



Modular metric spaces, I: Basic concepts[☆]

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ABSTRACT

The notion of a modular is introduced as follows. A (metric) *modular* on a set X is a function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ satisfying, for all $x, y, z \in X$, the following three properties: $x = y$ if and only if $w(\lambda, x, y) = 0$ for all $\lambda > 0$; $w(\lambda, x, y) = w(\lambda, y, x)$ for all $\lambda > 0$; $w(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, y, z)$ for all $\lambda, \mu > 0$. We show that, given $x_0 \in X$, the set $X_w = \{x \in X : \lim_{\lambda \rightarrow \infty} w(\lambda, x, x_0) = 0\}$ is a metric space with metric $d_w^\circ(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}$, called a *modular space*. The modular w is said to be *convex* if $(\lambda, x, y) \mapsto \lambda w(\lambda, x, y)$ is also a modular on X . In this case X_w coincides with the set of all $x \in X$ such that $w(\lambda, x, x_0) < \infty$ for some $\lambda = \lambda(x) > 0$ and is metrizable by $d_w^*(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq 1\}$. Moreover, if $d_w^\circ(x, y) < 1$ or $d_w^*(x, y) < 1$, then $(d_w^\circ(x, y))^2 \leq d_w^*(x, y) \leq d_w^\circ(x, y)$; otherwise, the reverse inequalities hold. We develop the theory of metric spaces, generated by modulars, and extend the results by H. Nakano, J. Musielak, W. Orlicz, Ph. Turpin and others for modulars on linear spaces.

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1. Introduction

The purpose of this paper, which is split into two parts, is to define the notion of a modular on an arbitrary set, develop the theory of metric spaces generated by modulars, called modular metric spaces and, on the basis of it, define new metric spaces of (multivalued) functions of bounded generalized variation of a real variable with values in metric semigroups and abstract convex cones. As an application we present an exhausting description of Lipschitz continuous and some other classes of superposition (or Nemytskii) operators, acting in these modular metric spaces.

In order to motivate our investigations, let us recall the notion of a function of bounded φ -variation (e.g., [1,2]), where $\varphi : \mathbb{R}^+ = [0, \infty) \rightarrow \mathbb{R}^+$ is a φ -function, i.e., a continuous nondecreasing unbounded function vanishing at zero only. Let X be the set of all real valued functions $x : I \rightarrow \mathbb{R}$ on the closed interval $I = [a, b] \subset \mathbb{R}$ with $a < b$ such that $x(a) = 0$. Clearly, X is a real linear space. The φ -variation of a function $x \in X$ is the quantity

$$\rho(x) = V_\varphi(x) = \sup \sum_{i=1}^m \varphi(|x(t_i) - x(t_{i-1})|) \in [0, \infty], \quad (1.1)$$

where the supremum is taken over all partitions $\{t_i\}_{i=0}^m$ of the interval I , i.e., $m \in \mathbb{N}$ and $a = t_0 < t_1 < \dots < t_{m-1} < t_m = b$. The linear subspace X_ρ of X of functions of bounded generalized φ -variation is defined by means of the functional ρ on X , which gives an example of a modular on X , as follows.

According to Orlicz [3], a *modular* on a real linear space X is a functional $\rho : X \rightarrow [0, \infty]$ satisfying the following four conditions: (A.1) $\rho(0) = 0$; (A.2) if $x \in X$ and $\rho(\alpha x) = 0$ for all numbers $\alpha > 0$, then $x = 0$; (A.3) $\rho(-x) = \rho(x)$ for all

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$x \in X$; and (A.4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$. A modular ρ on X is said to be convex [4] if, instead of the inequality in (A.4), the following inequality holds: $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$. The modular (1.1) is convex if the function φ is convex. It was shown in [5] that if ρ is a modular on X , then

$$X_\rho = \{x \in X : \lim_{\alpha \rightarrow +0} \rho(\alpha x) = 0\}, \quad (1.2)$$

called a *modular space*, is a linear subspace of X , and it can be equipped with an F -norm according to the rule:

$$|x|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \leq \varepsilon\}, \quad x \in X_\rho, \quad (1.3)$$

so that (by the definition of an F -norm) the functional $|\cdot|_\rho : X_\rho \rightarrow \mathbb{R}^+$ has the properties: (F.1) given $x \in X_\rho$, $|x|_\rho = 0$ if and only if $x = 0$; (F.2) $|-x|_\rho = |x|_\rho$ for all $x \in X_\rho$; (F.3) $|x+y|_\rho \leq |x|_\rho + |y|_\rho$ for all $x, y \in X_\rho$; and (F.4) if $c_n, c \in \mathbb{R}$ and $x_n, x \in X_\rho$ for $n \in \mathbb{N}$, $c_n \rightarrow c$ in \mathbb{R} and $|x_n - x|_\rho \rightarrow 0$ as $n \rightarrow \infty$, then $|c_n x_n - c x|_\rho \rightarrow 0$ as $n \rightarrow \infty$. Moreover, if the modular ρ on X is convex, then the modular space X_ρ coincides with

$$X_\rho^* = \{x \in X : \exists \text{ a number } \alpha = \alpha(x) > 0 \text{ such that } \rho(\alpha x) < \infty\}$$

and the functional

$$\|x\|_\rho = \inf\{\varepsilon > 0 : \rho(x/\varepsilon) \leq 1\}, \quad x \in X_\rho = X_\rho^*, \quad (1.4)$$

is an ordinary norm on X_ρ , which is equivalent to the F -norm $|\cdot|_\rho$ in the following sense [4]: given $x \in X_\rho$, the inequalities $|x|_\rho \leq 1$ and $\|x\|_\rho \leq 1$ are equivalent, and if at least one of them holds, then $\|x\|_\rho \leq |x|_\rho \leq \sqrt{\|x\|_\rho}$, whereas, otherwise, one has $\sqrt{\|x\|_\rho} \leq |x|_\rho \leq \|x\|_\rho$.

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [6,7] and were intensively developed by his mathematical school: Amemiya, Koshi, Shimogaki, Yamamuro [8,6,9] and others. Further and the most complete development of these theories are due to Orlicz, Mazur, Musielak, Luxemburg, Turpin [10,11,4,5,3,12] and their collaborators. In the present time the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [13–21,1,22,23], which in their turn have broad applications [24,19,21,25,22].

However, in spite of the significant generality of the modular spaces, in certain situations (e.g., connected with problems from multivalued analysis [26–31] such as the definition of metric functional spaces, description of the action of multivalued superposition operators) the notion of a modular on a linear space or on a space with an additional algebraic structure is too restrictive. So, the aim of the first part of this paper is to define and develop a new notion of a modular on an arbitrary set X coherent with the classical notion and to construct the corresponding theory of modular metric spaces adapted to the problems of description of (multivalued) superposition operators, which will be presented in the second part of the paper.

Returning back to the example of functions of finite φ -variation, suppose that the triple $(M, d, +)$ is a metric semigroup with zero $0 \in M$, i.e., an Abelian semigroup with respect to the addition operation $+$ equipped with metric d , which is translation invariant (for more details see Section 2.14). An example of M is the family of all nonempty compact convex subsets of a real normed linear space Z endowed with the Hausdorff metric d [32]. Denote by X the set of all functions $x : I \rightarrow M$ such that $x(a) = 0$. Noting that $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$, is a metric on \mathbb{R} and that the argument of the function φ from (1.1) can be rewritten for the difference $x - y$ of two functions $x, y : I \rightarrow \mathbb{R}$ in the form

$$|(x - y)(t) - (x - y)(s)| = d(x(t) + y(s), y(t) + x(s)), \quad t, s \in I,$$

and taking into account (1.3) or (1.4), in the general case of a metric semigroup $(M, d, +)$ -valued functions it is natural to set ([31, Section 3] and Section 2.15): given $\lambda > 0$ and $x, y \in X$,

$$w_\lambda(x, y) = \sup \sum_{i=1}^m \varphi \left(\frac{1}{\lambda} d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1})) \right), \quad (1.5)$$

where the supremum is taken over all partitions $\{t_i\}_{i=0}^m$ of the interval I . Then the function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ acting according to the rule $(\lambda, x, y) \mapsto w_\lambda(x, y)$ possesses certain basic properties, which are postulated as axioms of a metric modular on X (see Definition 2.1). The modular w defines an equivalence relation on X , which partitions X into equivalence classes, called *modular sets*. A modular set can be equipped with a metric turning it into a metric space. If the function φ is convex, then w from (1.5) is a convex metric modular (see Definition 3.3). The flexibility of our notion of a modular is due to the fact that, e.g., the space of functions $x : I \rightarrow M$ of bounded generalized φ -variation can be generated by different modulars and endowed with different metrics; for instance, replacing $1/\lambda$ by $1/\lambda^2$ in (1.5) we get a metric modular, which is no longer convex even if φ is a convex function. Although the notion of a metric modular is different from the classical notion of a modular on a linear space, they are coherent (cf. Theorem 3.11), and in what follows we present, on the whole, generalizations of classical linear results to modular metric spaces, which is motivated by applications to the theory of superposition operators (in part II). A more general variant of the theory of modular metric spaces is given in [33]. In short form some of the results of this paper were announced in [34] without proofs.

Part I is organized as follows. In Section 2 we define the notion of a metric modular on an arbitrary set and develop the theory of modular metric spaces. The main results of this section are Theorems 2.6, 2.8, 2.10, 2.13 and 2.17. In Section 3 we study the notion of a convex modular and construct the corresponding modular metric spaces, which have applications to the description of certain classes of superposition operators in the second part of the paper. The main results of Section 3 are Theorems 3.6–3.9, 3.11 and 3.14.

2. Metric modulars

Throughout the paper X is a nonempty set, $\lambda > 0$ is understood in the sense that $\lambda \in (0, \infty)$ and, due to the disparity of the arguments, functions $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ will be written as $w_\lambda(x, y) = w(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.1. A function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ is said to be a *metric modular on X* (or simply a *modular* if no ambiguity arises) if it satisfies the following three axioms:

- (i) given $x, y \in X$, $w_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
- (ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
- (iii) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If, instead of (i), we have only the condition

- (i') $w_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then w is said to be a (*metric*) *pseudomodular on X* .

2.2

Definition 2.1 appeared implicitly in [30, Section 4] and [31, Section 3] and explicitly in [34]. Condition $w : (0, \infty) \times X \times X \rightarrow (-\infty, \infty]$ in it does not lead to a greater generality: in fact, setting $x = y$ and $\mu = \lambda > 0$ in (iii) and taking into account (i'), for all $y, z \in X$, we find $0 = w_{2\lambda}(y, y) \leq 2w_\lambda(y, z)$. If $w_\lambda(x, y)$ does not depend on $x, y \in X$, then, by (i'), $w \equiv 0$ (although it may formally seem that (iii) in this case is a general subadditivity condition of the function $\lambda \mapsto w_\lambda$). Now, if $w_\lambda(x, y) = w(x, y)$ does not depend on $\lambda > 0$ and assumes *only finite* values, then axioms (i)–(iii) mean that w is a *metric on X* (*pseudometric* if (i) is replaced by (i')).

