Modular Forms, a Computational Approach

William Stein

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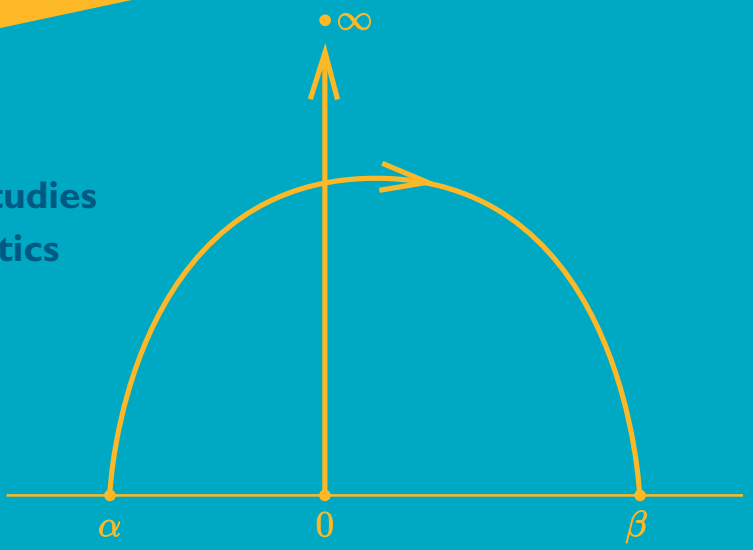

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with an appendix by
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To my grandmother, Annette Maurer

Contents

Preface

This is a graduate-level textbook about algorithms for computing with modular forms. It is nontraditional in that the primary focus is not on underlying theory; instead, it answers the question *"how do you use a computer to explicitly compute spaces of modular forms?"*

This book emerged from notes for a course the author taught at Harvard University in 2004, a course at UC San Diego in 2005, and a course at the University of Washington in 2006.

The author has spent years trying to find good practical ways to compute with classical modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and has implemented most of these algorithms several times, first in C++ [**Ste99b**], then in MAGMA [**BCP97**], and as part of the free open source computer algebra system SAGE (see [**Ste06**]). Much of this work has involved turning formulas and constructions buried in obscure research papers into precise computational recipes then testing these and eliminating inaccuracies.

The author is aware of no other textbooks on computing with modular forms, the closest work being Cremona's book [**Cre97a**], which is about computing with elliptic curves, and Cohen's book [**Coh93**] about algebraic number theory.

In this book we focus on how to compute *in practice* the spaces $M_k(N, \varepsilon)$ of modular forms, where $k \geq 2$ is an integer and ε is a Dirichlet character of modulus N (the appendix treats modular forms for higher rank groups). We spend the most effort explaining the general algorithms that appear so far to be the best (in practice!) for such computations. We will not discuss in any detail computing with quaternion algebras, half-integral weight forms, weight 1 forms, forms for noncongruence subgroups or groups other

than GL_2, Hilbert and Siegel modular forms, trace formulas, p-adic modular forms, and modular abelian varieties, all of which are topics for additional books. We also rarely analyze the complexity of the algorithms, but instead settle for occasional remarks about their practical efficiency.

For most of this book we assume the reader has some prior exposure to modular forms (e.g., [**DS05**]), though we recall many of the basic definitions. We cite standard books for proofs of the fundamental results about modular forms that we will use. The reader should also be familiar with basic algebraic number theory, linear algebra, complex analysis (at the level of [**Ahl78**]), and algorithms (e.g., know what an algorithm is and what big oh notation means). In some of the examples and applications we assume that the reader knows about elliptic curves at the level of [**Sil92**].

Chapter 1 is foundational for the rest of this book. It introduces congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and modular forms as functions on the complex upper half plane. We discuss q-expansions, which provide an important computational handle on modular forms. We also study an algorithm for computing with congruence subgroups. The chapter ends with a list of applications of modular forms throughout mathematics.

In Chapter 2 we discuss level 1 modular forms in much more detail. In particular, we introduce Eisenstein series and the cusp form Δ and describe their q-expansions and basic properties. Then we prove a structure theorem for level 1 modular forms and use it to deduce dimension formulas and give an algorithm for explicitly computing a basis. We next introduce Hecke operators on level 1 modular forms, prove several results about them, and deduce multiplicativity of the Ramanujan τ function as an application. We also discuss explicit computation of Hecke operators. In Section 2.6 we make some brief remarks on recent work on asymptotically fast computation of values of τ. Finally, we describe computation of constant terms of Eisenstein series using an analytic algorithm. We generalize many of the constructions in this chapter to higher level in subsequent chapters.

In Chapter 3 we turn to modular forms of higher level but restrict for simplicity to weight 2 since much is clearer in this case. (We remove the weight restriction later in Chapter 8.) We describe a geometric way of viewing cuspidal modular forms as differentials on modular curves, which leads to modular symbols, which are an explicit way to present a certain homology group. This chapter closes with methods for explicitly computing cusp forms of weight 2 using modular symbols, which we generalize in Chapter 9.

In Chapter 4 we introduce Dirichlet characters, which are important both in explicit construction of Eisenstein series (in Chapter 5) and in decomposing spaces of modular forms as direct sums of simpler spaces. The

main focus of this chapter is a detailed study of how to explicitly represent and compute with Dirichlet characters.

Chapter 5 is about how to explicitly construct the Eisenstein subspace of modular forms. First we define generalized Bernoulli numbers attached to a Dirichlet character and an integer then explain a new analytic algorithm for computing them (which generalizes the algorithm in Chapter 2). Finally we give without proof an explicit description of a basis of Eisenstein series, explain how to compute it, and give some examples.

