Modular symmetry by orbifolding magnetized $T^2 \times T^2$: realization of double cover of Γ_N

Shota Kikuchi,[a] **Tatsuo Kobayashi,**[a] **Hajime Otsuka,**[b] **Shintaro Takada**[a] **and Hikaru Uchida**[a]

[a]*Department of Physics, Hokkaido University,*
Sapporo 060-0810, Japan

[b]*KEK Theory Center, Institute of Particle and Nuclear Studies, KEK,*
1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan

E-mail: s-kikuchi@particle.sci.hokudai.ac.jp,
kobayashi@particle.sci.hokudai.ac.jp, hotsuka@post.kek.jp,
s-takada@particle.sci.hokudai.ac.jp,
h-uchida@particle.sci.hokudai.ac.jp

ABSTRACT: We study the modular symmetry of zero-modes on $T_1^2 \times T_2^2$ and orbifold compactifications with magnetic fluxes, M_1, M_2, where modulus parameters are identified. This identification breaks the modular symmetry of $T_1^2 \times T_2^2$, $\mathrm{SL}(2,\mathbb{Z})_1 \times \mathrm{SL}(2,\mathbb{Z})_2$ to $\mathrm{SL}(2,\mathbb{Z}) \equiv \Gamma$. Each of the wavefunctions on $T_1^2 \times T_2^2$ and orbifolds behaves as the modular forms of weight 1 for the principal congruence subgroup $\Gamma(N)$, N being 2 times the least common multiple of M_1 and M_2. Then, zero-modes transform each other under the modular symmetry as multiplets of double covering groups of Γ_N such as the double cover of S_4.

KEYWORDS: Discrete Symmetries, Flux compactifications

Contents

1   Introduction

The origin of the flavor structure such as the masses and the mixing angles of quarks and leptons is one of the significant mysteries of the standard model (SM). Many ideas have been proposed to understand the flavor structure. Among them, non-Abelian discrete flavor models [1–9] are attractive. In such flavor models, various non-Abelian discrete symmetries such as S_N, A_N, $\Delta(3N^2)$, $\Delta(6N^2)$ are assumed as symmetries of quark and lepton flavors. In those models, the realistic masses and mixing angles of quarks or leptons are obtained through breaking the flavor symmetries by vacuum expectation values (VEVs) of gauge singlet scalars, so-called flavons. However, a complicate vacuum alignment is required.

Extra dimensional theory such as superstring theory can lead to non-Abelian discrete flavor symmetries as geometrical symmetries. (See refs. [10, 11].) In particular, the two-dimensional (2D) torus T^2 and orbifolds have the geometrical symmetry, the so-called modular symmetry, $\Gamma \equiv \mathrm{SL}(2, Z)$ (or $\overline{\Gamma} \equiv \mathrm{SL}(2, Z)/\mathbb{Z}_2$). Zero-modes on such a geometry, corresponding to flavors of the SM quarks or leptons, transform under the modular transformation. It was investigated in magnetized D-brane models [12–16] and heterotic orbifold models [17–25]. (See also [26–28].) In this sense, the modular symmetry is regarded as a flavor symmetry. In particular, ref. [16] shows that $|M|$ zero-mode wavefunctions on

T^2 with magnetic flux M behave as modular forms of weight $1/2$ for $\widetilde{\Gamma}(2|M|)$, which is a normal subgroup of the double covering group of Γ, i.e. $\widetilde{\Gamma} \equiv \widetilde{SL}(2, Z)$, and then they are representations of the quotient group $\widetilde{\Gamma}'_{2|M|} \equiv \widetilde{\Gamma}/\widetilde{\Gamma}(2|M|)$. It is notable that the Yukawa couplings as well as higher order couplings also transform non-trivially under the modular transformation. In addition, instead of VEVs of flavons, the flavor symmetry coming from the modular symmetry is broken when the modulus is fixed through the modulus stabilization.

It is also interesting that the finite modular groups $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$ for $N = 2, 3, 4, 5$ are isomorphic to S_3, A_4, S_4, A_5 [29], respectively. Note that $\overline{\Gamma}(N)$ ($\Gamma(N)$) is a normal subgroup of $\overline{\Gamma}$ (Γ), so-called the principal congruence subgroup of level N. Recently, a lot of bottom-up approaches of flavor models inspired by these aspects have been studied [30–71]. In those models, the Yukawa couplings as well as higher order couplings are treated as modular forms of even-number weights. Furthermore, ref. [72] shows modular forms of odd-number weights are representations of $\Gamma'_N \equiv \Gamma/\Gamma(N)$, which is the double covering group of Γ_N. In the latest studies [73, 74], flavor models of $\Gamma'_4 \simeq S'_4$ with modular forms of weight integer were studied. Thus, it is important to study the modular flavor symmetry, Γ_N and its covering groups Γ'_N from both top-down and bottom-up approaches.

The moduli stabilization is a key issue. Three-form fluxes can stabilize complex structure modulus [75] as well as the dilaton. In ref. [28], for example, $T_1^2 \times T_2^2 \times T_3^2$ with three-form fluxes have been considered and the complex modulus of T_1^2, τ_1 and that of T_2^2, τ_2 have to be related each other, such as $\tau_1 = \tau_2 \equiv \tau$ due to the three-form fluxes. In other words, the modular symmetry on $T_1^2 \times T_2^2$, $\Gamma \times \Gamma$ is broken to Γ due to the three-form fluxes below the heavy mass scale of the stabilized moduli. A similar breaking $\Gamma \times \Gamma \to \Gamma$ can be realized by imposing a permutation symmetry between T_1^2 and T_2^2. Such a setup is quite interesting as follows. The zero-mode wavefunctions on T^2 with magnetic flux M behaves as the modular forms of the weight $1/2$ representing $\widetilde{\Gamma}'_{2|M|}$. Thus we expect that the above setup would lead to zero-mode wavefunctions behaving the modular forms of the weight 1 and representing double covering groups of Γ_N, i.e., Γ'_N.

Our purpose in this paper is to study the modular symmetry of zero-modes on $T_1^2 \times T_2^2$ with magnetic fluxes, where the complex structure moduli are identified as $\tau_1 = \tau_2 = \tau$. Furthermore, we also study its orbifolding by the $\mathbb{Z}_2$ twist, the $\mathbb{Z}_N$ shift, and also the $\mathbb{Z}_2$ permutation, which interchanges T_1^2 and T_2^2. These orbifoldings decompose a representation to smaller representations such as irreducible representations.

This paper is organized as follows. In section 2 we briefly review the zero-mode wavefunctions on T^2 with magnetic flux. In section 3, we give a review on the modular symmetry of zero-modes on T^2 and its orbifolding. In section 4, we apply them on a magnetized $T_1^2 \times T_2^2$ and its orbifolding by the $\mathbb{Z}_2$ twist, the $\mathbb{Z}_N$ shift, and the $\mathbb{Z}_2$ permutation. Here we identify $\tau_1 = \tau_2 \equiv \tau$. We find that the wavefunctions on the $T_1^2 \times T_2^2$ behave as modular forms of weight 1 for $\Gamma(N)$. The zore-modes are multiplets of Γ'_N and Γ_N. In section 5, we conclude this study. In appendix A, we explain a problem in models with odd magnetic fluxes.

2 Zero-mode wavefunctions on magnetized T^2

First, we review ten-dimensional (10D) $\mathcal{N} = 1$ super Yang-Mills theory on $M^4 \times T_1^2 \times T_2^2 \times T_3^2$ with magnetic fluxes, which is the low energy effective field theory of superstring theory. Also, our setup in section 4 is applicable to D7-brane models on $M^4 \times T_1^2 \times T_2^2$. The 10D Lagrangian is given by

$$S = \int_M d^4 x \prod_{i=1,2,3} \int_{T_i^2} d^2 z_i \left[-\frac{1}{4g^2} \text{Tr}(F^{MN} F_{MN}) + \frac{i}{2g} \text{Tr}(\bar{\lambda} \Gamma^M D_M \lambda) \right], \qquad (2.1)$$

where $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$ and $D_M \lambda = \partial_M \lambda - i[A_M, \lambda]$ with $M, N = 0, 1, \ldots, 9$. By Kaluza-Klein decomposition, ten-dimensional vector potential A_M and Majorana-Weyl spinors λ can be written as

$$A_M(x, z_1, z_2, z_3) = \sum_{n_1, n_2, n_3} \phi_{M, n_1 n_2 n_3}(x) \otimes \phi_{M, n_1}(z_1) \otimes \phi_{M, n_2}(z_2) \otimes \phi_{M, n_3}(z_3), \qquad (2.2)$$

$$\lambda(x, z_1, z_2, z_3) = \sum_{n_1, n_2, n_3} \psi_{n_1 n_2 n_3}(x) \otimes \psi_{n_1}(z_1) \otimes \psi_{n_2}(z_2) \otimes \psi_{n_3}(z_3). \qquad (2.3)$$

Here $\psi_{n_i}(z_i)$ is the n_i-th excited mode of 2D Weyl spinors on the i-th torus, T_i^2, and satisfies the following Dirac equation,

$$i \, \slashed{D}_2 \psi_{n_i}(z_i) = m_{n_i} \psi_n(z_i). \qquad (2.4)$$

Since m_{n_i} gives the compact scale mass of the four-dimensional Weyl spinor, we consider massless mode $\psi_0(z_i)$. In this section, we focus on one torus T^2, hence, we consider zero-mode wavefunctions on the magnetized T^2.

For simplicity, we study the wavefunctions on the torus T^2 with U(1) magnetic flux [76]. The torus can be regarded as the complex plane $\mathbb{C}$ divided by a two-dimensional lattice Λ, that is $T^2 \simeq \mathbb{C}/\Lambda$. Then, the lattice Λ is characterized by the complex modulus parameter $\tau \equiv e_2/e_1$ ($\text{Im}\tau > 0$), where e_1, e_2 are the basis that spans the lattice Λ. This torus has the metric such as

$$ds^2 = 2h_{\mu\nu} dz^\mu d\bar{z}^\nu, \quad h = |e_1|^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad (2.5)$$

and the U(1) magnetic flux is given by

$$F = \frac{i\pi M}{\text{Im}\tau} dz \wedge d\bar{z}. \qquad (2.6)$$

This flux leads to the vector potential one-form,

$$\begin{aligned} A(z, \bar{z}) &= \frac{\pi M}{\text{Im}\tau} \text{Im} \left((\bar{z} + \bar{\zeta}) dz \right) \\ &= -\frac{i\pi M}{2\text{Im}\tau} (\bar{z} + \bar{\zeta}) dz + \frac{i\pi M}{2\text{Im}\tau} (z + \zeta) d\bar{z} \\ &= A_z dz + A_{\bar{z}} d\bar{z}, \end{aligned} \qquad (2.7)$$

where ζ is a Wilson line phase. Since the complex coordinate z on T^2 is identified with $z + 1$ and $z + \tau$, the vector potential $A(z, \bar{z})$ obeys the following boundary conditions,

$$A(z + 1, \bar{z} + 1) = A(z, \bar{z}) + d\left(\frac{\pi M}{\operatorname{Im}\tau}\operatorname{Im}z\right) = A(z, \bar{z}) + d\chi_1(z, \bar{z}), \tag{2.8}$$

$$A(z + \tau, \bar{z} + \bar{\tau}) = A(z, \bar{z}) + d\left(\frac{\pi M}{\operatorname{Im}\tau}\operatorname{Im}\bar{\tau}z\right) = A(z, \bar{z}) + d\chi_2(z, \bar{z}), \tag{2.9}$$

which correspond to U(1) gauge transformation. Here, $\chi_1(z, \bar{z})$ and $\chi_2(z, \bar{z})$ are given by

$$\chi_1(z, \bar{z}) = \frac{\pi M}{\operatorname{Im}\tau}\operatorname{Im}(z + \zeta), \ \chi_2(z, \bar{z}) = \frac{\pi M}{\operatorname{Im}\tau}\operatorname{Im}[\bar{\tau}(z + \zeta)]. \tag{2.10}$$

