\limits_\Sigma \Omega\,. \tag{2.7}$$

Note that this is equal to the superpotential generated by a flux quantum along Σ.

By homological mirror symmetry [30, 31], the topological A-branes on W are related to B-branes on the mirror M. The latter correspond to elements in $D^b(M)$, the bounded derived category of coherent sheaves on M. Given a B-brane that corresponds to a complex $\mathcal{E}^\bullet \in D^b(M)$, the asymptotic behaviour of the central charge is

$$Z_B^{\text{asy}}(\mathcal{E}^\bullet) = \int_M e^J \Gamma_{\mathbb{C}}(M)\,(\mathrm{ch}\,\mathcal{E}^\bullet)^\vee\,, \tag{2.8}$$

where J is the Kähler class on M. The details of this formula will be discussed in the next section. The crucial fact is that the central charges of A- and B-branes are identified via the mirror map. While a construction for all objects in $D^b(M)$ is in general not available, the central charge only depends on the K-theory charge of a complex of sheaves.

Our approach to fix an integral basis for the period lattice will be to construct elements $\mathcal{E}^\bullet$ in $D^b(M)$ that generate the algebraic K-theory group $K^0_{\text{alg}}(M)$ and calculate the asymptotic behaviour of the central charges. Using the mirror map, these can be interpreted as the leading logarithmic terms of generators of the period lattice. The subleading terms are given by the corresponding solutions to the Picard-Fuchs equations.

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2.3 B-branes and the asymptotic behaviour of the central charge

For a Calabi-Yau manifold M, the topological B-branes and the open string states stretched between them are encoded in the bounded derived category of coherent sheaves $D^b(M)$. The objects of this category are equivalence classes of bounded complexes of coherent sheaves

$$\mathcal{E}^\bullet = \ldots \xrightarrow{d^{\mathcal{E}}_{-2}} \mathcal{E}^{-1} \xrightarrow{d^{\mathcal{E}}_{-1}} \mathcal{E}^0 \xrightarrow{d^{\mathcal{E}}_0} \mathcal{E}^1 \xrightarrow{d^{\mathcal{E}}_1} \ldots . \tag{2.9}$$

A set of maps $f_i : \mathcal{E}^i \to \mathcal{F}^i$, such that the f_i commute with the coboundary maps, corresponds to an element $f \in \mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. Objects as well as morphisms are identified under certain equivalence relations but a more detailed discussion of topological branes and $D^b(M)$ is outside the scope of this paper and can be found e.g. in [32].

However, we note that if there is an exact sequence

$$\ldots \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{F} \longrightarrow 0\,, \tag{2.10}$$

where $\mathcal{F}$ is a coherent sheaf and $\mathcal{E}^i$ are locally free sheaves, i.e. equivalent to vector bundles, then the complex

$$\mathcal{E}^\bullet = \ldots \longrightarrow \mathcal{E}^{-1} \longrightarrow \mathcal{E}^0 \longrightarrow 0\,, \tag{2.11}$$

is equivalent to $\mathcal{F}$ inside $D^b(M)$.

Now given the Kähler class J, the asymptotic charge of a B-brane that corresponds to the complex $\mathcal{E}^\bullet$ is given by

$$Z^{\mathrm{asy}}(\mathcal{E}^\bullet) = \int_M e^J \Gamma_{\mathbb{C}}(M) \left(\mathrm{ch}\, \mathcal{E}^\bullet\right)^\vee . \tag{2.12}$$

The characteristic class $\Gamma_{\mathbb{C}}(M)$ can be expressed in terms of the Chern classes of M and for a Calabi-Yau manifold the expansion reads

$$\Gamma_{\mathbb{C}}(M) = 1 + \frac{1}{24}c_2 - \frac{i\zeta(3)}{8\pi^3}c_3 + \frac{1}{5760}(7c_2^2 - 4c_4) + \ldots . \tag{2.13}$$

The Chern character of the complex is given by

$$\mathrm{ch}(\mathcal{E}^\bullet) = \ldots - \mathrm{ch}(E^{-1}) + \mathrm{ch}(E^0) - \mathrm{ch}(E^1) + \mathrm{ch}(E^2) - \ldots , \tag{2.14}$$

where E^i is the vector bundle corresponding to the locally free sheaf $\mathcal{E}^i$ and the involution $(\ldots)^\vee$ acts on an element $\beta \in H^{2k}(M)$ as $\beta^\vee = (-1)^k\beta$.

A general basis of 0-, 2-, 6- and 8-branes has been constructed in [28]. The 8-brane corresponds to the structure sheaf $\mathcal{O}_M$ and the 6-branes are generated by locally free resolutions of sheaves $\mathcal{O}_{J_i}$, where the divisors J_i generate the Kähler cone. The 0-brane is represented by the skyscraper sheaf $\mathcal{O}_{\mathrm{pt.}}$. A basis of 2-branes was constructed as

$$\mathcal{C}^\bullet_a = \iota_! \mathcal{O}_{\mathcal{C}^a}(K^{1/2}_{\mathcal{C}^a})\,, \tag{2.15}$$

where ι is the inclusion of the curve $\mathcal{C}^a$ that is part of a basis for the Mori cone and $K^{1/2}_{\mathcal{C}^a}$ is a spin structure on $\mathcal{C}^a$. The asymptotic charges have been calculated in [28] and for the readers convenience they are reproduced below.

We now describe a construction of 4-branes which in many cases leads to an integral basis. Given effective divisors D_i, $i \in I$ that correspond to codimension one subvarieties of M and $S = \bigcap_{i\in I} D_i$, the Koszul sequence

$$\begin{aligned}
&0 \longrightarrow \mathcal{O}_M\left(-\sum_{i\in I} D_i\right) \longrightarrow \bigoplus_{j\in I} \mathcal{O}_M\left(-\sum_{i\in I\setminus\{j\}} D_i\right) \longrightarrow \dots \\
&\longrightarrow \bigoplus_{i\in I} \mathcal{O}_M(-D_i) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_S \longrightarrow 0
\end{aligned} \tag{2.16}$$

is exact and provides a locally free resolution of the coherent sheaf $\mathcal{O}_S$. When I contains only one element, this is just the familiar short exact sequence

$$0 \longrightarrow \mathcal{O}_M(-D) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_D \longrightarrow 0\,. \tag{2.17}$$

The latter implies the equivalence

$$0 \longrightarrow \mathcal{O}_M(-D) \longrightarrow \mathcal{O}_M \longrightarrow 0 \quad \sim \quad 0 \longrightarrow \mathcal{O}_D \longrightarrow 0\,, \tag{2.18}$$

of complexes in $D^b(M)$. This is the locally free resolution employed in [28] to calculate the central charges for a basis of 6-branes.

More generally, we can use the Koszul sequence to describe branes wrapped on arbitrary cycles that are intersections of subvarieties of codimension one. If a basis of $H^{2,2}_V(M,\mathbb{C}) \cap H^4(M,\mathbb{Z})$ can be constructed this way, then, as we described above, this leads to an integral basis of the period lattice in the mirror. In particular the asymptotic behaviour then uniquely singles out a solution to the Picard-Fuchs system. For a Calabi-Yau hypersurface M in a toric variety $\mathbb{P}_\Delta$, the cohomology of the ambient spaces is generated by elements in $H^{1,1}(\mathbb{P}_\Delta)$. As was pointed out by the authors of [28], the quantum Lefschetz hyperplane theorem then guarantees that $H^{2,2}_V(M,\mathbb{C})$ is generated by restrictions of elements in $H^{2,2}(\mathbb{P}_\Delta,\mathbb{C})$.

The formula for the asymptotic central charge gives the following results:

- **8-brane:**

$$\begin{gathered}
Z^{\text{asy}}\left(\mathcal{O}_M\right) = \int_M e^J \Gamma_{\mathbb{C}}(M) = \frac{1}{4!} C^0_{ijkl} t^i t^j t^k t^l + \frac{1}{2} c_{ij} t^i t^j + c_i t^i + c_0\,, \\
C^0_{ijkl} = \int_M J_i J_j J_k J_l\,, \qquad c_{ij} = \frac{1}{24}\int_M c_2(M) J_i J_j\,, \\
c_i = -\frac{i\zeta(3)}{8\pi^3}\int_M c_3(M) J_i\,, \quad c_0 = \frac{1}{5760}\int_M \left[7c_2(M)^2 - 4c_4(M)\right]
\end{gathered} \tag{2.19}$$

- **6-brane wrapped on J_a:**

$$\begin{aligned}
Z^{\text{asy}}\left(\mathcal{O}_{J_a}\right) &= \int_M e^J \Gamma_{\mathbb{C}}(M) \left[1 - \text{ch}\left(\mathcal{O}_M(J_a)\right)\right] \\
&= -\frac{1}{3!} C^0_{aijk} t^i t^j t^k - \frac{1}{4} C^0_{aaij} t^i t^j - \left(\frac{1}{6} C^0_{aaai} + \frac{1}{24} c^a_i\right) t^i \\
&\quad - \left(\frac{1}{24} C^0_{aaaa} + c^a_0\right)\,, \\
c^a_i &= \int_M c_2(M) J_a J_i\,, \quad c^a_0 = \frac{1}{48}\int_M c_2(M) J_a^2 - \frac{\zeta(3)}{(2\pi i)^3}\int_M c_3(M) J_a
\end{aligned} \tag{2.20}$$

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- **4-brane wrapped on $H = D_a \cap D_b$:**

$$\begin{aligned} Z^{\text{asy}}\left(\mathcal{O}_{D_a\cap D_b}\right) &= \frac{1}{2}\int_M h_{ij}t^it^j + h_it^i + h\,, \\ h_{ij} &= \int_M D_aD_bJ_iJ_j\,, \quad h_i = \frac{1}{2}\int_M D_aD_b(D_a+D_b)J_i\,, \\ h &= \frac{1}{12}\int_M D_aD_b(2D_a^2+3D_aD_b+2D_b^2) + \frac{1}{24}\int_M c_2(M)D_aD_b \end{aligned} \tag{2.21}$$

- **2-brane wrapped on $\mathcal{C}^a$ dual to J_a:**

$$Z_{\text{asy}}(\mathcal{C}_a^\bullet) = -t_a \tag{2.22}$$

The charge of the **0-brane** is universally $Z^{\text{asy}}(\mathcal{O}_{\text{pt.}}) = 1$. We denoted the generators of the Kähler cone by J_i and the Kähler form is given by $J = t^iJ_i$.

Finally we need the intersection matrix of the 4-cycles mirror dual to the B-branes. They are not given by the classical intersection numbers in the A-model but rather by the open string index

$$\chi(\mathcal{E}^\bullet,\mathcal{F}^\bullet) = \int_M \text{Td}(M)\,(\text{ch}\,\mathcal{E}^\bullet)^\vee\,\text{ch}\,\mathcal{F}^\bullet\,. \tag{2.23}$$

The Todd class $\text{Td}(M)$ is for a Calabi-Yau fourfold given by

$$\text{Td}(M) = 1 + \frac{c_2(M)}{12} + 2V\,, \tag{2.24}$$

where V is the volume form. Note that if we construct a basis of B-branes

$$\vec{v} = (\mathcal{E}_1^\bullet,\ldots,\mathcal{E}_n^\bullet), \tag{2.25}$$

and introduce the intersection matrix $\eta_{ij} = \chi(v_i,v_j)$, the inverse matrix η^{-1} will act on the period vector Π corresponding to the mirror dual cycles. For example

$$\int_W \Omega\wedge\Omega = 0 \quad \to \quad \Pi^T\eta^{-1}\Pi = 0\,. \tag{2.26}$$

3 Elliptically fibered Calabi-Yau fourfolds

Although the methods to find integral generators of the period lattice are applicable to general Calabi-Yau manifolds we now restrict to elliptic fibrations

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & M \\ & & \big\downarrow \pi \\ & & B \end{array} \tag{3.1}$$

such that for a general choice of complex structure on M the fiber exhibits at most I_1 singularities over loci of codimension 1 in the base B. In particular we require the presence of a section. Fourfolds of this type have been previously studied in [17]. It turns out that the intersection ring and the relevant topological invariants are completely determined by the base. Note that this geometric setup is completely analogous to the threefolds studied in [19, 21].

3.1 Geometry of non-singular elliptic Calabi-Yau fourfolds

As far as it carries over to the fourfold case, we follow the notation in [21] which we now quickly review. The generators of the Mori cone of the base B are given by $\{[\tilde{\mathcal{C}}'^k]\}$, $k = 1, \ldots, h_{11}(B) = h_{11}(M) - 1$ and the dual basis of the Kähler cone is $\{[D'_k]\}$. In particular we assume that the Mori cone is simplicial. Let E be the section so that its divisor class is given by $[E]$.

We now obtain curves

$$\tilde{\mathcal{C}}^k = E \cdot \pi^{-1} \tilde{\mathcal{C}}'^k \,, \; k = 1, \ldots, h_{11}(B) \,, \tag{3.2}$$

on M for some representatives $\tilde{\mathcal{C}}'^k$ of $[\tilde{\mathcal{C}}'^k]$. A basis for the Mori cone on M is given by $\{[\tilde{\mathcal{C}}^k], [\tilde{\mathcal{C}}^e]\}$, where $[\tilde{\mathcal{C}}^e]$ is the class of the generic fiber. The Kähler cone of M is generated by the dual basis $\{[\tilde{D}_e], [\tilde{D}_k]\}$, where

$$[\tilde{D}_k] = \pi^*[D'_k] \,, \quad [\tilde{D}_e] = [E] + \pi^* c_1(B) \,. \tag{3.3}$$

In the following we will mostly drop the square brackets and assume that the distinction between subvarieties and corresponding classes is clear from the context. The intersection ring of M is determined in terms of intersections on B via

$$\begin{aligned} \int_M \tilde{D}_e \cdot P(\tilde{D}_e, \tilde{D}_1, \ldots, \tilde{D}_{h_{11}(B)}) &= \int_B P(c_1(B), D'_1, \ldots, D'_{h_{11}(B)}) \,, \\ \int_M P(1, \tilde{D}_1, \ldots, \tilde{D}_{h_{11}(B)}) &= 0 \,, \end{aligned} \tag{3.4}$$

where P is any polynomial in $h_{11}(B) + 1$ variables.

We denote the complexified areas of the curves in the base by

$$\tilde{T}^k = \int_{\tilde{\mathcal{C}}^k} \mathcal{B} + i\omega \,, \tag{3.5}$$

where ω is the Kähler class and $\mathcal{B}$ is the Neveu-Schwarz $\mathcal{B}$-field. The complexified area of the fiber will be called

$$\tilde{\tau} = \int_{\tilde{\mathcal{C}}^e} \mathcal{B} + i\omega \,. \tag{3.6}$$

The generators of the Mori cone and the dual generators of the Kähler cone provide a natural choice of basis for divisors and curves from the geometric perspective. However, as was already observed for elliptically fibered threefolds, the $\mathrm{SL}(2,\mathbb{Z})$ subgroup of the monodromy acts more naturally in a different choice of basis. We introduce

$$[\mathcal{C}^e] = [\tilde{\mathcal{C}}^e] \,, \quad [\mathcal{C}^k] = [\tilde{\mathcal{C}}^k] + \frac{a^k}{2}[\tilde{\mathcal{C}}^e] \,, \tag{3.7}$$

with

$$a^k = \int_{\tilde{\mathcal{C}}^k} c_1(B) \,, \tag{3.8}$$

and the dual basis

$$D_e = \tilde{D}_e - \frac{1}{2}\pi^* c_1(B) = E + \frac{1}{2}\pi^* c_1(B) \,, \quad D_k = \tilde{D}_k \,. \tag{3.9}$$

The complexified areas corresponding to $\mathcal{C}^k$ and $\mathcal{C}^e$ are now given by

$$\tau = \tilde{\tau} \quad \text{and} \quad T^k = \tilde{T}^k + \frac{a^k}{2}\tilde{\tau}\,, \tag{3.10}$$

respectively. Finally we introduce the exponentiated complexified areas

$$\tilde{Q}^k = \exp(2\pi i \tilde{T}^k)\,, \quad \tilde{q}_e = \exp(2\pi i \tilde{\tau})\,, \tag{3.11}$$

with similar definitions for Q^k and q_e.

We also define the topological invariants of the base

$$a = c_1(B)^3\,, \quad a_i = c_1(B)^2 \cdot D'_i\,, \quad a_{ij} = c_1(B) \cdot D'_i \cdot D'_j\,, \quad c_{ijk} = D'_i \cdot D'_j \cdot D'_k\,, \tag{3.12}$$

and denote the k-th degree component of $\mathrm{ch}(\mathcal{F}^\bullet)$ by $\mathrm{ch}_k(\mathcal{F}^\bullet)$.