Chapter 6 records a wide range of dimension formulas for spaces of modular forms, along with a few remarks about where they come from and how to compute them.

Chapter 7 is about linear algebra over exact fields, mainly the rational numbers. This chapter can be read independently of the others and does not require any background in modular forms. Nonetheless, this chapter occupies a central position in this book, because the algorithms in this chapter are of crucial importance to any actual implementation of algorithms for computing with modular forms.

Chapter 8 is the most important chapter in this book; it generalizes Chapter 3 to higher weight and general level. The modular symbols formulation described here is central to general algorithms for computing with modular forms.

Chapter 9 applies the algorithms from Chapter 8 to the problem of computing with modular forms. First we discuss decomposing spaces of modular forms using Dirichlet characters, and then explain how to compute a basis of Hecke eigenforms for each subspace using several approaches. We also discuss congruences between modular forms and bounds needed to provably generate the Hecke algebra.

Chapter 10 is about computing analytic invariants of modular forms. It discusses tricks for speeding convergence of certain infinite series and sketches how to compute every elliptic curve over $\mathbb{Q}$ with given conductor.

Chapter 11 contains detailed solutions to most of the exercises in this book. (Many of these were written by students in a course taught at the University of Washington.)

Appendix A deals with computational techniques for working with generalizations of modular forms to more general groups than $\mathrm{SL}_2(\mathbb{Z})$, such as $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. Some of this material requires more prerequisites than the rest of the book. Nonetheless, seeing a natural generalization of the material in the rest of this book helps to clarify the key ideas. The topics in the appendix are directly related to the main themes of this book: modular

symbols, Manin symbols, cohomology of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with various coefficients, explicit computation of modular forms, etc.

Software. We use SAGE, Software for Algebra and Geometry Experimentation (see [**Ste06**]), to illustrate how to do many of the examples. SAGE is completely free and packages together a wide range of open source mathematics software for doing much more than just computing with modular forms. SAGE can be downloaded and run on your computer or can be used via a web browser over the Internet. The reader is encouraged to experiment with many of the objects in this book using SAGE. We do not describe the basics of using SAGE in this book; the reader should read the SAGE tutorial (and other documentation) available at the SAGE website [**Ste06**]. All examples in this book have been automatically tested and should work exactly as indicated in SAGE version at least 1.5.

Acknowledgements. David Joyner and Gabor Wiese carefully read the book and provided a huge number of helpful comments.

John Cremona and Kevin Buzzard both made many helpful remarks that were important in the development of the algorithms in this book. Much of the mathematics (and some of the writing) in Chapter 10 is joint work with Helena Verrill.

Noam Elkies made remarks about Chapters 1 and 2. Sándor Kovács provided interesting comments on Chapter 1. Allan Steel provided helpful feedback on Chapter 7. Jordi Quer made useful remarks about Chapter 4 and Chapter 6.

The students in the courses that I taught on this material at Harvard, San Diego, and Washington provided substantial feedback: in particular, Abhinav Kumar made numerous observations about computing widths of cusps (see Section 1.4.1) and Thomas James Barnet-Lamb made helpful remarks about how to represent Dirichlet characters. James Merryfield made helpful remarks about complex analytic issues and about convergence in Stirling's formula. Robert Bradshaw, Andrew Crites (who wrote Exercise 7.5), Michael Goff, Dustin Moody, and Koopa Koo wrote most of the solutions included in Chapter 11 and found numerous typos throughout the book. Dustin Moody also carefully read through the book and provided feedback.

H. Stark suggested using Stirling's formula in Section 2.7.1, and Mark Watkins and Lynn Walling made comments on Chapter 3.

Parts of Chapter 1 follow Serre's beautiful introduction to modular forms [**Ser73**, Ch. VII] closely, though we adjust the notation, definitions, and order of presentation to be consistent with the rest of this book.

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Notation and Conventions. We denote canonical isomorphisms by $\cong$ and noncanonical isomorphisms by $\approx$. If V is a vector space and s denotes some sort of construction involving V, we let V_s denote the corresponding subspace and V^s the quotient space. E.g., if ι is an involution of V, then V_+ is $\mathrm{Ker}(\iota - 1)$ and $V^+ = V/\mathrm{Im}(\iota - 1)$. If A is a finite abelian group, then A_{tor} denotes the torsion subgroup and A/tor denotes the quotient A/A_{tor}. We denote right group actions using exponential notation. Everywhere in this book, N is a positive integer and k is an integer.

If N is an integer, a *divisor* t of N is a *positive* integer such that N/t is an integer.

Bibliography

[AB90] A. Ash and A. Borel, *Generalized modular symbols*, Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), Springer, Berlin, 1990, pp. 57–75.

[ADT04] Nadia Ben Atti and Gema M. Díaz-Toca, `http://hlombardi.free.fr/publis/ABMAvar.html` (2004).

[AG00] Avner Ash and Robert Gross, *Generalized non-abelian reciprocity laws: a context for Wiles' proof*, Bull. London Math. Soc. **32** (2000), no. 4, 385–397. MR 1760802 (2001h:11142)

[Aga00] A. Agashe, *The Birch and Swinnerton-Dyer formula for modular abelian varieties of analytic rank* 0, Ph.D. thesis, University of California, Berkeley (2000).

[AGG84] Avner Ash, Daniel Grayson, and Philip Green, *Computations of cuspidal cohomology of congruence subgroups of* $\mathrm{SL}(3, \mathbf{Z})$, J. Number Theory **19** (1984), no. 3, 412–436. MR 769792 (86g:11032)

[AGM] Avner Ash, Paul E. Gunnells, and Mark McConnell, *Cohomology of congruence subgroups of* $\mathrm{SL}_4(\mathbb{Z})$ *II*, in preparation.