Here and in what follows, we consider the 2D spinor with the U(1) charge $q = 1$. To preserve 2D Dirac equation, eq. (2.4), under U(1) gauge transformation, the boundary conditions for 2D Weyl spinors,

$$\psi(z) = \begin{pmatrix} \psi_+^M(z) \\ \psi_-^M(z) \end{pmatrix}, \quad \psi_-^M(z) = \overline{\psi_+^{-M}(z)}, \tag{2.11}$$

are obtained by

$$\psi(z + 1) = e^{i\chi_1(z)}\psi(z) = e^{i\pi M \frac{\operatorname{Im}(z+\zeta)}{\operatorname{Im}\tau}}\psi(z), \tag{2.12}$$

$$\psi(z + \tau) = e^{i\chi_2(z)}\psi(z) = e^{i\pi M \frac{\operatorname{Im}\bar{\tau}(z+\zeta)}{\operatorname{Im}\tau}}\psi(z). \tag{2.13}$$

Then, considering contractible loops on T^2, we obtain the Dirac quantization condition,

$$M \in \mathbb{Z}. \tag{2.14}$$

To determine the zero-mode of $\psi(z)$, we need to compute the 2D Dirac operator,

$$i\not{D} = i\gamma^z D_z + i\gamma^{\bar{z}} D_{\bar{z}} \tag{2.15}$$

$$= \begin{pmatrix} 0 & \frac{2i}{e_1}(\partial - iA_z) \\ \frac{2i}{\bar{e}_1}(\bar{\partial} - iA_{\bar{z}}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2i}{e_1}(\partial - \frac{\pi M}{2\operatorname{Im}\tau}(\bar{z} + \bar{\zeta})) \\ \frac{2i}{\bar{e}_1}(\bar{\partial} + \frac{\pi M}{2\operatorname{Im}\tau}(z + \zeta)) & 0 \end{pmatrix}, \tag{2.16}$$

where $\gamma^z, \gamma^{\bar{z}}$ are given by

$$\gamma^z = \frac{1}{e_1}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \gamma^{\bar{z}} = \frac{1}{\bar{e}_1}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \tag{2.17}$$

with $\{\gamma^z, \gamma^{\bar{z}}\} = 2h^{z\bar{z}}$. By solving the massless Dirac equation, $i\not{D}\psi(z) = 0$, for $M > 0$ ($M < 0$), we find only $\psi_+^M(z)$ ($\psi_-^M(z)$) has the $|M|$ number of degenerate zero-mode solutions,

$$\psi_{\pm,0}^{j,|M|}(z, \tau) = \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{\operatorname{Im}(z+\zeta)}{\operatorname{Im}\tau}}\sum_{l\in\mathbf{Z}} e^{i\pi|M|\tau\left(\frac{j}{|M|}+l\right)^2}e^{2\pi i|M|(z+\zeta)\left(\frac{j}{|M|}+l\right)} \tag{2.18}$$

$$= \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{\operatorname{Im}(z+\zeta)}{\operatorname{Im}\tau}}\vartheta\begin{bmatrix} \frac{j}{|M|} \\ 0 \end{bmatrix}(|M|z, |M|\tau),$$

where $j = 0, 1, \ldots, |M|-1$ and $\mathcal{A} = |e_1|^2 \mathrm{Im}\tau$ is the area of T^2. The ϑ function is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{l \in \mathbb{Z}} e^{\pi i (a+l)^2 \tau} e^{2\pi i (a+l)(\nu+b)}, \tag{2.19}$$

so-called the Jacobi theta function.

We can show that eq. (2.18) satisfies [77]

$$\psi_0^{j,|M|}(-z, \tau) = \psi_0^{|M|-j,|M|}(z, \tau), \tag{2.20}$$

as well as

$$\psi_0^{j,|M|}(z, \tau) = \psi_0^{j+|M|,|M|}(z, \tau), \tag{2.21}$$

and, later we will use these relations.

We can extend the above U(1) theory to U(N) super Yang-Mills theory. We introduce magnetic fluxes along the diagonal direction of U(N), $\mathrm{diag}(M, M', \cdots)$. Then, our theory has several zero-modes, whose Dirac equations include various magnetic fluxes, although their zero-mode wavefunctions are written by the above wavefunctions with corresponding magnetic fluxes. The product of such wavefuncitons, $\psi_{\pm,0}^{j,|M|}(z,\tau)\psi_{\pm,0}^{k,|M'|}(z,\tau)$ can be expanded by the wavefunctions $\psi_{\pm,0}^{\ell,|M|+|M'|}(z,\tau)$ [76],[1]

$$\psi_{\pm,0}^{j,|M|}(z, \tau)\ \psi_{\pm,0}^{k,|M'|}(z, \tau) = \sum_{\ell} C_{T^2}^{jk\ell}(\tau)\ \psi_{\pm,0}^{\ell,|M|+|M'|}(z, \tau), \tag{2.22}$$

where $\ell = j + k + |M|m$ with integer m. The τ-dependent coefficient $C_{T^2}^{jk\ell}(\tau)$ is written explicitly by

$$C_{T^2}^{jk\ell}(\tau) = \vartheta \begin{bmatrix} \frac{|M'|j-|M|k+|M||M'|m}{|M||M'|(|M|+|M'|)} \\ 0 \end{bmatrix} (0, |M||M'|(|M|+|M'|)\tau),$$

up to a constant factor. The coefficient provides us with three-point couplings because their couplings are obtained by wavefunction integrals in the compact space,

$$y_{T^2}^{jk\ell} = \int_{T^2} d^2z\, \psi_{\pm,0}^{j,|M|}(z, \tau)\ \psi_{\pm,0}^{k,|M'|}(z, \tau)(\psi_{\pm,0}^{\ell,|M|+|M'|}(z, \tau))^*. \tag{2.23}$$

Similarly, n-point couplings,

$$y_{j_1, j_2 \cdots, j_n} = \int_{T^2} d^2z\, \psi_{\pm,0}^{j_1,|M^{(1)}|}(z, \tau)\ \psi_{\pm,0}^{j_2,|M^{(2)}|}(z, \tau) \cdots (\psi_{\pm,0}^{j_n,|M^{(n)}|}(z, \tau))^*, \tag{2.24}$$

are also written by products of $C_{T^2}^{jk\ell}(\tau)$ [79].

3 Modular symmetry in magnetized T^2

In this section, we review the modular symmetry of the zero-mode wavefunctions on the magnetized T^2 and its orbifolding by $\mathbb{Z}_N$ twist and shift [16]. To simplify our analysis, we consider the torus with the magnetic U(1) flux and no Wilson lines. However, we can extend this analysis to the models with any flux and non-vanishing Wilson lines without any difficulty.

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[1]See also ref. [78].

3.1 T^2 models

First, we briefly review modular transformation of zero-mode wavefunctions on T^2. (See for the modular symmetry, e.g., [80–83].) The torus T^2 is constructed by $\mathbb{C}/\Lambda$, where Λ is spanned by the basis e_1, e_2 and characterized by the modulus parameter $\tau = e_2/e_1$ (Im$\tau >$ 0). Then, the same lattice with different modulus parameter is given by the following basis,

$$\begin{pmatrix} e_2' \\ e_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \equiv \Gamma. \tag{3.1}$$

This $\mathrm{SL}(2, \mathbb{Z})$ transformation is generated by two generators,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{3.2}$$

which satisfy the algebra:

$$S^2 = -\mathbb{I} \equiv Z, \ S^4 = (ST)^3 = \mathbb{I} = Z^2. \tag{3.3}$$

This gives following transformations:

$$\gamma : z \equiv \frac{u}{e_1} \to z' \equiv \frac{u}{e_1'} = \frac{z}{c\tau + d}, \tag{3.4}$$

$$\gamma : \tau \equiv \frac{e_2}{e_1} \to \tau' \equiv \frac{e_2'}{e_1'} = \frac{a\tau + b}{c\tau + d}, \tag{3.5}$$

where u is the complex coordinate of $\mathbb{C}$ and z is that of T^2. Note that the Wilson line ζ is transformed as in z. These are also generated by two generators S and T,

$$S : (z, \tau) \to \left(-\frac{z}{\tau}, -\frac{1}{\tau} \right), \quad T : (z, \tau) \to (z, \tau + 1). \tag{3.6}$$

Since $Z = -\mathbb{I}$ leaves τ invariant, $Z(z, \tau) = (-z, \tau)$, the transformation group for τ is isomorphic to $\bar{\Gamma} \equiv \Gamma/\{\pm\mathbb{I}\}$. Here, we introduce the principal congruence subgroup of level N defined by

$$\Gamma(N) \equiv \left\{ h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma \middle| \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\mathrm{mod} \ N) \right\}. \tag{3.7}$$

This is the normal subgroup of Γ, e.g., $\Gamma(1) \simeq \Gamma$. Its quotient group is given by

$$\Gamma_N' \equiv \Gamma/\Gamma(N) = \langle S, T | S^4 = (ST)^3 = T^N = \mathbb{I}, \ S^2 T = T S^2 \rangle. \tag{3.8}$$

Similarly, we can also introduce $\bar{\Gamma}(N) \equiv \Gamma(N)/\{\pm\mathbb{I}\}$ and obtain its quotient group as follows:

$$\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N) = \langle S, T | S^2 = (ST)^3 = T^N = \mathbb{I} \rangle. \tag{3.9}$$

The quotient Γ_N is isomorphic to $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$. In addition, Γ_N' is the double covering group of Γ_N. (See e.g., [72–74].)