2.3

The main property of a (pseudo)modular w on a set X is the following: given $x, y \in X$, the function $0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty]$ is *nonincreasing on $(0, \infty)$* . In fact, if $0 < \mu < \lambda$, then (iii), (i') and (ii) imply

$$w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(y, x) = w_\mu(y, x) = w_\mu(x, y).$$

It follows that at each point $\lambda > 0$ the right limit $w_{\lambda+0}(x, y) = \lim_{\mu \rightarrow \lambda+0} w_\mu(x, y)$ and the left limit $w_{\lambda-0}(x, y) = \lim_{\varepsilon \rightarrow +0} w_{\lambda-\varepsilon}(x, y)$ exist in $[0, \infty]$ and the following two inequalities hold:

$$w_{\lambda+0}(x, y) \leq w_\lambda(x, y) \leq w_{\lambda-0}(x, y). \tag{2.1}$$

Examples 2.4. The following indexed objects w are simple examples of (pseudo)modulars on a set X . Let $\lambda > 0$ and $x, y \in X$. We have:

- (a) $w_\lambda^a(x, y) = \infty$ if $x \neq y$, and $w_\lambda^a(x, y) = 0$ if $x = y$; and if (X, d) is a (pseudo)metric space with (pseudo)metric d , then we also have:
- (b) $w_\lambda^b(x, y) = d(x, y)/\varphi(\lambda)$, where $\varphi : (0, \infty) \rightarrow (0, \infty)$ is a nondecreasing function;
- (c) $w_\lambda^c(x, y) = \infty$ if $\lambda \leq d(x, y)$, and $w_\lambda^c(x, y) = 0$ if $\lambda > d(x, y)$;
- (d) $w_\lambda^d(x, y) = \infty$ if $\lambda < d(x, y)$, and $w_\lambda^d(x, y) = 0$ if $\lambda \geq d(x, y)$.

2.5. The modular set

Let w be a (pseudo)modular on a set X . The binary relation $\overset{w}{\sim}$ on X defined for $x, y \in X$ by

$$x \overset{w}{\sim} y \text{ if and only if } \lim_{\lambda \rightarrow \infty} w_\lambda(x, y) = 0 \tag{2.2}$$

is, by virtue of axioms (i'), (ii) and (iii), an equivalence relation (e.g., if $x \overset{w}{\sim} z$ and $z \overset{w}{\sim} y$, then $w_\lambda(x, y) \leq w_{\lambda/2}(x, z) + w_{\lambda/2}(y, z) \rightarrow 0$ as $\lambda \rightarrow \infty$, and so, $x \overset{w}{\sim} y$). Denote by $X/\overset{w}{\sim}$ the quotient-set of X with respect to $\overset{w}{\sim}$ and by $X_w^\circ(x) = \{y \in X : y \overset{w}{\sim} x\}$ the equivalence class of the element $x \in X$ in the quotient-set $X/\overset{w}{\sim}$. Note, in particular, that $x \in X_w^\circ(x)$ and that the transitivity property of $\overset{w}{\sim}$ implies $y \overset{w}{\sim} z$ if and only if $y, z \in X_w^\circ(x)$ for some $x \in X$ (e.g., $x = y$ or $x = z$).

It follows from Section 2.3 that the function $\tilde{d} : (X/\overset{w}{\sim}) \times (X/\overset{w}{\sim}) \rightarrow [0, \infty]$ given by $\tilde{d}(X_w^\circ(x), X_w^\circ(y)) = \lim_{\lambda \rightarrow \infty} w_\lambda(x, y)$, $x, y \in X$, is well defined (the limit at the right-hand side does not depend on the representatives of the equivalence classes) and satisfies the axioms of a metric, except, as **Example 2.4(a)** shows, that it may take infinite values.

In what follows we are interested in the equivalence classes $X_w^\circ(x)$. Note that the quotient-pair $(X/\overset{w}{\sim}, \tilde{d})$ may degenerate in interesting and important cases: e.g., in **Example 2.4(c)** we have $X_w^\circ(x) = X$ for all $x \in X$ and $\tilde{d} \equiv 0$.

Let us fix an element $x_0 \in X$ arbitrarily and set $X_w = X_w^\circ(x_0)$. The set X_w is called a *modular set*. We note that condition (2.2) in the definition of X_w is an analogue of the condition in the definition of X_ρ from (1.2), which is known in the literature as *condition (B.1) or B_1* (cf. [19,4,1,5] and the end of Section 2.15).

Theorem 2.6. *If w is a metric (pseudo)modular on X , then the modular set X_w is a (pseudo)metric space with (pseudo)metric given by*

$$d_w^\circ(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq \lambda\}, \quad x, y \in X_w.$$

Proof. Given $x, y \in X_w$, the value $d_w^\circ(x, y) \in \mathbb{R}^+$ is well defined: in fact, since $x \overset{w}{\sim} y$, then, by virtue of (2.2), there exists $\lambda_0 > 0$ such that $w_\lambda(x, y) \leq 1$ for all $\lambda \geq \lambda_0$, and so, setting $\lambda_1 = \max\{1, \lambda_0\}$, we get $w_{\lambda_1}(x, y) \leq 1 \leq \lambda_1$, which together with the definition of $d_w^\circ(x, y)$ gives $d_w^\circ(x, y) \leq \lambda_1 < \infty$.

Given $x \in X_w$, (i') implies $w_\lambda(x, x) = 0 < \lambda$ for all $\lambda > 0$, and so, $d_w^\circ(x, x) = 0$. Let w satisfy (i), $x, y \in X_w$ and $d_w^\circ(x, y) = 0$. Then $w_\mu(x, y)$ does not exceed μ for all $\mu > 0$. Hence for any $\lambda > 0$ and $0 < \mu < \lambda$, in view of Section 2.3, we have $w_\lambda(x, y) \leq w_\mu(x, y) \leq \mu \rightarrow 0$ as $\mu \rightarrow +0$. It follows that $w_\lambda(x, y) = 0$ for all $\lambda > 0$, and so, axiom (i) implies $x = y$.

Due to axiom (ii), the equality $d_w^\circ(x, y) = d_w^\circ(y, x)$, $x, y \in X_w$, is clear.

Let us show that $d_w^\circ(x, y) \leq d_w^\circ(x, z) + d_w^\circ(y, z)$ for all $x, y, z \in X_w$. In fact, by the definition of d_w° , for any $\lambda > d_w^\circ(x, z)$ and $\mu > d_w^\circ(y, z)$ we find $w_\lambda(x, z) \leq \lambda$ and $w_\mu(y, z) \leq \mu$, and so, axiom (iii) implies

$$w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(y, z) \leq \lambda + \mu.$$

It follows from the definition of d_w° that $d_w^\circ(x, y) \leq \lambda + \mu$, and it remains to pass to the limits as $\lambda \rightarrow d_w^\circ(x, z)$ and $\mu \rightarrow d_w^\circ(y, z)$. \square

The metric d_w° is a counterpart of the F -norm $|\cdot|_\rho$ from (1.3), and Theorem 2.6 generalizes the results from [5, Section 1.82] and [11, Section 1.21] for F -norms generated by classical modulars on linear spaces.

Examples 2.7. For modulars $w = w^a, w^b, w^c, w^d$ from Example 2.4 we have, respectively:

- (a) $X_w = \{x_0\}$ and $d_w^\circ(x, y) = 0$.
- (b) If the function φ is bounded from above, then $X_w = \{x_0\}$ and $d_w^\circ(x, y) = 0$. On the other hand, if $\varphi(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$, then $X_w = X$ and $d_w^\circ(x, y) = \psi^{-1}(d(x, y))$, where $\psi(\lambda) = \lambda\varphi(\lambda)$, $\lambda > 0$, is a strictly increasing function on $(0, \infty)$ such that $\psi(\lambda) \rightarrow 0$ as $\lambda \rightarrow +0$ and $\psi(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$ and ψ^{-1} is the inverse function for ψ . In particular, if $\varphi(\lambda) = \lambda^p$ with $p = \text{const} > 0$, then we get: $d_w^\circ(x, y) = (d(x, y))^{1/(p+1)}$.
- (c), (d) $X_w = X$ and $d_w^\circ(x, y) = d(x, y)$.

Theorem 2.8. *Let w be a (pseudo)modular on a set X . Put*

$$d_w^1(x, y) = \inf_{\lambda > 0} (\lambda + w_\lambda(x, y)), \quad x, y \in X_w.$$

Then d_w^1 is a (pseudo)metric on X_w such that $d_w^\circ \leq d_w^1 \leq 2d_w^\circ$ on $X_w \times X_w$.

Proof. Since, for $x, y \in X_w$, the value $w_\lambda(x, y)$ is finite due to (2.2) for $\lambda > 0$ large enough, then the set $\{\lambda + w_\lambda(x, y) : \lambda > 0\} \subset \mathbb{R}^+$ is nonempty and bounded from below, and so, $d_w^1(x, y) \in \mathbb{R}^+$.

If $x \in X_w$, then, by (i'), $\lambda + w_\lambda(x, x) = \lambda$ for all $\lambda > 0$, and so, $d_w^1(x, x) = 0$. Now let w satisfy (i), $x, y \in X_w$ and $d_w^1(x, y) = 0$. The equality $x = y$ will follow from (i) if we show that $w_\lambda(x, y) = 0$ for all $\lambda > 0$. On the contrary, suppose that $w_{\lambda_0}(x, y) > 0$ for some $\lambda_0 > 0$. Then for $\lambda \geq \lambda_0$ we find $\lambda + w_\lambda(x, y) \geq \lambda_0$, and if $0 < \lambda < \lambda_0$, then it follows from Section 2.3 that

$$0 < w_{\lambda_0}(x, y) \leq w_\lambda(x, y) \leq \lambda + w_\lambda(x, y).$$

Thus, $\lambda + w_\lambda(x, y) \geq \lambda_1 = \min\{\lambda_0, w_{\lambda_0}(x, y)\}$ for all $\lambda > 0$, and so, by the definition of d_w^1 , $d_w^1(x, y) \geq \lambda_1 > 0$, which contradicts the assumption.

Axiom (ii) implies the symmetry property of d_w^1 .