[AGM02] ———, *Cohomology of congruence subgroups of* $\mathrm{SL}_4(\mathbb{Z})$, J. Number Theory **94** (2002), no. 1, 181–212. MR 1904968 (2003f:11072)

[AGR93] Avner Ash, David Ginzburg, and Steven Rallis, *Vanishing periods of cusp forms over modular symbols*, Math. Ann. **296** (1993), no. 4, 709–723. MR 1233493 (94f:11044)

[Ahl78] Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR 510197 (80c:30001)

[AL70] A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.

[AO01] Scott Ahlgren and Ken Ono, *Addition and counting: the arithmetic of partitions*, Notices Amer. Math. Soc. **48** (2001), no. 9, 978–984. MR 1854533 (2002e:11136)

[AR79] Avner Ash and Lee Rudolph, *The modular symbol and continued fractions in higher dimensions*, Invent. Math. **55** (1979), no. 3, 241–250. MR 553998 (82g:12011)

[Art79] James Arthur, *Eisenstein series and the trace formula*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 253–274. MR 546601 (81b:10020)

[Ash77] Avner Ash, *Deformation retracts with lowest possible dimension of arithmetic quotients of self-adjoint homogeneous cones*, Math. Ann. **225** (1977), no. 1, 69–76. MR 0427490 (55 #522)

[Ash80] ———, *Cohomology of congruence subgroups* $\mathrm{SL}(n, \mathbb{Z})$, Math. Ann. **249** (1980), no. 1, 55–73. MR 82f:22010

[Ash84] ———, *Small-dimensional classifying spaces for arithmetic subgroups of general linear groups*, Duke Math. J. **51** (1984), no. 2, 459–468. MR 747876 (85k:22027)

[Ash86] ———, *A note on minimal modular symbols*, Proc. Amer. Math. Soc. **96** (1986), no. 3, 394–396. MR 822426 (87e:22024)

[Ash94] ———, *Unstable cohomology of* $\mathrm{SL}(n, \mathcal{O})$, J. Algebra **167** (1994), no. 2, 330–342. MR 1283290 (95g:20050)

[Bar57] E. S. Barnes, *The perfect and extreme senary forms*, Canad. J. Math. **9** (1957), 235–242. MR 0086834 (19,251e)

[Bar94] A. Barvinok, *A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, Math. Oper. Res. **19** (1994), no. 4, 769–779.

[Bas96] Jacques Basmaji, *Ein Algorithmus zur Berechnung von Hecke-Operatoren und Anwendungen auf modulare Kurven,* `http://modular.math.washington.edu/scans/papers/basmaji/`, 1996.

[BC06] S. S. Bullock and C. Connell, *Equivariant retracts of geometrically finite discrete groups acting on negatively pinched Hadamard manifolds*, in preparation, 2006.

[BCDT01] C. Breuil, B. Conrad, Fred Diamond, and R. Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939 (electronic). MR 2002d:11058

[BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3–4, 235–265, Computational algebra and number theory (London, 1993). MR 1 484 478

[BCS92] J. P. Buhler, R. E. Crandall, and R. W. Sompolski, *Irregular primes to one million*, Math. Comp. **59** (1992), no. 200, 717–722. MR 1134717 (93a:11106)

[BHKS06] K. Belebas, M. Van Hoeij, J. Klüners, and A. Steel, *Factoring polynomials over global fields*, preprint at `http://www.math.fsu.edu/~hoeij/papers.html` (2006).

[BI97] R. Baeza and M. I. Icaza, *On Humbert-Minkowski's constant for a number field*, Proc. Amer. Math. Soc. **125** (1997), no. 11, 3195–3202. MR 1403112 (97m:11092)

[Bir71] B. J. Birch, *Elliptic curves over* $\mathbf{Q}$*: A progress report*, 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York,

Stony Brook, N.Y., 1969), Amer. Math. Soc., Providence, R.I., 1971, pp. 396–400.

[BK90] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.

[BMS06] Yann Bugeaud, Maurice Mignotte, and Samir Siksek, *Classical and modular approaches to exponential Diophantine equations. II. The Lebesgue-Nagell equation*, Compos. Math. **142** (2006), no. 1, 31–62. MR 2196761

[Bro94] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, corrected reprint of the 1982 original. MR 1324339 (96a:20072)

[BS73] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault. MR 0387495 (52 #8337)

[BS02] K. Buzzard and W. A. Stein, *A mod five approach to modularity of icosahedral Galois representations*, Pacific J. Math. **203** (2002), no. 2, 265–282. MR 2003c:11052

[BT82] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982. MR 658304 (83i:57016)

[Bul00] S. S. Bullock, *Well-rounded retracts of rank one symmetric spaces*, preprint, 2000.

[Bum84] Daniel Bump, *Automorphic forms on* $\mathrm{GL}(3, \mathbf{R})$, Lecture Notes in Mathematics, vol. 1083, Springer-Verlag, Berlin, 1984. MR 765698 (86g:11028)

[Bum97] ———, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR 1431508 (97k:11080)

[Buz96] Kevin Buzzard, *On the eigenvalues of the Hecke operator* T_2, J. Number Theory **57** (1996), no. 1, 130–132. MR 96m:11033

[BW00] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. MR 1721403 (2000j:22015)

[Byg99] J. Bygott, *Modular forms and modular symbols over imaginary quadratic fields*, Ph.D. thesis, Exeter University, 1999.