We are now ready to construct the holomorphic functions of τ, the modular forms $f(\tau)$ of integer weight k for $\Gamma(N)$. First, we define the automorphy factor $J_k(\gamma, \tau)$ as

$$J_k(\gamma, \tau) = (c\tau + d)^k, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{3.10}$$

It is straightforward to show it satisfies

$$J_k(\gamma_1 \gamma_2, \tau) = J_k(\gamma_1, \gamma_2(\tau)) J_k(\gamma_2, \tau), \quad \gamma_1, \gamma_2 \in \Gamma. \tag{3.11}$$

Then, the modular forms $f(\tau)$ are defined as the functions satisfying the following relation:

$$f(\gamma(\tau)) = J_k(\gamma, \tau) \rho(\gamma) f(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{3.12}$$

where

$$\rho(\gamma_2 \gamma_1) = \rho(\gamma_2) \rho(\gamma_1), \quad \gamma_1, \gamma_2 \in \Gamma, \tag{3.13}$$

$$\rho(h) = \mathbb{I}, \qquad\qquad h \in \Gamma(N), \tag{3.14}$$

and therefore ρ is a unitary representation of the quotient group $\Gamma'_N = \Gamma/\Gamma(N)$. Since $f(Z(\tau)) = f(\tau)$, there is the constraint, $(-1)^k \rho(Z) = \mathbb{I}$. Thus, if $k = $ even, ρ becomes a representation of $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$. Here, we can extend the modular forms to the half integer weight $k/2$. (See e.g., [82, 84, 85].) We define the double covering group of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, $\widetilde{\Gamma} \equiv \widetilde{SL}(2, \mathbb{Z})$ as

$$\widetilde{\Gamma} \equiv \left\{ [\gamma, \epsilon] \big| \gamma \in \Gamma, \ \epsilon \in \{\pm 1\} \right\}. \tag{3.15}$$

This $\widetilde{SL}(2, \mathbb{Z})$ group is generated by two generators,

$$\widetilde{S} \equiv [S, 1], \ \widetilde{T} \equiv [T, 1], \tag{3.16}$$

which satisfy the algebra:

$$\widetilde{S}^2 = [-\mathbb{I}, 1] \equiv \widetilde{Z}, \ \widetilde{S}^4 = (\widetilde{S}\widetilde{T})^3 = [\mathbb{I}, -1] = \widetilde{Z}^2, \ \widetilde{S}^8 = (\widetilde{S}\widetilde{T})^6 = [\mathbb{I}, 1] \equiv \mathbb{I} = \widetilde{Z}^4, \ \widetilde{Z}\widetilde{T} = \widetilde{T}\widetilde{Z}. \tag{3.17}$$

The normal subgroup of $\widetilde{\Gamma}$, $\widetilde{\Gamma}(N)$ corresponding to $\Gamma(N)$ of Γ is defined by

$$\widetilde{\Gamma}(N) \equiv \{[h, \epsilon] \in \widetilde{\Gamma} | h \in \Gamma(N), \epsilon = 1\}. \tag{3.18}$$

Then, the new automorphy factor $\widetilde{J}_{k/2}(\widetilde{\gamma}, \tau)$ is given by

$$\widetilde{J}_{k/2}(\widetilde{\gamma}, \tau) \equiv \epsilon^k J_{k/2}(\gamma, \tau) = \epsilon^k (c\tau + d)^{k/2}, \quad k \in \mathbb{Z}, \quad \widetilde{\gamma} = \left[\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon \right] \in \tilde{\Gamma}, \tag{3.19}$$

where we take $(-1)^{k/2} = e^{-i\pi k/2}$. In this extension, the modular forms $\widetilde{f}(\tau)$ of half integer weight $k/2$ are defined as follows,

$$\widetilde{f}(\widetilde{\gamma}(\tau)) = \widetilde{J}_{k/2}(\widetilde{\gamma}, \tau) \rho(\widetilde{\gamma}) \widetilde{f}(\tau), \quad \widetilde{\gamma} \in \widetilde{\Gamma}, \tag{3.20}$$

where $\rho(h) = \mathbb{I}$, $h \in \widetilde{\Gamma}(N)$, that is, ρ is a unitary representation of the quotient group $\widetilde{\Gamma}'_N \equiv \widetilde{\Gamma}/\widetilde{\Gamma}(N)$. The algebra of $\widetilde{\Gamma}'_N$ is given by eq. (3.17) added the further relation $\widetilde{T}^N = \mathbb{I}$.

Next, we consider the modular transformation of zero-modes. Under S and T transformations in eq. (3.6), the equation of motion for 2D Weyl spinor $\psi(z)$, eq. (2.4), is preserved. The boundary conditions for $\psi(z)$, eqs. (2.12) and (2.13), however, are not preserved under T transformation unless $M = $ even. In appendix A, we show the inconsistency between the boundary conditions and the T transformation for $M = $ odd. Here and hereafter, we treat only $M = $ even case. Under S and T, the zero-modes in eq. (2.18) are transformed as

$$S : \psi_0^{j,|M|}(z,\tau) \rightarrow \psi_0^{j,|M|}\left(-\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-\tau)^{1/2} \sum_{k=0}^{|M|-1} e^{i\pi/4} \frac{1}{\sqrt{|M|}} e^{2\pi i \frac{jk}{|M|}} \psi_0^{k,|M|}(z,\tau), \quad (3.21)$$

$$T : \psi_0^{j,|M|}(z,\tau) \rightarrow \psi_0^{j,|M|}(z,\tau+1) = e^{i\pi \frac{j^2}{|M|}} \psi_0^{j,|M|}(z,\tau). \quad (3.22)$$

By using the modular forms of half integer weight in eq. (3.20), we can rewrite them as

$$\psi_0^{j,|M|}(\widetilde{\gamma}(z,\tau)) = \widetilde{J}_{1/2}(\widetilde{\gamma}, \tau) \sum_{k=0}^{|M|-1} \rho(\widetilde{\gamma})_{jk} \psi_0^{k,|M|}(z,\tau), \quad \widetilde{\gamma} \in \widetilde{\Gamma}, \quad (3.23)$$

$$\rho(\widetilde{S})_{jk} = e^{i\pi/4} \frac{1}{\sqrt{|M|}} e^{2\pi i \frac{jk}{|M|}}, \quad \rho(\widetilde{T})_{jk} = e^{i\pi \frac{j^2}{|M|}} \delta_{j,k}, \quad (3.24)$$

where $\rho(\widetilde{\gamma})$ is a unitary representation of the quotient group $\widetilde{\Gamma}'_{2|M|} \equiv \widetilde{\Gamma}/\widetilde{\Gamma}(2|M|)$:

$$\rho(\widetilde{S})^2 = \rho(\widetilde{Z}), \; \rho(\widetilde{S})^4 = [\rho(\widetilde{S})\rho(\widetilde{T})]^3 = -\mathbb{I}, \; \rho(\widetilde{Z})\rho(\widetilde{T}) = \rho(\widetilde{T})\rho(\widetilde{Z}), \; \rho(\widetilde{T})^{2|M|} = \mathbb{I}. \quad (3.25)$$

Thus, the zero-mode wavefunctions on T^2 behave as the modular forms of weight $1/2$ for $\widetilde{\Gamma}(2|M|)$.

3.2 $T^2/\mathbb{Z}_N$ twist orbifold models

Here, we review the modular symmetry for the wavefunctions on the magnetized $T^2/\mathbb{Z}_N$ twist orbifolds. The $T^2/\mathbb{Z}_N$ twist orbifolds are obtained by further identifying the complex coordinate of T^2, z with the $\mathbb{Z}_N$ discrete rotated points $\alpha_N^k z$, where

$$\alpha_N^k \equiv e^{2\pi i k/N}; \forall k \in \mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}. \quad (3.26)$$

In this identification, the wavefunctions on the magnetized $T^2/\mathbb{Z}_N$ twist orbifolds, $\psi_{T^2/\mathbb{Z}_N^m}^{j,|M|}(z)$, are required to satisfy the following boundary condition,

$$\psi_{T^2/\mathbb{Z}_N^m}^{j,|M|}(\alpha_N z) = \alpha_N^m \psi_{T^2/\mathbb{Z}_N^m}^{j,|M|}(z), \quad m \in \mathbb{Z}_N, \quad (3.27)$$

and, therefore, can be expressed by liner combinations of the wavefunctions on T^2 as [12, 77, 86, 87]

$$\psi_{T^2/\mathbb{Z}_N^m}^{j,|M|}(z) = \mathcal{N}_N^t \sum_{k=0}^{N-1} (\alpha_N^m)^{-k} \psi_{T^2}^{j,|M|}(\alpha_N^k z), \quad (3.28)$$

where $\mathcal{N}_N^t$ is the normalization factor. There exist only four consistent orbifolds such as $N = 2, 3, 4, 6$. However, except for $N = 2$, the modulus τ must be fixed to be a certain value for $N = 3, 4, 6$. Thus, any value of τ is allowed for $N = 2$, that is the full modular symmetry remains for only $N = 2$. Now, we focus on the $T^2/\mathbb{Z}_2$ twist orbifold although we can also consider others with the broken modular symmetry.

The zero-mode wavefunctions on $T^2/\mathbb{Z}_2$ twist orbifold, $\psi_{T^2/\mathbb{Z}_2^m}^{j,|M|}(\alpha_2 z, \tau)$, are obtained as follows. By eq. (2.20), zero-mode wavefunctions on T^2, $\psi_0^{j,|M|}$, satisfy the following relation,

$$\psi_0^{j,|M|}(\alpha_2^m z, \tau) = \psi_0^{j,|M|}((-1)^m z, \tau) = \psi_0^{|M|-j,|M|}(z, \tau), \quad m = 1. \tag{3.29}$$

Thus, using eq. (2.21), we can write $\psi_{T^2/\mathbb{Z}_2^m}^{j,|M|}(z, \tau)$ as

$$\psi_{T^2/\mathbb{Z}_2^m}^{j,|M|}(z, \tau) = \mathcal{N}_2^t \left(\psi_0^{j,|M|}(z, \tau) + (-1)^m \psi_0^{|M|-j,|M|}(z, \tau) \right), \quad \mathcal{N}_2^t = \begin{cases} 1/2 & (j = 0, |M|/2) \\ 1/\sqrt{2} & \text{(otherwise)} \end{cases}, \tag{3.30}$$

where $j = 0, 1, \ldots, |M|/2$ and $m = 0, 1$. There are the $(|M|/2 + 1)$ number of $\mathbb{Z}_2$-even modes $(m = 0)$ and $(|M|/2 - 1)$ -odd modes $(m = 1)$. Under these liner combinations, the formula of modular forms in eq. (3.23) becomes

$$\psi_{T^2/\mathbb{Z}_2^m}^{j,|M|}(z, \tau) = \widetilde{J}_{1/2}(\widetilde{\gamma}, \tau) \sum_{k=0}^{|M|/2} \rho_{T^2/\mathbb{Z}_2^m}(\widetilde{\gamma})_{jk} \psi_{T^2/\mathbb{Z}_2^m}^{k,|M|}(z, \tau). \tag{3.31}$$

The unitary representation $\rho_{T^2/\mathbb{Z}_2^m}$ is given by

$$\rho_{T^2/\mathbb{Z}_2^0}(\widetilde{S})_{jk} = e^{i\pi/4} \frac{2}{\sqrt{|M|}} \cos \left(\frac{2\pi jk}{|M|} \right), \quad \rho_{T^2/\mathbb{Z}_2^0}(\widetilde{T})_{jk} = e^{i\pi \frac{j^2}{|M|}} \delta_{j,k}, \tag{3.32}$$

$$\rho_{T^2/\mathbb{Z}_2^1}(\widetilde{S})_{jk} = e^{i\pi/4} \frac{2i}{\sqrt{|M|}} \sin \left(\frac{2\pi jk}{|M|} \right), \quad \rho_{T^2/\mathbb{Z}_2^1}(\widetilde{T})_{jk} = e^{i\pi \frac{j^2}{|M|}} \delta_{j,k}, \tag{3.33}$$

where $\rho_{T^2/\mathbb{Z}_2^0}(\widetilde{S})$ is multiplied by a further factor $1/\sqrt{2}$ for j or $k = 0, |M|/2$. We can directly check that they satisfy the algebra of $\widetilde{\Gamma}_{2|M|}'$, and the further algebraic relation,

$$\rho_{T^2/\mathbb{Z}_2^m}(\widetilde{S})^2 = \rho_{T^2/\mathbb{Z}_2^m}(\widetilde{Z}) = i(-1)^m. \tag{3.34}$$

Thus, the representations on the $T^2/\mathbb{Z}_2$ twist orbifold satisfy the same algebra with T^2. Note that we have not necessarily obtained the irreducible representation of $\widetilde{\Gamma}_{2|M|}'$. Actually, we will see the further decomposition in the end of this section.