Let us establish the triangle inequality: $d_w^1(x, y) \leq d_w^1(x, z) + d_w^1(y, z)$. By the definition of d_w^1 , for any $\varepsilon > 0$ we find $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda + w_\lambda(x, z) \leq d_w^1(x, z) + \varepsilon \quad \text{and} \quad \mu + w_\mu(y, z) \leq d_w^1(y, z) + \varepsilon,$$

whence, applying axiom (iii),

$$\begin{aligned} d_w^1(x, y) &\leq (\lambda + \mu) + w_{\lambda+\mu}(x, y) \leq \lambda + \mu + w_\lambda(x, z) + w_\mu(y, z) \\ &\leq d_w^1(x, z) + \varepsilon + d_w^1(y, z) + \varepsilon, \end{aligned}$$

and it remains to take into account the arbitrariness of $\varepsilon > 0$.

Let us prove that metrics d_w° and d_w^1 are equivalent on X_w . In order to obtain the left-hand side inequality, suppose that $\lambda > 0$ is arbitrary. If $w_\lambda(x, y) \leq \lambda$, then the definition of d_w° implies $d_w^\circ(x, y) \leq \lambda$. Now if $w_\lambda(x, y) > \lambda$, then $d_w^\circ(x, y) \leq w_\lambda(x, y)$: in fact, setting $\mu = w_\lambda(x, y)$ we find $\mu > \lambda$, and so, it follows from Section 2.3 that $w_\mu(x, y) \leq w_\lambda(x, y) = \mu$,

whence $d_w^o(x, y) \leq \mu = w_\lambda(x, y)$. Therefore, for any $\lambda > 0$ we have $d_w^o(x, y) \leq \max\{\lambda, w_\lambda(x, y)\} \leq \lambda + w_\lambda(x, y)$, and so, taking the infimum over all $\lambda > 0$, we arrive at the inequality $d_w^o(x, y) \leq d_w^1(x, y)$.

To obtain the right-hand side inequality, we note that, given $\lambda > 0$ such that $d_w^o(x, y) < \lambda$, by the definition of d_w^o , we get $w_\lambda(x, y) \leq \lambda$, and so, $d_w^1(x, y) \leq \lambda + w_\lambda(x, y) \leq 2\lambda$. Passing to the limit as $\lambda \rightarrow d_w^o(x, y)$, we get $d_w^1(x, y) \leq 2d_w^o(x, y)$. \square

We remark that the metric d_w^1 and Theorem 2.8 are more general variants of the F -norm and the result from [8] (see also [19, Theorem 1.1(b)]).

Examples 2.9. (a) Let us consider the (pseudo)modular $w = w^b$ from Example 2.4(b) with $\varphi(\lambda) = \lambda^p$ and $p = \text{const} > 0$. In order to calculate $d_w^1(x, y) = \inf_{\lambda > 0} f(\lambda)$, where $f(\lambda) = \lambda + (d(x, y)/\lambda^p)$, we note that the derivative $f'(\lambda) = 1 - (pd(x, y)/\lambda^{p+1})$ vanishes only at the point $\lambda_0 = (pd(x, y))^{1/(p+1)}$ and assumes negative values at the left of this point and positive values at the right, and so, λ_0 is the point where f attains the global minimum on $(0, \infty)$. It follows that

$$d_w^1(x, y) = f(\lambda_0) = (p + 1)p^{-p/(p+1)}(d(x, y))^{1/(p+1)}, \quad x, y \in X.$$

Note that this expression is coherent with Theorem 2.8 if we take into account Example 2.7(b) and the inequalities $1 \leq (p + 1)p^{-p/(p+1)} \leq 2$ for all $p > 0$. For classical modulars an example of this type was elaborated in [19, p. 5]. In particular, if $p = 1$, the expressions for $d_w^o(x, y)$ and $d_w^1(x, y)$ are of the form, respectively:

$$d_w^o(x, y) = \sqrt{d(x, y)} \quad \text{and} \quad d_w^1(x, y) = 2\sqrt{d(x, y)}, \quad x, y \in X.$$

(b) The results in Example 2.7(b) and 2.9(a) with $\varphi(\lambda) = \lambda^p$ are valid in a somewhat more general case when the (pseudo)modular w on the set X is p -homogeneous with $p > 0$, i.e., satisfies the condition:

$$w_\lambda(x, y) = w_1(x, y)/\lambda^p, \quad \lambda > 0, \quad x, y \in X.$$

If this is the case, $d(x, y)$ should be replaced by $w_1(x, y)$ in the expressions for $d_w^o(x, y)$ from Example 2.4(b) and for $d_w^1(x, y)$ from Example 2.9(a).

Another example of a p -homogeneous modular on a metric space (X, d) can be given as $w_\lambda(x, y) = (d(x, y)/\lambda)^p = w_1(x, y)/\lambda^p$.

(c) Given a (pseudo)metric space (X, d) and a convex function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ vanishing at zero only (it follows that φ is strictly increasing, continuous and admits the continuous inverse function φ^{-1}), we set:

$$w_\lambda(x, y) = \lambda\varphi(d(x, y)/\lambda), \quad \lambda > 0, \quad x, y \in X.$$

Then w is a metric (pseudo)modular on X (note that modulars of this type are not allowed in the classical theory of modulars on linear spaces). In order to see this, it suffices to verify axiom (iii):

$$\begin{aligned} w_{\lambda+\mu}(x, y) &\leq (\lambda + \mu)\varphi\left(\frac{\lambda}{\lambda + \mu} \cdot \frac{d(x, z)}{\lambda} + \frac{\mu}{\lambda + \mu} \cdot \frac{d(y, z)}{\mu}\right) \\ &\leq (\lambda + \mu)\left[\frac{\lambda}{\lambda + \mu}\varphi\left(\frac{d(x, z)}{\lambda}\right) + \frac{\mu}{\lambda + \mu}\varphi\left(\frac{d(y, z)}{\mu}\right)\right] \\ &= w_\lambda(x, z) + w_\mu(y, z). \end{aligned}$$

If, in addition, the function φ satisfies the condition $\varphi(\lambda) = o(\lambda)$ as $\lambda \rightarrow +0$, that is, $\varphi'(0) = \lim_{\lambda \rightarrow +0} \varphi(\lambda)/\lambda = 0$, then, given $x \in X$, we get:

$$\lim_{\lambda \rightarrow \infty} w_\lambda(x, x_0) = \lim_{\lambda \rightarrow \infty} \lambda\varphi\left(\frac{d(x, x_0)}{\lambda}\right) = \lim_{\mu \rightarrow +0} \frac{d(x, x_0)}{\mu} \varphi(\mu) = 0,$$

and so, $X_w = X$. Then $d_w^o(x, y)$ is of the form:

$$d_w^o(x, y) = \inf\{\lambda > 0 : \varphi(d(x, y)/\lambda) \leq 1\} = d(x, y)/\varphi^{-1}(1).$$

Let $\varphi(\lambda) = \lambda^p$ with $p > 1$. Then $d_w^o(x, y) = d(x, y)$. In order to calculate the value $d_w^1(x, y)$, we turn to Example 2.9(b) and note that

$$w_\lambda(x, y) = \lambda(d(x, y)/\lambda)^p = (d(x, y))^p/\lambda^{p-1} = w_1(x, y)/\lambda^{p-1}.$$

It follows that $d_w^1(x, y) = p(p - 1)^{(1-p)/p}d(x, y)$ for all $x, y \in X$.

(d) Setting $\varphi(\lambda) = e^\lambda$, $\lambda > 0$, and $w = w^b$ in Example 2.4(b), we find that $d_w^1(x, y) = d(x, y)$ if $d(x, y) \leq 1$, and $d_w^1(x, y) = 1 + \log(d(x, y))$ if $d(x, y) > 1$.

Now, let us exhibit specific relations between $\lambda > 0$, $w_\lambda(x, y)$ and $d_w^o(x, y)$.

Theorem 2.10. Given a (pseudo)modular w on X , $x, y \in X_w$ and $\lambda > 0$, we have:

- (a) if $d_w^o(x, y) < \lambda$, then $w_\lambda(x, y) \leq d_w^o(x, y) < \lambda$;
- (b) if $w_\lambda(x, y) = \lambda$, then $d_w^o(x, y) = \lambda$;

(c) if $\lambda = d_w^\circ(x, y) > 0$, then $w_{\lambda+0}(x, y) \leq \lambda \leq w_{\lambda-0}(x, y)$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous from the right on $(0, \infty)$, then along with (a)–(c) we have:

(d) $d_w^\circ(x, y) \leq \lambda$ if and only if $w_\lambda(x, y) \leq \lambda$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous from the left on $(0, \infty)$, then along with (a)–(c) we have:

(e) $d_w^\circ(x, y) < \lambda$ if and only if $w_\lambda(x, y) < \lambda$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous on $(0, \infty)$, then along with (a)–(e) we have:

(f) $d_w^\circ(x, y) = \lambda$ if and only if $w_\lambda(x, y) = \lambda$.

Proof. (a) For any $\mu > 0$ such that $d_w^\circ(x, y) < \mu < \lambda$, by the definition of d_w° and Section 2.3, we have $w_\mu(x, y) \leq \mu$ and $w_\lambda(x, y) \leq w_\mu(x, y)$, whence $w_\lambda(x, y) \leq \mu$, and it remains to pass to the limit as $\mu \rightarrow d_w^\circ(x, y)$.

(b) By the definition, $d_w^\circ(x, y) \leq \lambda$, and item (a) implies $d_w^\circ(x, y) = \lambda$.

(c) For any $\mu > \lambda = d_w^\circ(x, y)$, the definition of d_w° implies $w_\mu(x, y) \leq \mu$, and so, $w_{\lambda+0}(x, y) = \lim_{\mu \rightarrow \lambda+0} w_\mu(x, y) \leq \lim_{\mu \rightarrow \lambda+0} \mu = \lambda$.

For any $0 < \mu < \lambda$ we find $w_\mu(x, y) > \mu$ (otherwise, by the definition of d_w° , we have $\lambda = d_w^\circ(x, y) \leq \mu$), and so, $w_{\lambda-0}(x, y) = \lim_{\mu \rightarrow \lambda-0} w_\mu(x, y) \geq \lim_{\mu \rightarrow \lambda-0} \mu = \lambda$.

(d) The implication \Leftarrow follows from the definition of d_w° . Let us prove the reverse implication. If $d_w^\circ(x, y) < \lambda$, then, by virtue of item (a), $w_\lambda(x, y) < \lambda$, and if $d_w^\circ(x, y) = \lambda$, then $w_\lambda(x, y) = w_{\lambda+0}(x, y) \leq \lambda$, which is a consequence of the continuity from the right of the function $\mu \mapsto w_\mu(x, y)$ and item (c).