The definitions above are straightforward extensions of the corresponding threefold expressions introduced in [21]. For Calabi-Yau fourfolds a basis of middle-dimensional cycles has to be specified as well. It turns out that for elliptically fibered fourfolds with at most I_1 singularities in the fibers such a basis is given by

$$H_k = E \cdot \pi^{-1} D'_k = E \cdot \tilde{D}_k\,, \quad H^k = \pi^{-1}\tilde{\mathcal{C}}'^k\,, \tag{3.13}$$

with

$$H_i \cdot H_j = -a_{ij}\,, \quad H_i \cdot H^j = \delta_i^j\,, \quad H^i \cdot H^j = 0\,. \tag{3.14}$$

We call the 4-cycles $H^k = \pi^{-1}\tilde{\mathcal{C}}'^k$, $k = 1, \ldots, h_{11}(B)$ that result from lifting a curve in the base to a 4-cycle in M the π-vertical 4-cycles. As we will see in section 4.5.1 the genus zero amplitudes that correspond to π–vertical 4-cycles have particularly simple modular properties. Using the Koszul sequence (2.16) we calculate

$$\begin{aligned} \mathrm{ch}(\mathcal{O}_{H_i}) &= H_i - \frac{1}{2}\tilde{\mathcal{C}}^k(c_{kii} - a_{ki}) + \frac{1}{12}V(2a_i - 3a_{ii} + 2c_{iii})\,, \\ \mathrm{ch}(\mathcal{O}_{H^i}) &= H^i - \tilde{\mathcal{C}}^e \cdot h^i\,, \end{aligned} \tag{3.15}$$

with the volume form V and

$$h^i = \int_M E\,\mathrm{ch}_3(\mathcal{O}_{H^i}) = \sum_{a,b} \frac{1}{2}\lambda_{a,b} E \cdot (D_a \cdot D_b) \cdot (D_a + D_b)\,, \tag{3.16}$$

where we assume that

$$H^i = \sum_{a,b} \lambda_{a,b} \bar{D}_a \cdot \bar{D}_b\,, \tag{3.17}$$

for effective divisors $\bar{D}_a$. The Chern characters of the 6-branes are given by

$$\begin{aligned} \mathrm{ch}(\mathcal{O}_{\tilde{D}_i}) &= \tilde{D}_i - \frac{1}{2}H^k c_{kii} + \frac{1}{6}\tilde{\mathcal{C}}^e c_{iii}\,, \\ \mathrm{ch}(\mathcal{O}_E) &= E + \frac{1}{2}H_i \cdot a^i + \frac{1}{6}\tilde{\mathcal{C}}^i a_i + \frac{1}{24}V \cdot a\,. \end{aligned} \tag{3.18}$$

Moreover, $\mathrm{ch}(\mathcal{O}_M) = 1$, $\mathrm{ch}(\tilde{\mathcal{C}}^{e\bullet}) = \tilde{\mathcal{C}}^e$ and $\mathrm{ch}(\tilde{\mathcal{C}}^{k\bullet}) = \tilde{\mathcal{C}}^k$.

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3.2 Fourier-Mukai transforms and the $\mathbf{SL(2,\mathbb{Z})}$ monodromy

The B-model periods are multi-valued and experience monodromies along paths encircling special divisors in the complex structure moduli space. Homological mirror symmetry [30] implies that the corresponding monodromies in the A-model lift to auto-equivalences of the derived category [30, 33, 34]. Furthermore, an important theorem by Orlov states that every equivalence of derived categories of coherent sheaves of smooth projective varieties is a Fourier-Mukai transform.

A Fourier-Mukai transform $\Phi_{\mathcal{E}} : D^b(X) \to D^b(Y)$ is determined by an object $\mathcal{E} \in D^b(X \times Y)$ and acts as [33, 34][3]

$$\mathcal{F}^\bullet \mapsto R\pi_{1*}(\mathcal{E} \otimes_L L\pi_2^* \mathcal{F}^\bullet)\,, \tag{3.19}$$

where π_1 and π_2 are the projections from $X \times Y$ to Y and X respectively. The object $\mathcal{E}$ is called the kernel and R and L indicate that one has to take the left- or right derived functor in place of π_*, π^* or $\otimes$.

For our purpose the nice property of this picture is that certain general monodromies correspond to generic Fourier-Mukai kernels. This allows us to write down closed forms not only for the large complex structure monodromies but also for a certain generic conifold monodromy and a third type that is special to elliptically fibered Calabi-Yau.

Let D be one of the generators of the Kähler cone and C the dual curve. The limit in which C becomes large corresponds to a divisor in the Kähler moduli space. It is well known [33] that the Fourier-Mukai transform corresponding to the monodromy around this large radius divisor acts as

$$\mathcal{E}^\bullet \mapsto \mathcal{O}(D) \otimes \mathcal{E}^\bullet\,. \tag{3.20}$$

We choose a basis of branes

$$\left(\mathcal{O}_M, \mathcal{O}_E, \mathcal{O}_{D_i}, \mathcal{O}_{H_i}, \mathcal{O}_{H^i}, \tilde{\mathcal{C}}^i, \tilde{\mathcal{C}}^e, \mathcal{O}_{\text{pt.}}\right)\,, \tag{3.21}$$

and calculate the monodromy for the large radius divisor corresponding to D_j,

$$\tilde{T}_j = \begin{pmatrix} 1 & 0 & -\delta_j^k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\delta_j^k & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_i^k & 0 & -c_{jik} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_i^k & 0 & -c_{jik} & 0 & \frac{1}{2}(c_{jii}+c_{jji}-a_{ji}) \\ 0 & 0 & 0 & 0 & \delta_k^i & 0 & -\delta_j^i & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_k^i & 0 & -\delta_j^i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\,, \tag{3.22}$$

acting on the vector of charges. One can obtain a similar expression for the monodromy $\tilde{T}_e$, corresponding to $\tilde{D}_e$.

[3] An accessible explanation for physicists of how these calculations are performed can be found in [35].

Another auto-equivalence, the Seidel-Thomas twist, corresponds to the locus where, given a suitable loop based on the point of large radius, the D8-brane becomes massless. Its action on the brane charges is given by

$$Z(\mathcal{E}^\bullet) \mapsto Z(\mathcal{E}^\bullet) - \chi(\mathcal{E}^\bullet, \mathcal{O}_M) Z(\mathcal{O}_M) \,. \tag{3.23}$$

As was explained in [12], for a Calabi-Yau fourfold $\chi(\mathcal{O}_M, \mathcal{O}_M) = 2$. This implies that $Z(\mathcal{O}_M)$ transforms into $-Z(\mathcal{O}_M)$ and this monodromy is of order two.

Elliptically fibered Calabi-Yau manifolds with at most I_1 singularities exhibit yet another type of auto-equivalence. Physically it corresponds to T-duality along both circles of the fiber torus. The corresponding action Φ on the derived category was first studied by Bridgeland [36] in the context of elliptic surfaces. Calculations for Calabi-Yau threefolds can be found in [37] and were elaborated on in the subsequent review [38]. In full generality the auto-equivalences and their implications for the modularity of the amplitudes on elliptic Calabi-Yau threefolds with I_1 singularities [39] have been presented in [40].

We can decompose the Chern character of a general brane $\mathcal{E}^\bullet$ as

$$\begin{aligned}
\mathrm{ch}_0(\mathcal{E}^\bullet) &= n, \\
\mathrm{ch}_1(\mathcal{E}^\bullet) &= n_E\, E + F_1, \\
\mathrm{ch}_2(\mathcal{E}^\bullet) &= E \cdot B_1 + F_2, \\
\mathrm{ch}_3(\mathcal{E}^\bullet) &= E \cdot B_2 + n_e\, \tilde{\mathcal{C}}^e, \\
\mathrm{ch}_4(\mathcal{E}^\bullet) &= s\, V.
\end{aligned} \tag{3.24}$$

Here we introduced $n, n_E, n_e, s \in \mathbb{Q}$, and F_i, B_i are pullbacks of forms in $H^{i,i}(B, \mathbb{C})$. The volume form on M is denoted by V. Adapting the calculation in [38] to Calabi-Yau fourfolds, we find that the Chern character of the transformed brane is given by

$$\begin{aligned}
\mathrm{ch}_0(\Phi(\mathcal{E}^\bullet)) &= n_E \,, \\
\mathrm{ch}_1(\Phi(\mathcal{E}^\bullet)) &= B_1 - \frac{1}{2} n_E\, c_1 - n \cdot E \,, \\
\mathrm{ch}_2(\Phi(\mathcal{E}^\bullet)) &= B_2 - \frac{1}{2} B_1 \cdot c_1 + \frac{1}{12} n_E\, c_1^2 - F_1 \cdot E + \frac{1}{2} n\, c_1 \cdot E \,, \\
\mathrm{ch}_3(\Phi(\mathcal{E}^\bullet)) &= -\frac{1}{2} B_2 \cdot c_1 + \frac{1}{12} B_1 \cdot c_1^2 + s\, \tilde{\mathcal{C}}^e + \frac{1}{2}\, c_1 \cdot F_1 \cdot E - F_2 \cdot E - \frac{1}{6} n\, c_1^2 \cdot E \,, \\
\mathrm{ch}_4(\Phi(\mathcal{E}^\bullet)) &= -n_e\, V - \frac{1}{6}\, c_1^2 \cdot F_1 \cdot E + \frac{1}{2} c_1 \cdot F_2 \cdot E + \frac{1}{24} n\, c_1^3 \cdot E \,,
\end{aligned} \tag{3.25}$$

with $c_1 = \pi^* c_1(B)$. Using the formulae for the Chern characters of the basis of branes introduced above, this translates into the matrix

$$\tilde{S} = \begin{pmatrix} 0 & -1 & 0 & a^k & 0 & \frac{1}{2}\left(c_{kii}a^i - a_k\right) & 0 & \frac{1}{12}(3a_{ii}a^i - 2c_{iii}a^i - a) \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_i^k & 0 & a_{ki} & 0 & -\frac{1}{2}a_{ii} \\ 0 & 0 & \delta_i^k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta_k^i & 0 & h^i + \frac{1}{2}a^i \\ 0 & 0 & 0 & 0 & \delta_k^i & 0 & h^i - \frac{1}{2}a^i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.26}$$

for the corresponding monodromy.

We can now explicitly calculate that

$$\left(\prod_{i=1}^{h_{11}(B)} \tilde{T}_i^{-a^i}\right) \tilde{S} \cdot \tilde{S} = -\mathbb{I}\,, \tag{3.27}$$

and another careful calculation reveals

$$(\tilde{S} \cdot \tilde{T}_e^{-1})^3 = -\mathbb{I}\,. \tag{3.28}$$

It follows that

$$S = \left(\prod_{i=1}^{h^{11}(B)} \tilde{T}_i^{-a^i/2}\right) \tilde{S}\,, \quad T = \left(\prod_{i=1}^{h^{11}(B)} \tilde{T}_i^{a^i/2}\right) \tilde{T}_e^{-1}\,, \tag{3.29}$$

generate a group isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$, the modular group. In particular, Q^k, q are invariant under T, while some of the $\tilde{Q}^k$ obtain a sign under T-transformations if the canonical class of the base is not even. As was already noted by [21], this makes Q^k and q the correct expansion parameters for the topological string amplitudes to exhibit modular properties.

3.3 Toric construction of mirror pairs

To fix conventions we will briefly review the Batyrev construction of Calabi-Yau n-fold mirror pairs (M, W) as hypersurfaces in toric ambient spaces [41].

The data of the mirror pair is encoded in an $n+1$-dimensional reflexive lattice polytope $\Delta \subset \Gamma$ and the choice of a regular star triangulation of Δ and the polar polytope

$$\Delta^* = \{p \in \Gamma^*_{\mathbb{R}} \,|\, \langle q, p\rangle \geq -1, \forall q \in \Delta\}\,, \tag{3.30}$$

that is embedded in the dual lattice Γ^*. We denoted the real extensions of the lattices by $\Gamma_{\mathbb{R}}$ and $\Gamma^*_{\mathbb{R}}$ respectively. The triangulation of Δ^* leads to a fan by taking the cones over the facets that in turn is associated to a toric variety $\mathbb{P}_\Delta$. The family M of Calabi-Yau n-folds is given by the vanishing loci of sections $P_\Delta \in \mathcal{O}(K_{\Delta^*})$

$$P_\Delta = \sum_{\nu\in\Delta\cap\Gamma} \prod_{\nu^*\in\Delta^*\cap\Gamma^*} a_\nu x_{\nu^*}^{\langle\nu,\nu^*\rangle+1} = 0\,. \tag{3.31}$$

The mirror family W is obtained by exchanging $\Delta \leftrightarrow \Delta^*$.

Even for a generic choice of section the Calabi-Yau varieties thus constructed might be singular. For $n \leq 3$ the singularities can be resolved by blowing up the ambient space. This is not always possible for fourfolds. However, all models studied in this paper can be fully resolved by toric divisors.

3.4 Toric geometry of elliptic fibrations

For F-theory we need Calabi-Yau manifolds that are elliptically fibered. One way to construct these is by taking a torically fibered ambient space such that the hypersurface constraint cuts out a genus one curve from the fiber [42]. Toric fibrations can be understood in terms of toric morphisms. A toric morphism $\phi : \mathbb{P}_\Delta \to \mathbb{P}_{\Delta_B}$ in turn is encoded in a lattice morphisms

$$\phi : \Gamma \to \Gamma_B\,, \tag{3.32}$$

such that the image of every cone in Σ is completely contained inside a cone of Σ_B. We obtain a fibration with the fan of the generic fiber given by $\Sigma_F \in \Gamma_F$ if the morphism $\phi : \Gamma \to \Gamma_B$ is surjective and the sequence

$$0 \to \Gamma_F \hookrightarrow \Gamma \xrightarrow{\phi_B} \Gamma_B \to 0\,, \tag{3.33}$$

is exact.

We can now obtain elliptically fibered mirror pairs (M, W) from the following construction [12]. First we combine a base polytope Δ^B and a reflexive fiber polytope Δ^F and embed them into a $n+1$-dimensional polytope Δ as follows:

$$\nu^* \in \Delta^* \left\{ \left| \begin{array}{c:c|c:c} \Delta^{B*} & \begin{matrix}\nu_i^{F*}\\ \vdots \\ \nu_i^{F*}\end{matrix} & s_{ij}\Delta^B & \begin{matrix}\nu_j^{F}\\ \vdots \\ \nu_j^{F}\end{matrix} \\ 0 & \Delta^{F*} & 0 & \Delta^F \end{array} \right| \right\} \nu \in \Delta \tag{3.34}$$

For a fixed $\nu_i^{F*} \in \Delta^{F*}$ and $\nu_j^F \in \Delta^F$ we introduced $s_{ij} = \langle \nu_j^F, \nu_i^{F*}\rangle + 1 \in \mathbb{Z}_{>0}$. This describes a reflexive pair of polytopes (Δ, Δ^*) given by the convex hulls of the points appearing in (3.34). Using the Batyrev construction one gets an n-fold M from the locus given by (3.31) on the ambient space $\mathbb{P}_\Delta$. As mentioned above, M inherits a fibration structure from the ambient spaces $\mathbb{P}_\Delta \to \mathbb{P}_{\Delta^B}$ and we can identify a map

$$M = \{\underline{x} \subset \mathbb{P}_\Delta | P_\Delta(\underline{x}) = 0\} \xrightarrow{\pi} B = \mathbb{P}_{\Delta^B} \tag{3.35}$$

In the following we will consider fibers constructed as E_8 hypersurfaces

$$\mathcal{E}_{E_8} : \quad X_6(1,2,3) = \{(x,y,z) \subset \mathbb{P}^2(1,2,3) \,:\, x^6 + y^3 + z^2 - sxyz = 0\}. \tag{3.36}$$

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One can obtain a fibration using the E_8 fiber and a base B from the following toric data:

$$
\begin{array}{|c|ccccc|cc|}
\text{div.} & & & \bar{\nu}_i^* & & & l^{(e)} & l^{\bullet} \\
K_M & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\
D_1 & & & & -2 & -3 & 0 & * \\
\vdots & & \Delta_B & & \vdots & \vdots & 0 & * \\
D_n & & & & -2 & -3 & 0 & * \\
E & 0 & 0 & 0 & -2 & -3 & 1 & -\sum * \\
2\tilde{D}_e & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
3\tilde{D}_e & 0 & 0 & 0 & 0 & 1 & 3 & 0
\end{array}
\tag{3.37}
$$

In particular, fibrations of this type have a section and at most I_1 singularities in the fiber. We will denote as $M_B^{E_8}$ the elliptically fibered n-fold given by the fibration $\mathcal{E}_{E_8} \hookrightarrow M_B^{E_8} \to B$.

3.5 Picard-Fuchs operators

The periods of the holomorphic n-form on a Calabi-Yau n-fold are annihilated by a set of differential operators, the Picard-Fuchs system. For Calabi-Yau varieties constructed as hypersurfaces in a toric ambient space it is easy to write down differential equations for which the solution set is in general larger than that spanned by the periods. However, in many cases the solution sets are equal and it is sufficient to study the so-called GKZ-system. How to derive the GKZ-system from the toric data and the relation to the Picard-Fuchs system is explained e.g. in [43].

4 Amplitudes, geometric invariants and modular forms

The topological string A-model encodes *Gromov-Witten invariants*, counting holomorphic maps

$$f : \Sigma_{g,\bar{p}} \to M \,, \tag{4.1}$$

from pointed curves $\Sigma_{g,\bar{p}}$ of genus g into M. The general formula for the virtual dimension of the moduli stack of stable maps[4] into a Calabi-Yau M is given by

$$\operatorname{vir\,dim} \bar{M}_{g,n}(M,\beta) = (\dim M - 3)(1-g) + n \,, \tag{4.2}$$

where n is the number of marked points and we require $f_*[\Sigma] = \beta \in H_2(M)$ for $f \in \bar{M}_{g,n}(M,\beta)$ and Σ the domain of f.

While for Calabi-Yau threefolds the virtual dimension is zero at all genera with $n = 0$, in the case of fourfolds it is non-negative only when $g = 0, 1$. A positive virtual dimension

[4] A map is stable if it has at most a finite number of non-trivial automorphisms that preserve marked and nodal points.

can be compensated by intersecting with classes on M pulled back along the evaluation maps

$$\mathrm{ev}_i = f(p_i) : \bar{M}_{g,n} \to M\,, \quad i \in 0, \dots, n\,. \tag{4.3}$$

On the other hand, intersecting with the pull-back of the fundamental class $[M]$ leads to vanishing invariants. The latter property of Gromov-Witten invariants is called the Fundamental class axiom. It follows that for fourfolds the invariants with $g \geq 2$ vanish.