3.3 $T^2/\mathbb{Z}_N$ shift orbifold models

As another example, we now review the magnetized $T^2/\mathbb{Z}_N$ shift orbifolds. The $T^2/\mathbb{Z}_N$ shift orbifolds are obtained by further identifying the complex coordinate of T^2, z with the $\mathbb{Z}_N$ discrete shift points $z + ke_N^{(m,n)}$ [88], where

$$ke_N^{(m,n)} \equiv (m + n\tau)/N; \forall k, \exists m, \exists n \in \mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}. \tag{3.35}$$

Since the full modular symmetry remains only on the $T^2/\mathbb{Z}_N$ shift orbifolds obtained by further identifying z with $z + k e_N^{(m,n)}$ for all $m, n \in \mathbb{Z}_N$, we consider such full shift orbifolds.

Any $\mathbb{Z}_N$ shift is generated by two shifts, $e_N^{(1,0)} = 1/N$ and $e_N^{(0,1)} = \tau/N$. We should consider the identifications with these shifts. In these identifications, the wavefunctions on the magnetized $T^2/\mathbb{Z}_N$ shift orbifolds, $\psi^{j,|M|}_{T^2/\mathbb{Z}_N^{(\ell_1, \ell_2)}}(z)$, are required to satisfy the following further boundary conditions,

$$\psi^{j,|M|}_{T^2/\mathbb{Z}_N^{(\ell_1, \ell_2)}}(z + e_N^{(1,0)}) = \alpha_N^{\ell_1} e^{i\chi_N^{(1,0)}(z)} \psi^{j,|M|}_{T^2/\mathbb{Z}_N^{(\ell_1, \ell_2)}}(z), \tag{3.36}$$

$$\psi^{j,|M|}_{T^2/\mathbb{Z}_N^{(\ell_1, \ell_2)}}(z + e_N^{(0,1)}) = \alpha_N^{\ell_2} e^{i\chi_N^{(0,1)}(z)} \psi^{j,|M|}_{T^2/\mathbb{Z}_N^{(\ell_1, \ell_2)}}(z), \tag{3.37}$$

with the $\mathbb{Z}_N$ phase $\alpha_N^\ell = e^{2\pi i \ell/N}$, $\ell \in \mathbb{Z}_N$ and

$$\chi_N^{(m,n)}(z) = \pi |M| \left(\frac{\mathrm{Im}(\bar{e}_N^{(m,n)}(z + \tau))}{\mathrm{Im}\tau} + \frac{mn}{N} \right). \tag{3.38}$$

The exponential factor $e^{i\chi_N^{(m,n)}(z)}$ is required to be consistent with the torus boundary conditions in eqs. (2.12) and (2.13). Moreover, to generate any shift from these two shifts, it should be satisfied that

$$\alpha_N^{\ell_1 + \ell_2} e^{i\chi_N^{(1,0)}(z + e_N^{(0,1)}) + i\chi_N^{(0,1)}(z)} = \alpha_N^{\ell_1 + \ell_2} e^{i\chi_N^{(0,1)}(z + e_N^{(1,0)}) + i\chi_N^{(1,0)}(z)} = \alpha_N^{\ell_1 + 2} e^{i\chi_N^{(1,1)}(z)}. \tag{3.39}$$

This shows the conditions for both the magnetic flux and the $\mathbb{Z}_N$ phase as follows:

$$\begin{cases} M/N^2 \equiv s \in \mathbb{Z}, \ \ell_{1+2} = \ell_1 + \ell_2 \ (\mathrm{mod} \ N) & (\text{for } N \in \mathbb{Z}) \\ M/N^2 \equiv s \in 2\mathbb{Z} + 1, \ \ell_{1+2} = \ell_1 + \ell_2 + N/2 \ (\mathrm{mod} \ N) & (\text{for } N \in 2\mathbb{Z}) \end{cases}. \tag{3.40}$$

Remembering the assumption $M \in 2\mathbb{Z}$, the case of $s \in 2\mathbb{Z} + 1$, $N \in 2\mathbb{Z} + 1$ is rejected from the above. Taking these into account, the boundary condition for any $\mathbb{Z}_N$ shift is induced as

$$\psi^{j,|M|}_{T^2/\mathbb{Z}_N^\ell}(z + k e_N^{(m,n)}) = (\alpha_N^\ell)^k e^{ik\chi_N^{(m,n)}(z)} \psi^{j,|M|}_{T^2/\mathbb{Z}_N^\ell}(z), \quad \ell = m\ell_1 + n\ell_2 \ (\mathrm{mod} \ N). \tag{3.41}$$

According to eq. (3.40), for $s \in 2\mathbb{Z} + 1$, $N \in 2\mathbb{Z}$, the extra factor $mnN/2$ is added to ℓ. Then, the consistency of the contractible loops on T^2 gives the further magnetic flux condition $M/N \equiv t \in \mathbb{Z}$, but it has been already satisfied. Thus, the eigenfunctions for $\exists e_N^{(m,n)}$-shift can be expressed by liner combinations of the wavefunctions on T^2 as

$$\psi^{j,|M|}_{T^2/\mathbb{Z}_N^\ell}(z) = \mathcal{N}_N^s \sum_{k=0}^{N-1} (\alpha_N^\ell)^{-k} e^{-ik\chi_N^{(m,n)}(z)} \psi^{j,|M|}_{T^2}(z + k e_N^{(m,n)}), \tag{3.42}$$

where $\mathcal{N}_N^s$ is the normalization factor. Since $e^{-ik\chi_N^{(m,n)}(z)} \psi^{j,|M|}_{T^2}(z + k e_N^{(m,n)})$ satisfies the same equation of motion and boundary conditions with $\psi^{j,|M|}_{T^2}(z)$, it is shown that

$$\psi^{j,|M|}_{T^2}(z + k e_N^{(m,n)}) = e^{ik\chi_N^{(m,n)}(z)} e^{i\pi km(2j - (N-k)nN|s|)/N} \psi^{j+knN|s|,|M|}_{T^2}(z). \tag{3.43}$$

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Eventually, we obtain the eigenfunctions for $^\exists e_N^{(m,n)}$-shift as

$$\psi_{T^2/\mathbb{Z}_N^\ell}^{j,|M|}(z,\tau) = \mathcal{N}_N^s \sum_{k=0}^{N-1} e^{-2\pi i k(\ell-mj)/N} e^{-i\pi k(N-k)mn|s|} \psi_{T^2}^{j+knN|s|,|M|}(z,\tau), \qquad (3.44)$$

where $j = 0, \ldots, N|s| - 1$. To construct the eigenfunctions for $^\forall e_N^{(m,n)}$-shifts, we have to consider the simultaneous eigenfunctions for both $e_N^{(1,0)}$ and $e_N^{(0,1)}$-shifts under the conditions in eqs. (3.36), (3.37), and (3.40). Since the boundary condition for $e_N^{(1,0)}$, eq. (3.36), gives the constraint $\ell_1 = j \pmod{N}$, we can obtain the eigenfunctions for $^\forall e_N^{(m,n)}$-shifts as follows:

$$\Psi_{T^2/\mathbb{Z}_N^{(\ell_1,\ell_2)}}^{r,|s|}(z,\tau) \equiv \psi_{T^2/\mathbb{Z}_N^{(\ell_1,\ell_2)}}^{j,|M|}(z,\tau) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-2\pi i k \ell_2/N} \psi_{T^2}^{j+kN|s|,|M|}(z,\tau), \qquad (3.45)$$

$$j = Nr + \ell_1 \in \mathbb{Z}_{N|s|}, \ r \in \mathbb{Z}_{|s|}, \ \ell_1, \ell_2 \in \mathbb{Z}_N,$$

where M and s can only take the values allowed in eq. (3.40). There are $|s|$ number of the $\mathbb{Z}_N$ shifts invariant modes ($\ell_1 = \ell_2 = 0$) and ($|M| - |s|$) not invariant modes. Under these liner combinations, the formula of modular forms in eq. (3.23) becomes

$$\Psi_{T^2/\mathbb{Z}_N^{(\ell_1,\ell_2)}}^{r,|s|}(\widetilde{\gamma}(z,\tau)) = \widetilde{J}_{1/2}(\widetilde{\gamma},\tau) \sum_{r'=0}^{|s|-1} \sum_{\ell_1',\ell_2'=0}^{N-1} \rho_{T^2/\mathbb{Z}_N^{(\ell_1,\ell_2)}}(\widetilde{\gamma})_{rr',(\ell_1,\ell_2)(\ell_1',\ell_2')} \Psi_{T^2/\mathbb{Z}_N^{(\ell_1',\ell_2')}}^{r',|s|}(z,\tau),$$

$$(3.46)$$

for $\widetilde{\gamma} \in \widetilde{\Gamma}$, and unitary matrices are represented by

$$\rho_{T^2/\mathbb{Z}_N^{(\ell_1,\ell_2)}}(\widetilde{S})_{rr',(\ell_1,\ell_2)(\ell_1',\ell_2')} = e^{i\pi/4} \frac{1}{\sqrt{|s|}} e^{2\pi i \left(\frac{\ell_1}{N}+r\right)\left(\frac{\ell_1'}{N}+r'\right)/|s|} \delta_{\ell_2,\ell_1'} \delta_{N-\ell_1,\ell_2'}, \qquad (3.47)$$

$$\rho_{T^2/\mathbb{Z}_N^{(\ell_1,\ell_2)}}(\widetilde{T})_{rr',(\ell_1,\ell_2)(\ell_1',\ell_2')} = e^{i\pi\left(\frac{\ell_1}{N}+r\right)^2/|s|} \delta_{r,r'} \delta_{\ell_1,\ell_1'} \delta_{\ell_2-\ell_1,\ell_2'}, \qquad (3.48)$$

where $\delta_{\ell_2-\ell_1,\ell_2'}$ in $\rho_{T^2/\mathbb{Z}_N^{(\ell_1,\ell_2)}}(\widetilde{T})$ is modified into $\delta_{\ell_2-\ell_1+N/2,\ell_2'}$ for $s \in 2\mathbb{Z}+1$, $N \in 2\mathbb{Z}$. Then, for $s \in 2\mathbb{Z}$, we can directly show that the $\mathbb{Z}_N$ shifts invariant modes behave as modular forms for $\widetilde{\Gamma}(2|M|/N^2)$ and variant modes are that for $\widetilde{\Gamma}(2|M|)$. Although it may seem these give the same result for $s \in 2\mathbb{Z}+1$, the modular transformation does not close in the $\mathbb{Z}_N$ shift invariant modes, but they transform to $\mathbb{Z}_N$ shift variant modes. Note that the invariant modes correspond to modes on $T^{2'} \simeq \mathbb{C}/\Lambda'$, $\Lambda' \equiv \Lambda/N$ with the magnetic flux $M/N^2 = s$. Thus, we can understand this from the fact that the wavefunctions on torus with the magnetic flux $M = 2\mathbb{Z}+1$ are not consistent with T transformation $\tau \to \tau + 1$.