(e) By virtue of (a), it suffices to prove the implication \Leftarrow . The definition of d_w° gives $d_w^\circ(x, y) \leq \lambda$, but if, on the contrary, $\lambda = d_w^\circ(x, y)$, then, by item (c), we would have $w_\lambda(x, y) = w_{\lambda-0}(x, y) \geq \lambda$, which contradicts the assumption.

(f) \Leftarrow follows from (b). For the reverse assertion, the two inequalities $w_\lambda(x, y) \leq \lambda \leq w_\lambda(x, y)$ follow from (c). \square

Example 2.11. The limit $w_{\lambda+0}(x, y)$ in Theorem 2.10(c) cannot be replaced by $w_\lambda(x, y)$, in general. To see this, consider the (pseudo)modular $w = w^c$ from Example 2.4(c) (see also 2.7(c)). It is clear that the function $\mu \mapsto w_\mu(x, y)$ is continuous from the left on $(0, \infty)$, but not from the right (at the point $\lambda = d(x, y)$). If $x \neq y$, then for $\lambda = d_w^\circ(x, y) = d(x, y) > 0$ we find

$$w_{\lambda+0}(x, y) = 0 < \lambda = d(x, y) < \infty = w_{\lambda-0}(x, y),$$

whereas $w_\lambda(x, y) = \infty > \lambda$. This example shows also that item (d) may fail without the assumption of the right continuity of the function $\mu \mapsto w_\mu(x, y)$.

A similar situation holds with respect to the left limit in Theorem 2.10(c), (e); an appropriate example is 2.4(d).

2.12

In addition to Theorem 2.10 let us recall two implications, which are always valid: if $w_\lambda(x, y) \leq \lambda$, then $d_w^\circ(x, y) \leq \lambda$, and if $w_\lambda(x, y) > \lambda$, then $d_w^\circ(x, y) \leq w_\lambda(x, y)$; they are established in the proof of Theorem 2.8.

In the next theorem we show that the convergence in metric d_w° and modular convergence of sequences from X_w are equivalent.

Theorem 2.13. Let w be a (pseudo)modular on a set X . Given a sequence $\{x_n\}_{n=1}^\infty \subset X_w$ and $x \in X_w$, we have: $d_w^\circ(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $w_\lambda(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda > 0$. A similar assertion holds for Cauchy sequences.

Proof. *Sufficiency.* Given arbitrary $\varepsilon > 0$, the assumption implies that $w_\varepsilon(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, and so, there is a number $n_0(\varepsilon)$ such that $w_\varepsilon(x_n, x) \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$, whence $d_w^\circ(x_n, x) \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$.

Necessity. Let us fix $\lambda > 0$ arbitrarily. Then, for each $\varepsilon > 0$, we have: either (a) $0 < \varepsilon < \lambda$, or (b) $\varepsilon \geq \lambda$. In case (a), by the assumption, there is a number $n_0(\varepsilon)$ such that $d_w^\circ(x_n, x) < \varepsilon$ for all $n \geq n_0(\varepsilon)$, and so, by Theorem 2.10(a), we get $w_\varepsilon(x_n, x) < \varepsilon$ for all $n \geq n_0(\varepsilon)$. Since $\varepsilon < \lambda$, then, in view of Section 2.3, we find $w_\lambda(x_n, x) \leq w_\varepsilon(x_n, x) < \varepsilon$ for all $n \geq n_0(\varepsilon)$. In case (b) we set $n_1(\varepsilon) = n_0(\lambda/2)$. From Section 2.3 and the just established fact (when $\varepsilon = \lambda/2 < \lambda$), we get:

$$w_\lambda(x_n, x) \leq w_{\lambda/2}(x_n, x) < \lambda/2 \leq \varepsilon/2 < \varepsilon \quad \text{for all } n \geq n_1(\varepsilon).$$

Hence, $w_\lambda(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda > 0$. \square

In order to present the example in Section 2.15, we need the following notion.

2.14. Metric semigroup

([30, Section 4], [31, Section 3]). The triple $(M, d, +)$ is said to be a *metric semigroup* if (M, d) is a metric space with metric d , $(M, +)$ is an Abelian semigroup with respect to the addition operation $+$ and the metric d is translation invariant: $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in M$. The element $0 \in M$ such that $x+0 = 0+x = x$ for all $x \in M$ is called the *zero*. The metric semigroup $(M, d, +)$ is said to be *complete*, if the metric space (M, d) is complete.

Given a metric semigroup $(M, d, +)$ and any $x, y, \bar{x}, \bar{y} \in M$, we have:

$$d(x, y) \leq d(x + \bar{x}, y + \bar{y}) + d(\bar{x}, \bar{y}), \tag{2.3}$$

$$d(x + \bar{x}, y + \bar{y}) \leq d(x, y) + d(\bar{x}, \bar{y}). \tag{2.4}$$

If sequences $\{x_n\}, \{y_n\}, \{\bar{x}_n\}$ and $\{\bar{y}_n\}$ of elements from M converge in M to elements x, y, \bar{x} and \bar{y} as $n \rightarrow \infty$, respectively, then, by virtue of (2.4),

$$d(x_n + \bar{x}_n, y_n + \bar{y}_n) \rightarrow d(x + \bar{x}, y + \bar{y}) \quad \text{as } n \rightarrow \infty, \tag{2.5}$$

and, in particular, the addition operation $(x, y) \mapsto x + y$ in M is a *continuous* mapping from $M \times M$ into M .

2.15. The modular generating functions of finite φ -variation

Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a φ -function (cf. Introduction), $(M, d, +)$ be a metric semigroup, $I = [a, b]$ a closed interval in \mathbb{R} and $X = M^I$ the set of all functions mapping I into M . Given $\lambda > 0$ and $x, y \in X$, we define a function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ by the rule:

$$w_\lambda(x, y) = \sup \sum_{i=1}^m \varphi \left(\frac{1}{\lambda} d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1})) \right), \tag{2.6}$$

where the supremum is taken over all partitions $\{t_i\}_{i=0}^m$ of the interval I .

Let us show that w is a pseudomodular on X , having verified only axiom (iii). We make use of the following simple observation: if $\alpha, \beta \geq 0, \alpha + \beta \leq 1$ and $A, B \geq 0$, then

$$\varphi(\alpha A + \beta B) \leq \max\{\varphi(A), \varphi(B)\} \leq \varphi(A) + \varphi(B).$$

If $\lambda, \mu > 0, x, y, z \in X, m \in \mathbb{N}, a = t_0 < t_1 < \dots < t_{m-1} < t_m = b$ and $i \in \{1, \dots, m\}$, then, by virtue of (2.3) and the translation invariance of d , we have:

$$\begin{aligned} C_i &\equiv d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1})) \\ &\leq d(x(t_i) + y(t_{i-1}) + y(t_i) + z(t_{i-1}), y(t_i) + x(t_{i-1}) + z(t_i) + y(t_{i-1})) + d(y(t_i) + z(t_{i-1}), z(t_i) + y(t_{i-1})) \\ &= d(x(t_i) + z(t_{i-1}), z(t_i) + x(t_{i-1})) + d(y(t_i) + z(t_{i-1}), z(t_i) + y(t_{i-1})) \\ &\equiv A_i + B_i, \end{aligned}$$

and so, the monotonicity of φ and the observation above imply

$$\varphi\left(\frac{C_i}{\lambda + \mu}\right) \leq \varphi\left(\frac{\lambda}{\lambda + \mu} \cdot \frac{A_i}{\lambda} + \frac{\mu}{\lambda + \mu} \cdot \frac{B_i}{\mu}\right) \leq \varphi\left(\frac{A_i}{\lambda}\right) + \varphi\left(\frac{B_i}{\mu}\right).$$

Summing over $i = 1, \dots, m$ and taking the supremum over all partitions of the interval I , we obtain the inequality in axiom (iii).

Let us establish that, given $x, y \in X$, the function $\lambda \mapsto w_\lambda(x, y)$ is *continuous from the right* on $(0, \infty)$. Let $\lambda > 0$. By virtue of (2.1), it suffices to show that $w_\lambda(x, y) \leq w_{\lambda+0}(x, y)$. For each partition $\{t_i\}_{i=0}^m$ of the interval I and any $\mu > \lambda$ we have:

$$\sum_{i=1}^m \varphi\left(\frac{1}{\mu} d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1}))\right) \leq w_\mu(x, y),$$

and so, as $\mu \rightarrow \lambda + 0$, the continuity of φ implies

$$\sum_{i=1}^m \varphi\left(\frac{1}{\lambda} d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1}))\right) \leq w_{\lambda+0}(x, y).$$

It remains to take the supremum over all partitions $\{t_i\}_{i=0}^m$ of the interval I .

For a constant function $x_0(t) = x_0$ for all $t \in I$, where $x_0 \in M$, the definition (2.6) of the pseudomodular w gives (note that the expression on the right does not depend on x_0):

$$w_\lambda(x, x_0) = \sup \sum_{i=1}^m \varphi\left(\frac{1}{\lambda} d(x(t_i), x(t_{i-1}))\right), \quad x \in M^I.$$

If $x \in X_w = X_w^\circ(x_0)$, then x is said to be a *function of bounded generalized φ -variation in the sense of Wiener and Young* (if φ is convex, more details can be found in [31, Section 3]). To the best of the author's knowledge, if the function φ is nonconvex, the question concerning the complete description of the space X_w remains open even in the case when $M = \mathbb{R}$ (cf. (1.1) and (1.2)); in particular, it is not known whether the modular space X_w coincides with the larger set X_w^* defined in Section 3.1 below. For a certain discussion and a partial solution of this problem when $M = \mathbb{R}$ see [16,35].

2.16. The right inverse (pseudo)modular

Let w be a (pseudo)modular on a set X . The *right inverse* for w is the function $w^+ : (0, \infty) \times X \times X \rightarrow [0, \infty]$ defined, for all $\mu > 0$ and $x, y \in X$, by the rule:

$$w_\mu^+(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq \mu\} \quad (\inf \emptyset = \infty).$$

The properties of w^+ are gathered in the following theorem.

Theorem 2.17. *Let w be a (pseudo)modular on X . Then w^+ is also a (pseudo)modular on X such that the following (in)equalities in $[0, \infty]$ hold:*

- (a) $w_{\mu+0}^+(x, y) = w_\mu^+(x, y)$ for all $\mu > 0$ and $x, y \in X$;
 (b) $w_{\lambda+0}(x, y) = w_\lambda^{++}(x, y) \leq w_\lambda(x, y)$ for all $\lambda > 0$ and $x, y \in X$, where $w^{++} = (w^+)^+$.