We will now review the Calabi-Yau fourfold invariants for $g = 0, 1$ and how they are encoded in various observables of the topological A-model.

4.1 Review of genus zero invariants

From the general virtual dimension formula we find $\mathrm{vir}\dim \bar{M}_{0,1} = 2$, and given $\gamma \in H^{2,2}(M, \mathbb{Z})$ we obtain well-defined invariants

$$N_{0,\beta}(\gamma) = \int_{\xi} \mathrm{ev}_1^*(\gamma)\,, \tag{4.4}$$

with $\xi = [\bar{M}_{0,1}(M, \beta)]_{\mathrm{virt.}}$. From the topological string theory perspective they are encoded in the instanton part of the normalized double-logarithmic quantum periods

$$F_{\gamma}^{(0)} = \text{classical} + \sum_{\beta \geq 0} N_{0,\beta}(\gamma) q^{\beta}\,. \tag{4.5}$$

In particular, the classical terms corresponding to $F_{\gamma}^{(0)}$ are determined by $Z^{\mathrm{asy}}(\mathcal{O}_{\gamma})$. While the Gromov-Witten invariants are in general rational numbers, they are conjecturally related to integral *instanton numbers* $n_{0,\beta}$ via

$$\sum_{\beta \geq 0} N_{0,\beta}(\gamma) q^{\beta} = \sum_{\beta \geq 0} n_{0,\beta}(\gamma) \sum_{d=1}^{\infty} \frac{q^{d\beta}}{d^2}\,. \tag{4.6}$$

The Gromov-Witten invariants can also be related to *meeting invariants* m_{β_1,β_2} [27], which for $\beta_1, \beta_2 \in H_2(M, \mathbb{Z})$ virtually enumerate rational curves of class β_1 meeting rational curves of class β_2. They are recursively defined via the following rules.

1. The invariants are symmetric,

$$m_{\beta_1,\beta_2} = m_{\beta_2,\beta_1}\,. \tag{4.7}$$

2. If either $\deg(\beta_1) \leq 0$ or $\deg(\beta_2) \leq 0$, then $m_{\beta_1,\beta_2} = 0$.

3. If $\beta_1 \neq \beta_2$, then

$$m_{\beta_1,\beta_2} = \sum_{i,j} n_{0,\beta_1}(\gamma_i) \eta^{(2),ij} n_{0,\beta_2}(\gamma_j) + m_{\beta_1,\beta_2-\beta_1} + m_{\beta_1-\beta_2,\beta_2}\,, \tag{4.8}$$

where $\gamma_i \in H_V^4(M, \mathbb{Z})$ form a basis mod torsion and

$$\eta_{ij}^{(2)} = \int_M \gamma_i \cup \gamma_j\,. \tag{4.9}$$

4. If $\beta_1 = \beta_2 = \beta$, then

$$m_{\beta,\beta} = n_{0,\beta}(c_2(T_M)) + \sum_{i,j} n_{0,\beta}(\gamma_i)\eta^{(2),ij} n_{0,\beta}(\gamma_j) - \sum_{\beta'+\beta''=\beta} m_{\beta',\beta''} \,. \tag{4.10}$$

Genus one invariants for Calabi-Yau fourfolds haven been calculated for example in [27, 28].

4.2 Genus one invariants

At genus one, the virtual dimension vanishes for Calabi-Yau manifolds of any dimension. The corresponding invariants are encoded in the holomorphic limit of the genus one free energy

$$F^{(1)} = \text{classical} + \sum_{\beta \geq 0} N_{1,\beta} q^\beta \,. \tag{4.11}$$

Assuming $h^{2,1} = 0$ it has the general form

$$F^{(1)} = \left(\frac{\chi}{24} - h^{1,1} - 2\right) \log X_0 + \log\det\left(\frac{1}{2\pi i}\frac{\partial z}{\partial t}\right) + \sum_i b_i \log z_i - \frac{1}{24}\log\Delta \,. \tag{4.12}$$

In this expression χ is the Euler characteristic of M, Δ is the discriminant and $z(t)$ is the mirror map in terms of the algebraic coordinates z and the flat coordinates t. The coefficients b_i can be fixed by the limiting behaviour of $F^{(1)}$ in the moduli space.

Assuming that the coordinates z are chosen such that $z_i(t) = t_i + \mathcal{O}(t^2)$, the large radius limit

$$\lim_{t\to\infty} F^{(1)} = -\frac{1}{24}\sum_i \left(\int_M c_3(M) \cup J_i\right) t_i + \text{regular}\,, \tag{4.13}$$

implies

$$b_i = -\frac{1}{24}\int_M c_3(M) \cup J_i - 1 \,. \tag{4.14}$$

At genus one, the conjectured relation of the Gromow-Witten numbers to integral invariants $n_{1,\beta}$ is more involved and has been worked out in [27]. It involves the meeting invariants as well as the genus zero Gromov-Witten invariants and is given by

$$\begin{aligned}\sum_{\beta>0} N_{1,\beta} q^\beta = &\sum_{\beta>0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} \\ &+ \frac{1}{24}\sum_{\beta>0} n_{0,\beta}\left(c_2(T_M)\right)\log(1-q^\beta) \\ &- \frac{1}{24}\sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2}\log(1-q^{\beta_1+\beta_2}) \,.\end{aligned} \tag{4.15}$$

In appendix A.1 we provide genus one invariants of the one parameter fourfold geometries discussed in [12]. In the following, apart from studying the modular properties of the amplitudes, we calculate the integral invariants for E_8 fibrations with bases $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^2$. We provide some of the invariants in appendix A.3. To our knowledge the latter case has not been studied in the literature before and provides further evidence supporting the conjectured relations.

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4.3 Quasi modular forms and holomorphic anomaly equations

In this section we further explore aspects of modularity on elliptically fibered fourfolds (with at most I_1 singular fibers) that have previously been observed by [17]. The latter authors have proven modularity of the 4-point function with all legs in the base and found modular expansions for the genus zero string amplitudes discussed in section 4.1. We aim here to derive corresponding modular anomaly equations. For the K3 case this was done in [44] and for elliptic threefolds in [19, 20]. Our strategy will be the following: we study the generic degree 24 hypersurface X_{24} in $\mathbb{P}(1,1,1,1,8,12)$ that has been used by [17] to illustrate the modular structure and we find the modular anomaly equations. We borrow the differential operators of [39, 45] and find their corresponding version for CY fourfolds by comparing with the observed modular anomaly equations. Then, we conjecture the general form of such differential equations for multiparameter families of elliptically fibered toric CY fourfolds with at most I_1 singularities in the fiber. This leads to a generalized version of the modular anomaly equations that we observed for X_{24}. At the end of the day we provide data for another CY fourfold supporting our conjecture.

For X_{24} it was found in [17] that the instanton parts of the genus zero free energies $F^{(0)}_{\gamma}$ admit an expansion

$$F^{(0),\text{inst}}_{\gamma}(\tau,\underline{\widetilde{T}}) = \sum_{\beta\in H_2(B,\mathbb{Z})} F^{(0)}_{\gamma,\beta}(\tau)\widetilde{Q}^{\beta}, \quad F^{(0)}_{\gamma,\beta} = \left(\frac{q^{\frac{1}{24}}}{\eta}\right)^{12c_1(B)\cdot\beta} P^{(0)}_{\beta}(\gamma)\,, \tag{4.16}$$

where $P^{(0)}_{\beta}(\gamma)$ is a polynomial in the ring of quasimodular forms $\mathbb{C}[E_2,E_4,E_6]$ [46]. Note that an analogous ansatz can be used for the genus one string amplitudes.[5] For the latter case the modular weight of each polynomial coefficient $P^{(1)}_{\beta}$ is given by $w^{(1)}_{\beta} = 6c_1(B)\cdot\beta$.

For the genus zero case we make a special distinction for the two observed kind of amplitudes. Given $\gamma \in H^4_V(M,\mathbb{Z})$ we might have

(a) $F^{(0)}_{\gamma}$ transforming under $\mathrm{SL}(2,\mathbb{Z})$ with pure modular weight -2.

(b) $F^{(0)}_{\gamma}$ transforming under $\mathrm{SL}(2,\mathbb{Z})$ with one component of modular weight -2 and another one of modular weight 0.

It turns out that corresponding 4-cycles can be directly related to the basis introduced above (3.13).

First consider the π–vertical 4-cycles H^i. Note that we can express them as

$$H^i = a^{ij}a^k\widetilde{D}_j\widetilde{D}_k\,, \tag{4.17}$$

which satisfies the intersection relations (3.14), where a^{ij} is the inverse of a_{ij}. Now the asymptotic part of the corresponding genus zero amplitudes follows from the computations of sections 2.3 and 3.1, and reads

$$F^{(0)}_{H^i} = \tau T^i + h^i\tau + \frac{1}{2}a^i + F^{(0),\text{inst}}_{H^i}(q,\underline{Q})\,. \tag{4.18}$$

[5]In this case the entries for $\gamma\in H^4_V(M,\mathbb{Z})$ are ommited.

Notice that the double logarithmic part is proportional to τ and only the i-th (twisted) base Kähler parameter T^i appears in the classical part of the amplitude. The first property is analogous to the behaviour of the periods $\partial_{\widetilde{T}^i}\mathcal{F}^0$ for CY threefolds, where $\mathcal{F}^0$ is the prepotential. The second property can be satisfied by choosing a special basis of the threefold periods. Following the same lines of the analysis that has been carried out for $\partial_{\widetilde{T}^i}\mathcal{F}^0$ in [39, 45], we find that the corresponding polynomials $P^{(0)}_\beta(H^i)$ for $F^{(0)}_{H^i}$ have modular weight $w^{(0)}_\beta(H^i) = 6c_1(B)\cdot\beta - 2$. Hence we expect the full $F^{(0)}_{H^i}$ amplitudes to transform with modular weight -2.

On the other hand, the leading behaviour of the periods over the cycles H_i is of the form

$$F^{(0)}_{H_i} = \frac{c_{ijk}}{2}\widetilde{T}^j\widetilde{T}^k + \frac{1}{2}(c_{iij} - a_{ij})\widetilde{T}^j + s_i + F^{(0),\text{inst}}_{H_i}(q,\underline{Q})\,, \tag{4.19}$$

where the constant s_i can be determined as

$$s_i = \frac{1}{24}\left(\int c_2(B)\cdot D'_i - a_i\right) + \frac{1}{12}(2a_i - 3a_{ii} + 2c_{iii})\,. \tag{4.20}$$

We find that $P^{(0)}_\beta(H_i) \in \widetilde{M}_{6c_1(B)\cdot\beta-2}(\Gamma_1) \oplus \widetilde{M}_{6c_1(B)\cdot\beta}(\Gamma_1)$. Therefore $F^{(0)}_{H_k}$ belongs to the case (b). This can be seen from the factorization of the Yukawa coupling $C_{\widetilde{T}^i\widetilde{T}^j\widetilde{T}^k\widetilde{T}^l}$ in (1.6), which has modular weight -2. In the next section, to illustrate the reason for the different modular behaviour of the $F^{(0)}_{H^k}$ and $F^{(0)}_{H_k}$ amplitudes, we study the fourfold X_{24}.

As a special remark, recall the monodromy transformations $\tilde{T} := \tilde{T}_e^{-1}$ and $\tilde{S}$ introduced in section 3.2. By introducing the factors in (3.29), we find that they generate the modular group. However, T and S do not belong to the monodromy group of the geometry. On the other hand, $\tilde{T}$ and $\tilde{S}$ act on the fiber parameter as the modular group, since

$$\tilde{T}:\tau\mapsto\tau+1\,,\quad \tilde{S}:\tau\mapsto -\frac{1}{\tau}\,,\quad \tilde{S}^2:\tau\mapsto\tau\,,\quad (\tilde{S}\tilde{T})^3:\tau\mapsto\tau. \tag{4.21}$$

We note that the coordinates introduced (3.10) transform under $\tilde{T}$ and $\tilde{S}$ as

$$\tilde{T}: Q^k \mapsto (-1)^{a^k}Q^k\,,\quad \tilde{S}: Q^k\mapsto (-1)^{a^k}Q^k\,. \tag{4.22}$$

This has been explained already in [39, 40] for CY threefolds. We find that the same argument holds for CY fourfolds. On the one hand, $\tilde{T}$ acts on $\tilde{T}^k$ trivially. On the other hand, straightforward calculations show that up to exponentially small terms $\tilde{S}$ acts as $T^k \mapsto T^k + \frac{a^k}{2}$. This leads to (4.22). Moreover, note that the Dedekind eta function transforms as

$$\eta^{12a^k}(\tilde{T}\tau) = (-1)^{a^k}\eta^{12a^k}(\tau)\,,\quad \eta^{12a^k}(\tilde{S}\tau) = (-1)^{a^k}\tau^{6a^k}\eta^{12a^k}(\tau). \tag{4.23}$$

It follows that Q^k and η^{12a^k} are modular objects with the same multiplier system. In particular, we can rewrite the instanton part of the string amplitudes in the coordinates (3.10) as

$$F^{(0),\text{inst}}_\gamma = \sum_{\beta\in H_2(B,\mathbb{Z})} P^{(0)}_\beta(\gamma)\left(\frac{Q^\beta}{\eta^{12c_1(B)\cdot\beta}}\right)\,, \tag{4.24}$$

where each factor in the parenthesis transforms as a modular form of weight -$6c_1(B)\cdot\beta$. A similar expansion can be obtained for the genus one string amplitudes $F^{(1)}$.

4.4 Modularity on the fourfold $X_{24}(1,1,1,1,8,12)$

We now study the fourfold X_{24} which has been introduced in [9]. Its integrality and modular properties have been further discussed in [17, 27]. Following the construction of section (3.4) we pick the polytopes

$$\begin{aligned}\Delta^{*B} &= \mathrm{conv}(\{(-1,0,0),(0,-1,0),(0,0,-1),(1,1,1)\}),\\ \Delta^{*F} &= \mathrm{conv}(\{(-1,0),(0,-1),(2,3)\}).\end{aligned} \tag{4.25}$$

Here Δ^{*B} is the polytope for the base $\mathbb{P}^3$ and Δ^{*F} the fiber polytope for the E_8 fiber with special inner point $\nu_3^{*F} = (2,3)$. We summarize the toric data in the following table which provides the points of the polytope Δ^* of $\mathbb{P}_{\Delta^*}$ together with the corresponding toric divisors $D_{x_i} = \{x_i = 0\}$:

div.	coord.	$\bar{\nu}_i^*$						$l^{(e)}$	$l^{(b)}$
K_M	x_0	1	0	0	0	0	0	−6	0
D_1	x	1	−1	0	0	0	0	2	0
D_2	y	1	0	−1	0	0	0	3	0
E	z	1	2	3	0	0	0	1	−4
L	u_1	1	2	3	1	1	1	0	1
L	u_2	1	2	3	−1	0	0	0	1
L	u_3	1	2	3	0	−1	0	0	1
L	u_4	1	2	3	0	0	−1	0	1

. (4.26)

We use the Sage — Mathematics Software System [47] to calculate toric intersection numbers and the Mori cone. We also provide a worksheet to illustrate the use of Sage for determining the topological invariants and the asymptotic expansions of the integral periods. It can be downloaded from the page [48]. The intersections of the divisors $\widetilde{D}_b = L$ and $\widetilde{D}_e = L + 4E$ determine the constants defined in section 3.1,

$$a = 64, \quad a^b = 4, \quad a_b = 16, \quad a_{bb} = 4, \quad c_{bbb} = 1. \tag{4.27}$$

The Polytope Δ^* describes a degree 24 hypersurface X_{24} given by the locus P_Δ in $\mathbb{P}_{\Delta^*} = \mathbb{P}(1,1,1,1,8,12)$. Let X_{24}^* be the mirror manifold of X_{24} defined by the locus $P_{\Delta^*} = 0$ in $\mathbb{P}_\Delta$, where

$$P_{\Delta^*} = x_0\Big(z^6(\alpha_1 u_1^{24} + \alpha_1 u_2^{24} + \alpha_3 u_3^{24} + \alpha_4 u_4^{24}) + \alpha_0(u_1u_2u_3u_4)xyz + \alpha_6 x^3 + \alpha_7 y^2\Big). \tag{4.28}$$

Here the α_i parametrize the complex structure of X_{24}^*.