Instead, there are the $|s|$ number of the $\mathbb{Z}_N$ shift $(\ell_1,\ell_2) = (N/2,N/2)$ modes and the other $(|M| - |s|)$ modes transformed independently under the modular transformation. Also we can check that the former modes behave as the modular forms for $\widetilde{\Gamma}(8|M|/N^2)$ and they satisfy the further algebraic relation,

$$\rho_{T^2/\mathbb{Z}_N^{(N/2,N/2)}}(\widetilde{T})^{2|M|/N^2} = i\mathbb{I}. \qquad (3.49)$$

The latter modes just behave as that for $\widetilde{\Gamma}(2|M|)$.

3.4 $T^2/\mathbb{Z}_N$ twist and shift orbifold models

As the end of the review of the $T^2/\mathbb{Z}_N$ orbifolds, we study the $T^2/\mathbb{Z}_N$ twist and shift orbifold models. The full modular symmetry remains only on the combination of the $T^2/\mathbb{Z}_2$ twist orbifold and the full $T^2/\mathbb{Z}_2$ shift orbifold, that is the full $T^2/\mathbb{Z}_2$ twist and shift orbifold, since the full $T^2/\mathbb{Z}_2$ shift orbifold only satisfies the consistency condition with the $T^2/\mathbb{Z}_2$ twist orbifold, i.e., $N - \ell_{1,2} \equiv \ell_{1,2} \pmod{N}$. For $M/4 = s \in 2\mathbb{Z}$, the wavefunctions on the above orbifold are given by

$$
\begin{aligned}
\Psi^{r,|s|}_{T^2/\mathbb{Z}_2^{(m;\ell_1,\ell_2)}} &= \mathcal{N}_2^{st} \left(\Psi^{r,|s|}_{T^2/\mathbb{Z}_2^{(\ell_1,\ell_2)}} + (-1)^{m+\ell_2} \Psi^{|s|-r-\ell_1,|s|}_{T^2/\mathbb{Z}_2^{(\ell_1,\ell_2)}} \right) \\
&= \mathcal{N}_2^{st} \left(\psi^{2r+\ell_1,4|s|}_{T^2/\mathbb{Z}_2^m} + (-1)^{\ell_2} \psi^{2r+\ell_1+2|s|,4|s|}_{T^2/\mathbb{Z}_2^m} \right) \\
&= \mathcal{N}_2^{st} \Big(\psi^{2r+\ell_1,4|s|}_{T^2} + (-1)^{\ell_2+m} \psi^{2(|s|-r-\ell_1)+\ell_1,4|s|}_{T^2} + (-1)^{\ell_2} \psi^{2(|s|+r)+\ell_1,4|s|}_{T^2} \\
&\quad + (-1)^m \psi^{2(2|s|-r-\ell_1)+\ell_1,4|s|}_{T^2} \Big), \\
&\qquad s \in 2\mathbb{Z}, \ r \in \mathbb{Z}_{\frac{|s|}{2}+1-\ell_1}, \ m, \ell_1, \ell_2 \in \mathbb{Z}_2,
\end{aligned}
\tag{3.50}
$$

where $\mathcal{N}_2^{st}$ is the normalization factor. There are the $(|M|/8+1)$ and $(|M|/8-1)$ numbers of the full $\mathbb{Z}_2$ shifts invariant modes $(m; \ell_1, \ell_2) = (0; 0, 0)$ and $(1; 0, 0)$, respectively. Under these liner combinations, the unitary representations in eq. (3.24) become

$$
\rho_{T^2/\mathbb{Z}_2^{(0;\ell_1,\ell_2)}}(\widetilde{S})_{rr',(\ell_1,\ell_2)(\ell_1',\ell_2')} = e^{i\pi/4} \frac{2}{\sqrt{|s|}} \cos\left(2\pi(\ell_1/2+r)(\ell_1'/2+r')/|s|\right) \delta_{\ell_2,\ell_1'} \delta_{\ell_1,\ell_2'},
\tag{3.51}
$$

$$
\rho_{T^2/\mathbb{Z}_2^{(0;\ell_1,\ell_2)}}(\widetilde{T})_{rr',(\ell_1,\ell_2)(\ell_1',\ell_2')} = e^{i\pi\left(\frac{\ell_1}{N}+r\right)^2/|s|} \delta_{r,r'} \delta_{\ell_1,\ell_1'} \delta_{\ell_2-\ell_1,\ell_2'},
\tag{3.52}
$$

$$
\rho_{T^2/\mathbb{Z}_2^{(1;\ell_1,\ell_2)}}(\widetilde{S})_{rr',(\ell_1,\ell_2)(\ell_1',\ell_2')} = e^{i\pi/4} \frac{2i}{\sqrt{|s|}} \sin\left(2\pi(\ell_1/2+r)(\ell_1'/2+r')/|s|\right) \delta_{\ell_2,\ell_1'} \delta_{\ell_1,\ell_2'},
\tag{3.53}
$$

$$
\rho_{T^2/\mathbb{Z}_2^{(1;\ell_1,\ell_2)}}(\widetilde{T})_{rr',(\ell_1,\ell_2)(\ell_1',\ell_2')} = e^{i\pi\left(\frac{\ell_1}{N}+r\right)^2/|s|} \delta_{r,r'} \delta_{\ell_1,\ell_1'} \delta_{\ell_2-\ell_1,\ell_2'}.
\tag{3.54}
$$

Then, we can directly show that the $\mathbb{Z}_2$ shifts invariant modes $(m; \ell_1, \ell_2) = (m; 0, 0)$ behave as the modular forms for $\widetilde{\Gamma}(|M|/2)$ and variant modes are that for $\widetilde{\Gamma}(2|M|)$. Moreover, they satisfy the further algebraic relation such as eq. (3.34).

The same argument is also possible for $s \in 2\mathbb{Z}+1$, but we show only the results here. There are the $(|M|/4-1)$ and $(|M|/4+1)$ numbers of the $\mathbb{Z}_2$ twist and full shifts $(m; \ell_1, \ell_2) = (0; 1, 1)$ and $(1, 1, 1)$ modes, respectively. They are transformed independently under the modular transformation. The $\mathbb{Z}_2$ twist and full shifts $(m; 1, 1)$ modes behave as the modular forms for $\widetilde{\Gamma}(2|M|)$ and they satisfy the further algebraic relation such as eqs. (3.34) and (3.49). Other modes just behave as that for $\widetilde{\Gamma}(2|M|)$ and satisfy eq. (3.34).

4 Modular symmetry in magnetized $T_1^2 \times T_2^2$

In the previous section, we have seen the modular symmetry on the magnetized T^2 and its orbifolds by the $\mathbb{Z}_N$ twist and shift. In this section, let us consider the modular symmetry of the zero-mode wavefunctions on the magnetized $T_1^2 \times T_2^2$ and orbifolds where the complex modulus parameters are identified as $\tau_1 = \tau_2 \equiv \tau$. As in the previous analyses on T^2, we assume the even magnetic fluxes and focus on the zero-mode wavefunctions on the orbifolds.

4.1 $T_1^2 \times T_2^2$ models

Since the wavefunctions on T^2 behave like the modular forms of weight $1/2$ for $\widetilde{\Gamma}(2|M|)$, we can treat the wavefunctions on $T_1^2 \times T_2^2$ as the modular forms of weight 1 as follows:

$$
\psi_{0,T_1^2}^{j,|M_1|}(\gamma(z_1,\tau))\psi_{0,T_2^2}^{k,|M_2|}(\gamma(z_2,\tau))
$$

$$
= J_1(\gamma,\tau) \sum_{m=0}^{|M_1|-1} \rho(\gamma)_{jm} \sum_{n=0}^{|M_2|-1} \rho(\gamma)_{kn} \psi_{0,T_1^2}^{m,|M_1|}(z_1,\tau)\psi_{0,T_2^2}^{n,|M_2|}(z_2,\tau)
$$

$$
\equiv J_1(\gamma,\tau) \sum_{m=0}^{|M_1|-1}\sum_{n=0}^{|M_2|-1} \rho(\gamma)_{(jk)(mn)} \psi_{0,T_1^2}^{m,|M_1|}(z_1,\tau)\psi_{0,T_2^2}^{n,|M_2|}(z_2,\tau), \quad (4.1)
$$

$$
\rho(S)_{(jk)(mn)} = \prod_{t=1,2} \rho_{T_t^2}(\widetilde{S})_{j_t m_t} = \rho_{T_1^2}(\widetilde{S})_{jm}\rho_{T_2^2}(\widetilde{S})_{kn} = \frac{i}{\sqrt{|M_1 M_2|}} e^{2i\pi(\frac{jm}{|M_1|}+\frac{kn}{|M_2|})}, \quad (4.2)
$$

$$
\rho(T)_{(jk)(mn)} = \prod_{t=1,2} \rho_{T_t^2}(\widetilde{T})_{j_t m_t} = \rho_{T_1^2}(\widetilde{T})_{jm}\rho_{T_2^2}(\widetilde{T})_{kn} = e^{i\pi(\frac{j^2}{|M_1|}+\frac{k^2}{|M_2|})}\delta_{j,m}\delta_{k,n}, \quad (4.3)
$$

$$
j,m \in \mathbb{Z}_{|M_1|}, k,n \in \mathbb{Z}_{|M_2|}, \gamma \in \Gamma,
$$

where the lower indices 1 and 2 of the coordinates z and the magnetic fluxes M denote the tori T_1^2 and T_2^2, respectively. Note that the modular symmetry on $T_1^2 \times T_2^2$, $\Gamma \times \Gamma$, is broken to Γ by the identification $\tau_1 = \tau_2 \equiv \tau$. Similarly, the unitary representation $\rho(\gamma)_{(jk)(mn)}$ is broken from $\widetilde{\Gamma}'_{2|M_1|} \times \widetilde{\Gamma}'_{2|M_2|}$ to its subgroup. Since $\rho(\gamma)_{(jk)(mn)}$ is given by the tensor products of the representations on T_1^2 and T_2^2 such as above, by multiplying both algebraic relations in eq. (3.25) for T_1^2 and T_2^2, we can obtain the following relations for $T_1^2 \times T_2^2$:

$$
\rho(T)_{(jk)(mn)}^{2\mathrm{lcm}(|M_1|,|M_2|)} = \rho(S)_{(jk)(mn)}^4 = [\rho(S)\rho(T)]_{(jk)(mn)}^3 = \delta_{(jk),(mn)},
$$
$$
[\rho(S)^2\rho(T)]_{(jk)(mn)} = [\rho(T)\rho(S)^2]_{(jk)(mn)}, \quad \rho(S)_{(jk)(mn)}^2 = -\delta_{j,|M_1|-m}\delta_{k,|M_2|-n}. \quad (4.4)
$$

This is just the algebra of $\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$.[2] Thus, the zero-mode wavefunctions on $T_1^2 \times T_2^2$ behave as the modular forms of weight 1 for $\Gamma(2\mathrm{lcm}(|M_1|,|M_2|))$. This argument on the algebraic relations is valid for other orbifolds unless we study orbifolding across the tori T_1^2 and T_2^2. Such orbifolding by permutation may affect the algebraic relations since their representations cannot be written as the tensor products. The remaining of this section, we consider orbifolding $T_1^2 \times T_2^2$ by the $\mathbb{Z}_2$ twist, the full $\mathbb{Z}_N$ shifts and the $\mathbb{Z}_2$ permutation that interchanges the two tori coordinates, $z_1 \leftrightarrow z_2$.

---

[2]$2\mathrm{lcm}(a,b)$ denotes two times the least common multiple of a and b.