Proof. Let us verify only axioms (i) and (iii) for w^+ . (i) Let $x, y \in X$ and $w_\mu^+(x, y) = 0$ for all $\mu > 0$. Then $\lambda > w_\mu^+(x, y)$ for all $\lambda > 0$, and so, the definition of w^+ implies $w_\lambda(x, y) \leq \mu$ for all $\mu > 0$, whence $w_\lambda(x, y) = 0$ for all $\lambda > 0$, and so, by (i) for w , we get $x = y$. (iii) Let us show that $w_{\lambda+\mu}^+(x, y) \leq w_\lambda^+(x, z) + w_\mu^+(y, z)$. If $w_\lambda^+(x, z) = \infty$ or $w_\mu^+(y, z) = \infty$, then the inequality is obvious. So, suppose that $w_\lambda^+(x, z) < \infty$ and $w_\mu^+(y, z) < \infty$. Then, given $\xi > w_\lambda^+(x, z)$ and $\eta > w_\mu^+(y, z)$, the definition of w^+ implies $w_\xi(x, z) \leq \lambda$ and $w_\eta(y, z) \leq \mu$, and so,

$$w_{\xi+\eta}(x, y) \leq w_\xi(x, z) + w_\eta(y, z) \leq \lambda + \mu.$$

It follows that $\xi + \eta \in \{\gamma > 0 : w_\gamma(x, y) \leq \lambda + \mu\}$, and so, $w_{\lambda+\mu}^+(x, y) \leq \xi + \eta$, and it remains to pass to the limits as $\xi \rightarrow w_\lambda^+(x, z)$ and $\eta \rightarrow w_\mu^+(y, z)$.

(a) By virtue of (2.1), we have $w_{\mu+0}^+(x, y) \leq w_\mu^+(x, y)$. If $w_{\mu+0}^+(x, y) = \infty$, then, by the last inequality, $w_\mu^+(x, y) = \infty$. Now, let $w_{\mu+0}^+(x, y) < \infty$ and $\lambda > 0$ be any number such that $w_{\mu+0}^+(x, y) < \lambda$. Then there exists $\varepsilon_0 > 0$ such that $w_{\mu+\varepsilon}^+(x, y) < \lambda$ for all $0 < \varepsilon < \varepsilon_0$. From the definition of w^+ we find $w_\lambda(x, y) \leq \mu + \varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$, whence we get $w_\lambda(x, y) \leq \mu$. It follows from Section 2.16 that $w_\mu^+(x, y) \leq \lambda$. Passing to the limit as $\lambda \rightarrow w_{\mu+0}^+(x, y)$ we conclude that $w_\mu^+(x, y) \leq w_{\mu+0}^+(x, y)$.

(b) First, given $\lambda > 0$ and $x, y \in X$, we obtain the inequality in (b). If $w_\lambda(x, y) = \infty$, the inequality is clear. Now if $w_\lambda(x, y) < \infty$, then, for each $\mu > w_\lambda(x, y)$, the definition of w^+ implies $w_\mu^+(x, y) \leq \lambda$. Applying the definition of w^{++} we get $w_\lambda^{++}(x, y) \leq \mu$ for all $\mu > w_\lambda(x, y)$, and so, $w_\lambda^{++}(x, y) \leq w_\lambda(x, y)$.

Now we prove the equality in (b). In the last paragraph we have shown that $w_\xi^{++}(x, y) \leq w_\xi(x, y)$ for all $\xi > 0$, whence, as $\xi \rightarrow \lambda + 0$, we get $w_{\lambda+0}^{++}(x, y) \leq w_{\lambda+0}(x, y)$. Since, by item (a), $w_{\lambda+0}^{++}(x, y) = w_\lambda^{++}(x, y)$, then $w_\lambda^{++}(x, y) \leq w_{\lambda+0}(x, y)$.

If $w_\lambda^{++}(x, y) = \infty$, then the last inequality in the previous paragraph implies $w_{\lambda+0}(x, y) = \infty$. So, assume that $w_\lambda^{++}(x, y) < \infty$. Then, for any $\mu > w_\lambda^{++}(x, y)$, the definition of w^{++} gives $w_\mu^+(x, y) \leq \lambda$. If $w_\mu^+(x, y) < \lambda$, then $w_\lambda(x, y) \leq \mu$, and so, $w_{\lambda+0}(x, y) \leq w_\lambda(x, y) \leq \mu$. Now if $w_\mu^+(x, y) = \lambda$, then, for each $\xi > \lambda = w_\mu^+(x, y)$, the definition of w^+ implies $w_\xi(x, y) \leq \mu$, whence $w_{\lambda+0}(x, y) = \lim_{\xi \rightarrow \lambda+0} w_\xi(x, y) \leq \mu$. Thus, we have shown that $w_{\lambda+0}(x, y) \leq \mu$ for all $\mu > w_\lambda^{++}(x, y)$, and so, $w_{\lambda+0}(x, y) \leq w_\lambda^{++}(x, y)$. \square

Remark 2.18. Let w be a (pseudo)modular on a set X .

Note that $w^{+++} = w^+$; in fact, by Theorem 2.17(a), (b), we have $w_\mu^+(x, y) = w_{\mu+0}^+(x, y) = w_\mu^{+++}(x, y)$. Moreover, if the function $\lambda \mapsto w_\lambda(x, y)$ is continuous from the right on $(0, \infty)$ for all $x, y \in X$, then $w^{+++} = w$.

The term the ‘right inverse (pseudo)modular’ for w^+ can be motivated as follows. For the sake of convenience we temporarily employ also the notation $w(\xi; x, y) = w_\xi(x, y)$. If $\lambda > 0$ and $w_\lambda(x, y) \in (0, \infty)$, then the inequalities $w^+(w_\lambda(x, y); x, y) \leq \lambda$ and $w_\lambda^{++}(x, y) \leq w_\lambda(x, y)$ are equivalent (cf. the inequality in Theorem 2.17(b)). On the other hand, if $\mu > 0$ and $w_\mu^+(x, y) \in (0, \infty)$, then we have the inequalities: $w(w_\mu^+(x, y) + 0; x, y) \leq \mu$ and $\mu \leq w(w_\mu^+(x, y) - 0; x, y)$. Therefore, if the function $\lambda \mapsto w_\lambda(x, y)$ is continuous on $(0, \infty)$, then $w(w_\mu^+(x, y); x, y) = \mu, \mu > 0$.

Examples 2.19. (a) If w is the (pseudo)modular w^b from Example 2.4(b) with $\varphi(\lambda) = \lambda^p$ and $p > 0$, then $w_\mu^+(x, y) = (d(x, y)/\mu)^{1/p}$. In particular, if $p = 1$, then $w^+ = w$.

(b) Applying the notation from Example 2.4(c), (d), we have: if $x, y \in X, \mu > 0$ and $\lambda > 0$, then $w_\mu^{c+}(x, y) = d(x, y), w_\lambda^{c++}(x, y) = w_\lambda^d(x, y)$, and so, the inequality in Theorem 2.17(b) is sharp. Note also that $X_{w^c} = X$, whereas $X_{w^{c+}} = \{x_0\}$.

3. Convex metric modulars

3.1. The set X_w^*

Given a metric (pseudo)modular w on a set X , along with the modular set X_w from Section 2.5 we also put

$$X_w^* \equiv X_w^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } w_\lambda(x, x_0) < \infty\} \quad (x_0 \in X).$$

It is clear from (2.2) that $X_w \subset X_w^*$, and this inclusion is, in general, strict: it is seen from Example 2.7(b) with bounded function φ that $X_w = \{x_0\}$ and $X_w^* = X$. Now let us expose a more subtle example of this type.

Example 3.2. Let (M, d) be a metric space and $X = M^{\mathbb{N}}$ be the set of all sequences $x : \mathbb{N} \rightarrow M$. Define a function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ by

$$w_\lambda(x, y) = \sup_{n \in \mathbb{N}} \left(\frac{d(x(n), y(n))}{\lambda} \right)^{1/n}, \quad \lambda > 0, x, y \in X.$$

Then w is a metric modular on X , for which the function $\lambda \mapsto w_\lambda(x, y)$ is continuous from the right on $(0, \infty)$ for all $x, y \in X$. Fixing some elements $x_0, x \in M, x \neq x_0$, let $x_0, x \in X$ be the corresponding constant functions: $x_0(n) = x_0$ and $x(n) = x$ for all $n \in \mathbb{N}$. It follows that $x \in X_w^*(x_0) \setminus X_w^o(x_0)$: in fact, if $\lambda > d(x, x_0)$, then

$$w_\lambda(x, x_0) = \sup_{n \in \mathbb{N}} (d(x, x_0)/\lambda)^{1/n} = \lim_{n \rightarrow \infty} (d(x, x_0)/\lambda)^{1/n} = 1.$$

Definition 3.3. A function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ is said to be a *convex (metric) modular* on a set X if it satisfies the axioms (i) and (ii) from Definition 2.1 as well as the following axiom

$$(iv) \quad w_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \frac{\mu}{\lambda+\mu} w_\mu(y, z) \quad \forall \lambda, \mu > 0, x, y, z \in X.$$

If, instead of (i), we have only condition (i') from Definition 2.1, then w is called a *convex (metric) pseudomodular* on X .

Clearly, (iv) implies (iii), and so, convex (pseudo)modulars have all the properties presented in Section 2. However, convex (pseudo)modulars have some additional specific properties, which will be studied below. Rewriting the inequality in (iv) in the form $(\lambda + \mu)w_{\lambda+\mu}(x, y) \leq \lambda w_\lambda(x, z) + \mu w_\mu(y, z)$ we find that the function w is a convex (pseudo)modular on X if and only if the function $\widehat{w}_\lambda(x, y) = \lambda w_\lambda(x, y), \lambda > 0, x, y \in X$, is simply a (pseudo)modular on X . The last somewhat unusual property means that the term 'convex modular' needs certain justifications—this will be done at the end of this section in Theorem 3.11.

Examples 3.4. (a) The (pseudo)modulars from Example 2.4 are convex, except item (b); this is true in item (b) as well only if $\varphi(\lambda) = \lambda$ (and not $\varphi(\lambda) \equiv 1$). In Section 2.15 the pseudomodular w is convex if the function φ is convex, and in Example 3.2 the modular w is not convex (cf. (3.2)).

(b) Given a (pseudo)metric space (X, d) and a convex φ -function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we set $w_\lambda(x, y) = \varphi(d(x, y)/\lambda)$ for all $\lambda > 0$ and $x, y \in X$. Since the function \widehat{w} coincides with the (pseudo)modular from Example 2.9(c), the (pseudo)modular w is convex.