By considering the torus action on the homogeneous coordinates of $\mathbb{P}_\Delta$, $x_i \to \lambda_a^{l_i^{(a)}} x_i$, the set of complex structure parameters can be reduced to the local coordinates for $\mathcal{M}_{cs}(W)$ given by

$$z^a = (-1)^{l_0^{(a)}} \prod_{k=1}^{|\Delta^*|} \alpha_k^{l_k^{(a)}}, \quad a = 1, \ldots, h_{21}(W). \tag{4.29}$$

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In particular the large complex structure limit is defined to be the point at $z = 0$, this is the maximal degeneration point which corresponds to a large radius limit for the mirror manifold M [41]. For the case of X^*_{24} we have the following two large complex structure variables

$$z_e = \frac{\alpha_5\alpha_6^2\alpha_7^3}{\alpha_0^6}, \quad z_b = \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{\alpha_5^4}. \tag{4.30}$$

Essential for the B-model description is the nowhere vanishing holomorphic (4,0) form. This can be written as the residuum

$$\Omega(z) = \mathrm{Res}_{P_{\Delta^*}=0}\frac{1}{P_{\Delta^*}(z)}\prod_i \frac{dX_i}{X_i}, \tag{4.31}$$

where X_i are inhomogeneous coordinates on $\mathbb{P}_\Delta$. Using the methods of [43], one can obtain the GKZ differential operators from the Mori cone vectors of M. From the GKZ operators one can extract the Picard-Fuchs operators $\mathcal{L}_i\Pi(z) = 0$. Solving the latter differential equations, we obtain the periods $\Pi_\kappa(z) = \int_{\Gamma^\kappa}\Omega(z)$ in (1.8). In our present case we find the Picard-Fuchs operators

$$\begin{aligned} \mathcal{L}_1 &= \theta_e(\theta_e - 4\theta_b) - 12z_e(6\theta_e - 5)(6\theta_e - 1), \\ \mathcal{L}_2 &= \theta_b^4 - z_b(4\theta_b - \theta_e)(4\theta_b - \theta_e + 1)(4\theta_b - \theta_e + 2)(4\theta_b - \theta_e + 3), \end{aligned} \tag{4.32}$$

where $\theta_a = z^a\partial_{z^a}$. The components of the discriminant of these Picard-Fuchs operators are

$$\begin{aligned} \Delta_1 &= 1 - 256z_b \\ \Delta_2 &= (1 - 432z_e)^4 - z_bz_e^4. \end{aligned} \tag{4.33}$$

Using (3.13), we can determine a basis $\{H^b, H_b\}$ of 4-cycles on X_{24} given by

$$\begin{aligned} H^b &= \widetilde{D}_b^2 \\ H_b &= E\cdot\widetilde{D}_b. \end{aligned} \tag{4.34}$$

For later convenience we introduce a special basis $\{H^b, H_b^\circ\}$ and refer to this as a 'pure modular basis', where H_b° is given by

$$H_b^\circ = H_b + 2H^b. \tag{4.35}$$

The respective genus zero string amplitudes in the basis $\{H^b, H_b\}$ given by (4.18) and (4.19) are

$$\begin{aligned} F^{(0)}_{H^b} &= 2\tau^2 + \tau t + \tau + 2 + F^{(0),\mathrm{inst}}_{H^b}(q,\widetilde{Q}) \\ F^{(0)}_{H_b} &= \frac{1}{2}t^2 - \frac{3}{2}t + \frac{17}{12} + F^{(0),\mathrm{inst}}_{H_b}(q,\widetilde{Q}). \end{aligned} \tag{4.36}$$

Here τ and t are the Kähler moduli corresponding to the flat coordinates, which appear in the leading order of the mirror map of z_e and z_b respectively. For H_b° the associated amplitude is given by $F^{(0)}_{H_b^\circ} = F^{(0)}_{H_b} + 2F^{(0)}_{H^b}$. In [17] it has been observed that $F^{(0)}_{H^b}$ is of modular weight $k_{H^b} = -2$ while $F^{(0)}_{H_b}$ has a component of modular weight 0 and another of weight -2. On the other hand $C_{tttt} = \eta^{\alpha\beta}\partial_t^2 F_\alpha^{(0)}\partial_t^2 F_\beta^{(0)}$ has modular weight $k_{C_{tttt}} = -2$. The intersection matrix of 4-cycles $\eta'^{(2)}$ in the pure modular basis takes the form

$$\begin{array}{cc} & \begin{array}{cc} H^b & H_b \end{array} \\ \eta^{(2)} = & \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \end{array} \begin{array}{c} \\ H^b \\ H_b \end{array} \quad \longrightarrow \quad \begin{array}{cc} & \begin{array}{cc} H^b & H_b^\circ \end{array} \\ \eta'^{(2)} = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ H^b \\ H_b^\circ \end{array} .$$

Then from $C_{tttt} = 2\partial_t^2 F_{H^b}^{(0)}\partial_t^2 F_{H_b^\circ}^{(0)}$ it follows that $F_{H_b^\circ}^{(0)}$ is of modular weight $k_{H_b^\circ} = 0$. Moreover, in [17], there have appeared signs of a modular anomaly equation for $F_{H^b}^{(0)}$. We find that the periods $F_{H^b}^{(0)}$ satisfy the relation

$$\frac{\partial F_{H^b,d}^{(0)}}{\partial E_2} = -\frac{1}{12}\sum_{s=1}^{d-1} s F_{H^b,d-s}^{(0)} F_{H^b,s}^{(0)} . \tag{4.37}$$

For the case of $F_{H_b^\circ}^{(0)}$ we find that it does not follow the relation in (4.37), but another kind of recursive relation given by

$$\frac{\partial F_{H_b^\circ,d}^{(0)}}{\partial E_2} = -\frac{1}{12d}\left(\sum_{s=1}^{d-1} s^2 F_{H_b^\circ,s}^{(0)} F_{H^b,d-s}^{(0)} + F_{H^b,d}^{(0)}\right) . \tag{4.38}$$

On the other hand, the genus one string amplitude can be easily computed from (4.11) and (4.14). For X_{24} this reads

$$F^{(1)} = 968\log X^0 + \log\det\left(\frac{\partial(z_e,z_b)}{\partial(\tau,t)}\right) - \frac{1}{24}\log(\Delta_1\Delta_2) + \frac{959}{24}\log z_e + 39\log z_b . \tag{4.39}$$

We find that $F^{(1)}$ follows an expansion of the form (4.16) with polynomial coefficients $P_d^{(1)}$ of modular weight $w_d^{(1)} = 24d$, i.e. $F^{(1)}$ transforms with modular weight $k_1 = 0$ as expected. Moreover, we observe a recursive relation for $F^{(1)}$ in terms of the amplitude $F_{H^b}^{(0)}$

$$\frac{\partial F_d^{(1)}}{\partial E_2} = -\frac{1}{12}\left(\sum_{s=1}^{d-1} s F_s^{(1)} F_{H^b,d-s}^{(0)} + \left(\frac{5}{2}a_b + d\right) F_{H^b,d}^{(0)}\right) . \tag{4.40}$$

As a special remark, the π-vertical period $F_{H^b}^{(0)}$ in X_{24} closely resembles the quadratic logarithmic solution of the Picard-Fuchs operators in the elliptically fibered Calabi-Yau threefold given by an E_8 fibration over $\mathbb{P}^2$. In the following section we make use of this similarity and extend it to the language of differential operators introduced in [39, 45]. We find that (4.37), (4.38) and (4.40) can be derived from such special differential relations. Then we give a conjectural, generalized version of the modular anomaly equations for fourfolds. In appendix A.4 we provide data supporting the modular anomaly equations (4.37), (4.38) and (4.40).

4.5 Derivation of modular anomaly equations

In this section we use the approach of [39, 40, 45] to derive modular anomaly equations for general, non-singular elliptic Calabi-Yau fourfolds with E_8 fibers as described by the toric data in (3.37). In particular, we find recursive relations satisfied by the periods over π-vertical cycles and the genus one free energies. On the other hand, we argue that the relation (4.38) is special to X_{24} and stems from a holomorphic anomaly equation satisfied by the 4-point couplings that we derive in 4.5.3.

Recall that z_e, z_b are complex structure parameters that can be expressed in terms of the mirror map as

$$z_e = q\Big(1 + \mathcal{O}(q, \widetilde{Q})\Big) \quad \text{and} \quad z_b = \widetilde{Q}\Big(1 + \mathcal{O}(q, \widetilde{Q})\Big)\,. \tag{4.41}$$

When taking derivatives with respect to the Eisenstein series $E_2(q)$, we can keep either z_b fixed or t fixed. In the first case one has to account for the q dependence of z_b. To distinguish between these operations $\mathcal{L}_{E_2(q)}$ is defined in [39, 45] to be the derivative with z_b held constant. A derivative where t is fixed is denoted by $\partial_{E_2(q)}$, i.e.

$$\mathcal{L}_{E_2} f := \partial_{E_2(q)} f(q, z_b), \quad \partial_{E_2} f := \partial_{E_2(q)} f(q, \widetilde{Q})\,. \tag{4.42}$$

One immediately obtains the relations

$$\mathcal{L}_{E_2} z_b = 0, \quad \mathcal{L}_{E_2} \tau = 0\,. \tag{4.43}$$

In [21], the following non-trivial results for the elliptic threefold $X_{18} \to \mathbb{P}^2$ have been derived

$$X_{18}: \quad \mathcal{L}_{E_2} z_e = 0, \quad \mathcal{L}_{E_2} X^0 = 0, \quad \mathcal{L}_{E_2} t = \frac{1}{12}\partial_t \mathcal{F}^{(0),\text{inst}}\,. \tag{4.44}$$

As we noted above, the asymptotic behavior of the periods over π-vertical cycles closely resembles that of the double logarithmic periods for elliptic Calabi-Yau threefolds. Indeed we verified that for $X_{24} \to \mathbb{P}^3$ the relations

$$X_{24}: \quad \mathcal{L}_{E_2} z_e = 0, \quad \mathcal{L}_{E_2} X^0 = 0, \quad \mathcal{L}_{E_2} t = \frac{1}{12} F^{(0),\text{inst}}_{H^b}\,, \tag{4.45}$$

hold. Moreover, for any rational or logarithmic functions $f(z_e, z_b)$ and $g(X^0)$ one finds

$$\mathcal{L}_{E_2} f(z_e, z_b) = 0, \quad \mathcal{L}_{E_2} g(X^0) = 0\,. \tag{4.46}$$

We can relate the two differential operators in (4.42) by making use of (4.45) and the chain rule to obtain

$$\mathcal{L}_{E_2} f = \partial_{E_2} f + \frac{1}{12}(\partial_t f)(F^{(0)}_{H^b})\,. \tag{4.47}$$

Once again we replace $\partial_t \mathcal{F}^0 \leftrightarrow F^{(0)}_{H^b}$ in the analogous threefold relation and find

$$\mathcal{L}_{E_2} F^{(0),\text{inst}}_{H^b} = 0\,. \tag{4.48}$$

Together these relations immediately imply the recursive relation observed in (4.37),

$$\partial_{E_2} F^{(0),\text{inst.}}_{H^b} + \frac{1}{12} F^{(0),\text{inst.}}_{H^b} \partial_t F^{(0),\text{inst.}}_{H^b} = 0\,. \tag{4.49}$$

We are now ready to generalize the discussion,

4.5.1 Modular anomaly equations for periods over π-vertical 4-cycles

We now consider a general non-singular elliptic Calabi-Yau fourfold M with E_8 fiber as described by the toric data in (3.37). The definition of the differential operators (4.42) can be extended to multiparameter families as

$$\mathcal{L}_{E_2(q)} f := \partial_{E_2(q)} f(q, \underline{z}), \quad \partial_{E_2(q)} f(q, \underline{\widetilde{Q}}) . \tag{4.50}$$

Furthermore the relations (4.43) now read $\mathcal{L}_{E_2(q)} z^i = \mathcal{L}_{E_2(q)} \tau = 0$. We conjecture the generalization of (4.45) to be given by

$$\mathcal{L}_{E_2} z_e = \mathcal{L}_{E_2} f(z_e, \underline{z}) = \mathcal{L}_{E_2} g(X^0) = 0, \quad \mathcal{L}_{E_2} t^i = \frac{1}{12} F^{(0),\text{inst}}_{H^i} . \tag{4.51}$$

Note that $F^{(0),\text{inst}}_{H^i}$ on the right hand side of the last equation is singled out as the unique π-vertical period which only involves t^i and τ. Using the chain rule and (4.51), $\mathcal{L}_{E_2}$ can be expressed as

$$\mathcal{L}_{E_2} f = \partial_{E_2} f + \frac{1}{12} (\partial_{t^i} f) F^{(0),\text{inst}}_{H^i} . \tag{4.52}$$

Another useful relation we borrow from [39, 45] by replacing a linear combination of $\partial_{\tilde{T}^i} \mathcal{F}^{(0)}$ that matches the leading asymptotic behaviour of $F^{(0)}_{H^i}$ is

$$\mathcal{L}_{E_2} \partial_{t_i} z^a = -\frac{1}{12} \delta^{i' j'} (\partial_{t_{i'}} z^a)(\partial_{t_{j'}} F^{(0),\text{inst}}_{H^i}) . \tag{4.53}$$

We also assume $\mathcal{L}_{E_2} F^{(0),\text{inst}}_{H^k} = 0$ for the instanton contributions to (4.18). This determines the multiparameter version of the recursive relation (4.37) for the amplitudes $F^{(0)}_{H^k}$ associated to the π-vertical 4-cycles H^k

$$\frac{\partial F^{(0)}_{H^k,\beta}}{\partial E_2} = -\frac{1}{12} \sum_{\beta' + \beta'' = \beta} \beta'_j F^{(0)}_{H^k,\beta'} F^{(0)}_{H^j,\beta''} . \tag{4.54}$$

We provide evidence of this relation for the geometry with base $\mathbb{P}_1 \times \mathbb{P}_2$ in appendix A.5.

4.5.2 Genus one modular anomaly equation

For the same Calabi-Yau fourfold M described in section 4.5.1, we discuss now the modular anomaly equation for the genus one string amplitude. Recall the form of $F^{(1)}$ given in (4.11). Due to (4.50), we find that $\mathcal{L}_{E_2}$ acts non-trivially only on the determinant contribution,

$$\mathcal{L}_{E_2} F^{(1)} = \mathcal{L}_{E_2} \log \left(\det \left(\frac{\partial z^b}{\partial t^a} \right) \right) = \sum_{a,b} (\partial_{z^b} t^a) \mathcal{L}_{E_2} (\partial_{t^a} z^b) = -\frac{1}{12} \delta^{ij} \partial_{t^i} F^{(0),\text{inst}}_{H^j} . \tag{4.55}$$

However, we acted on both the classical and the instanton contributions. Denote the classical part by $P^{(1)}_{\text{class}}(\underline{t}) = \sum_{a=1}^{h_{11}(M)} (b_a + 1) t^a$, which is the linear polynomial appearing in (4.13). This gives a non-trivial contribution when acting with the differential operator $\mathcal{L}_{E_2}$ on $F^{(1)}$

$$\mathcal{L}_{E_2} F^{(1)} = \mathcal{L}_{E_2} P^{(1)}_{\text{class}} + \mathcal{L}_{E_2} F^{(1),\text{inst}} , \tag{4.56}$$

where

$$\mathcal{L}_{E_2} P^{(1)}_{\text{class}} = \sum_{i=1}^{h_{11}(B)} (b_i + 1) \mathcal{L}_{E_2} t^i . \tag{4.57}$$

Using both results (4.55) and (4.56) together with the expressions (4.52) and (4.45), we find the genus one modular anomaly equation for elliptically fibered Calabi-Yau fourfolds following the construction in section 3.4,

$$\frac{\partial F^{(1)}_\beta}{\partial E_2} = -\frac{1}{12} \left(\sum_{\beta' + \beta'' = \beta} \beta'_i F^{(1)}_{\beta'} F^{(0)}_{H^i, \beta''} + \left(\frac{5}{2} a_i + \beta_i \right) F^{(0)}_{H^i, \beta} \right). \tag{4.58}$$

Again we provide the corresponnding data for the case that $B = \mathbb{P}_1 \times \mathbb{P}_2$ in appendix A.5 which provides a non-trivial check.

4.5.3 4-point coupling modular anomaly equation

From the B-model perspective the 4-point couplings C_{pqrs} are rational functions in the complex structure variables z_e, z^i. The A-model 4-point couplings can be expressed in the mirror coordinates $\underline{t}$ and are related to these via

$$C_{abcd}(\underline{t}) = \frac{1}{(X^0)^2} C_{pqrs}(\underline{z}) \frac{\partial z^p(\underline{t})}{\partial t^a} \frac{\partial z^q(\underline{t})}{\partial t^b} \frac{\partial z^r(\underline{t})}{\partial t^c} \frac{\partial z^s(\underline{t})}{\partial t^d} , \quad a, b, c, d = 1, \ldots, h_{11}(M) . \tag{4.59}$$

As we reviewed in the introduction, the 4-point coupling can be factorized in terms of the 3-point couplings C^γ_{ab}. On the A-side the latter are derivatives of the string amplitudes $C^\gamma_{ab} = \partial_{t^a} \partial_{t^b} F^{(0)}_\gamma$. The factorization of the 4-point function is given by

$$C_{abcd}(\underline{t}) = \partial_{t^a} \partial_{t^b} F^{(0)}_\gamma(\underline{t}) \eta^{(2),\gamma\delta} \partial_{t^c} \partial_{t^d} F^{(0)}_\delta(\underline{t}) . \tag{4.60}$$

Now we act with $\mathcal{L}_{E_2}$ on the A-model 4-point coupling with all legs in the base, i.e. $C_{ijkl}(\tau, \underline{\tilde{T}})$, with $i, j, k, l = 1, \ldots h_{11}(B)$. This leads to the relation

$$\begin{aligned} \mathcal{L}_{E_2} C_{ijkl} = -\frac{1}{12} \delta^{i'}_{j'} \Big(& C_{i'jkl} \partial_{t_i} F^{(0),\text{inst}}_{H^{j'}} + C_{ii'kl} \partial_{t_j} F^{(0),\text{inst}}_{H^{j'}} \\ & + C_{iji'l} \partial_{t_k} F^{(0),\text{inst}}_{H^{j'}} + C_{ijki'} \partial_{t_l} F^{(0),\text{inst}}_{H^{j'}} \Big), \end{aligned} \tag{4.61}$$

where we have used (4.53). We can now insert (4.52) to get a recursive relation of C_{ijkl} with respect to the Eisenstein series E_2, i.e. a modular anomaly equation for the 4-point coupling.