[Car59a] L. Carlitz, *Arithmetic properties of generalized Bernoulli numbers*, J. Reine Angew. Math. **202** (1959), 174–182. MR 0109132 (22 #20)

[Car59b] ———, *Some arithmetic properties of generalized Bernoulli numbers*, Bull. Amer. Math. Soc. **65** (1959), 68–69. MR 0104630 (21 #3383)

[CDT99] Brian Conrad, Fred Diamond, and Richard Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), no. 2, 521–567. MR 1639612 (99i:11037)

[CF67] George E. Cooke and Ross L. Finney, *Homology of cell complexes*, Based on lectures by Norman E. Steenrod, Princeton University Press, Princeton, N.J., 1967. MR 0219059 (36 #2142)

[Che05] Imin Chen, *A Diophantine equation associated to* $X_0(5)$, LMS J. Comput. Math. **8** (2005), 116–121 (electronic). MR 2153792 (2006b:11052)

[CL04] J. Cremona and M. P. Lingham, *Finding all elliptic curves with good reduction outside a given set of primes*, in progress (2004).

[CO77] H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, 69–78. Lecture Notes in Math., Vol. 627. MR 57 #12396

[Coh93] H. Cohen, *A course in computational algebraic number theory*, Springer-Verlag, Berlin, 1993. MR 94i:11105

[Cou01] Renaud Coulangeon, *Voronoï theory over algebraic number fields*, Réseaux euclidiens, designs sphériques et formes modulaires, Monogr. Enseign. Math., vol. 37, Enseignement Math., Geneva, 2001, pp. 147–162. MR 1878749 (2002m:11064)

[Cre] J. E. Cremona, personal communication.

[Cre84] ———, *Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields*, Compositio Math. **51** (1984), no. 3, 275–324.

[Cre92] ———, *Modular symbols for $\Gamma_1(N)$ and elliptic curves with everywhere good reduction*, Math. Proc. Cambridge Philos. Soc. **111** (1992), no. 2, 199–218.

[Cre97a] ———, *Algorithms for modular elliptic curves*, second ed., Cambridge University Press, Cambridge, 1997, `http://www.maths.nott.ac.uk/personal/jec/book/`.

[Cre97b] ———, *Computing periods of cusp forms and modular elliptic curves*, Experiment. Math. **6** (1997), no. 2, 97–107.

[Cre06] ———, Proceedings of the 7th International Symposium (ANTS-VII) (2006).

[CS88] J. H. Conway and N. J. A. Sloane, *Low-dimensional lattices. III. Perfect forms*, Proc. Roy. Soc. London Ser. A **418** (1988), no. 1854, 43–80. MR 953277 (90a:11073)

[CW94] J. E. Cremona and E. Whitley, *Periods of cusp forms and elliptic curves over imaginary quadratic fields*, Math. Comp. **62** (1994), no. 205, 407–429.

[CWZ01] Janos A. Csirik, Joseph L. Wetherell, and Michael E. Zieve, *On the genera of $X_0(N)$*, `http://www.csirik.net/papers.html` (2001).

[Dar97] H. Darmon, *Faltings plus epsilon, Wiles plus epsilon, and the generalized Fermat equation*, C. R. Math. Rep. Acad. Sci. Canada **19** (1997), no. 1, 3–14. MR 1479291 (98h:11034a)

[Dem04] L. Dembélé, *Quaternionic Manin symbols, Brandt matrices and Hilbert modular forms*, preprint, 2004.

[Dem05] L. Dembélé, *Explicit computations of Hilbert modular forms on $\mathbb{Q}(\sqrt{5})$*, Experiment. Math. **14** (2005), no. 4, 457–466. MR 2193808

[DI95] F. Diamond and J. Im, *Modular forms and modular curves*, Seminar on Fermat's Last Theorem, Providence, RI, 1995, pp. 39–133.

[Dia96] F. Diamond, *On deformation rings and Hecke rings*, Ann. of Math. (2) **144** (1996), no. 1, 137–166. MR 1405946 (97d:11172)

[Dix82] John D. Dixon, *Exact solution of linear equations using p-adic expansions*, Numer. Math. **40** (1982), no. 1, 137–141. MR 681819 (83m:65025)

[Dok04] Tim Dokchitser, *Computing special values of motivic L-functions*, Experiment. Math. **13** (2004), no. 2, 137–149.

[DP04] H. Darmon and R. Pollack, *The efficient calculation of Stark-Heegner points via overconvergent modular symbols.*

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005.

[DVS05] M. Dutour, F. Vallentin, and A. Schürmann, *Classification of perfect forms in dimension* 8, talk at Oberwolfach meeting *Sphere packings: Exceptional structures and relations to other fields*, November 2005.

[Ebe02] Wolfgang Ebeling, *Lattices and codes*, revised ed., Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2002, a course partially based on lectures by F. Hirzebruch.

[ECdJ+06] Bas Edixhoven, Jean-Marc Couveignes, Robin de Jong, Franz Merkl, and Johan Bosman, *On the computation of coefficients of modular form*, `http://www.arxiv.org/abs/math.NT/0605244` (2006).