(c) Let (M, d) be a metric space, $\varphi_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be convex φ -functions for all $i \in \mathbb{N}$ and $X = M^{\mathbb{N}}$. Setting

$$w_\lambda(x, y) = \sum_{i=1}^{\infty} \varphi_i \left(\frac{d(x_i, y_i)}{\lambda} \right), \quad \lambda > 0, x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in M^{\mathbb{N}},$$

we find that w is a convex modular on X .

3.5

The main property of a convex (pseudo)modular w on a set X can be expressed as follows: given $x, y \in X$, the functions $\lambda \mapsto w_\lambda(x, y)$ and $\lambda \mapsto \widehat{w}_\lambda(x, y) = \lambda w_\lambda(x, y)$ are *nonincreasing* on $(0, \infty)$:

$$\text{if } 0 < \mu \leq \lambda, \text{ then } w_\lambda(x, y) \leq \frac{\mu}{\lambda} w_\mu(x, y) \leq w_\mu(x, y). \tag{3.1}$$

Moreover, condition (iv) is equivalent to the following condition:

$$(iv') \quad w_\nu(x, y) \leq \frac{\lambda}{\nu} w_\lambda(x, z) + \frac{\mu}{\nu} w_\mu(y, z) \quad \forall \lambda, \mu, \nu > 0, \lambda + \mu \leq \nu, x, y, z \in X.$$

If w is a convex (pseudo)modular on a set X , then

$$X_w = X_w^*; \tag{3.2}$$

in fact (cf. Section 3.1), if $x \in X_w^*$, then $w_\mu(x, x_0) < \infty$ for some number $\mu > 0$, and so, by virtue of (3.1), for any $\lambda > \mu$ we find $w_\lambda(x, x_0) \leq (\mu/\lambda)w_\mu(x, x_0) \rightarrow 0$ as $\lambda \rightarrow \infty$, implying $x \in X_w$. In this case the (pseudo)modular \widehat{w} is well defined as well, and the following relations hold:

$$X_w^* = X_w^* = X_w \supset X_{\widehat{w}}. \tag{3.3}$$

However, the last inclusion may be strict as Example 2.4(b) with $\varphi(\lambda) = \lambda$ shows. In order to define a metric on the set X_w^* , note that, by Theorem 2.6, we have the metric d_w^o on $X_{\widehat{w}}$ given by the rule:

$$d_w^o(x, y) = \inf\{\lambda > 0 : \widehat{w}_\lambda(x, y) \leq \lambda\} = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}.$$

The next theorem shows that a metric d_w^* on X_w^* can be introduced by the same rule independently of whether the last inclusion in (3.3) is strict or not.

Theorem 3.6. *Given a convex (pseudo)modular w on a set X , put:*

$$d_w^*(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}, \quad x, y \in X_w^*.$$

Then (X_w^*, d_w^*) is a (pseudo)metric space.

Proof. The value $d_w^*(x, y) \in \mathbb{R}^+$ is well defined for all $x, y \in X_w^*$. In fact, there are numbers $\lambda = \lambda(x) > 0$ and $\mu = \mu(y) > 0$ such that $w_\lambda(x, x_0) < \infty$ and $w_\mu(y, x_0) < \infty$, and so, applying the inequality (iv') with $z = x_0$ we get $\lim_{\nu \rightarrow \infty} w_\nu(x, y) = 0$. It follows that there exists $\nu_0 > 0$ such that $w_\nu(x, y) \leq 1$ for all $\nu \geq \nu_0$, whence the definition of d_w^* implies $d_w^*(x, y) \leq \nu_0 < \infty$.

The remaining part of the proof coincides essentially with the proof of Theorem 2.6, and so, we verify only two axioms of metric d_w^* . If $x, y \in X_w^*$ and $d_w^*(x, y) = 0$, then $w_\mu(x, y) \leq 1$ for all $\mu > 0$. It follows from (3.1) that, given $\lambda > 0$ and $0 < \mu < \lambda$, $w_\lambda(x, y) \leq (\mu/\lambda)w_\mu(x, y) \leq \mu/\lambda \rightarrow 0$ as $\mu \rightarrow 0$, and so, $w_\lambda(x, y) = 0$ for all $\lambda > 0$, whence $x = y$. Now, let $x, y, z \in X_w^*$. If $\lambda > d_w^*(x, z)$ and $\mu > d_w^*(y, z)$, the definition of d_w^* gives $w_\lambda(x, z) \leq 1$ and $w_\mu(y, z) \leq 1$, and so, by axiom (iv),

$$w_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu}w_\lambda(x, z) + \frac{\mu}{\lambda + \mu}w_\mu(y, z) \leq \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} = 1.$$

Thus, $d_w^*(x, y) \leq \lambda + \mu$, and the arbitrariness of λ and μ as above implies $d_w^*(x, y) \leq d_w^*(x, z) + d_w^*(y, z)$. \square

The metric d_w^* is a counterpart of the norm (1.4), and Theorem 3.6 is a generalization of the results from [10,4,6,3] (see also [19, Theorem 1.2] and [21, Theorem 1.5]).

In the next theorem we present a generalization of the norm, which is known in the classical theory of modulars as the *Amemiya norm* (cf. [17], [19, Theorem 1.2 and Remark 2], [21, p. 165], [7, p. 218]).

Theorem 3.7. *Let w be a convex (pseudo)modular on X . Set*

$$d_w^A(x, y) = \inf_{\lambda > 0}(\lambda + \lambda w_\lambda(x, y)), \quad x, y \in X_w^*.$$

Then d_w^A is a (pseudo)metric on X_w^* and $d_w^*(x, y) \leq d_w^A(x, y) \leq 2d_w^*(x, y)$ for all $x, y \in X_w^*$.

Proof. Since $0 < \lambda + \lambda w_\lambda(x, y) < \infty$ for λ large enough (see the beginning of the proof of Theorem 3.6), the set $\{\lambda + \lambda w_\lambda(x, y) : \lambda > 0\}$ is nonempty and bounded from below, and so, d_w^A is well defined: $0 \leq d_w^A(x, y) < \infty$ for all $x, y \in X_w^*$.

Let us verify only the nondegeneracy of d_w^A and the triangle inequality for d_w^A . Suppose that w satisfies (i), $x, y \in X_w^*$ and $d_w^A(x, y) = 0$, and show that $w_\lambda(x, y) = 0$ for all $\lambda > 0$. If this is not so, then $w_{\lambda_0}(x, y) > 0$ for some $\lambda_0 > 0$. We have: if $\lambda \geq \lambda_0$, then $\lambda + \lambda w_\lambda(x, y) \geq \lambda_0$, and if $0 < \lambda < \lambda_0$, then, by virtue of (3.1), we find $w_{\lambda_0}(x, y) \leq (\lambda/\lambda_0)w_\lambda(x, y)$, and so, $\lambda_0 w_{\lambda_0}(x, y) \leq \lambda w_\lambda(x, y) \leq \lambda + \lambda w_\lambda(x, y)$. It follows that, for all $\lambda > 0$, we have $\lambda + \lambda w_\lambda(x, y) \geq \min\{\lambda_0, \lambda_0 w_{\lambda_0}(x, y)\} \equiv \lambda_1 > 0$ implying $d_w^A(x, y) \geq \lambda_1 > 0$, which contradicts the assumption.

Now we show that $d_w^A(x, y) \leq d_w^A(x, z) + d_w^A(y, z)$, $x, y, z \in X_w^*$. By the definition of d_w^A , given $\varepsilon > 0$, there exist $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda + \lambda w_\lambda(x, z) \leq d_w^A(x, z) + \varepsilon \quad \text{and} \quad \mu + \mu w_\mu(y, z) \leq d_w^A(y, z) + \varepsilon,$$

and so, the convexity of w implies

$$\begin{aligned} d_w^A(x, y) &\leq (\lambda + \mu) + (\lambda + \mu)w_{\lambda+\mu}(x, y) \\ &\leq \lambda + \mu + (\lambda + \mu)\left[\frac{\lambda}{\lambda + \mu}w_\lambda(x, z) + \frac{\mu}{\lambda + \mu}w_\mu(y, z)\right] \\ &= \lambda + \lambda w_\lambda(x, z) + \mu + \mu w_\mu(y, z) \\ &\leq d_w^A(x, z) + \varepsilon + d_w^A(y, z) + \varepsilon, \end{aligned}$$

and it remains to pass to the limit as $\varepsilon \rightarrow +0$.

Let us prove the equivalence of metrics d_w^A and d_w^* . Given $\lambda > 0$, we have: if $w_\lambda(x, y) \leq 1$, then, by the definition of d_w^* , $d_w^*(x, y) \leq \lambda$, and if $w_\lambda(x, y) > 1$, then $d_w^*(x, y) \leq \lambda w_\lambda(x, y)$; in fact, setting $\mu = \lambda w_\lambda(x, y)$, we find $\mu > \lambda$, and so, (3.1) implies $w_\mu(x, y) \leq (\lambda/\mu)w_\lambda(x, y) = \lambda w_\lambda(x, y)/\mu = 1$, whence $d_w^*(x, y) \leq \mu = \lambda w_\lambda(x, y)$. It follows that $d_w^*(x, y) \leq \max\{\lambda, \lambda w_\lambda(x, y)\} \leq \lambda + \lambda w_\lambda(x, y)$ for all $\lambda > 0$, and the inequality at the left-hand side follows. In order to establish the right-hand side inequality, note that the definition of d_w^* implies $w_\lambda(x, y) \leq 1$ for all $\lambda > d_w^*(x, y)$, and so, $d_w^A(x, y) \leq \lambda + \lambda w_\lambda(x, y) \leq 2\lambda$, and it remains to pass to the limit as $\lambda \rightarrow d_w^*(x, y)$. \square

For the sake of applications in part II we reformulate **Theorem 2.10** somehow: in the next theorem we establish certain relations between a number $\lambda > 0$, convex modular $w_\lambda(x, y)$ and metric $d_w^*(x, y)$.

Theorem 3.8. *Let w be a convex metric (pseudo)modular on a set X , $\lambda > 0$ and $x, y \in X_w^*$. We have:*

- (a) if $d_w^*(x, y) < \lambda$, then $w_\lambda(x, y) \leq d_w^*(x, y)/\lambda < 1$;
- (b) if $w_\lambda(x, y) = 1$, then $d_w^*(x, y) = \lambda$;
- (c) if $\lambda = d_w^*(x, y) > 0$, then $w_{\lambda+0}(x, y) \leq 1 \leq w_{\lambda-0}(x, y)$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous from the right on $(0, \infty)$, then along with (a)–(c) we have:

- (d) $d_w^*(x, y) \leq \lambda$ if and only if $w_\lambda(x, y) \leq 1$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous from the left on $(0, \infty)$, then along with (a)–(c) we have:

- (e) $d_w^*(x, y) < \lambda$ if and only if $w_\lambda(x, y) < 1$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous on $(0, \infty)$, then along with (a)–(e) we have:

- (f) $d_w^*(x, y) = \lambda$ if and only if $w_\lambda(x, y) = 1$.