Before the end of this subsection, we comment on products of wavefunctions and couplings. Using eq. (2.22), we can expand the product of wavefunctions on $T_1^2 \times T_2^2$ as

$$\psi_{0,T^2 \times T^2}^{j,k}(z_1, z_2, \tau)\, \psi_{0,T^2 \times T^2}^{j',k'}(z_1, z_2, \tau) = \sum_{j'',k'''} C^{(j,j',j''),(k,k',k'l')}(\tau)\, \psi_{0,T^2 \times T^2}^{j'',k''}(z_1, z_2, \tau), \quad (4.5)$$

where

$$\psi_{0,T^2 \times T^2}^{j,k}(z_1, z_2, \tau) = \psi_{0,T_1^2}^{j,|M_1|}(z_1, \tau)\psi_{0,T_2^2}^{k,|M_2|}(z_2, \tau), \quad (4.6)$$

$$C^{(j,j',j''),(k,k',k'')}(\tau) = C_{T^2}^{jj'j''}(\tau)C_{T^2}^{kk'k''}(\tau). \quad (4.7)$$

The modular transformation behaviors in left and right hand sides in eq. (4.5) must be the same. The τ-dependent coefficient $C^{(j,j',j''),(k,k',k'')}(\tau)$ is the modular form of weight 1. In particular, when the magnetic fluxes for $\psi_{0,T^2 \times T^2}^{j,k}(z_1, z_2, \tau)$ and $\psi_{0,T^2 \times T^2}^{j',k'}(z_1, z_2, \tau)$ are the same, $C^{(j,j',j''),(k,k',k'')}(\tau)$ are multiplets under $\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$. That is, the three-point couplings are the modular form of weight 1 with a non-trivial representation of $\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$. The n-point couplings are also obtained by products of $C^{(j,j',j''),(k,k',k'')}(\tau)$, and they are modular forms of weight $(n-2)$.

4.2 $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist and shift orbifold models

First of all, we consider orbifolding by the $\mathbb{Z}_2$ twist and the full $\mathbb{Z}_N$ shift, where the algebraic relations in the previous section is valid. On the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist and shift orbifolds, in general, the tensor product of the representations of $\widetilde{\Gamma}'_{2|M_1|}$ and $\widetilde{\Gamma}'_{2|M_2|}$ gives the representation of $\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$.

Since the wavefunctions on the above orbifolds are obtained by the tensor products of each orbifold, for example, the wavefunctions on the pair of T_1^2 and T_2^2 with $M_1 = M_2 = 2$ are obtained as

$$\begin{pmatrix} \psi_{T_1^2 \times T_2^2}^{(00)}(z_1, z_2) \\ \psi_{T_1^2 \times T_2^2}^{(10)}(z_1, z_2) \\ \psi_{T_1^2 \times T_2^2}^{(01)}(z_1, z_2) \\ \psi_{T_1^2 \times T_2^2}^{(11)}(z_1, z_2) \end{pmatrix} = \begin{pmatrix} \psi_{0,T_1^2}^{0,2}(z_1)\psi_{0,T_2^2}^{0,2}(z_2) \\ \psi_{0,T_1^2}^{1,2}(z_1)\psi_{0,T_2^2}^{0,2}(z_2) \\ \psi_{0,T_1^2}^{0,2}(z_1)\psi_{0,T_2^2}^{1,2}(z_2) \\ \psi_{0,T_1^2}^{1,2}(z_1)\psi_{0,T_2^2}^{1,2}(z_2) \end{pmatrix}, \quad (4.8)$$

while the $\mathbb{Z}_N$-shift even modes on the pair of T_1^2 and the $T_2^2/\mathbb{Z}_N$ shift orbifold with $M_1 = 2$, $M_2 = 2N^2$ are

$$\begin{pmatrix} \psi_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}^{(00)}(z_1, z_2) \\ \psi_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}^{(10)}(z_1, z_2) \\ \psi_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}^{(01)}(z_1, z_2) \\ \psi_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}^{(11)}(z_1, z_2) \end{pmatrix} = \begin{pmatrix} \psi_{0,T_1^2}^{0,2}(z_1)\Psi_{T_2^2/\mathbb{Z}_N^{(0,0)}}^{0,2}(z_2) \\ \psi_{0,T_1^2}^{1,2}(z_1)\Psi_{T_2^2/\mathbb{Z}_N^{(0,0)}}^{0,2}(z_2) \\ \psi_{0,T_1^2}^{0,2}(z_1)\Psi_{T_2^2/\mathbb{Z}_N^{(0,0)}}^{1,2}(z_2) \\ \psi_{0,T_1^2}^{1,2}(z_1)\Psi_{T_2^2/\mathbb{Z}_N^{(0,0)}}^{1,2}(z_2) \end{pmatrix}. \quad (4.9)$$

Then, the representations of the S, T transformations are same on both wavefunctions,

$$\rho(S) = \frac{i}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.10}$$

They generate the group $\Gamma_4' \simeq S_4'$ which has the order 48. From the correspondence of the $\mathbb{Z}_N$-shift even modes to the $1/N$ torus with the magnetic flux $M_2/N^2 = 2$, we can understand these equalities. Now, it is straightforward to confirm that they satisfy the above general rule for the algebraic relations on the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist and shift orbifolds, hence, the product representation of $\widetilde{\Gamma}_4'$ and $\widetilde{\Gamma}_4'$ gives the representation of Γ_4'. The matrices $\rho(S)$ and $\rho(T)$ in eq. (4.10) correspond to a reducible representation $\Gamma_4' \simeq S_4'$. They can be decomposed into a triplet and a singlet. The triplet corresponds to

$$\begin{pmatrix} \psi^{(00)}_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}(z_1, z_2) \\ \frac{1}{\sqrt{2}} \left(\psi^{(10)}_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}(z_1, z_2) + \psi^{(01)}_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}(z_1, z_2) \right) \\ \psi^{(11)}_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}(z_1, z_2), \end{pmatrix}, \tag{4.11}$$

where S and T are expressed as follows:

$$\rho(S) = \frac{i}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.12}$$

In addition, the singlet corresponds to

$$\frac{1}{\sqrt{2}} \left(\psi^{(10)}_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}(z_1, z_2) - \psi^{(01)}_{T_1^2 \times (T_2^2/\mathbb{Z}_N^0)}(z_1, z_2) \right), \tag{4.13}$$

where S and T are expressed as

$$\rho(S) = -i, \qquad \rho(T) = i. \tag{4.14}$$

The wavefunction of the singlet vanishes at $z_1 = z_2 = 0$, while the other do not vanish. Thus, the singlet is trivial as the conventional modular form $f(\tau)$.

Similarly, we can study other types of orbifolding. However, there are exceptions on the pair of the $\mathbb{Z}_2$ twist orbifolds and that of the $\mathbb{Z}_N$ shift orbifolds. In the former case, since the representations on the $T^2/\mathbb{Z}_2$ twist orbifolds satisfy the relation in eq. (3.34), the tensor product of these obeys

$$\rho(T)^{2\mathrm{lcm}(|M_1|,|M_2|)} = \rho(S)^4 = [\rho(S)\rho(T)]^3 = \mathbb{I}, \ \rho(S)^2 = -(-1)^{m_1+m_2}\mathbb{I}, \tag{4.15}$$

where m_1, m_2 denote the $\mathbb{Z}_2$-twist eigenmodes $0, 1$ on T_1^2 and T_2^2, respectively. Therefore, the products of the same $\mathbb{Z}_2$-twist eigenmodes ($m_1 = m_2$) correspond to $\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$, while the different $\mathbb{Z}_2$-twist eigenmodes ($m_1 \neq m_2$) correspond to $\Gamma_{2\mathrm{lcm}(|M_1|,|M_2|)}$.

In the latter case, only on the pair of the $\mathbb{Z}_N$ shift orbifolds with the magnetic fluxes $M = N^2 s$, $s \in 2\mathbb{Z}+1$, $N \in 2\mathbb{Z}$, there are the further relation in eq. (3.49) for the $\mathbb{Z}_N$-shift $(\ell_1, \ell_2) = (N/2, N/2)$ modes. Then, the tensor products of the algebraic relations obey

$$\rho(T)^{4\mathrm{lcm}(|s_1|,|s_2|)} = \rho(S)^4 = [\rho(S)\rho(T)]^3 = \mathbb{I}, \ \rho(S)^2\rho(T) = \rho(T)\rho(S)^2. \qquad (4.16)$$

This means that the products of the $\mathbb{Z}_N$-shift $(\ell_1, \ell_2) = (N/2, N/2)$ modes correspond to $\Gamma'_{4\mathrm{lcm}(|s_1|,|s_2|)}$, where $4\mathrm{lcm}(a,b)$ denotes 4 times the least common multiple of a and b.

With these in mind, we show the algebraic relations for the unitary representation on the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist and shift orbifolds in table 1. The dimension of each normal subspace (eigenmode) is given by the products of the number of modes on each $T^2/\mathbb{Z}_N$ orbifold discussed in section 3. It is shown in table 2. For simplicity, we omit the results of the $\mathbb{Z}_N$ shift orbifolds with the magnetic fluxes $M = N^2 s$, $s \in 2\mathbb{Z}+1$, $N \in 2\mathbb{Z}$, but we can obtain them from the arguments up to now.

4.3 $(T_1^2 \times T_2^2)/\mathbb{Z}_2$ permutation orbifold models

Next, we consider the $(T_1^2 \times T_2^2)/\mathbb{Z}_2$ permutation orbifolds. It is obtained by further identifying the complex coordinates of $T_1^2 \times T_2^2$, (z_1, z_2) with the $\mathbb{Z}_2$ discrete interchanged points $I_2^n(z_1, z_2) \equiv (z_{1+n}, z_{2+n})$, $n \in \mathbb{Z}_2 = \{0, 1\}$, $z_{\mathrm{odd}} \equiv z_1$, $z_{\mathrm{even}} \equiv z_2$. We can easily check $I_2^n \circ I_2^m = I_2^{n+m}$. In this identification, the wavefunctions on the magnetized $(T_1^2 \times T_2^2)/\mathbb{Z}_2$ permutation orbifolds, $\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}^{J,|M|}(z_1, z_2)$, are required to satisfy the following further boundary condition,

$$\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}^{J,|M|}(I_2^1(z_1, z_2)) = \psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}^{J,|M|}(z_2, z_1) = e^{i\pi n}\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}^{J,|M|}(z_1, z_2), \quad n \in \mathbb{Z}_2, \ (4.17)$$

and, therefore, can be expressed by liner combinations of the wavefunctions on $T_1^2 \times T_2^2$ as

$$\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}^{J,|M|}(z_1, z_2) = \mathcal{N}_2^p \sum_{\ell=0,1} (e^{i\pi n})^{-\ell}\psi_{0,T_1^2}^{j,|M|}(I_2^\ell(z_1), \tau)\psi_{0,T_2^2}^{k,|M|}(I_2^\ell(z_2), \tau), \ j \geq k, \ j,k \in \mathbb{Z}_{|M|},$$

$$(4.18)$$

where the normalization factor $\mathcal{N}_2^p = 1/2$ and $1/\sqrt{2}$ for $j = k$ and $j \neq k$, respectively. Note that there exists the magnetic flux condition $M_1 = M_2 \equiv M$ to identify two tori T_1^2 and T_2^2. Furthermore, the modulus parameters are also required to satisfy the condition $\tau_1 = \tau_2 \equiv \tau$, but it has been already assumed. There are the $|M|(|M| + 1)/2$ number of $\mathbb{Z}_2$-even modes ($n = 0$) and $|M|(|M| - 1)/2$-odd modes ($n = 1$). Under these liner combinations, the formula of modular forms in eq. (4.1) becomes

$$\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}^{(jk),|M|}(\gamma(z_1, z_2, \tau)) = J_1(\gamma, \tau) \sum_{m=0}^{|M|-1} \sum_{\ell=0}^{m} \rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}(\gamma)_{(jk)(m\ell)} \psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}^{(m\ell),|M|}(z_1, z_2, \tau).$$

$$(4.19)$$

The unitary representation $\rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}$ is given by

$$\rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^n}(\gamma)_{(jk)(m\ell)} = \left(\rho(\gamma)_{(jk)(m\ell)} + (-1)^n \rho(\gamma)_{(jk)(\ell m)}\right), \ j \geq k, \ m \geq \ell, \qquad (4.20)$$