As an example we go back to the E_8 fibration over $\mathbb{P}^3$. We apply the modular anomaly equation (4.61) to $C_{tttt} \equiv C^{(4)}_b$, which reduces to

$$\frac{\partial}{\partial E_2} C^{(4)}_b = -\frac{1}{12} \left[\left(\partial_t C^{(4)}_b \right) F^{(0),\text{inst}}_{H^b} + 4 C^{(4)}_b \left(\partial_t F^{(0),\text{inst}}_{H^b} \right) \right] . \tag{4.62}$$

It turns out that this implies the recursive relation (4.38). To see this we insert the factorization of $C^{(4)}_b$ given in (4.60). We choose the basis $\{H^b, H^\circ_b\}$ introduced in (4.35). In such a basis the equation we found in (4.62) can be brought into the form

$$\frac{\partial}{\partial E_2} \left(\partial^2_t F^{(0),\text{inst}}_{H^\circ_b} \right) \partial^2_t F^{0,\text{inst}}_{H^b} = -\frac{1}{12} \left(\partial_t F^{(0),\text{inst}}_{H^b} + \partial_t \left(F^{(0),\text{inst}}_{H^b} \cdot \partial^2_t F^{(0),\text{inst}}_{H^\circ_b} \right) \right) \partial^2_t F^{(0),\text{inst}}_{H^b} . \tag{4.63}$$

We can now cancel $\partial_t^2 F_{H^b}^{(0),\text{inst}}$ on both sides of the equation and integrate with respect to t. The result is the modular anomaly equation (4.38) satisfied by $F_{H_b^\circ}^{(0)}$

$$\frac{\partial}{\partial E_2}\partial_t F_{H_b^\circ}^{(0),\text{inst}} = -\frac{1}{12}\Big(F_{H^b}^{(0),\text{inst}}\partial_t^2 F_{H_b^\circ}^{(0),\text{inst}} + F_{H^b}^{(0),\text{inst}}\Big). \tag{4.64}$$

It is immediately clear that this does not generalize to multiparameter families and periods $F_{H_i}^{(0)}$ (4.19). We can always obtain a basis $\{H^i, H_i^\circ\}$ such that $F_{H_i^\circ}^{(0)}$ has modular weight 0. This basis only has to satisfy that $\eta^{(2)}$ is anti-block-diagonal. Then the 4-point coupling with all legs in the base is given by

$$C_{ijkl} = 2\sum_{m=1}^{h_{11}(B)} \partial_{t_i}\partial_{t_j} F_{H^m}^{(0),\text{inst}}\left(c_{ikl} + \partial_{t_k}\partial_{t_l} F_{H_m^\circ}^{(0),\text{inst}}\right), \quad i,j,k,l = 1,\ldots h_{11}(B)\,. \tag{4.65}$$

Acting with $\mathcal{L}_{E_2}$ leads to the relation

$$\mathcal{L}_{E_2} C_{ijkl} = \cdots + 2\sum_{m=1}^{h_{11}(B)} (\partial_{t_i}\partial_{t_j} F_{H^m}^{(0),\text{inst}})\partial_{E_2}\Big(\partial_{t_k}\partial_{t_l} F_{H_m^\circ}^{(0),\text{inst}}\Big), \tag{4.66}$$

which cannot be factorized as was possible in the case of X_{24}.

5 Horizontal flux vacua for X_{24}^*

We will now use the integral period basis for the mirror X_{24}^* of $X_{24}(1,1,1,1,8,12)$ to study the admissible horizontal fluxes and the corresponding vacua. To this end we analytically continue the basis to various special loci in the complex structure moduli space. Note that the structure of the moduli space is similar to that of the mirror of the threefold $X_{18}(1,1,1,6,9)$ which has been studied in [49].

Recall the defining equation of X_{24}^*,

$$z_b\, u_1^{24} + u_2^{24} + u_3^{24} + u_4^{24} + (u_1u_2u_3u_4)^6 + u_1u_2u_3u_4xy + z_e^{\frac{1}{2}}\, x^3 + y^2 = 0\,. \tag{5.1}$$

The two components of the discriminant are given by the vanishing loci of

$$\Delta_1 = 1 - 2^8\cdot z_b\,, \quad \Delta_2 = 2^{24}3^{12}\cdot z_e^4 z_b - \left(1 - 2^4 3^3\cdot z_e\right)^4\,. \tag{5.2}$$

First we introduce a new set of complex structure variables by rescaling the homogeneous coordinates on $\mathbb{P}(1,1,1,1,8,12)$. The defining equation (5.1) becomes

$$u_1^{24} + u_2^{24} + u_3^{24} + u_4^{24} + 4\,\phi\,(u_1u_2u_3u_4)^6 + 2\sqrt{3}\,\psi\, u_1u_2u_3u_4xy + x^3 + y^2 = 0\,, \tag{5.3}$$

and the new complex structure variables ϕ, ψ are related to z_e, z_b via

$$z_b = \frac{1}{256}\frac{1}{\phi^4}\,, \quad z_e = \frac{1}{432}\frac{\phi}{\psi^6}\,. \tag{5.4}$$

In these variables the components of the conifold become

$$\Delta_1' = (\phi-1)(\phi+1)(1+\phi^2)\,, \quad \Delta_2' = (\phi'-1)(\phi'+1)(1+\phi'^2)\,, \tag{5.5}$$

where we introduced $\phi' = \phi - \psi^6$.

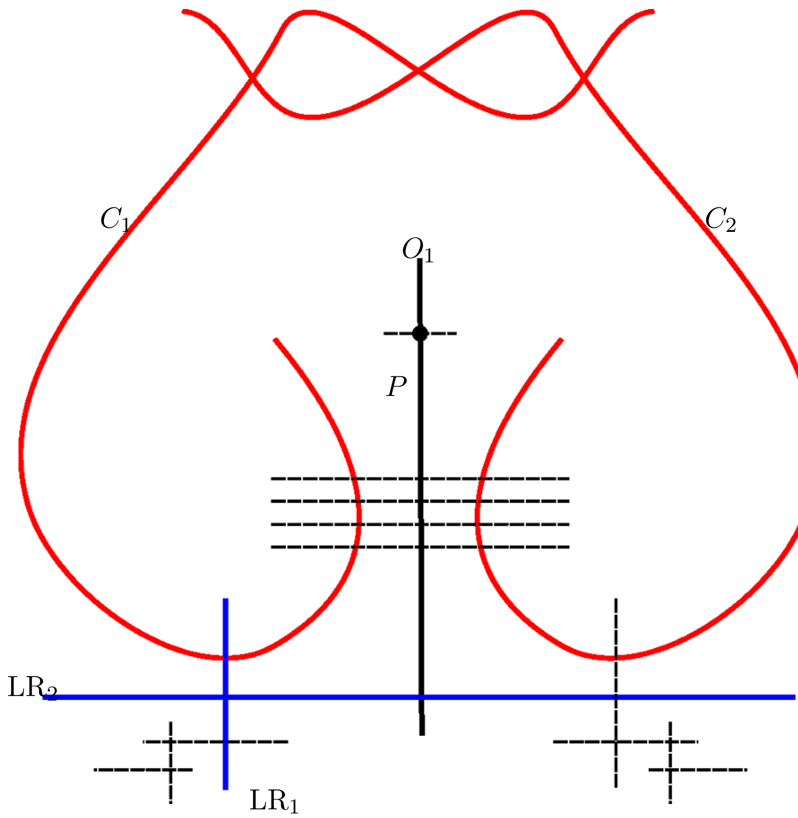

Figure 1. Schematic structure of the resolved complex structure moduli space of X^*_{24}. The large complex structure divisors are shown in blue and the conifold components are red. Exceptional divisors resolving non-normal crossing intersections are indicated with dashed lines.

The general structure of the moduli space is sketched in figure 1. Note that z_e and z_b are the Batyrev variables and the large complex structure divisors LR_1 and LR_2 correspond to $z_e = 0$ and $z_b = 0$ respectively. On the other hand, using ϕ and ψ as variables, both $\Delta'_1 = 0$ and $\Delta'_2 = 0$ have a forth-order tangency with LR_2. Only after resolving $\mathrm{LR}_2 \cap \{\Delta'_1 = 0\}$ we get LR_1 as one of the exceptional divisors. This is reflected in the fact that the point $\{z_e = 0\} \cap \{z_b = 0\}$ corresponds to a double-scaling limit in ϕ and ψ. The two divisors that correspond to the components of the conifold are labelled with C_1 and C_2 respectively. Furthermore, we will analyze solutions around the orbifold divisor O_1 that is given by $\psi = 0$.

Finally note that Δ_1 and Δ_2 as well as LR_1 and LR_2 are exchanged under the involution

$$z_e = 2^{-4}3^{-3} - z'_e\,, \quad z_b = \left(\frac{2^4 3^3 z'_e}{1 - 2^4 3^3 z'_e}\right)^4 z'_b\,. \tag{5.6}$$

Physically this involution can be seen as the result of T-dualizing along both cycles of the fiber and the corresponding transformation of the A-brane charges is given by $\tilde{S}$, (5.18).

5.1 Conifold C_1

First we study the possible fluxes around $C_1 \cap \mathrm{LR}_1$. To this end we choose local coordinates

$$c_1 = z_b + \frac{1}{256}\,, \tag{5.7}$$

and z_e. We transform and solve the Picard-Fuchs equations to obtain a vector of eight solutions with asymptotic behaviour given by

$$\Pi_c = \left(1,\, c_1,\, z_e,\, \log\left(z_e\right), \log^2\left(z_e\right),\, \log^3\left(z_e\right),\, \log^4\left(z_e\right),\, c_1^{3/2}\right) + \mathcal{O}(c^2, z^2)\,. \tag{5.8}$$

We demand that the leading monomial of each period is absent from the other solutions to specify the vector uniquely.

This is related to the integral basis at large complex structure via

$$\Pi_{LR} = T_c \cdot \Pi_c\,. \tag{5.9}$$

The matrix T_c can be obtained by numerical analytic continuation and is given by

$$\begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \frac{54\pi^6 r_4^2-91}{24\pi^2} & r_4 & \frac{1}{6\pi^4} & 0 \\ f_{2,1} & \left(1+i\sqrt{2}\right) r_3 & f_{2,3} & 0 & 0 & 0 & 0 & \frac{10240 i\sqrt{2}}{3\pi^2} \\ f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \frac{1-6i\pi^3 r_4}{4\pi^2} & -\frac{i}{3\pi^3} & 0 & 0 \\ r_1+ir_2+1 & \frac{1}{4}\left(2+3i\sqrt{2}\right) r_3 & f_{4,3} & 0 & 0 & 0 & 0 & \frac{2048 i\sqrt{2}}{\pi^2} \\ f_{5,1} & f_{5,2} & f_{5,3} & -\frac{3\pi^3 r_4+i}{2\pi} & -\frac{1}{2\pi^2} & 0 & 0 & \frac{1024 i\sqrt{2}}{3\pi^2} \\ \frac{2ir_2}{3} & \frac{ir_3}{\sqrt{2}} & f_{6,3} & 0 & 0 & 0 & 0 & \frac{4096 i\sqrt{2}}{3\pi^2} \\ f_{7,1} & -\frac{i\left(\sqrt{2}\pi r_3-256\right)}{8\pi} & f_{7,3} & \frac{i}{2\pi} & 0 & 0 & 0 & -\frac{1024 i\sqrt{2}}{3\pi^2} \\ 1 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the algebraic constraint

$$\int \Omega \wedge \Omega = 0 \quad \Leftrightarrow \quad \Pi_c^T T_c^T \eta^{-1} T_c \Pi_c = 0\,, \tag{5.10}$$

and the integral monodromies corresponding to LR_1 and C_1[6] we reduced the numerical uncertainty to five real values r_i, $i = 1, \ldots, 5$. Due to the size of the expressions we relegated the elements $f_{*,*}$ and the numerical values into appendix A.5.

To further simplify the analysis we will move away from $z_e = 0$ and introduce

$$c_2 = z_e - \frac{1}{1728}\,. \tag{5.11}$$

[6]Since we performed the analytic continuation to very high precision, the integral monodromy matrix corresponding to $c_1 \to e^{2\pi i} c_1$ is essentially determined by the numerical value.

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The corresponding vector of solutions is given by

$$\Pi_{c'} = \begin{pmatrix} 1 - 3840c_1c_2 + 430080c_1^2c_2 \\ c_1 - 1920c_1^2c_2 \\ c_1^2 \\ c_1^3 \\ c_2 + 32c_1c_2 - 29568c_2^2c_1 - \frac{13216}{3}c_1^2c_2 \\ c_2^2 + \frac{1}{18}c_1c_2 + 64c_2^2c_1 - \frac{40}{9}c_1^2c_2 \\ c_2^3 + \frac{1}{12}c_2^2c_1 + \frac{1}{432}c_1^2c_2 \\ c_1^{3/2} - \frac{2024}{9}c_1^{5/2} \end{pmatrix} + \mathcal{O}(c^4)\,. \tag{5.12}$$

This is related to the integral basis at large complex structure via

$$\Pi_{LR} = T_c \cdot T_{c'} \cdot \Pi_{c'}\,. \tag{5.13}$$

The numerical value of $T_{c'}$ as well as those of the other continuation matrices in this section are provided in a Mathematica worksheet that can be downloaded from [48].

We now obtain the monodromy action

$$M_c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9 & 0 & 20 & 0 & -10 & 0 & -10 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 13 & 0 & -6 & 0 & -6 \\ 0 & -1 & 0 & 2 & 1 & -1 & 0 & -1 \\ 0 & -4 & 0 & 8 & 0 & -3 & 0 & -4 \\ 0 & 1 & 0 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.14}$$

on Π_{LR} when transported along a lasso wrapping C_1. Using the algebraic constraints

$$\int \Omega \wedge \Omega = 0\,, \quad \int \Omega \wedge \partial_{c_1}\Omega = 0\,, \quad \int \Omega \wedge \partial_{c_2}\Omega = 0\,, \tag{5.15}$$

we find the analytic expression for $(T_cT_{c'})^T\eta^{-1}T_cT_{c'}$. Unfortunately we are unable to solve the resulting equation for $T_{c'}$.

However, note that

$$M_c = \mathbb{I} - \vec{v}\cdot\vec{v}^T\cdot\eta^{-1}\,, \tag{5.16}$$

where $\vec{v} = \pm\,(0,\,10,\,0,\,6,\,1,\,4,\,-1,\,0)$. In other words, the monodromy M_c corresponds to a Seidel-Thomas twist, where the charge of the shrinking brane is given by

$$\pi_c = \vec{v}\eta^{-1}T_cT_{c'}\Pi_{c'} = \frac{2048\sqrt{2}}{3\pi^2}\left(c_1^{\frac{3}{2}} - \frac{2024}{9}c_1^{\frac{5}{2}} + \mathcal{O}(c^4)\right)\,. \tag{5.17}$$

Let us insert the topological invariants (3.12) into (3.26) to obtain the action of the Bridgeland type involution on Π_{LR},

$$\tilde{S} = \begin{pmatrix} 0 & -1 & 0 & 4 & 0 & -6 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 4 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{5.18}$$

Then we observe

$$\vec{v} \cdot \eta^{-1} \cdot \tilde{S} \cdot \tilde{T}_2^{-2} = (1, 0, 0, 0, 0, 0, 0, 0) , \tag{5.19}$$

where $\tilde{T}_2$ is the monodromy corresponding to LR_2. The involution exchanges C_1 and C_2 and transforms the brane vanishing at C_1, up to large complex structure monodromies, into a D_8-brane.

To obtain a vanishing superpotential at $c_1 = 0$, we can turn on $n \in \mathbb{Z}$ units of flux along the cycle with period π_c. For this to be a supersymmetric minimum we also have to check that $D_i W = 0$. In flat coordinates t_c^i this condition reads

$$(\partial_i + K_i) W = 0 , \tag{5.20}$$

where $K_i = \partial_i K$ and K is the Kähler potential

$$e^{-K} = \int \bar{\Omega} \wedge \Omega = \Pi_{LR}^{\dagger} \eta^{-1} \Pi_{LR} . \tag{5.21}$$

As flat coordinates we can use the normalized periods

$$\begin{aligned} t_c^1 &= \frac{\Pi_{c',2}}{\Pi_{c',1}} = c_1 + 1920 c_1^2 c_2 + \mathcal{O}(c^4) , \\ t_c^2 &= \frac{\Pi_{c',5}}{\Pi_{c',1}} = c_2 + 32 c_1 c_2 - \frac{13216}{3} c_1^2 c_2 - 25728 c_1 c_2^2 + \mathcal{O}(c^4) . \end{aligned} \tag{5.22}$$

In terms of these, the vanishing period reads

$$\pi_c = \frac{2048\sqrt{2}}{3\pi^2} \left[(t_c^1)^{\frac{3}{2}} - \frac{2024}{9} (t_c^1)^{\frac{5}{2}} + \mathcal{O}(t_c^4) \right] . \tag{5.23}$$

Using the numerical result for $T_c \cdot T_{c'}$ we find that $\partial_i K$ are regular at $c_1 = 0$ and therefore $D_i \pi_c \sim (t_c^1)^{i-1/2}$.