[EGM98] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, Harmonic analysis and number theory. MR 1483315 (98g:11058)

[Eil47] Samuel Eilenberg, *Homology of spaces with operators. I*, Trans. Amer. Math. Soc. **61** (1947), 378–417; errata, 62, 548 (1947). MR 0021313 (9,52b)

[Elk98] Noam D. Elkies, *Elliptic and modular curves over finite fields and related computational issues*, Computational perspectives on number theory (Chicago, IL, 1995), AMS/IP Stud. Adv. Math., vol. 7, Amer. Math. Soc., Providence, RI, 1998, pp. 21–76. MR 1486831 (99a:11078)

[EVGS02] Philippe Elbaz-Vincent, Herbert Gangl, and Christophe Soulé, *Quelques calculs de la cohomologie de* $\mathrm{GL}_N(\mathbb{Z})$ *et de la* K*-théorie de* $\mathbb{Z}$, C. R. Math. Acad. Sci. Paris **335** (2002), no. 4, 321–324. MR 1931508 (2003h:19002)

[FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249 (93a:20069)

[FJ02] D. W. Farmer and K. James, *The irreducibility of some level 1 Hecke polynomials*, Math. Comp. **71** (2002), no. 239, 1263–1270 (electronic). MR 2003e:11046

[FL] D. W. Farmer and Stefan Lemurell, *Maass forms and their L-functions*, AIM 2005-15, arXiv:math.NT/0506102.

[FM99] G. Frey and M. Müller, *Arithmetic of modular curves and applications*, Algorithmic algebra and number theory (Heidelberg, 1997), Springer, Berlin, 1999, pp. 11–48.

[Fra98] J. Franke, *Harmonic analysis in weighted* L_2*-spaces*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 2, 181–279.

[FT93] A. Fröhlich and M. J. Taylor, *Algebraic number theory*, Cambridge University Press, Cambridge, 1993.

[FvdG] C. Faber and G. van der Geer, *Sur la cohomologie des Systémes Locaux sur les Espaces des Modules des Courbes de Genus 2 and des Surfaces Abéliennes*, arXiv:math.AG/0305094.

[Gel75] Stephen S. Gelbart, *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J., 1975, Annals of Mathematics Studies, No. 83. MR 0379375 (52 #280)

[GH81] M. J. Greenberg and J. R. Harper, *Algebraic topology*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1981, A first course. MR 83b:55001

[GLQ04] Josep González, Joan-Carles Lario, and Jordi Quer, *Arithmetic of $\mathbb{Q}$-curves*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 125–139. MR 2058647 (2005c:11068)

[GM03] P. E. Gunnells and M. McConnell, *Hecke operators and $\mathbb{Q}$-groups associated to self-adjoint homogeneous cones*, J. Number Theory **100** (2003), no. 1, 46–71.

[Gol05] Dorian Goldfeld, *Automorphic forms and L-functions on the general linear group*, to appear, 2005.

[Gon97] A. B. Goncharov, *The double logarithm and Manin's complex for modular curves*, Math. Res. Lett. **4** (1997), no. 5, 617–636.

[Gon98] ———, *Multiple polylogarithms, cyclotomy and modular complexes*, Math. Res. Lett. **5** (1998), no. 4, 497–516.

[Gor93] D. Gordon, *Discrete logarithms in* GF(p) *using the number field sieve*, SIAM J. Discrete Math. **6** (1993), no. 1, 124–138. MR 94d:11104

[Gor04] ———, *Discrete logarithm problem,* `http://www.win.tue.nl/~henkvt/content.html`.

[GP05] Benedict H. Gross and David Pollack, *On the Euler characteristic of the discrete spectrum*, J. Number Theory **110** (2005), no. 1, 136–163. MR 2114678 (2005k:11100)

[Gre83] Ralph Greenberg, *On the Birch and Swinnerton-Dyer conjecture*, Invent. Math. **72** (1983), no. 2, 241–265. MR 700770 (85c:11052)

[Gri05] G. Grigorov, *Kato's Euler System and the Main Conjecture*, Harvard Ph.D. Thesis (2005).

[Gro98] Benedict H. Gross, *On the Satake isomorphism*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 223–237. MR 1696481 (2000e:22008)

[GS81] F. Grunewald and J. Schwermer, *A nonvanishing theorem for the cuspidal cohomology of SL_2 over imaginary quadratic integers*, Math. Ann. **258** (1981), 183–200.

[GS02] Mark Giesbrecht and Arne Storjohann, *Computing rational forms of integer matrices*, J. Symbolic Comput. **34** (2002), no. 3, 157–172. MR 1935075 (2003j:15016)

[Gun99] P. E. Gunnells, *Modular symbols for $\mathbb{Q}$-rank one groups and Voronoĭ reduction*, J. Number Theory **75** (1999), no. 2, 198–219.

[Gun00a] ———, *Computing Hecke eigenvalues below the cohomological dimension*, Experiment. Math. **9** (2000), no. 3, 351–367. MR 1 795 307

[Gun00b] ———, *Symplectic modular symbols*, Duke Math. J. **102** (2000), no. 2, 329–350.

[Hara] G. Harder, *Congruences between modular forms of genus 1 and of genus 2*, Arbeitstagung.

[Harb] ———, *Kohomologie arithmetischer Gruppen*, lecture notes, Universität Bonn, 1987–1988.

[Har87] ———, *Eisenstein cohomology of arithmetic groups. The case* GL_2, Invent. Math. **89** (1987), no. 1, 37–118. MR 892187 (89b:22018)

[Har91] ———, *Eisenstein cohomology of arithmetic groups and its applications to number theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990) (Tokyo), Math. Soc. Japan, 1991, pp. 779–790. MR 1159264 (93b:11057)

[Har05] ———, *Modular symbols and special values of automorphic L-functions*, preprint, 2005.

[HC68] Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62, Springer-Verlag, Berlin, 1968. MR 0232893 (38 #1216)

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, corrected reprint of the 1978 original. MR 1834454 (2002b:53081)

[Hij74] H. Hijikata, *Explicit formula of the traces of Hecke operators for* $\Gamma_0(N)$, J. Math. Soc. Japan **26** (1974), no. 1, 56–82.