T_1^2	T_2^2	Magnetic flux	Algebra for each mode
T^2	T^2, $\mathbb{Z}_2$ twist	M_1, M_2	$\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$
T^2	$\mathbb{Z}_N$ shift, $\mathbb{Z}_2$ twist & shift	$M_1, M_2 = N_2^2 s_2$ ($s_2 \in 2\mathbb{Z}$)	$\begin{cases} \Gamma'_{2\mathrm{lcm}(|M_1|,|s_2|)}(+_2^s) \\ \Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}(-_2^s) \end{cases}$
$\mathbb{Z}_2$ twist	$\mathbb{Z}_2$ twist	M_1, M_2	$\begin{cases} \Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}(\pm_1^t \pm_2^t) \\ \Gamma_{2\mathrm{lcm}(|M_1|,|M_2|)}(\pm_1^t \mp_2^t) \end{cases}$
$\mathbb{Z}_2$ twist	$\mathbb{Z}_N$ shift	$M_1, M_2 = N_2^2 s_2$ ($s_2 \in 2\mathbb{Z}$)	$\begin{cases} \Gamma'_{2\mathrm{lcm}(|M_1|,|s_2|)}(+_2^s) \\ \Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}(-_2^s) \end{cases}$
$\mathbb{Z}_2$ twist	$\mathbb{Z}_2$ twist & shift	$M_1, M_2 = 4s_2$ ($s_2 \in 2\mathbb{Z}$)	$\begin{cases} \Gamma'_{2\mathrm{lcm}(|M_1|,|s_2|)}(\pm_1^t \pm_2^t +_2^s) \\ \Gamma_{2\mathrm{lcm}(|M_1|,|s_2|)}(\pm_1^t \mp_2^t +_2^s) \\ \Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}(\pm_1^t \pm_2^t -_2^s) \\ \Gamma_{2\mathrm{lcm}(|M_1|,|M_2|)}(\pm_1^t \mp_2^t -_2^s) \end{cases}$
$\mathbb{Z}_N$ shift	$\mathbb{Z}_N$ shift, $\mathbb{Z}_2$ twist & shift	$M_1 = N_1^2 s_1,$ $M_2 = N_2^2 s_2$ ($s_1, s_2 \in 2\mathbb{Z}$)	$\begin{cases} \Gamma'_{2\mathrm{lcm}(|s_1|,|s_2|)}(+_1^s +_2^s) \\ \Gamma'_{2\mathrm{lcm}(|s_1|,|M_2|)}(+_1^s -_2^s) \\ \Gamma'_{2\mathrm{lcm}(|M_1|,|s_2|)}(-_1^s +_2^s) \\ \Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}(-_1^s -_2^s) \end{cases}$
$\mathbb{Z}_2$ twist & shift	$\mathbb{Z}_2$ twist & shift	$M_1 = 4s_1,$ $M_2 = 4s_2$ ($s_1, s_2 \in 2\mathbb{Z}$)	$\begin{cases} \Gamma'_{2\mathrm{lcm}(|s_1|,|s_2|)}(\pm_1^t +_1^s \pm_2^t +_2^s) \\ \Gamma_{2\mathrm{lcm}(|s_1|,|s_2|)}(\pm_1^t +_1^s \mp_2^t +_2^s) \\ \Gamma'_{2\mathrm{lcm}(|s_1|,|M_2|)}(\pm_1^t +_1^s \pm_2^t -_2^s) \\ \Gamma_{2\mathrm{lcm}(|s_1|,|M_2|)}(\pm_1^t +_1^s \mp_2^t -_2^s) \\ \Gamma'_{2\mathrm{lcm}(|M_1|,|s_2|)}(\pm_1^t -_1^s \pm_2^t +_2^s) \\ \Gamma_{2\mathrm{lcm}(|M_1|,|s_2|)}(\pm_1^t -_1^s \mp_2^t +_2^s) \\ \Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}(\pm_1^t -_1^s \pm_2^t -_2^s) \\ \Gamma_{2\mathrm{lcm}(|M_1|,|M_2|)}(\pm_1^t -_1^s \mp_2^t -_2^s) \end{cases}$

Table 1. The algebraic relations for the unitary representation on the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist and shift orbifolds. The first (second) column shows the types of orbifolds from T_1^2 (T_2^2), which also include T_1^2 (T_2^2) itself. The third column shows the flux condition on each orbifold. The last column shows the algebraic relations that the unitary representation on the orbifolds at least satisfies. The sign $(+/-_{1/2}^{t/s})$ means the invariant $(+)$/variant $(-)$ modes under the $\mathbb{Z}_2$ twist $(^t)/\mathbb{Z}_N$ shift $(^s)$ on T_1^2 $(_1)/T_2^2$ $(_2)$.

T_1^2	T_2^2	Normal subspaces	# of zero-modes
T^2	T^2	-	$|M_1||M_2|$
T^2	$\mathbb{Z}_2$ twist	$\pm_2^t$	$|M_1|(|M_2| \pm 2)/2$
T^2	$\mathbb{Z}_N$ shift	$+_2^s$	$|M_1||s_2|$
		$-_2^s$	$|M_1||s_2|(N_2^2 - 1)$
T^2	$\mathbb{Z}_2$ twist & shift	$\pm_2^t +_2^s$	$|M_1|(|s_2| \pm 2)/2$
		$\pm_2^t -_2^s$	$3|M_1||s_2|/2$
$\mathbb{Z}_2$ twist	$\mathbb{Z}_2$ twist	$\pm_1^t \pm_2^t$	$(|M_1| \pm 2)(|M_2| \pm 2)/4$
		$\pm_1^t \mp_2^t$	$(|M_1| \pm 2)(|M_2| \mp 2)/4$
$\mathbb{Z}_2$ twist	$\mathbb{Z}_N$ shift	$\pm_1^t +_2^s$	$(|M_1| \pm 2)|s_2|/2$
		$\pm_1^t -_2^s$	$(|M_1| \pm 2)|s_2|(N_2^2 - 1)/2$
$\mathbb{Z}_2$ twist	$\mathbb{Z}_2$ twist & shift	$\pm_1^t \pm_2^t +_2^s$	$(|M_1| \pm 2)(|s_2| \pm 2)/4$
		$\pm_1^t \pm_2^t -_2^s$	$3(|M_1| \pm 2)|s_2|/4$
		$\pm_1^t \mp_2^t +_2^s$	$(|M_1| \pm 2)(|s_2| \mp 2)/4$
		$\pm_1^t \mp_2^t -_2^s$	$3(|M_1| \pm 2)|s_2|/4$
$\mathbb{Z}_N$ shift	$\mathbb{Z}_N$ shift	$+_1^s +_2^s$	$|s_1||s_2|$
		$+_1^s -_2^s$	$|s_1||s_2|(N_2^2 - 1)$
		$-_1^s +_2^s$	$|s_1||s_2|(N_1^2 - 1)$
		$-_1^s -_2^s$	$|s_1|s_2|(N_1^2 - 1)(N_2^2 - 1)$
$\mathbb{Z}_N$ shift	$\mathbb{Z}_2$ twist & shift	$+_1^s \pm_2^t +_2^s$	$|s_1|(|s_2| \pm 2)/2$
		$+_1^s \pm_2^t -_2^s$	$3|s_1||s_2|/2$
		$-_1^s \pm_2^t +_2^s$	$|s_1|(N_1^2 - 1)(|s_2| \pm 2)/2$
		$-_1^s \pm_2^t -_2^s$	$3|s_1||s_2|(N_1^2 - 1)/2$
$\mathbb{Z}_2$ twist & shift	$\mathbb{Z}_2$ twist & shift	$\pm_1^t +_1^s \pm_2^t +_2^s$	$(|s_1| \pm 2)(|s_2| \pm 2)/4$
		$\pm_1^t +_1^s \pm_2^t -_2^s$	$3(|s_1| \pm 2)|s_2|/4$
		$\pm_1^t -_1^s \pm_2^t +_2^s$	$3|s_1|(|s_2| \pm 2)/4$
		$\pm_1^t -_1^s \pm_2^t -_2^s$	$9|s_1||s_2|/4$
		$\pm_1^t +_1^s \mp_2^t +_2^s$	$(|s_1| \pm 2)(|s_2| \mp 2)/4$
		$\pm_1^t +_1^s \mp_2^t -_2^s$	$3(|s_1| \pm 2)|s_2|/4$
		$\pm_1^t -_1^s \mp_2^t +_2^s$	$3|s_1|(|s_2| \mp 2)/4$
		$\pm_1^t -_1^s \mp_2^t -_2^s$	$9|s_1||s_2|/4$

Table 2. The number of zero-modes on each normal subspace for the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist and shift orbifolds. The first (second) column shows the types of orbifolds from T_1^2 (T_2^2), which also include T_1^2 (T_2^2) itself. The third column shows the normal subspaces (eigenmodes) labeled by the $\mathbb{Z}_2$-twist eigenmodes $\pm^t$ and the $\mathbb{Z}_N$-shift eigenmodes $\pm^s$ for $(T_1^2 \times T_2^2)/\mathbb{Z}_N$. The notation is same with table 1. The last column shows the number of zero-modes on each normal subspace.

where it is multiplied by further factor $1/2$ for $m = \ell$. It satisfies the same algebraic relations with $\Gamma'_{2|M|}$. Thus, the representations on the $(T_1^2 \times T_2^2)/\mathbb{Z}_2$ permutation orbifold obey the same algebra with that on $T_1^2 \times T_2^2$, but their dimensions are smaller. For example, the $\mathbb{Z}_2$-permutation even modes on the $(T_1^2 \times T_2^2)/\mathbb{Z}_2$ permutation orbifold with the magnetic flux $M = 2$ is given by

$$
\begin{pmatrix}
\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^0}^{(00),2}(z_1, z_2) \\
\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^0}^{(10),2}(z_1, z_2) \\
\psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^0}^{(11),2}(z_1, z_2)
\end{pmatrix}
=
\begin{pmatrix}
\psi_{0,T_1^2}^{0,2}(z_1)\psi_{0,T_2^2}^{0,2}(z_2) \\
\frac{1}{\sqrt{2}}\left(\psi_{0,T_1^2}^{1,2}(z_1)\psi_{0,T_2^2}^{0,2}(z_2) + \psi_{0,T_1^2}^{0,2}(z_1)\psi_{0,T_2^2}^{1,2}(z_2)\right) \\
\psi_{0,T_1^2}^{1,2}(z_1)\psi_{0,T_2^2}^{1,2}(z_2)
\end{pmatrix}.
\tag{4.21}
$$

The unitary representations of the S and T are expressed as follows:

$$
\rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^0}(S) = \frac{i}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad \rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_2^0}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\tag{4.22}
$$

These matrices are the same as those in eq. (4.12). They generate the group $\Gamma'_4 \simeq S'_4$ which has the order 48. The three zero-modes correspond to a triplet of $\Gamma'_4 \simeq S'_4$. Thus, orbifolding by twist, shift, and permutation can decompose reducible representations into smaller one such as a reducible representation by their eigenvalues.