The scalar potential is given by

$$v = e^K \left[(D_i W)(D_{\bar{j}} \bar{W}) G^{i\bar{j}} - 3 W \bar{W} \right] , \tag{5.24}$$

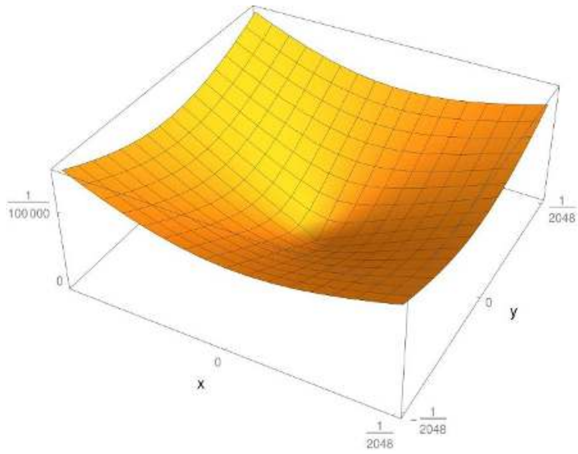

Figure 2. The scalar potential generated by aligned flux, depending on the distance to the conifold C_1 in flat coordinates $t_c^1 = x + Iy$, $t_c^2 = 0$.

where $G^{i\bar{j}}$ is the inverse of the metric $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. We restrict to $t_c^2 = 0$ and introduce $\mathrm{Re}(t_c^2) = x$, $\mathrm{Im}(t_c^2) = y$. Then the leading terms of the scalar potential are

$$v = 0.020174\sqrt{x^2+y^2} + 0.31715x^2 + 0.31715y^2 - 2.8019x\sqrt{x^2+y^2} + \mathcal{O}(x^3, y^3)\,. \quad (5.25)$$

A plot is shown in figure 2. We checked that this is the dominant contribution at least up to order seven, where we calculated the coefficients to a precision of twenty digits. Deep inside the radius of convergence $|t_c^1| \approx |c_1| < 1/256$ the potential is well approximated by the leading order $v \approx 0.020174 \cdot |c_1|$. Our findings are in agreement with [12] where it was argued that for Calabi-Yau fourfolds the Conifold is generically stabilized by aligned flux.

5.2 Orbifold O_1

To expand around $O_1 \cap \mathrm{LR}_2$ we use the variables z_b and

$$o_1 = \frac{1}{z_b^6}\,. \quad (5.26)$$

We find a vector of solutions to the transformed Picard-Fuchs system with leading terms

$$\begin{aligned}\Pi_o = \big(&o_1^5,\, o_1^5 \log(z_b)\,,\, o_1^5 \log^2(z_b)\,,\, o_1^5 \log^3(z_b)\,,\\ &o_1,\, o_1 \log(z_b)\,,\, o_1 \log^2(z_b)\,,\, o_1 \log^3(z_b)\big) + \mathcal{O}(o_1^7, z)\,.\end{aligned} \quad (5.27)$$

It is related to the integral basis at large complex structure via

$$\Pi_{\mathrm{LR}} = T_o \cdot \Pi_o\,. \quad (5.28)$$

However, in contrast to the analytic continuation matrix to the conifold, T_o can be determined exactly with the help of the Barnes integral method. The latter has been discussed for one-parameter models in [50] and can be adapted to this two-parameter model. We give the analytic expression in the Mathematica worksheet that can be found online [48].

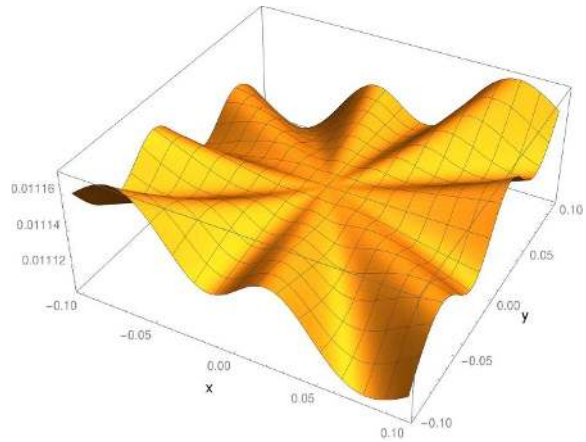

Figure 3. The scalar potential generated by a generic choice of flux, depending on the distance to the orbifold O_1 in coordinates $o_1 = x + Iy$, $o_2 = 0$.

The monodromy acting on Π_{LR} when transported along a lasso wrapping O_1 is of order six and given by

$$M_o = \begin{pmatrix} 1 & 1 & 0 & -4 & 0 & 6 & 0 & 2 \\ -1 & 0 & -4 & 0 & -10 & 0 & -20 & 0 \\ 0 & 0 & 1 & 1 & 0 & -4 & 0 & 2 \\ 0 & 0 & -1 & 0 & -4 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} . \tag{5.29}$$

To analyze the possible fluxes we will again move away from the large complex structure divisor and introduce the variable

$$o_2 = z_b - \frac{1}{512} . \tag{5.30}$$

Solutions in the new variables are

$$\Pi_{o'} = \left(o_1,\, o_1^7,\, o_1^{12},\, o_1^{19},\, o_1^5,\, o_1^{11},\, o_1^{17},\, o_1^{23}\right) + \mathcal{O}(o_1^2, o_2)\,, \tag{5.31}$$

We demand that the leading monomial of each period is absent from the other solutions to specify the vector uniquely. It is related to the previous basis via

$$\Pi_o = T_{o'} \cdot \Pi_{o'}\,, \tag{5.32}$$

where the numerical expression for $T_{o'}$ has been calculated with a precision of around fifty digits.

From the solution vector it follows that every choice of flux leads to a vanishing superpotential at $o_1 = 0$. Moreover, our numerical analysis shows that $D_{\sigma_i} W = 0$, $i = 1, 2$ is generically satisfied at O_1. If one chooses the flux superpotential

$$
\begin{aligned}
W = T_{o'}^{-1} T_o^{-1} \Pi_{\mathrm{LR},0} = &-(0.237201 \; - 0.907908i) o_1 \\
&+ (97.5605 \; - 9.49343i) o_1 o_2 - (24181.7 + 1211.32i) o_1 o_2^2 + \mathcal{O}(o^4)\,,
\end{aligned}
\tag{5.33}
$$

this leads to the scalar potential

$$
v = 0.0111391615585497874 39 + \mathcal{O}(x^2, y^2)\,. \tag{5.34}
$$

in terms of $o_1 = x + Iy$ at $o_2 = 0$. A plot of the potential, expanded to order eleven, is shown in figure 3. Note that the radius of convergence is $o_1 < 216 \cdot (2 - 2^{3/4}) \approx 69$.

We did a Monte Carlo scan over non-vanishing flux vectors and found that the scalar potential was always positive at $x = y = 0$. Moreover, the behaviour close to the origin was qualitatively the same in that the gradient vanished at $x = y = 0$ but the Hessian was undefined.

We also performed an analytic continuation to the special locus P where the Calabi-Yau becomes a Gepner model. However, the behaviour of the scalar potential was qualitatively the same as for a generic point on $\mathcal{O}_1$. For some recent discussion of moduli stabilization at the point of large complex structure in a particular example see also [51].

6 Conclusions and outlook

We described a very efficient method to obtain the integral flux superpotential using the central charge formula defined in terms of the $\hat{\Gamma}$ class . This method is simple enough to be applied to multi moduli cases. In particular if the Calabi-Yau fourfold is embedded in a toric ambient space it is in general straightforward to find a basis by toric intersection calculus and the Frobenius method for constructing the periods at the points of maximal unipotent monodromy. Example calculations in Sage can be found on our homepage [48].

We then restrict to non-singular elliptic Calabi-Yau fourfolds and study universal monodromies in the integral basis of the horizontal cohomology and the dual homology. Using this basis we provide general expressions for the monodromies corresponding to T_i-shifts, that act as $t_i \to t_i + 1$ on the Kähler moduli. In the derived category these correspond to the auto-equivalences induced by tensoring with the line bundles of the dual divisor. Physically this is the integral Neveu-Schwarz B-field shift and the action on the periods follows directly from their leading logarithms which are determined again by A-model intersection numbers. In particular, the T_e-shift acts as the parabolic operator T in $\mathrm{SL}(2, \mathbb{Z})$ on the fiber parameter.

More non-trivially we extend Bridgelands construction of an auto-equivalence of elliptic surfaces to the class of elliptic Calabi-Yau fourfolds with at most I_1 singularities in the fibers. This provides an action of the order two element S in $\mathrm{SL}(2, \mathbb{Z})$ on the fiber parameter. Apart from being a non-trivial check of the integrality of our periods, these auto-equivalences generate the full $\mathrm{PSL}(2,\mathbb{Z})$ action on the elliptic parameter. This gives rise to modular properties of the genus zero and holomorphic genus one amplitudes as well as a holomorphic anomaly that we analyze in detail.

Let us summarize the types of the amplitudes and the results. The virtual dimension formula (4.2) is positive for genus zero. Therefore we need a meeting condition for rational curves with $\gamma \in H^4(M,\mathbb{Z})$ (mod torsion) and get different amplitudes $F^{(0)}_\gamma(q)$ for each γ, whose geometry with respect to the fibration structure plays an important role. In genus one the virtual dimension is zero and we get a universal amplitude $F^{(1)}$. For $g > 1$ the dimension is negative and hence all higher genus amplitudes vanish. Finally one can also consider the modular properties of the 4-point functions. The clearest situation arises for the genus zero amplitudes associated to π-vertical 4-cycles H^k and for the genus one amplitude as well as for the 4-point functions with all legs in the base. In each case we get a complete and universal answer for the holomorphic anomaly equations which can be derived using the methods in [21, 39].

For genus zero amplitudes over 4-cycles that are not π-vertical we observe a modular anomaly equation only for the E_8 fibration over $\mathbb{P}^3$. However, we argue that this is a consequence of the modular anomaly equation of the 4-point function which factorizes for two-parameter families. We also check the integrality of the curve counting invariants of [27] at genus one for various new cases.

In order to study the global properties of the horizontal flux superpotential relevant for F-theory compactifications, we analytically continued the periods of the mirror X^*_{24} to the following critical divisors displayed in figure 1, whose symmetry implies that we only need to consider the left half of it. We first studied the conifold divisor C_1. Here we could determine an analytic expression for the 8×8 continuation matrix T_c in (5.9) up to five numerical coefficients.[7] We also generalized the result of [12] that flux along the vanishing cycle stabilizes the theory at this divisor. We further analyzed the possible flux superpotentials at the generic orbifold divisor O_1 and its special locus P.

The most obvious generalization of this work is to include singular elliptic fibrations. Our formalism for fixing the integral periods explained in section 2.3 will work essentially unchanged and with the same technical tools as long as we have Calabi-Yau spaces embedded into toric varieties and the resolutions of the singularities can be described torically. This will be essential to probe in a quantitative way the flux stabilization mechanism of realistic F-theory vacua. The generalization of the construction of the Bridgeland auto-equivalence should also be possible in principle. In fact at least in the Calabi-Yau threefold case the results for the all genus amplitudes which can be expressed in terms of Weyl-invariant Jacobi-Forms [52, 53] indicate that the affine Weyl-group of the singularity will appear as part of the auto-equivalences of the derived category of the A-model.

[7]Further details about this highly non-trivial analytic continuation can be found at [48].

$M/n_{0,d}(J^2)$	$d=1$	2	3	4
$X_6(1^6)$	60480	440884080	6255156277440	117715791990353760
$X_{10}(1^5,5)$	1582400	791944986400	783617464399966400	031333248042176116592000
$X_{3,4}(1^7)$	16128	17510976	36449586432	100346754888576
$X_{2,5}(1^7)$	24500	48263250	181688069500	905026660335000
$X_{4,4}(1^6,2)$	27904	71161472	354153540352	2336902632563200
$X_{2,2,4}(1^8)$	11776	7677952	9408504320	15215566524416
$X_{2,3,3}(1^8)$	9396	4347594	3794687028	4368985908840
$X_{2,2,2,4}(1^9)$	6912	1919808	988602624	669909315456
$X_{2,2,2,2,2}(1^{10})$	5120	852480	259476480	103646279680

Table 1. Genus 0 invariants in $F^{(0)}_{J^2}$ for nine hypergeometric one parameter CY fourfold geometries.

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A Supplementary data

A.1 Curve counting invariants for one parameter fourfolds

Here we report the genus zero and genus one curve counting invariants for nine one parameter fourfolds in toric ambient spaces with generalized hypergeometric type Picard-Fuchs equations. The genus zero invariants agree with the ones calculated in [12]. The genus one invariants provide a new test for the multi covering formula derived in [27]. Similar checks for one parameter Calabi-Yau spaces in Grassmannian ambient spaces with Apery type Picard-Fuchs operators were provided in [28].

$M/n_{1,d}$	$d=1$	2	3	4	5
$X_6(1^6)$	0	0	2734099200	387176346729900	26873294164654597632
$X_{10}(1^5,5)$	0	30044000	3559247945776000	22569533194514770326000	88310003296637165555077889280
$X_{3,4}(1^7)$	0	0	2813440	81906297984	1006848150400512
$X_{2,5}(1^7)$	0	0	9058000	845495712250	20201716419250520
$X_{4,4}(1^6,2)$	0	1280	146150912	5670808217856	132534541018149888
$X_{2,2,4}(1^8)$	0	0	47104	4277292544	42843921424384
$X_{2,3,3}(1^8)$	0	0	53928	1203128235	7776816583356
$X_{2,2,2,3}(1^9)$	0	0	1024	65526084	338199639552
$X_{2,2,2,2,2}(1^{10})$	0	0	3779200	15090827264	27474707200000

Table 2. Genus 1 invariants for several one parameter CY fourfold geometries.

A.2 Toric data for $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$

Here we consider the hypersurface $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$. This arises from an E_8 fibration over the base $B=\mathbb{P}^1\times\mathbb{P}^2$. The base polytope Δ^{*B} of $\mathbb{P}^1\times\mathbb{P}^2$ is given by

$$\begin{array}{|c|rrr|rr|}\hline \text{div.} & & \bar\nu_i^{*B} & & l'^{(1)} & l'^{(2)} \\ D'_0 & 0 & 0 & 0 & -2 & -3 \\ D'_2 & 1 & 1 & 0 & 0 & 0 \\ D'_2 & -1 & 0 & 0 & 0 & 0 \\ D'_2 & 0 & -1 & 0 & 0 & 1 \\ D'_1 & 0 & 0 & 1 & 1 & 1 \\ D'_1 & 0 & 0 & -1 & 1 & 1 \\ \hline \end{array}\,. \tag{A.1}$$

Hence the polytope Δ^* corresponding to the fibration over Δ^{*E_8} is given by

$$\begin{array}{|cc|rrrrrr|rrr|}\hline \text{div.} & \text{coord.} & & & \bar\nu_i^{*} & & & & l^{(e)} & l^{(1)} & l^{(2)} \\ K_M & x_0 & 1 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ 2\tilde D_e & x & 1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3\tilde D_e & y & 1 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\ E & z & 1 & 2 & 3 & 0 & 0 & 0 & 1 & -2 & -3 \\ \tilde D_2 & u_1 & 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 & 1 \\ \tilde D_2 & u_2 & 1 & 2 & 3 & -1 & 0 & 0 & 0 & 0 & 1 \\ \tilde D_2 & u_3 & 1 & 2 & 3 & 0 & -1 & 0 & 0 & 0 & 1 \\ \tilde D_1 & u_4 & 1 & 2 & 3 & 0 & 0 & 1 & 0 & 1 & 0 \\ \tilde D_1 & u_5 & 1 & 2 & 3 & 0 & 0 & -1 & 0 & 1 & 0 \\ \hline \end{array}\,. \tag{A.2}$$

The intersections among divisors lead to the constants in (3.12),

$$\begin{aligned}
&c_{ijk} = \begin{cases} c_{122} = c_{212} = c_{221} = 1\,, \\ 0 \text{ otherwise}\,, \end{cases} \\
&a = 54, \quad a^i = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\,, \quad a_i = \begin{pmatrix} 9 & 12 \end{pmatrix}\,, \quad a_{ij} = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}\,.
\end{aligned} \tag{A.3}$$

The Picard-Fuchs equations read

$$\begin{aligned}
\mathcal{L}_1 &= \theta_e(\theta_e - 2\theta_1 - 3\theta_2) - 12 z_e (6\theta_e + 5)(6\theta_e + 1)\,, \\
\mathcal{L}_2 &= \theta_1^2 - z_1(\theta_e - 2\theta_1 - 3\theta_2)(\theta_e - 2\theta_1 - 3\theta_2 - 1)\,, \\
\mathcal{L}_3 &= \theta_2^3 - z_2(\theta_e - 2\theta_1 - 3\theta_2)(\theta_e - 2\theta_1 - 3\theta_2 - 1)(\theta_e - 2\theta_1 - 3\theta_2 - 2)\,.
\end{aligned} \tag{A.4}$$

Here the coordinates z^a are determined by (4.29). The discriminants of the Picard-Fuchs equations are given by

$$\begin{aligned}
\Delta_1 &= (-1 + 4z_1)^3 - 54(1 + 12 z_1) z_2 - 729 z_2^2\,, \\
\Delta_2 &= -\Big[-1 + 864 z_e \Big(1 + 216 z_e(-1 + 4z_1)\Big)\Big]^3 + 4738381338321616896 z_e^6 z_2^2 \\
&\quad + 4353564672 z_e^3 (-1 + 432 z_e) \Big[1 + 864 z_e \Big(-1 + 216 z_e (1 + 12 z_1)\Big)\Big] z_2\,.
\end{aligned}$$

Using the choice of basis for 4-cycles in (3.13) we obtain

$$H_1 = E \cdot \tilde{D}_1, \quad H_2 = E \cdot \tilde{D}_2, \quad H^1 = \tilde{D}_2^2, \quad H^2 = \tilde{D}_1 \tilde{D}_2\,. \tag{A.5}$$

Hence the (3.14) intersections follow as

$$\eta^{(2)} = \begin{pmatrix} 0 & -3 & 1 & 0 \\ -3 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\,. \tag{A.6}$$