Proof. (a) If μ is arbitrary such that $d_w^*(x, y) < \mu < \lambda$, then the definition of d_w^* and (3.1) imply $w_\mu(x, y) \leq 1$ and $w_\lambda(x, y) \leq (\mu/\lambda)w_\mu(x, y) \leq \mu/\lambda$.

(b) The definition d_w^* and item (a) imply, respectively, $d_w^*(x, y) \leq \lambda$ and $d_w^*(x, y) = \lambda$.

(c) Since $w_\mu(x, y) \leq 1$ for all $\mu > \lambda = d_w^*(x, y)$, we have $w_{\lambda+0}(x, y) \leq 1$, and since $w_\mu(x, y) > 1$ for all $0 < \mu < \lambda = d_w^*(x, y)$ (otherwise, $w_\mu(x, y) \leq 1$ implies $\lambda = d_w^*(x, y) \leq \mu$), we have $w_{\lambda-0}(x, y) \geq 1$.

Items (d)–(f) are established in the same manner as the corresponding items in the proof of **Theorem 2.10** taking into account the convexity of the (pseudo)modular. \square

Now we establish a specific equivalence of metrics d_w° and d_w^* for a convex modular w , which is similar to the one encountered in the Introduction. In particular, it will imply an analogue of **Theorem 2.13** for metric d_w^* on the modular set $X_w^* = X_w$.

Theorem 3.9. *Let w be a convex (pseudo)modular on a set X . Then, given $x, y \in X_w^*$, for the (pseudo)metrics d_w^* and d_w° we have:*

- (a) conditions $d_w^\circ(x, y) < 1$ and $d_w^*(x, y) < 1$ are equivalent, and if at least one of them holds, then $d_w^*(x, y) \leq d_w^\circ(x, y) \leq \sqrt{d_w^*(x, y)}$;
- (b) conditions $d_w^\circ(x, y) \geq 1$ and $d_w^*(x, y) \geq 1$ are equivalent, and if at least one of them holds, then $\sqrt{d_w^*(x, y)} \leq d_w^\circ(x, y) \leq d_w^*(x, y)$.

Proof. 1. First, we show that $d_w^\circ(x, y) < 1$ implies $d_w^*(x, y) \leq d_w^\circ(x, y)$. In fact, for any λ such that $d_w^\circ(x, y) < \lambda < 1$ the definition of d_w° gives $w_\lambda(x, y) \leq \lambda < 1$, whence, by virtue of the definition of d_w^* , $d_w^*(x, y) \leq \lambda$, and it remains to pass to the limit as $\lambda \rightarrow d_w^\circ(x, y)$.

2. Now we show that if $d_w^*(x, y) < 1$, then $d_w^\circ(x, y) \leq \sqrt{d_w^*(x, y)}$. In fact, in this case $d_w^*(x, y) \leq \sqrt{d_w^*(x, y)} < 1$, and so, by **Theorem 3.8(a)**, for any λ such that $\sqrt{d_w^*(x, y)} < \lambda < 1$ we find $w_\lambda(x, y) \leq d_w^*(x, y)/\lambda < \lambda$, whence the definition of d_w° implies $d_w^\circ(x, y) \leq \lambda$, and it suffices to let λ go to $\sqrt{d_w^*(x, y)}$.

Thus, in steps 1 and 2 we have established the desired inequalities and showed that the inequalities $d_w^\circ(x, y) < 1$ and $d_w^*(x, y) < 1$ are equivalent, and so, as a consequence, the inequalities $d_w^\circ(x, y) \geq 1$ and $d_w^*(x, y) \geq 1$ are equivalent as well.

3. The inequality $d_w^*(x, y) \geq 1$ implies $d_w^\circ(x, y) \leq d_w^*(x, y)$: by the definition of d_w^* , $w_\lambda(x, y) \leq 1$ for all $\lambda > d_w^*(x, y)$, but $\lambda > 1$, and so, $w_\lambda(x, y) < \lambda$. It follows from the definition of d_w° that $d_w^\circ(x, y) \leq \lambda$.

4. Finally, we show that if $d_w^\circ(x, y) \geq 1$, then $\sqrt{d_w^*(x, y)} \leq d_w^\circ(x, y)$. In fact, given $\lambda > d_w^\circ(x, y)$, we find $w_\lambda(x, y) \leq \lambda$, but $\lambda > 1$, and so, $\lambda^2 > \lambda > 1$. By the convexity of w and (3.1), we get

$$w_{\lambda^2}(x, y) \leq (\lambda/\lambda^2)w_\lambda(x, y) \leq (\lambda/\lambda^2) \cdot \lambda = 1,$$

whence $d_w^*(x, y) \leq \lambda^2$, and it remains to let $\lambda \rightarrow d_w^\circ(x, y)$. \square

Examples 3.10. (a) If the modular w is not convex, **Theorem 3.9** may be false. The idea of the following example is borrowed from the classical theory of modulars (cf. [19, p. 8]). Let (X, d) be a metric space. We set

$$w_\lambda(x, y) = \frac{d(x, y)}{\lambda + d(x, y)}, \quad \lambda > 0, \quad x, y \in X.$$

Then w is a nonconvex metric modular on X and $d_w^*(x, y) \equiv 0$, whereas

$$d_w^\circ(x, y) = \frac{1}{2}(\sqrt{(d(x, y))^2 + 4d(x, y)} - d(x, y)), \quad x, y \in X.$$

Note that in the proof of **Theorem 3.9** the implications in steps 1 and 3 do not rely on the convexity of w :

$$d_w^\circ(x, y) < 1 \Rightarrow d_w^*(x, y) \leq d_w^\circ(x, y), \quad \text{and} \quad d_w^*(x, y) \geq 1 \Rightarrow d_w^\circ(x, y) \leq d_w^*(x, y).$$

The example above corresponds to the former implication.

(b) Consider the p -homogeneous convex (pseudo)modular $w_\lambda(x, y) = (d(x, y)/\lambda)^p$ from **Example 3.4(b)**, where $\lambda > 0$, $x, y \in X$ and $p = \text{const} \geq 1$. Then $X_w^* = X_w = X$ and, by virtue of **Example 2.9(b)**, (c), we have:

$$d_w^\circ(x, y) = (d(x, y))^{p/(p+1)}, \quad d_w^1(x, y) = (p+1)p^{-p/(p+1)}(d(x, y))^{p/(p+1)},$$

$$d_w^*(x, y) = d(x, y), \quad d_w^A(x, y) = \begin{cases} d(x, y) & \text{if } p = 1, \\ p(p-1)^{(1-p)/p}d(x, y) & \text{if } p > 1. \end{cases}$$

The last two formulas hold for $0 < p < 1$ as well.

(c) The inequalities in **Theorem 3.9** are the best possible: in fact, if w is the (pseudo)modular from the previous example, then for $p = 1$ we have $d_w^\circ(x, y) = \sqrt{d_w^*(x, y)}$, and if $p > 1$, then

$$d_w^\circ(x, y) = (d_w^*(x, y))^{p/(p+1)} \rightarrow d_w^*(x, y) \quad \text{as } p \rightarrow \infty.$$

Now we expose the coherence between our theory and the classical theory of modulars and modular linear spaces (cf. [34, Theorem 3]). The following theorem is a straightforward consequence of the corresponding definitions and axioms.

Theorem 3.11. *Let X be a real linear space.*

(a) *Given a functional $\rho : X \rightarrow [0, \infty]$, we set*

$$w_\lambda(x, y) = \rho\left(\frac{x-y}{\lambda}\right), \quad \lambda > 0, x, y \in X. \tag{3.4}$$

Then we have: ρ is a modular (convex modular) on X in the sense of classical axioms (A.1)–(A.4) from the Introduction if and only if w is a metric modular (convex metric modular, respectively) on X .

(b) *On the other hand, let the function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ satisfy the following two conditions:*

- (I) $w_\lambda(\mu x, 0) = w_{\lambda/\mu}(x, 0)$ for all $\lambda, \mu > 0$ and $x \in X$;
- (II) $w_\lambda(x+z, y+z) = w_\lambda(x, y)$ for all $\lambda > 0$ and $x, y, z \in X$.

Given $x \in X$, we set $\rho(x) = w_1(x, 0)$. Then we have: w is a metric modular (convex metric modular) on X if and only if ρ is a classical modular (convex modular, respectively) on X . Moreover, the equality (3.4) holds, the set $X_\rho = X_w^\circ(0)$ is a linear subspace of X and the functional $\|x\|_\rho = d_w^\circ(x, 0)$, $x \in X_\rho$, is an F -norm on X_ρ (and if w is convex, then $X_\rho^ \equiv X_w^*(0) = X_\rho$ is a linear subspace of X and the functional $\|x\|_\rho = d_w^*(x, 0)$, $x \in X_\rho^*$, is a norm on X_ρ^* , respectively). Similar assertions hold if we replace the word ‘modular’ by ‘pseudomodular’. \square*

In the second part of the paper we will need a certain observation (cf. **Theorem 3.14**), whose construction is encountered several times. It is based on the notions from Section 2.14 and the following Section 3.12.

3.12. Abstract convex cone

([30, Section 4], [31, Section 3], [36, Section 2], [37]). The quadruple $(M, d, +, \cdot)$ is said to be an *abstract convex cone* if the triple $(M, d, +)$ is a metric semigroup with zero $0 \in M$ and the operation $\cdot : \mathbb{R}^+ \times M \rightarrow M$ of multiplication of elements from M by nonnegative numbers, given by $(\alpha, x) \mapsto \alpha x$, for all $\alpha, \beta \in \mathbb{R}^+$ and $x, y \in M$ has the following properties: $d(\alpha x, \alpha y) = \alpha d(x, y)$ and

$$\alpha(x+y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta x, \quad \alpha(\beta x) = (\alpha\beta)x, \quad 1 \cdot x = x \tag{3.5}$$

(cf. examples in **Example 3.13**). If the metric space (M, d) is complete, then the corresponding abstract convex cone is called *complete*.

The following equality holds in an abstract convex cone $(M, d, +, \cdot)$:

$$d(\alpha x + \beta y, \alpha y + \beta x) = |\alpha - \beta|d(x, y), \quad \alpha, \beta \in \mathbb{R}^+, x, y \in M. \tag{3.6}$$

It follows that $d(\alpha x, \beta y) \leq \alpha d(x, y) + |\alpha - \beta|d(y, 0)$, and so, the operation of multiplication of numbers from \mathbb{R}^+ by elements from M is a continuous mapping from $\mathbb{R}^+ \times M$ into M .