[Hsu96] Tim Hsu, *Identifying congruence subgroups of the modular group*, Proc. Amer. Math. Soc. **124** (1996), no. 5, 1351–1359. MR 1343700 (96k:20100)

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, with an appendix by Vladimir G. Berkovich. MR 1876802 (2002m:11050)

[Hum80] James E. Humphreys, *Arithmetic groups*, Lecture Notes in Mathematics, vol. 789, Springer, Berlin, 1980. MR 584623 (82j:10041)

[Ica97] M. I. Icaza, *Hermite constant and extreme forms for algebraic number fields*, J. London Math. Soc. (2) **55** (1997), no. 1, 11–22. MR 1423282 (97j:11034)

[Jaq91] David-Olivier Jaquet, *Classification des réseaux dans* $\mathbf{R}^7$ *(via la notion de formes parfaites)*, Astérisque (1991), no. 198-200, 7–8, 177–185 (1992), Journées Arithmétiques, 1989 (Luminy, 1989). MR 1144322 (93g:11071)

[JBS03] A. Jorza, J. Balakrishna, and W. Stein, *The Smallest Conductor for an Elliptic Curve of Rank Four is Composite,* `http://modular.math.washington.edu/rank4/`.

[JC93] David-Olivier Jaquet-Chiffelle, *Énumération complète des classes de formes parfaites en dimension* 7, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 1, 21–55. MR 1209694 (94d:11048)

[Kan00] Masanobu Kaneko, *The Akiyama-Tanigawa algorithm for Bernoulli numbers*, J. Integer Seq. **3** (2000), no. 2, Article 00.2.9, 6 pp. (electronic). MR 1800883 (2001k:11026)

[Kel06] Bernd C. Kellner, *Bernoulli numbers*, `http://www.bernoulli.org` (2006).

[Kna92] A. W. Knapp, *Elliptic curves*, Princeton University Press, Princeton, NJ, 1992.

[Knu] Donald E. Knuth, *The art of computer programming. Vol. 2*, third ed., Addison-Wesley Publishing Co., Reading, Mass., Seminumerical algorithms, Addison-Wesley Series in Computer Science and Information Processing.

[Kob84] N. Koblitz, *Introduction to elliptic curves and modular forms*, Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1984. MR 86c:11040

[Kri90] Aloys Krieg, *Hecke algebras*, Mem. Amer. Math. Soc. **87** (1990), no. 435, x+158. MR 1027069 (90m:16024)

[Laf02] Laurent Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002), no. 1, 1–241. MR 1875184 (2002m:11039)

[Lan66] R. P. Langlands, *Eisenstein series*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 235–252. MR 0249539 (40 #2784)

[Lan76] Robert P. Langlands, *On the functional equations satisfied by Eisenstein series*, Springer-Verlag, Berlin, 1976, Lecture Notes in Mathematics, Vol. 544. MR 0579181 (58 #28319)

[Lan95] S. Lang, *Introduction to modular forms*, Springer-Verlag, Berlin, 1995, with appendixes by D. Zagier and W. Feit, corrected reprint of the 1976 original.

[Lem01] Dominic Lemelin, *Mazur-tate type conjectures for elliptic curves defined over quadratic imaginary fields*.

[Leo58] Heinrich-Wolfgang Leopoldt, *Eine Verallgemeinerung der Bernoullischen Zahlen*, Abh. Math. Sem. Univ. Hamburg **22** (1958), 131–140. MR 0092812 (19,1161e)

[Li75] W-C. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285–315.

[Lin05] M. Lingham, *Modular forms and elliptic curves over imaginary quadratic fields*, Ph.D. thesis, Nottingham, 2005.

[LLL82] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann. **261** (1982), no. 4, 515–534. MR 682664 (84a:12002)

[LS76] Ronnie Lee and R. H. Szczarba, *On the homology and cohomology of congruence subgroups*, Invent. Math. **33** (1976), no. 1, 15–53. MR 0422498 (54 #10485)

[LS90] J.-P. Labesse and J. Schwermer (eds.), *Cohomology of arithmetic groups and automorphic forms*, Lecture Notes in Mathematics, vol. 1447, Berlin, Springer-Verlag, 1990. MR 1082959 (91h:11033)

[LS02] Joan-C. Lario and René Schoof, *Some computations with Hecke rings and deformation rings*, Experiment. Math. **11** (2002), no. 2, 303–311, with an appendix by Amod Agashe and William Stein. MR 1959271 (2004b:11072)

[LS04] Jian-Shu Li and Joachim Schwermer, *On the Eisenstein cohomology of arithmetic groups*, Duke Math. J. **123** (2004), no. 1, 141–169. MR 2060025 (2005h:11108)

[Lub94] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994, with an appendix by Jonathan D. Rogawski.

[Man72] J. I. Manin, *Parabolic points and zeta functions of modular curves*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 19–66. MR 47 #3396

[Mar01] François Martin, *Périodes de formes modulaires de poids 1*.

[Mar03] Jacques Martinet, *Perfect lattices in Euclidean spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 327, Springer-Verlag, Berlin, 2003. MR 1957723 (2003m:11099)

[Mar05] Greg Martin, *Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$*, J. Number Theory **112** (2005), no. 2, 298–331. MR 2141534 (2005m:11069)

[Maz73] B. Mazur, *Courbes elliptiques et symboles modulaires*, Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 414, Springer, Berlin, 1973, pp. 277–294. Lecture Notes in Math., Vol. 317. MR 55 #2930

[McC91] M. McConnell, *Classical projective geometry and arithmetic groups*, Math. Ann. **290** (1991), no. 3, 441–462. MR 92k:22020

[Men79] Eduardo R. Mendoza, *Cohomology of* PGL_2 *over imaginary quadratic integers*, Bonner Mathematische Schriften [Bonn Mathematical Publications], 128, Universität Bonn Mathematisches Institut, Bonn, 1979, Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1979. MR 611515 (82g:22012)

[Mer94] L. Merel, *Universal Fourier expansions of modular forms*, On Artin's conjecture for odd 2-dimensional representations, Springer, 1994, pp. 59–94.