The genus zero amplitudes in the basis (A.5) read

$$\begin{aligned}
F^{(0)}_{H_1} &= \frac{1}{2} t_2^2 - \frac{3}{2} t_2 + \frac{5}{4} + F^{(0),\text{inst}}_{H_1}(q_e, \widetilde{Q}_1, \widetilde{Q}_2)\,, \\
F^{(0)}_{H_2} &= t_1 t_2 - t_1 - t_2 + \frac{5}{4} + F^{(0),\text{inst}}_{H_2}(q_e, \widetilde{Q}_1, \widetilde{Q}_2)\,, \\
F^{(0)}_{H^1} &= \tau^2 + \tau t_1 + 1 + F^{(0),\text{inst}}_{H^1}(q_e, \widetilde{Q}_1, \widetilde{Q}_2)\,, \\
F^{(0)}_{H^2} &= \frac{3}{2}\tau^2 + \tau t_2 + \frac{1}{2}\tau + \frac{3}{2} + F^{(0),\text{inst}}_{H^2}(q_e, \widetilde{Q}_1, \widetilde{Q}_2)\,.
\end{aligned} \tag{A.7}$$

In appendix A.5 we show some of the instanton expansions of the above expressions in terms of quasi-modular forms. Note that we make use of a ‘pure modular’ basis — as in the case of X_{24} — to compute the modular weight zero components of the $F_{H_i}^{(0)}$ periods. We define such a basis as follows

$$\begin{aligned} F_{H_1^\circ}^{(0)} &\equiv F_{H_1}^{(0)} + \frac{3}{2} F_{H^2}^{(0)}\,, \\ F_{H_2^\circ}^{(0)} &\equiv F_{H_2}^{(0)} + F_{H^2}^{(0)} + \frac{3}{2} F_{H^1}^{(0)}\,. \end{aligned} \tag{A.8}$$

The second Chern class of $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$ can be written in terms of the basis (A.5) as

$$c_2\Big(T_{M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}}\Big) = 24H_1 + 36H_2 + 102H^1 + 138H^2\,. \tag{A.9}$$

We compute the genus zero Gromov-Witten invariants of (A.9) in appendix A.3. Further constants related to the Chern classes are

$$\begin{aligned} b_1 &= -\frac{1}{24} c_3\Big(T_{M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}}\Big)\cdot \tilde{D}_1 - 1 = \frac{43}{2}, \quad b_2 = -\frac{1}{24} c_3\Big(T_{M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}}\Big)\cdot \tilde{D}_2 - 1 = 29\,, \\ b_e &= -\frac{1}{24} c_3\Big(T_{M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}}\Big)\cdot \tilde{D}_e - 1 = \frac{539}{4}, \quad \chi = 19728\,. \end{aligned} \tag{A.10}$$

This leads to the genus one amplitude

$$F^{(1)} = \frac{543}{4}\tau + \frac{45}{2} t_1 + 30 t_2 + F^{(1),\text{inst}}(q_e, \widetilde{Q}_1, \widetilde{Q}_2)\,, \tag{A.11}$$

where we give part of the expansion of $F^{(1),\text{inst}}$ in terms of quasi-modular forms in appendix A.5.

A.3 Geometric invariants for $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$

$n_{0,(0,d_1,d_2)}(H_1)$	$d_2=0$	1	2	3	4
$d_1=0$	$*$	-9	36	-243	2304
1	9	-153	2745	-49734	904500
2	0	-738	43506	-1719756	56117574
3	0	-2250	353916	-27555633	1515365226
4	0	-5355	1951704	-277450434	24502800744
$n_{0,(1,d_1,d_2)}(H_1)$	$d_2=0$	1	2	3	4
$d_1=0$	540	2160	-13500	138240	-1698840
1	-1620	55080	-1456380	34833240	-786936060
2	0	320760	-27424980	1396005840	-55422152100
3	0	1090800	-252097380	25003580040	-1654348658580
4	0	2786400	-1521167040	274895998560	-29038118214600
$n_{0,(2,d_1,d_2)}(H_1)$	$d_2=0$	1	2	3	4
$d_1=0$	1080	-143370	2298240	-35363790	578799000
1	249480	-11734470	409114800	-12410449830	342447273720
2	-3240	-74598570	9085010220	-583905569940	27847911802680
3	0	-271666710	92772238680	-11648976938100	920958991711200
4	0	-731942730	605426932980	-139049122837500	17515925402297760

Table 3. Genus 0 invariants associated to H_1 of $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$ for degree $d_e = 0,1,2$ of the elliptic parameter.

$n_{0,(0,d_1,d_2)}(H_2)$	$d_2=0$	1	2	3	4
$d_1=0$	$*$	-24	114	-864	8808
1	6	-192	4440	-93744	1898622
2	0	-744	55050	-2528040	92087760
3	0	-2040	390744	-34977312	2139264666
4	0	-4560	1973472	-318919680	31152820512
$n_{0,(1,d_1,d_2)}(H_2)$	$d_2=0$	1	2	3	4
$d_1=0$	720	6120	-43920	495360	-6528960
1	-720	67680	-2349360	65718720	-1654942320
2	0	314280	-34350480	2043688320	-90803818800
3	0	961920	-274751280	31523616000	-2326758388560
4	0	2313000	-1517061600	313418304000	-36732061356480
$n_{0,(2,d_1,d_2)}(H_2)$	$d_2=0$	1	2	3	4
$d_1=0$	1440	-1036800	8217540	-131045040	2264001480
1	0	-13718160	660289320	-23552058960	724510733760
2	-1440	-69796080	11223041760	-851198459760	45595230845400
3	0	-230700960	99434663640	-14568373007280	1290110994869760
4	0	-588578400	593689222980	-157013407044000	22030115559925320

Table 4. Genus 0 invariants associated to H_2 of $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$ for degree $d_e = 0,1,2$ of the elliptic parameter.

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$n_{0,(0,d_1,d_2)}(H^1)$	$d_2=0$	1	2	3	4
$d_1=0$	$*$	6	-30	234	-2424
1	0	30	-870	20196	-431874
2	0	84	-8682	460512	-18225348
3	0	180	-51600	5535630	-376340394
4	0	330	-224112	44650908	-4939206672
$n_{0,(1,d_1,d_2)}(H^1)$	$d_2=0$	1	2	3	4
$d_1=0$	0	-1800	12240	-138240	1833120
1	0	-12240	493920	-14789520	388121760
2	0	-39960	5793120	-389357280	18571150800
3	0	-93600	38578320	-5210146800	422999503920
4	0	-181800	182164320	-45722836800	6013372484160
$n_{0,(2,d_1,d_2)}(H^1)$	$d_2=0$	1	2	3	4
$d_1=0$	0	377460	-2483820	38068380	-651769560
1	0	2668140	-147669480	5533834140	-175351411440
2	0	9566100	-1999807560	168934134600	-9623706319080
3	0	24142860	-14689968840	2502807844680	-241840328961600
4	0	49469940	-74741749380	23760824553000	-3714780571613640

Table 5. Genus 0 invariants associated to H^1 of $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$ for degree $d_e = 0, 1, 2$ of the elliptic parameter.

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$n_{0,(0,d_1,d_2)}(H^2)$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	0	3	-12	81	-768
1	-3	51	-915	16578	-301500
2	0	246	-14502	573252	-18705858
3	0	750	-117972	9185211	-505121742
4	0	1785	-650568	92483478	-8167600248
$n_{0,(1,d_1,d_2)}(H^2)$	$d_2 = 0$	1	2	3	4
$d_1 = 0)$	0	-1080	5400	-51840	617760
1	720	-21240	537120	-12547080	279335520
2	0	-116640	9797760	-492862320	19402918200
3	0	-386640	88556760	-8722465560	574019167320
4	0	-973800	528827040	-95139102240	10012773524400
$n_{0,(2,d_1,d_2)}(H^2)$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	0	143370	-1149120	15155910	-231519600
1	424332	4966110	-164102760	4798354950	-129069932760
2	1440	29183490	-3444863940	217072351980	-10203218591040
3	0	101974950	-34165780560	4236009888780	-331677657148320
4	0	267877530	-218885967900	49833948532500	-6234396678989640

Table 6. Genus 0 invariants associated to H^2 of $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$ for degree $d_e = 0, 1, 2$ of the elliptic parameter.

$n_{1,(0,d_1,d_2)}$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	$*$	0	0	70	-1602
1	0	0	0	-990	52884
2	0	0	1161	-183402	12496941
3	0	0	15174	-4442538	487139904
4	0	0	110151	-56477430	199225723852
$n_{1,(1,d_1,d_2)}$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	-18	36	-90	-30744	706572
1	-18	288	-5166	698400	-43754310
2	0	972	-681210	138997944	-11571378390
3	0	2304	-10098990	3786456528	-504941463486
4	0	4500	-80360496	52885199952	-218756626565280
$n_{1,(2,d_1,d_2)}$	$d_2 = 0$	1	2	3	4
$d_1 = 0$	18	-11772	47052	6547608	-221005710
1	4266	-123228	2995704	-257783256	18293975928
2	18	-498420	208221228	-54043640640	5483304374166
3	0	-1313172	3364230240	-1643238648792	265936355246088
4	0	-2743308	965359376676	-273025875142044	36253634952195918

Table 7. Genus 1 invariants of $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$ for degree $d_e = 0, 1, 2$ of the elliptic parameter.

m_{β_1,β_2}	β_2								
β_1	$(0,0,1)$	$(0,1,0)$	$(1,0,0)$	$(0,0,2)$	$(0,1,1)$	$(0,2,0)$	$(1,0,1)$	$(1,1,0)$	$(2,0,0)$
$(0,0,1)$	-180	378	72	-1422	0	5400	36720	-11880	10800
$(0,1,0)$		-2016	-342	6894	0	-24840	-138240	57240	-49680
$(1,0,0)$			0	648	0	-2160	-18360	0	-4320
$(0,0,2)$				-15012	0	52920	376920	-85320	105840
$(0,1,1)$					0	0	0	0	0
$(0,2,0)$						38800	-1744200	516240	38880
$(1,0,1)$							-7069680	3656800	-3499200
$(1,1,0)$								38800	1036800
$(2,0,0)$									38800

Table 8. Meeting invariants for $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$.

A.4 Modular expressions for $X_{24}(1,1,1,1,8,12)$

$$
\begin{aligned}
F^{(0),\mathrm{inst}}_{H^b} &= P^{(0)}_{22}(H^b)\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right) + P^{(0)}_{46}(H^b)\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right)^2 + P^{(0)}_{70}(H^b)\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right)^3 + \cdots \qquad \text{(A.12)}\\
P^{(0)}_{22}(H^b) &= -\frac{5}{18}E_4E_6(35E_4^3+37E_6^2)\\
P^{(0)}_{46}(H^b) &= -\frac{1}{12}E_2\left(P^{(0)}_{22}(H^b)\right)^2 - \frac{5E_4E_6}{2985984}\Big(29908007E_4^9+207234483E_4^6E_6^2\\
&\quad +208392741E_4^3E_6^4+27245569E_6^6\Big)\\
P^{(0)}_{70}(H^b) &= \frac{1}{1671768834048}5E_4E_6\bigg(-129361397672887E_4^{15}-2336995567194997E_4^{12}E_6^2\\
&\quad -8349302045771014E_4^9E_6^4-8287506676944650E_4^6E_6^6-2198284344978035E_4^3E_6^8\\
&\quad -104063870681681E_6^{10}-74649600E_2^2E_4^2E_6^2\left(35E_4^3+37E_6^2\right)^3\\
&\quad -38880E_2E_4E_6\left(35E_4^3+37E_6^2\right)\Big(29908007E_4^9+207234483E_4^6E_6^2\\
&\quad +208392741E_4^3E_6^4+27245569E_6^6\Big)\bigg)\\
F^{(0),\mathrm{inst}}_{H_b^\circ} &= P^{(0)}_0(H_b^\circ) + P^{(0)}_{24}(H_b^\circ)\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right) + P^{(0)}_{48}(H_b^\circ)\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right)^2 + P^{(0)}_{72}(H_b^\circ)\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right)^3 + \cdots \qquad \text{(A.13)}\\
P^{(0)}_0(H_b^\circ) &= 960\sum_{d=1}^{\infty}\frac{\sigma_3(d)}{d^2}q_e^d,\\
P^{(0)}_{24}(H_b^\circ) &= \frac{5}{10368}\left(10321E_4^6+1680E_2E_4^4E_6+59182E_4^3E_6^2+1776E_2E_4E_6^3+9985E_6^4\right)\\
P^{(0)}_{48}(H_b^\circ) &= \frac{5}{13759414272}\bigg(34974695189E_4^{12}+955855257580E_4^9E_6^2+2375228903358E_4^6E_6^4\\
&\quad +958823179372E_4^3E_6^6+33221332181E_6^8+737280E_2^2E_4^2E_6^2\left(35E_4^3+37E_6^2\right)^2\\
&\quad +576E_2E_4E_6\left(19602269E_4^9+134498081E_4^6E_6^2+137176487E_4^3E_6^4+18933723E_6^6\right)\bigg)\\
P^{(0)}_{72}(H_b^\circ) &= \frac{5}{12999674453557248}\bigg(169868512046891311E_4^{18}+10991441298020921814E_4^{15}E_6^2\\
&\quad +81579781878072712593E_4^{12}E_6^4+152135959047477825460E_4^9E_6^6\\
&\quad +81740383608791276385E_4^6E_6^8+10747159517985301398E_4^3E_6^{10}\\
&\quad +154580302495588543E_6^{12}+16124313600E_2^3E_4^3E_6^3\left(35E_4^3+37E_6^2\right)^3\\
&\quad +8398080E_2^2E_4^2E_6^2\left(35E_4^3+37E_6^2\right)\Big(44357407E_4^9+305364363E_4^6E_6^2+309961101E_4^3E_6^4\\
&\quad +42023369E_6^6\Big)+432E_2E_4E_6\Big(188980289153801E_4^{15}+4041754304722571E_4^{12}E_6^2\\
&\quad +14159768366734202E_4^9E_6^4+14323384784691190E_4^6E_6^6+4070877046013005E_4^3E_6^8\\
&\quad +171728558335663E_6^{10}\Big)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
F^{(1),\mathrm{inst}} &= P_0^{(1)} + P_{24}^{(1)}\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right) + P_{48}^{(1)}\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right)^2 + P_{72}^{(1)}\left(\frac{q_e^2}{\eta^{48}}\widetilde{Q}\right)^3 + \cdots \qquad \text{(A.14)}\\
P_0^{(1)} &= -2\left(\frac{\chi}{24} - h_{11}\right)\sum_{d=1}\frac{\sigma_1(d)}{d} q_e^d,\\
P_{24}^{(1)} &= \frac{5}{5184}\Big(-10321E_4^6 + 34440E_2E_4^4E_6 - 59182E_4^3E_6^2 + 36408E_2E_4E_6^3 - 9985E_6^4\Big)\\
P_{48}^{(1)} &= \frac{5}{1719926784}\Bigg(-8718461011E_4^{12} - 238460285300E_4^9E_6^2 - 592848334770E_4^6E_6^4\\
&\quad - 239525096180E_4^3E_6^6 - 8301513619E_6^8 + 7649280E_2^2E_4^2E_6^2(35E_4^3+37E_6^2)^2\\
&\quad + 96E_2E_4E_6\Big(599169347E_4^9 + 4155664383E_4^6E_6^2 + 4173110841E_4^3E_6^4 + 542601349E_6^6\Big)\Bigg)\\
P_{72}^{(1)} &= \frac{5}{2166612408926208}\Bigg(-5449494372519982E_4^{18} - 3526301098569327294E_4^{15}E_6^2\\
&\quad - 2618749414216735613 7E_4^{12}E_6^4 - 48905698228868539588E_4^9E_6^6 -\\
&\quad 26341691595249846705E_4^6E_6^8 - 3475678553808910878E_4^3E_6^{10}\\
&\quad - 504932196408524 71E_6^{12} + 337714790400E_2^3E_4^3E_6^3\Big(35E_4^3+37E_6^2\Big)^3\\
&\quad + 1399680E_2^2E_4^2E_6^2(35E_4^3+37E_6^2)(3711620489E_4^9 + 25730000061E_4^6E_6^2\\
&\quad + 25856467947E_4^3E_6^4 + 3371520463E_6^6) + 108E_2E_4E_6\Big(5231073695092861E_4^{15}\\
&\quad + 93698918450783911E_4^{12}E_6^2 + 335105155725269122E_4^9E_6^4\\
&\quad + 332061543066849710E_4^6E_6^6 + 87589262461920785E_4^3E_6^8 + 4161976813713563E_6^{10}\Big)\Bigg)
\end{aligned}
$$