Examples 3.13. A simple example of an abstract convex cone is a normed linear space $(Z, |\cdot|)$ with the induced metric $d(y, z) = |y - z|$, $y, z \in Z$, and operations of addition $+$ and multiplication \cdot of elements from Z by nonnegative numbers. If $K \subset Z$ is a convex cone (i.e., $y + z, \alpha y \in K$ for all $y, z \in K$ and $\alpha \geq 0$), then $(K, d, +, \cdot)$ is an abstract convex cone, which is complete if Z is a Banach space and K is closed in Z .

Let $(Z, |\cdot|)$ be a real normed linear space. Denote by $\text{cbc}(Z)$ the family of all nonempty closed bounded convex subsets of Z and, given $P, Q \in \text{cbc}(Z)$, set: $P + Q = \{p + q : p \in P, q \in Q\}$ (Minkowski's sum), $\alpha P = \{\alpha p : p \in P\}$ for $\alpha \in \mathbb{R}^+$ and $P +^* Q = \text{cl}(P + Q)$, where $\text{cl}(R)$ designates the closure in Z of the set $R \subset Z$. The operations in $\text{cbc}(Z)$ have the following properties [38,39]: $P +^* Q = \text{cl}(\text{cl}(P) + \text{cl}(Q))$, $\alpha(P +^* Q) = \alpha P +^* \alpha Q$, $(\alpha + \beta)P = \alpha P +^* \beta P$, $\alpha(\beta P) = (\alpha\beta)P$ and $1 \cdot P = P$ for all $\alpha, \beta \in [0, \infty)$. The Abelian semigroup $\text{cbc}(Z)$ is endowed with the Hausdorff metric d_H , generated by the norm $|\cdot|$ in Z : if $P, Q \in \text{cbc}(Z)$, then we set

$$d_H(P, Q) = \max \left\{ \sup_{p \in P} \inf_{q \in Q} |p - q|, \sup_{q \in Q} \inf_{p \in P} |p - q| \right\} \\ = \inf \{ \alpha > 0 : P \subset Q + \alpha S \text{ and } Q \subset P + \alpha S \},$$

where $S = \{z \in Z : |z| \leq 1\}$. The properties of d_H are as follows ([40, Lemma 2.2], [32, Lemma 3]): if $P, Q, R \in \text{cbc}(Z)$ and $\alpha \geq 0$, then $d_H(\alpha P, \alpha Q) = \alpha d_H(P, Q)$ and

$$d_H(P +^* R, Q +^* R) = d_H(P + R, Q + R) = d_H(P, Q).$$

Consequently, $(\text{cbc}(Z), d_H, +, \cdot)$ is an abstract convex cone, which is complete if Z is a Banach space (this follows from the properties of the Hausdorff metric d_H , cf. [41, Theorems II-9 and II-14]). Note that, by the above (Sections 2.14 and 3.12), the operations of $+^*$ -addition in $\text{cbc}(Z)$ and multiplication by numbers from \mathbb{R}^+ are continuous. More examples of metric semigroups and abstract convex cones, necessary for our purposes, will be presented in the second part of the paper.

Theorem 3.14. (a) Let $(X, +)$ be an Abelian semigroup with zero 0 and w be a metric (pseudo)modular on X , which is translation invariant:

$$w_\lambda(x + z, y + z) = w_\lambda(x, y) \quad \text{for all } \lambda > 0 \text{ and } x, y, z \in X. \tag{3.7}$$

Set $X_w = X_w^\circ(0)$. Then the triple $(X_w, d_w^\circ, +)$ is a (pseudo)metric semigroup. A similar assertion holds for a convex (pseudo)modular w on X if we replace X_w by $X_w^* = X_w^*(0)$ and the function d_w° —by d_w^* .

(b) In addition to conditions in (a) suppose that an operation of multiplication by numbers from \mathbb{R}^+ is defined in X satisfying (3.5) and w is a convex (pseudo)modular on X , which is homogeneous in the sense:

$$w_\lambda(\mu x, \mu y) = w_{\lambda/\mu}(x, y) \quad \text{for all } \lambda, \mu > 0 \text{ and } x, y \in X. \tag{3.8}$$

Set $X_w^* = X_w^*(0)$. Then the quadruple $(X_w^*, d_w^*, +, \cdot)$ is an abstract convex (pseudo)cone.

Proof. (a) If $x, y \in X_w$, then, by virtue of Definition 2.1(iii) and (3.7), we have, as $\lambda \rightarrow \infty$:

$$w_\lambda(x + y, 0) \leq w_{\lambda/2}(x + y, y) + w_{\lambda/2}(0, y) = w_{\lambda/2}(x, 0) + w_{\lambda/2}(y, 0) \rightarrow 0,$$

whence $x + y \in X_w$, and so, $(X_w, +)$ is a semigroup. The translation invariance of d_w° follows from the corresponding property of the (pseudo)modular w . If w is a modular on X , then, as was shown in Theorem 2.6, d_w° is a metric on X_w , and so, $(X_w, d_w^\circ, +)$ is a metric semigroup in the sense of Section 2.14.

Now if we suppose that w is convex, then $x, y \in X_w^*$ implies $x + y \in X_w^*$; in fact, $w_\lambda(x, 0)$ and $w_\mu(y, 0)$ are finite for some positive numbers $\lambda = \lambda(x)$ and $\mu = \mu(y)$, and so,

$$w_{\lambda+\mu}(x + y, 0) \leq \frac{\lambda}{\lambda + \mu} w_\lambda(x + y, y) + \frac{\mu}{\lambda + \mu} w_\mu(0, y) \\ = \frac{\lambda}{\lambda + \mu} w_\lambda(x, 0) + \frac{\mu}{\lambda + \mu} w_\mu(y, 0) < \infty.$$

(b) By item (a), the triple $(X_w^*, d_w^*, +)$ is a (pseudo)metric semigroup. Let us show that $\alpha x \in X_w^*$ for all $\alpha \in \mathbb{R}^+$ and $x \in X_w^*$. From (3.7) and properties (3.5), given $\lambda > 0$, we find

$$w_\lambda(0 \cdot x, 0) = w_\lambda(0 \cdot x + x, 0 + x) = w_\lambda((0 + 1)x, x) = w_\lambda(x, x) = 0, \tag{3.9}$$

and so, $0 \cdot x \in X_w^*$ (moreover, if w is a convex modular on X , then (3.9) implies $0 \cdot x = 0$). Let $\alpha > 0$. Applying (3.7) and (3.8), we get:

$$w_\lambda(\alpha \cdot 0, 0) = w_\lambda(\alpha \cdot 0 + \alpha \cdot 0, 0 + \alpha \cdot 0) = w_\lambda(\alpha(1 + 1) \cdot 0, \alpha \cdot 0) \\ = w_{\lambda/\alpha}(1 \cdot 0 + 1 \cdot 0, 1 \cdot 0) = w_{\lambda/\alpha}(0, 0) = 0, \quad \lambda > 0, \tag{3.10}$$

and so, $\alpha \cdot 0 \in X_w^*$ (moreover, if w is a convex modular on X , then (3.10) implies $\alpha \cdot 0 = 0$). Now, if $\alpha > 0$ and $x \in X_w^*$, so that $w_\mu(x, 0) < \infty$ for some $\mu > 0$, then the definition of a convex (pseudo)modular implies

$$w_{2\mu\alpha}(\alpha x, 0) \leq \frac{1}{2} w_{\mu\alpha}(\alpha x, \alpha \cdot 0) + \frac{1}{2} w_{\mu\alpha}(\alpha \cdot 0, 0) = \frac{1}{2} w_{(\mu\alpha)/\alpha}(x, 0) \leq w_\mu(x, 0) < \infty,$$

and so, $\alpha x \in X_w^*$ (if w is a convex modular, then $\alpha \cdot 0 = 0$ and

$$w_{\mu\alpha}(\alpha x, 0) = w_{\mu\alpha}(\alpha x, \alpha \cdot 0) = w_{(\mu\alpha)/\alpha}(x, 0) = w_\mu(x, 0) < \infty).$$

In addition to the translation invariance, the (pseudo)metric d_w^* on X_w^* is homogeneous, i.e., if $\alpha > 0$ and $x, y \in X_w^*$, then

$$\begin{aligned} d_w^*(\alpha x, \alpha y) &= \inf\{\lambda > 0 : w_\lambda(\alpha x, \alpha y) \leq 1\} = \inf\{\lambda > 0 : w_{\lambda/\alpha}(x, y) \leq 1\} \\ &= \inf\{\alpha\mu : \mu > 0 \text{ and } w_\mu(x, y) \leq 1\} = \alpha d_w^*(x, y). \end{aligned}$$

And in the case when $\alpha = 0$, for all $\lambda > 0$ we have:

$$w_\lambda(0 \cdot x, 0 \cdot y) \leq \frac{1}{2}w_{\lambda/2}(0 \cdot x, 0) + \frac{1}{2}w_{\lambda/2}(0 \cdot y, 0) = 0,$$

and so, $d_w^*(0 \cdot x, 0 \cdot y) = 0 = 0 \cdot d_w^*(x, y)$.

Finally, note that if w is a convex pseudomodular on X , then the quadruple $(X_w^*, d_w^*, +, \cdot)$ satisfies all the conditions of Section 3.12, but d_w^* is only a pseudometric, and so, this quadruple was called an abstract convex pseudocone in the formulation of Theorem 3.14. Now, if w is a convex modular on X , then d_w^* is a metric on X_w^* and the above quadruple is an abstract convex cone in the sense of Section 3.12. \square

Remark 3.15. The inclusion $\alpha x \in X_w$ for all $\alpha \in \mathbb{R}^+$ and $x \in X_w$ in item (b) of Theorem 3.14 holds also if the (pseudo)modular w is not necessarily convex. However, for the (pseudo)metric d_w^o the equality $d_w^o(\alpha x, \alpha y) = \alpha d_w^o(x, y)$ for all $\alpha \geq 0$ and $x, y \in X_w$ may fail even if w is convex. To see this, let $(X, d, +, \cdot)$ be an abstract convex cone (e.g., $X = \mathbb{R}$) and a convex modular w on X is of the form $w_\lambda(x, y) = d(x, y)/\lambda$. Then, given $\alpha \geq 0$ and $x, y \in X_w^* = X$, we have:

$$d_w^o(\alpha x, \alpha y) = \sqrt{\alpha d(x, y)} = \sqrt{\alpha} \cdot \sqrt{d(x, y)} = \sqrt{\alpha} d_w^o(x, y).$$

Thus, Theorem 3.14 fails for the pair (X_w, d_w^o) .

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