[Mer99] ———, *Arithmetic of elliptic curves and Diophantine equations*, J. Théor. Nombres Bordeaux **11** (1999), no. 1, 173–200, Les XXèmes Journées Arithmétiques (Limoges, 1997). MR 1730439 (2000j:11084)

[Mes86] J.-F. Mestre, *La méthode des graphes. Exemples et applications*, Proceedings of the international conference on class numbers and fundamental units of algebraic number fields (Katata) (1986), 217–242.

[Miy89] T. Miyake, *Modular forms*, Springer-Verlag, Berlin, 1989, translated from the Japanese by Yoshitaka Maeda.

[MM89] R. MacPherson and M. McConnell, *Classical projective geometry and modular varieties*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 237–290. MR 98k:14076

[MM93] ———, *Explicit reduction theory for Siegel modular threefolds*, Invent. Math. **111** (1993), no. 3, 575–625. MR 94a:32052

[MTT86] B. Mazur, J. Tate, and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), no. 1, 1–48.

[MW94] Colette Mœglin and Jean-Loup Waldspurger, *Décomposition spectrale et séries d'Eisenstein*, Progress in Mathematics, vol. 113, Birkhäuser Verlag, Basel, 1994, Une paraphrase de l'Écriture [A paraphrase of Scripture]. MR 1261867 (95d:11067)

[Nec94] V. I. Nechaev, *On the complexity of a deterministic algorithm for a discrete logarithm*, Mat. Zametki **55** (1994), no. 2, 91–101, 189. MR 96a:11145

[Ong86] Heidrun E. Ong, *Perfect quadratic forms over real-quadratic number fields*, Geom. Dedicata **20** (1986), no. 1, 51–77. MR 823160 (87f:11023)

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, MA, 1994, translated from the 1991 Russian original by Rachel Rowen. MR 1278263 (95b:11039)

[Que06] J. Quer, *Dimensions of spaces of modular forms for* $\Gamma_H(N)$, Preprint.

[Rib92] K. A. Ribet, *Abelian varieties over* $\mathbf{Q}$ *and modular forms*, Algebra and topology 1992 (Taejŏn), Korea Adv. Inst. Sci. Tech., Taejŏn, 1992, pp. 53–79. MR 94g:11042

[Ros86] M. Rosen, *Abelian varieties over* $\mathbf{C}$, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 79–101.

[RS01] K. A. Ribet and W. A. Stein, *Lectures on Serre's conjectures*, Arithmetic algebraic geometry (Park City, UT, 1999), IAS/Park City Math. Ser., vol. 9, Amer. Math. Soc., Providence, RI, 2001, pp. 143–232. MR 2002h:11047

[Sap97] Leslie Saper, *Tilings and finite energy retractions of locally symmetric spaces*, Comment. Math. Helv. **72** (1997), no. 2, 167–202. MR 1470087 (99a:22019)

[Sar03] Peter Sarnak, *Spectra of hyperbolic surfaces*, Bull. Amer. Math. Soc. (N.S.) **40** (2003), no. 4, 441–478 (electronic). MR 1997348 (2004f:11107)

[SC03] Samir Siksek and John E. Cremona, *On the Diophantine equation* $x^2+7 = y^m$, Acta Arith. **109** (2003), no. 2, 143–149. MR 1980642 (2004c:11109)

[Sch86] Joachim Schwermer, *Holomorphy of Eisenstein series at special points and cohomology of arithmetic subgroups of* $\mathrm{SL}_n(\mathbf{Q})$, J. Reine Angew. Math. **364** (1986), 193–220. MR 817646 (87h:11048)

[Sch90] A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), no. 2, 419–430.

[Sch95] R. Schoof, *Counting points on elliptic curves over finite fields*, J. Théor. Nombres Bordeaux **7** (1995), no. 1, 219–254, Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993). MR 1413578 (97i:11070)

[Ser73] J-P. Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7.

[Ser87] ———, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.

[Shi59] G. Shimura, *Sur les intégrales attachées aux formes automorphes*, J. Math. Soc. Japan **11** (1959), 291–311.

[Shi94] ———, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press, Princeton, NJ, 1994, reprint of the 1971 original, Kan Memorial Lectures, 1.

[Sho80a] V. V. Shokurov, *Shimura integrals of cusp forms*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 670–718, 720. MR 582162 (82b:10029)

[Sho80b] ———, *A study of the homology of Kuga varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 2, 443–464, 480. MR 571104 (82f:14023)

[Sho97] Victor Shoup, *Lower bounds for discrete logarithms and related problems*, Advances in cryptology—EUROCRYPT '97 (Konstanz), Lecture Notes in Comput. Sci., vol. 1233, Springer, Berlin, 1997, pp. 256–266. MR 98j:94023

[Sil92] J. H. Silverman, *The arithmetic of elliptic curves*, Springer-Verlag, New York, 1992, corrected reprint of the 1986 original.

[Sou75] Christophe Soulé, *Cohomologie de* $SL_3(Z)$, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), no. 5, Ai, A251–A254. MR 0396849 (53 #709)

[Sta79] R. E. Staffeldt, *Reduction theory and* K_3 *of the Gaussian integers*, Duke Math. J. **46** (1979), no. 4, 773–798. MR 552526 (80m:22014)

[Ste] Allan Steel, *Advanced matrix algorithms*, Seminar Talk at Harvard University.

