MODULE CATEGORIES WITH INFINITE RADICAL SQUARE ZERO ARE OF FINITE TYPE

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It is well known that an artin algebra A is of finite representation type if and only if $\mathrm{rad}^{\infty}(\mathrm{mod}A) = 0$. In this note we deepen this result by showing that $(\mathrm{rad}^{\infty}(\mathrm{mod}A))^2 = 0$ implies that A is of finite representation type.

1 Introduction.

Let A be an artin algebra over a commutative artin ring R. By an A-module we mean a finitely generated, right A-module. We denote by mod$A$ the category of all A-modules, by ind$A$ the full subcategory of mod$A$ whose objects are the indecomposable A-modules, and then rad(mod$A$) is the Jacobson radical of mod$A$, that is, the ideal in mod$A$ generated by all non-invertible morphisms in ind$A$. The *infinite radical* $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ of mod$A$ is the intersection of all powers $\mathrm{rad}^{i}(\mathrm{mod}\ A)$, $i \geq 1$, of rad(mod$A$). The algebra A is said to be of *finite representation type* if ind$A$ has only finitely many non-isomorphic A-modules.

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Recent investigations showed that $\text{rad}^\infty(\text{mod}A)$ contains important informations on the category mod$A$ (see the survey article [**S3**]). We are interested in describing artin algebras A with $\text{rad}^\infty(\text{mod}A)$ nilpotent. It is well known that an artin algebra A is of finite representation type if and only if $\text{rad}^\infty(\text{mod}A)= 0$ (see [**KS**] and [**S3**]). On the other hand, for every hereditary algebra of infinite representation type H, we have that $(\text{rad}^\infty(\text{mod}H))^2 \neq 0$. The aim of this paper is to show the following:

THEOREM. *Let* A *be an artin algebra such that* $(\text{rad}^\infty(\text{mod}A))^2 = 0$. *Then* A *is of finite representation type.*

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2 Proof of the theorem.

We denote by D the *standard duality* $\text{Hom}_R(-,\overline{I})$ on mod$A$, where $\overline{I}$ is the injective envelope of $R/\text{rad}\ R$ in mod R. We use the notations Γ_A for the Auslander-Reiten quiver of A, and $\tau_A = DTr$ and $\tau_A^{-1} = TrD$ for the Auslander-Reiten translations in Γ_A. Also, as usual, we shall not distinguish vertices of Γ_A from the corresponding indecomposable modules. A path

$$X_o \to X_1 \to \cdots \to X_n$$

in Γ_A is said to be *sectional* if $\tau_A X_i \neq X_{i-2}$, for $2 \leq i \leq n$. Further, such a path is an *oriented cycle* if $X_o = X_n$. A connected quiver is said to be *non trivial* if it contains at least two vertices (and hence at least one arrow).

Let X be in ind$A$. Then X is called *periodic* if for some $n > 0$ we have that $\tau_A^n X = X$. Besides, such a module X is called *left stable* (respectively *right stable*) if $\tau_A^n X \neq 0$ for all positive (respectively negative) integers n and it is called *stable* if it is both left and right stable. The τ_A-orbit of X is the set of all possible modules of the form $\tau_A^i X$, $i \in \mathbb{Z}$.

For basic facts about Auslander-Reiten theory we refer to [**AR3**, **AR4**] and for tilting theory to [**As**], [**Ri1**] and [**Ri2**].

Proof of the theorem. Suppose that A is an artin algebra of infinite representation type such that $(\mathrm{rad}^\infty(\mathrm{mod}A))^2 = 0$. Since A is of infinite representation type, by a theorem of Auslander, [**A2**](3.1), there exists an infinite sequence of proper epimorphisms

$$\cdots \to M_r \xrightarrow{f_r} M_{r-1} \to \cdots \to M_1 \xrightarrow{f_1} M_o$$

with $M_i \in \mathrm{ind}A$. Let $\mathcal{M} = \{M_i : i \in \mathbb{N}\}$. Fix $r \geq 0$ and consider a projective cover $h_r : \mathbf{P}(M_r) \to M_r$ of M_r. Then, for each $t \geq r+1$ there exists a morphism $g_t : \mathbf{P}(M_r) \to M_t$ such that $h_r = f_{r+1} \cdots f_t g_t$. Hence, $h_r \in \mathrm{rad}^\infty(\mathbf{P}(M_r), M_r)$. Since $(\mathrm{rad}^\infty(\mathrm{mod}A))^2 = 0$ and h_r is an epimorphism, we deduce that $\mathrm{rad}^\infty(M_r, -) = 0$.

We claim that, for each $r \geq 0$, there exists a natural number s_r such that $\mathrm{rad}^{s_r}(M_r, -) = 0$. Indeed, since $\mathrm{rad}^\infty(M_r, -) = 0$, we infer that $\mathrm{rad}^\infty(M_r, DA) = 0$. Observe that there exists an s_r such that $\mathrm{rad}^{s_r}(M_r, DA) = \mathrm{rad}^\infty(M_r, DA)$ because $\mathrm{Hom}(M_r, DA)$ is an artinian R-module. Since for each A-module X there is a monomorphism of the form $X \to (DA)^m$, for some $m \geq 1$, we get our claim.