A.5 Modular expressions for $M_{\mathbb{P}^1\times\mathbb{P}^2}^{E_8}$

$$
\begin{aligned}
F_{H^i}^{(0),\mathrm{inst}} &= \sum_{d_1,d_2} P^{(0)}_{6(a^1d_1+a^2d_2)-2}(H^i)\left(\frac{q_e^{\frac{1}{2}}}{\eta^{12}}\right)^{a^1d_1+a^2d_2}\widetilde{Q}_1^{d_1}\widetilde{Q}_2^{d_2} \qquad \text{(A.15)}\\
P_{10}(H^1) &= 0, \qquad P_{10}(H^2) = -3E_4E_6,\\
P_{16}(H^1) &= \frac{3}{2}E_4 + \frac{9}{2}E_4E_6^2, \qquad P_{16}(H^2) = \frac{31}{48}E_4^2 + \frac{113}{48}E_4E_6^2,\\
P_{22}(H^1) &= 0, \qquad P_{22}(H^2) = -\frac{17}{32}E_4^2E_6 - \frac{7}{32}E_4E_6^3,\\
P_{28}(H^1) &= \frac{85}{48}E_4^7 + \frac{3}{8}E_2E_4^5E_6 + \frac{109}{6}E_4^4E_6^2 + \frac{9}{8}E_2E_4^2E_6^3 + \frac{137}{16}E_4E_6^4\\
P_{28}(H^2) &= \frac{1}{192}E_4\Big(587E_4^6 + 103E_2E_4^4E_6 + 5907E_4^3E_6^2 + 329E_2E_4E_6^3 + 2866E_6^4\Big)\\
P_{34}(H^1) &= -\frac{1}{9216}E_4\Big(48359E_4^6E_6 + 161426E_4^3E_6^3 + 39047E_6^5 + 24E_2E_4\Big(E_4^3+3E_6^2\Big)\Big(31E_4^3+113E_6^2\Big)\Big)\\
P_{34}(H^2) &= -\frac{1}{110592}E_4\left(208991E_4^6E_6 + 755906E_4^3E_6^3 + 196319E_6^5 + 4E_2E_4\Big(31E_4^3+113E_6^2\Big)^2\right)\\
F_{H_i^\circ}^{(0),\mathrm{inst}} &= \sum_{d_1,d_2} P^{(0)}_{6(a^1d_1+a^2d_2)}(H_i^\circ)\left(\frac{q_e^{\frac{1}{2}}}{\eta^{12}}\right)^{a^1d_1+a^2d_2}\widetilde{Q}_1^{d_1}\widetilde{Q}_2^{d_2} \qquad \text{(A.16)}
\end{aligned}
$$

$$P_0(H_1^\circ) = -24(b_1+1)\sum_{d=1}^{\infty}\frac{\sigma_3(d)}{d^2}q_e^d, \quad P_0(H_2^\circ) = -24(b_2+1)\sum_{d=1}^{\infty}\frac{\sigma_3(d)}{d^2}q_e^d,$$

$$P_{12}(H_1^\circ) = \frac{9}{4}E_4^3 + \frac{9}{4}E_6^2, \qquad P_{12}(H_2^\circ) = \frac{3}{2}E_4^3 + \frac{1}{2}E_2E_4E_6 + E_6^2,$$

$$P_{18}(H_1^\circ) = \frac{1}{288}\Big(-31E_2E_4^4 - 926E_4^3E_6 - 113E_2E_4E_6^2 - 226E_6^3\Big),$$

$$P_{18}(H_2^\circ) = \frac{1}{16}\left(-137E_4^3E_6 - 47E_6^3 - 2E_2\Big(E_4^4 + 3E_4E_6^2\Big)\right),$$

$$P_{24}(H_1^\circ) = \frac{1}{128}\left(33E_4^6 - 24E_2E_4^4E_6 + 2E_4^2\Big(-2E_2^2 + 71E_4\Big)E_6^2 - 16E_2E_4E_6^3 + 13E_6^4\right),$$

$$P_{24}(H_2^\circ) = \frac{1}{384}\Big(51E_4^6 + 17E_2E_4^4E_6 + 199E_4^3E_6^2 + 7E_2E_4E_6^3 + 14E_6^4\Big),$$

$$P_{30}(H_1^\circ) = \frac{1}{2304}\left(-648E_2E_4^7 - E_4^5\Big(31E_2^2 + 52747E_4\Big)E_6 - 4444E_2E_4^4E_6^2 \right.$$
$$\left. - E_4^2\Big(113E_2^2 + 105455E_4\Big)E_6^3 - 2396E_2E_4E_6^4 - 10422E_6^5\right),$$

$$P_{30}(H_2^\circ) = \frac{1}{1152}\left(-386E_2E_4^7 - E_4^5\Big(72E_2^2 + 32849E_4\Big)E_6 - 5002E_2E_4^4E_6^2 \right.$$
$$\left. - 24E_4^2\Big(9E_2^2 + 2659E_4\Big)E_6^3 - 2100E_2E_4E_6^4 - 6151E_6^5\right),$$

$$P_{36}(H_1^\circ) = \frac{1}{1327104}\left(628895E_4^9 + 438639E_2E_4^7E_6 + 9743040E_4^6E_6^2 + 1649058E_2E_4^4E_6^3 \right.$$
$$\left. + 3E_4^2\Big(25538E_2^2 + 3005857E_4\Big)E_6^4 + 400623E_2E_4E_6^5 + 392638E_6^6\right),$$

$$P_{36}(H_2^\circ) = \frac{1}{221184}\Big(333303E_4^9 + 54875E_2E_4^7E_6 + 5411350E_4^6E_6^2 + 220490E_2E_4^4E_6^3$$
$$+ 5526895E_4^3E_6^4 + 70235E_2E_4E_6^5 + 326788E_6^6\Big).$$

$$F^{(1),\text{inst}} = \sum_{d_1,d_2} P^{(1)}_{6(a^1d_1+a^2d_2)}\left(\frac{q_e^{\frac{1}{2}}}{\eta^{12}}\right)^{a^1d_1+a^2d_2}\widetilde{Q}_1^{d_1}\widetilde{Q}_2^{d_2} \tag{A.17}$$

$$P_0^{(1)} = -2\left(\frac{\chi}{24} - h_{11}\right)\sum_{d=1}^{\infty}\frac{\sigma_1(d)}{d}q_e^d,$$

$$P_{12}^{(1)} = \frac{3}{4}\Big(-6E_4^3 + 10E_2E_4E_6 - 5E_6^2\Big),$$

$$P_{18}^{(1)} = \frac{1}{576}\Big(-2581E_2E_4^4 + 9250E_4^3E_6 - 8363E_2E_4E_6^2 + 2990E_6^3\Big),$$

$$P_{24}^{(1)} = \frac{1}{6144}\Big(-2565E_4^6 + 8160E_2E_4^4E_6 - 10454E_4^3E_6^2 + 3360E_2E_4E_6^3 - 805E_6^4\Big)$$

$$P_{30}^{(1)} = \frac{1}{2304}\left(-24891E_2E_4^7 + E_4^5\Big(-4813E_2^2 + 151361E_4\Big)E_6 - 250867E_2E_4^4E_6^2 \right.$$
$$\left. + E_4^2\Big(-15059E_2^2 + 297124E_4\Big)E_6^3 - 121250E_2E_4E_6^4 + 28875E_6^5\right),$$

$$P_{36}^{(1)}=\frac{1}{7962624}\Big(-22238425E_4^9-356475921E_4^6E_6^2-357370707E_4^3E_6^4-20115395E_6^6+$$
$$108E_2^2E_4^2\left(31E_4^3+113E_6^2\right)\left(577E_4^3+1871E_6^2\right)+36E_2\Big(3099607E_4^7E_6$$
$$+10537042E_4^4E_6^3+2578903E_4E_6^5\Big)\Big).$$

A.6 Analytic continuation data for $X_{24}(1,1,1,1,8,12)$

We provide the numerical and — as far as we know them — analytic expressions for the continuation matrices T_c, T_c', T_o, T_o' in a Mathematica worksheet on the webpage [48]. Due to their special importance we reproduce here the intersection matrix at $c_1 = c_2 = 0$ as well as the entries of the continuation matrix to the point $z_e = c_1 = 0$:

$$(T_cT_{c'})^T\eta^{-1}T_cT_{c'}=\kappa$$
$$\cdot\begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & \frac{204448}{135} & 0 & 0 & -\frac{60466176}{5} & 0 \\ 0 & -6 & \frac{33392}{15} & -\frac{3091952128}{6075} & 0 & \frac{2519424}{5} & \frac{24428335104}{25} & 0 \\ \frac{2}{3} & \frac{204448}{135} & -\frac{3091952128}{6075} & \frac{283662214756352}{2460375} & -\frac{46656}{5} & -\frac{3083774976}{25} & -\frac{15768933728256}{125} & 0 \\ 0 & 0 & 0 & -\frac{46656}{5} & 0 & 0 & -\frac{2176782336}{5} & 0 \\ 0 & 0 & \frac{2519424}{5} & -\frac{3083774976}{25} & 0 & \frac{3265173504}{5} & -\frac{12224809598976}{25} & 0 \\ 0 & -\frac{60466176}{5} & \frac{24428335104}{25} & -\frac{15768933728256}{125} & -\frac{2176782336}{5} & -\frac{12224809598976}{25} & \frac{13420960199737344}{125} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} \end{pmatrix},$$

where

$$\kappa=\frac{1}{1572864\pi^4}. \tag{A.18}$$
$$f_{1,1}=\frac{-2916\pi^{12}r_4^4+10260\pi^6r_4^2-27\pi^4r_5r_4+144r_1^2-32r_2^2-7129}{1152}$$
$$f_{2,1}=\frac{1}{3}(6r_1+4ir_2+3)$$
$$f_{3,1}=\frac{1}{384}\left(216\pi^6r_4^2+48r_1+32ir_2+3i\pi r_5-404\right)$$
$$f_{7,1}=-\frac{1}{12}i(2r_2-9\pi^3r_4)$$
$$f_{5,1}=\frac{1}{48}\left(-54\pi^6r_4^2-36i\pi^3r_4-12r_1+8ir_2+101\right)$$
$$f_{1,2}=\frac{1}{24}\left(3r_1r_3-\sqrt{2}r_2r_3+384\pi^2r_4-24r_5\right)$$
$$f_{3,2}=\frac{-2304i\pi^6r_4^2+768\pi^3r_4+i\sqrt{2}\pi r_3+\pi r_3+3968i}{16\pi}$$
$$f_{5,2}=-\frac{-i\sqrt{2}\pi r_3+\pi r_3+768\pi^3r_4+256i}{8\pi}$$
$$f_{1,3}=-\big(116640\pi^{16}r_3r_4^4-410400\pi^{10}r_3r_4^2-207360\pi^8r_3r_4^2-71424\pi^6r_3r_4+276480\pi^4r_3r_4$$

$$+1080\pi^8 r_3 r_5 r_4-45\pi^4 r_1 r_3^2+15\sqrt{2}\pi^4 r_2 r_3^2+11796480 r_1+3932160\sqrt{2} r_2$$
$$-5760\pi^4 r_1^2 r_3+1280\pi^4 r_2^2 r_3+285160\pi^4 r_3+349440\pi^2 r_3+4464\pi^4 r_3 r_5)/(768\pi^4 r_3)$$

$$f_{2,3}=\frac{5\left(3i\sqrt{2}\pi^4 r_3^2+3\pi^4 r_3^2+768\pi^4 r_1 r_3+512 i\pi^4 r_2 r_3+384\pi^4 r_3+786432 i\sqrt{2}-786432\right)}{32\pi^4 r_3}$$

$$f_{3,3}=\Big(15i\sqrt{2}\pi^4 r_3^2+15\pi^4 r_3^2+17280\pi^{10} r_4^2 r_3-428544 i\pi^9 r_4^2 r_3+3840\pi^4 r_1 r_3+2560 i\pi^4 r_2 r_3$$
$$+142848\pi^6 r_4 r_3-92160 i\pi^5 r_4 r_3+240 i\pi^5 r_5 r_3-32320\pi^4 r_3+738048 i\pi^3 r_3+15360\pi^2 r_3$$
$$+61440 i\pi r_3+3932160 i\sqrt{2}-3932160\Big)/(512\pi^4 r_3)$$

$$f_{4,3}=\frac{15\left(3i\sqrt{2}\pi^4 r_3^2+2\pi^4 r_3^2+512\pi^4 r_1 r_3+512 i\pi^4 r_2 r_3+512\pi^4 r_3+786432 i\sqrt{2}-524288\right)}{128\pi^4 r_3}$$

$$f_{5,3}=-\Big(-15i\sqrt{2}\pi^4 r_3^2+15\pi^4 r_3^2+17280\pi^{10} r_4^2 r_3+3840\pi^4 r_1 r_3-2560 i\pi^4 r_2 r_3+11520 i\pi^7 r_4 r_3$$
$$+142848\pi^6 r_4 r_3-32320\pi^4 r_3+47616 i\pi^3 r_3+15360\pi^2 r_3-3932160 i\sqrt{2}$$
$$-3932160)/(256\pi^4 r_3)$$

$$f_{6,3}=\frac{5i\left(3\sqrt{2}\pi^4 r_3^2+512\pi^4 r_2 r_3+786432\sqrt{2}\right)}{64\pi^4 r_3}$$

$$f_{7,3}=\frac{i\left(-15\sqrt{2}\pi^4 r_3^2-2560\pi^4 r_2 r_3+11520\pi^7 r_4 r_3+47616\pi^3 r_3-3932160\sqrt{2}\right)}{256\pi^4 r_3}$$

$$f_{1,4}=\frac{1}{64}(16\pi^2 r_4-r_5)$$

$$f_{3,4}=\frac{-18 i\pi^6 r_4^2+6\pi^3 r_4+32 i}{8\pi}$$

$$r_1=0.0333238838392332919265429398082$$
$$r_2=-1.\ 29219644630091977480074761037$$
$$r_3=74.0860643209298158454123721134$$
$$r_4=-0.00948778220735050311547607017424$$
$$r_5=122.\ 032462442689559692241449686$$

B General genus zero modular anomaly equation

After releasing a pre-print of this paper it was brought to our attention by Georg Oberdieck that there is a conjectured modular anomaly equation for elliptic Calabi-Yau n-folds in [23, 24]. For $n=4$ the conjecture implies the modular anomaly equations for the Gromov-Witten potentials associated to π-vertical cycles and for genus one free energies that we derived in this paper. For the non π-vertical cycles the conjectured anomaly equation agrees with our results for $M^{E_8}_{\mathbb{P}^3}$ and we also checked it for the Gromov-Witten potentials of $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$. Our results on the modular structure can therefore be seen as a partial derivation and non-trivial check of the holomorphic anomaly equation conjectured in [23, 24] for Calabi-Yau fourfolds. We will now briefly describe the general form of the holomorphic anomaly equations for genus zero Gromov-Witten potentials.

Let $F^{(g)}_{\gamma_1,\ldots,\gamma_m}$ be the string amplitude associated to the Gromov-Witten invariants

$$N_{g,\kappa}(\gamma_1,\ldots,\gamma_m)=\int_{[\overline{M}_{g,n}(M,\kappa)]^{vir}}\prod_i \mathrm{ev}_i^*(\gamma_i)\,, \tag{B.1}$$

where $\kappa\in H_2(M,\mathbb{Z})$ and $\gamma_i\in H^*(M,\mathbb{Z})$. On the one hand, given $\beta\in H_2(B,\mathbb{Z})$ conjecture A in [23, 24] implies that

$$\mathrm{Coeff}(F^{(g)}_{\gamma_1,\ldots,\gamma_m},Q^\beta)\in\frac{1}{\eta^{12c_1(B)\cdot\beta}}\mathbb{C}[E_2,E_4,E_6]\,, \tag{B.2}$$

which matches with our Ansatz given in expression (4.24). On the other hand, conjecture B of [23, 24] implies a general modular anomaly equation for $F^{(g)}_{\gamma_1,\ldots,\gamma_m}$.

Following the discussion of sections 3.1 and 4.4, we can make a generalization of the pure modular basis by taking the 4-cycles,

$$H^i=a^{ij}a^k\tilde{D}_j\tilde{D}_k\,,\quad H^\circ_i=D_e\tilde{D}_i\,,\quad i=1,\ldots,h^{1,1}(B), \tag{B.3}$$

as these fulfill the intersection relations

$$H^i\cdot H^\circ_j=\delta^i_j\,,\quad H^i\cdot H^j=0\,,\quad H^\circ_i\cdot H^\circ_j=0\,. \tag{B.4}$$

Note that $H^i\in H_4(M,\mathbb{Z})$ while in general $H^\circ_i\notin H_4(M,\mathbb{Z})$. Let $\ell\in H^2(B)$ such that $\langle\beta,\ell\rangle\neq 0$. Then for a given $\gamma\in H^{2,2}(M,\mathbb{C})$ Georg Oberdieck pointed out to us that conjecture B of [23, 24] implies a modular anomaly equation for $F^{(0)}_\gamma$, which in the modular basis (B.3) reads

$$\begin{aligned}\frac{\partial F^{(0)}_{\gamma,\beta}}{\partial E_2}=-\frac{1}{12}\Bigg[&\sum_{\beta=\beta'+\beta''}\left(\beta'_iF^{(0)}_{\gamma,\beta'}F^{(0)}_{H^i,\beta''}-\frac{\langle\beta',\ell\rangle^2\langle\beta'',\pi_*\gamma\rangle+\langle\beta'',\ell\rangle^2\langle\beta',\pi_*\gamma\rangle}{\langle\beta,\ell\rangle^2}F^{(0)}_{H^i,\beta'}F^{(0)}_{H^\circ_i,\beta''}\right)\\&+\frac{2}{\langle\beta,\ell\rangle}F^{(0)}_{\pi^*(\pi_*(\gamma)\cup\ell),\beta}-\frac{\langle\pi_*\gamma,\beta\rangle}{\langle\beta,\ell\rangle^2}F^{(0)}_{\pi^*\ell^2,\beta}\Bigg]\,.\end{aligned} \tag{B.5}$$

From the properties of the Gysin morphisms it follows that $\pi_*H^\circ_i=D'_i$ and $\pi_*H^i=0$. Hence for a π-vertical 4-cycle H^i, the modular anomaly equation (B.5) of its corresponding string amplitude $F^{(0)}_{H^i}$ reduces to equation (4.54).

Now we consider the 4-cycles H°_i where equation (B.5.4) becomes more involved. It is easy to verify for $M^{E_8}_{\mathbb{P}^3}$ that (B.5) reduces to (4.38). Moreover, when $h^{1,1}(B)\geq 2$ equation (B.5) in general implies multiple relations, since it depends on the choice of ℓ. We checked equation (B.5) for $M^{E_8}_{\mathbb{P}^1\times\mathbb{P}^2}$ of which we include the toric data in appendix A.2. We also provide some modular expressions for the corresponding amplitudes $F^{(0)}_{H^\circ_1}$ and $F^{(0)}_{H^\circ_2}$ in appendix A.5.

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