4.4 $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist, shift and permutation orbifold models

Now, we are ready to write down the algebraic relations for the unitary representations of the zero-mode wavefunctions on the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist, shift, and permutation orbifolds. As we saw in the previous subsection, the $\mathbb{Z}_2$ permutation does not affect the algebraic relations for the representations, although it is the permutation across the two tori T_1^2 and T_2^2. Thus, the algebraic relations in section 4.1 is valid for orbifolds including the $\mathbb{Z}_2$ permutation. Note that to identify two tori T_1^2 and T_2^2, the only pairs of the same $\mathbb{Z}_2$-twist ($\mathbb{Z}_N$-shift) eigenmodes are allowed on the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist, shift and permutation orbifolds.

As shown in the previous subsection, we can construct the wavefunctions on the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist, shift and permutation orbifolds as

$$
\Psi_{(T_1^2 \times T_2^2)/\mathbb{Z}_N^{(m;\ell_1,\ell_2;n)}}^{J,|M|}(z_1, z_2) = \frac{1}{\sqrt{2}}\sum_{p=0,1}(e^{i\pi n})^{-p}\psi_{T_1^2/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}^{j,|M|}(I_2^p(z_1), \tau)\psi_{T_2^2/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}^{k,|M|}(I_2^p(z_2), \tau),
\tag{4.23}
$$

where $\psi_{T^2/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}^{j,|M|}$ denotes the wavefunctions on the $T^2/\mathbb{Z}_N$ twist and shift orbifolds. The unitary representation $\rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_N^{(m;\ell_1,\ell_2;n)}}$ is also given by

$$
\begin{aligned}
&\rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_N^{(m;\ell_1,\ell_2;n)}}(\gamma)_{(jk)(pq)} \\
&= \left(\rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}(\gamma)_{(jk)(pq)} + (-1)^n \rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}(\gamma)_{(jk)(qp)}\right),
\end{aligned}
\tag{4.24}
$$

Orbifolds	Magnetic flux	Algebra for each mode
$\mathbb{Z}_2$ permutation	$M_1 = M_2 = M$	$\Gamma'_{2|M|}$
$\mathbb{Z}_2$ twist & permutation	$M_1 = M_2 = M$	$\Gamma'_{2|M|}(\pm_1^t \pm_2^t)$
$\mathbb{Z}_N$ shift & $\mathbb{Z}_2$ permutation	$M_1 = M_2 = M = N^2 s, \ s \in 2\mathbb{Z}$	$\begin{cases} \Gamma'_{2|s|}(+_1^s +_2^s) \\ \Gamma'_{2|M|}(-_1^s -_2^s) \end{cases}$
$\mathbb{Z}_2$ twist & shift & permutation	$M_1 = M_2 = M = N^2 s, \ s \in 2\mathbb{Z}$	$\begin{cases} \Gamma'_{2|s|}(\pm_1^t \pm_2^t, +_1^s +_2^s) \\ \Gamma'_{2|M|}(\pm_1^t \pm_2^t, -_1^s -_2^s) \end{cases}$

Table 3. The algebraic relations for the unitary representation on the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist, shift and permutation orbifolds. The first column shows the types of orbifolds. The second column shows the magnetic fluxes. The last column shows the algebraic relations for the unitary representation. The notation is same with table 1.

Orbifolds	Normal subspaces	# of zero − modes
$\mathbb{Z}_2$ permutation	$\pm^i$	$|M|(|M| \pm 1)/2$
$\mathbb{Z}_2$ twist & permutation	$\pm_1^t \pm_2^t \pm^i$	$(|M| \pm 2)(|M| \pm 4)/8$
	$\pm_1^t \pm_2^t \mp^i$	$|M|(|M| \pm 2)/8$
$\mathbb{Z}_N$ shift & $\mathbb{Z}_2$ permutation	$+_1^s +_2^s \pm^i$	$|s|(|s| \pm 1)/2$
	$-_1^s -_2^s \pm^i$	$|s|(N^2-1)(|s|(N^2-1) \pm 1)/2$
$\mathbb{Z}_2$ twist & shift & permutation	$\pm_1^t +_1^s \pm_2^t +_2^s \pm^i$	$(|s| \pm 2)(|s| \pm 4)/8$
	$\pm_1^t +_1^s \pm_2^t +_2^s \mp^i$	$|s|(|s| \pm 2)/8$
	$\pm_1^t -_1^s \pm_2^t -_2^s \pm^i$	$3|s|(3|s| \pm 2)/8$
	$\pm_1^t -_1^s \pm_2^t -_2^s \mp^i$	$3|s|(3|s| \mp 2)/8$

Table 4. The number of zero-modes on each normal subspace for the $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ twist, shift and permutation orbifolds. The first column shows the types of orbifolds. The second column shows the normal subspaces labelled by the $\mathbb{Z}_2$-twist eigenmodes $\pm^t$, the $\mathbb{Z}_N$-shift eigenmodes $\pm^s$ and the $\mathbb{Z}_2$-permutation eigenmodes $\pm^i$ for $(T_1^2 \times T_2^2)/\mathbb{Z}_N$. The last column shows the number of zero-modes on each normal subspace. The notation is same with table 1.

where

$$\rho_{(T_1^2 \times T_2^2)/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}(\gamma)_{(jk)(pq)} \equiv \rho_{T_1^2/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}(\widetilde{\gamma})_{jp} \rho_{T_2^2/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}(\widetilde{\gamma})_{kq}, \qquad (4.25)$$

and $\rho_{T^2/\mathbb{Z}_N^{(m;\ell_1,\ell_2)}}$ denotes the representations on the $T^2/\mathbb{Z}_N$ twist and shift orbifolds. Then, as shown in table 3, we can obtain the algebraic relations for the unitary representation on each orbifold. The dimension of each normal subspace (eigenmode) is shown in table 4.

5 Conclusion

In this paper, we have discussed the modular symmetry on $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ assuming the modulus parameters are identified. This identification allows us to regard the zero-mode wavefunctions on $(T_1^2 \times T_2^2)/\mathbb{Z}_N$ as the modular forms of weight 1. Moreover, the modular symmetry on $T_1^2 \times T_2^2$, $\Gamma \times \Gamma$, is broken to Γ. Zero-modes are multiplets of the favor symmetry $\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$ and $\Gamma_{2\mathrm{lcm}(|M_1|,|M_2|)}$ depending on orbifolding. Only if we consider the pair of the $T^2/\mathbb{Z}_2$ twist orbifolds, the flavor symmetry of zero-modes are $\Gamma_{2\mathrm{lcm}(|M_1|,|M_2|)}$. In the case of other orbifolds obtained by the $\mathbb{Z}_2$ twist, the full $\mathbb{Z}_N$ shift and the $\mathbb{Z}_2$ permutation, the flavor symmetries are given by $\Gamma'_{2\mathrm{lcm}(|M_1|,|M_2|)}$. Especially, we have shown the realization of the double cover of S_4, i.e., S_4'. That would be interesting from the recent bottom-up approach of model building [73, 74]. Also the flavor symmetry $\Gamma_{2\mathrm{lcm}(|M_1|,|M_2|)}$ would be interesting.

Orbifolding decomposes zero-modes by eigenvalues of the $\mathbb{Z}_2$ twist, the $\mathbb{Z}_N$ shift and the $\mathbb{Z}_2$ permutation, and reduce the number of zero-modes, namely the generation number of quarks and leptons. Three-generation models on twist orbifolds $T^2/\mathbb{Z}_2$ with magnetic fluxes have been classified in refs. [89, 90]. Combinations of orbifolding by the $\mathbb{Z}_2$ twist, the $\mathbb{Z}_N$ shift and the $\mathbb{Z}_2$ permutation provide us with the further possibility to construct three-generation models. We would study such model building and its phenomenological aspects elsewhere.

A T transformation for $M = $ odd

Here, we show the inconsistency between the boundary conditions, eqs. (2.12) and (2.13), and the T transformation, eq. (3.22), for $M = $ odd. First, we note that the boundary conditions mean the wavefunctions at the point, z, which denotes the complex coordinate on T^2, are identified with ones at the translated points, $z + n\tau + m$, $n, m \in \mathbb{Z}$, up to the phase factor related to the gauge transformation. It can be easily checked that the equation of motion, eq. (2.4), is preserved under the T transformation:

$$(i \slashed{D}_2 - m)\psi(z, \tau) = 0 \xrightarrow{T} (i \slashed{D}_2 - m)\psi(z, \tau + 1) = 0. \tag{A.1}$$

However, the boundary conditions for wavefunctions are changed as follows:

$$\psi(z+1, \tau) = e^{i\pi M \frac{\mathrm{Im}(z+\zeta)}{\mathrm{Im}\tau}} \psi(z, \tau) \xrightarrow{T} \psi(z+1, \tau+1) = e^{i\pi M \frac{\mathrm{Im}(z+\zeta)}{\mathrm{Im}\tau}} \psi(z, \tau+1), \tag{A.2}$$

$$\psi(z+\tau, \tau) = e^{i\pi M \frac{\mathrm{Im}\bar{\tau}(z+\zeta)}{\mathrm{Im}\tau}} \psi(z, \tau) \xrightarrow{T} \psi(z+\tau+1, \tau+1) = e^{i\pi M \frac{\mathrm{Im}[(\bar{\tau}+1)(z+\zeta)]}{\mathrm{Im}\tau}} \psi(z, \tau+1). \tag{A.3}$$

These lead to the following conditions for $\psi(z, \tau+1)$:

$$\psi(z+1, \tau+1) = e^{i\pi M \frac{\text{Im}(z+\zeta)}{\text{Im}\tau}} \psi(z, \tau+1), \tag{A.4}$$

$$\psi(z+\tau, \tau+1) = e^{i\pi M} \cdot e^{i\pi M \frac{\text{Im}\bar{\tau}(z+\zeta)}{\text{Im}\tau}} \psi(z, \tau+1). \tag{A.5}$$

Here, the only second condition is different with the boundary conditions for $\psi(z, \tau)$ by the exponential factor $e^{i\pi M}$. If we focus on $M = $ even, $e^{i\pi M}$ drops and the boundary conditions for $\psi(z, \tau+1)$ are the same with $\psi(z, \tau)$. While if we focus on $M = $ odd, $e^{i\pi M}$ equals to -1. Then, $\psi(z, \tau+1)$ cannot be expanded by $\psi(z, \tau)$ since it obeys the same equation of motion and the different boundary conditions with $\psi(z, \tau)$. This inconsistency is caused from the gaps of the phase factor between the wavefunctions at z and $z + n\tau + m$.

Thus, for $M = $ odd, we cannot define the T transformation as the map among the wavefunctions at the point z, $\psi(z, \tau)$. (See for details [16].)

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