[Ste97] ———, *A new algorithm for the computation of canonical forms of matrices over fields*, J. Symbolic Comput. **24** (1997), no. 3-4, 409–432, Computational algebra and number theory (London, 1993). MR 1484489 (98m:65070)

[Ste99a] Norman Steenrod, *The topology of fibre bundles*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, reprint of the 1957 edition, Princeton Paperbacks. MR 1688579 (2000a:55001)

[Ste99b] W. A. Stein, `HECKE`*: The Modular Symbols Calculator*, software (available online) (1999).

[Ste00] ———, *Explicit approaches to modular abelian varieties*, Ph.D. thesis, University of California, Berkeley (2000).

[Ste06] ———, `SAGE`: *Software for Algebra and Geometry Experimentation,* `http://sage.scipy.org/sage`.

[Str69] Volker Strassen, *Gaussian elimination is not optimal*, Numerische Mathematik **13** (1969), 354–356.

[Stu87] J. Sturm, *On the congruence of modular forms*, Number theory (New York, 1984–1985), Springer, Berlin, 1987, pp. 275–280.

[SV01] W. A. Stein and H. A. Verrill, *Cuspidal modular symbols are transportable*, LMS J. Comput. Math. **4** (2001), 170–181 (electronic). MR 1 901 355

[SV03] B. Speh and T. N. Venkataramana, *Construction of some generalised modular symbols*, preprint, 2003.

[SW02] William A. Stein and Mark Watkins, *A database of elliptic curves—first report*, Algorithmic number theory (Sydney, 2002), Lecture Notes in Comput. Sci., vol. 2369, Springer, Berlin, 2002, pp. 267–275. MR 2041090 (2005h:11113)

[SW05] Jude Socrates and David Whitehouse, *Unramified Hilbert modular forms, with examples relating to elliptic curves*, Pacific J. Math. **219** (2005), no. 2, 333–364. MR 2175121

[Tat75] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1975, pp. 33–52. Lecture Notes in Math., Vol. 476. MR 52 #13850

[Tho89] J. G. Thompson, *Hecke operators and noncongruence subgroups*, Group theory (Singapore, 1987), de Gruyter, Berlin, 1989, including a letter from J.-P. Serre, pp. 215–224. MR 981844 (90a:20105)

[Tot05] A. Toth, *On the Steinberg module of Chevalley groups*, Manuscripta Math. **116** (2005), no. 3, 277–295.

[TW95] R. Taylor and A. J. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553–572.

[vdG] Gerard van der Geer, *Siegel Modular Forms*, arXiv:math.AG/0605346.

[vGvdKTV97] Bert van Geemen, Wilberd van der Kallen, Jaap Top, and Alain Verberkmoes, *Hecke eigenforms in the cohomology of congruence subgroups of* $\mathrm{SL}(3, \mathbf{Z})$, Experiment. Math. **6** (1997), no. 2, 163–174. MR 1474576 (99a:11059)

[Vig77] Marie-France Vignéras, *Séries thêta des formes quadratiques indéfinies*, Séminaire Delange-Pisot-Poitou, 17e année (1975/76), Théorie des nombres: Fasc. 1, Exp. No. 20, Secrétariat Math., Paris, 1977, p. 3. MR 0480352 (58 #521)

[Vog85] K. Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), no. 3, 399–419.

[Vog97] David A. Vogan, Jr., *Cohomology and group representations*, Representation theory and automorphic forms (Edinburgh, 1996), Proc. Sympos. Pure Math., vol. 61, Amer. Math. Soc., Providence, RI, 1997, pp. 219–243. MR 1476500 (98k:22064)

[Vor08] G. Voronoĭ, *Nouvelles applications des paramétres continus à la théorie des formes quadratiques, I. Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math. **133** (1908), 97–178.

[VZ84] David A. Vogan, Jr. and Gregg J. Zuckerman, *Unitary representations with nonzero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90. MR 762307 (86k:22040)

[Wan82] Kai Wang, *A proof of an identity of the Dirichlet L-function*, Bull. Inst. Math. Acad. Sinica **10** (1982), no. 3, 317–321. MR 679019 (84c:10040)

[Wan95] Xiang Dong Wang, 2-*dimensional simple factors of* $J_0(N)$, Manuscripta Math. **87** (1995), no. 2, 179–197. MR 1334940 (96h:11059)

[Wes] U. Weselman, personal communication.

[Whi90] E. Whitley, *Modular symbols and elliptic curves over imaginary quadratic number fields*, Ph.D. thesis, Exeter University, 1990.

[Wie05] Gabor Wiese, *Modular Forms of Weight One Over Finite Fields*, Ph.D. thesis (2005).

[Wil95] A. J. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551. MR 1333035 (96d:11071)

[Wil00] ———, *The Birch and Swinnerton-Dyer Conjecture*, `http://www.claymath.org/prize_problems/birchsd.htm`.

[Yas05a] D. Yasaki, *On the cohomology of* $\mathrm{SU}(2,1)$ *over the Gaussian integers*, preprint, 2005.

[Yas05b] ———, *On the existence of spines for* $\mathbf{Q}$*-rank 1 groups*, preprint, 2005.

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— ***John E. Cremona, University of Nottingham***

William Stein is an associate professor of mathematics at the University of Washington at Seattle. He earned a PhD in mathematics from UC Berkeley and has held positions at Harvard University and UC San Diego. His current research interests lie in modular forms, elliptic curves, and computational mathematics.

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