We have then that none of the morphisms f_r can belong to $\mathrm{rad}^\infty(\mathrm{mod}A)$, and so, that all modules in $\mathcal{M}$ are in the same connected component of Γ_A. Furthermore, for each $r \geq 0$, any sectional path starting at M_r has length bounded by s_r. Indeed, if there is such a path

$$M_r = Z_o \to Z_1 \to \cdots \to Z_l$$

with $l > s_r$, choosing irreducible morphisms f_i for each arrow $Z_{i-1} \to Z_i$, it follows from [**B**] and [**IT**] that the composition $f_l \cdots f_1$ is not zero and clearly belongs to $\mathrm{rad}^l(M_r, Z_l)$, a contradiction.

Let now $\mathcal{C}$ be the connected component of Γ_A which contains all modules of $\mathcal{M}$ and let us denote by $\mathcal{C}_l$ (respectively, by $\mathcal{C}_r$) the left (respectively, the right) stable part of $\mathcal{C}$. It is obtained by deleting from $\mathcal{C}$ the τ_A-orbits of the indecomposable projective (respectively, injective) modules.

Since $\mathcal{C}$ is infinite, either $\mathcal{C}_l$ or $\mathcal{C}_r$ has a connected component which is non-trivial. In fact, if there is, say, an infinite family of left stable trivial components $\{\tau_A^i X\}, i \geq 0$, with X non-periodic, then, for N big enough and $i \geq N$, we will have that $\tau_A^i X$ is not a neighbor of a projective module, which leads to a contradiction. Further, if that is not the case, there are infinitely many periodic orbits of trivial components that would be neighbors of orbits of projective or injective indecomposable modules, which is again a contradiction.

We claim now that every non-trivial connected component of $\mathcal{C}_l$ (respectively, of $\mathcal{C}_r$) contains an oriented cycle. Indeed, let $\mathcal{C}'$ be a non-trivial connected component of $\mathcal{C}_l$ that does not contain an oriented cycle. Then, by [**L**](3.6), there exists a valued quiver Δ, containing no oriented cycle, such that $\mathcal{C}'$ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta$ which is closed under predecessors. Let us fix a copy of Δ in $\mathcal{C}'$ such that no module in Δ is a successor of a projective module in $\mathcal{C}$. Let $\mathcal{D}$ be the full translation subquiver of $\mathcal{C}'$ whose vertices are all predecessors of Δ in $\mathcal{C}'$. Note that $\mathcal{D}$ is also closed under predecessors in $\mathcal{C}$. Let I be the annihilator of $\mathcal{D}$ in A, $B = A/I$ and M the direct sum of the modules in $\tau_A\Delta$. We claim that $\mathrm{Hom}_A(M, \tau_A M) = 0$. Indeed, if this were not so, there would exist direct summands Y and Z of M and a non-zero morphism $f : Y \to \tau_A Z$. Observe that such a morphism would belong to $\mathrm{rad}^\infty(\mathrm{mod}A)$, because $\mathcal{D}$ is closed under predecessors and has no oriented cycles. Now, if $\pi : \mathbf{P}(Y) \to Y$ is a projective cover, our choice of Δ implies that $\pi \in \mathrm{rad}^\infty(\mathrm{mod}A)$, and hence $f\pi$ is a non-zero morphism in $(\mathrm{rad}^\infty(\mathrm{mod}A))^2$, a contradiction. Consequently, $\mathrm{Hom}_A(M, \tau_A M) = 0$ and, by [**S1**], Lemma 2, Δ is finite. Then, $I = \mathrm{ann}M$ (see [**S2**], Lemma 3) and hence M is a faithful B-module. Observe also that $\mathcal{D}$ consists of B-modules, so that $\tau_B X = \tau_A X$ for any X in $\mathcal{D}$. Therefore, $\mathrm{Hom}_B(M, \tau_B M) = 0$ and, similarly, $\mathrm{Hom}_B(\tau_B^{-1} M, M) = 0$. Moreover, if $\mathrm{Hom}_B(M, X) \neq 0$ for an $X \in \mathrm{ind}B$ which is not a direct summand of M, then $\mathrm{Hom}_B\ (\tau_B^{-1} M, X) \neq 0$. Therefore, by [**RSS**] (1.5) and (1.6) (see also [**S3**](3.2)), M is a tilting B-module. Further, by [**S3**](3.4), $H = \mathrm{End}_B(M)$ is a hereditary algebra. This means that B is a tilted algebra and that $\mathcal{D}$ is a full translation subquiver of the connecting component Σ of Γ_B which is closed under predecessors and Δ is a slice in Σ. Since Σ has no projective modules, we infer that B is given by a tilting module without preinjective direct summands (see [**Ri2**], p. 42). Then, by a result of Strauss [**St**] (7.5), there exists a factor algebra C of B which is concealed. Observe then that, since Γ_C has regular components, $(\mathrm{rad}^\infty(\mathrm{mod}\ C))^2 \neq 0$. Indeed, let Z be a vertex of a regular connected component of Γ_C and let us consider a projective cover $\mathbf{P}(Z) \to Z$ and an injective envelope $Z \to \mathbf{I}(Z)$ of Z. Then their composite is clearly a non-zero morphism in $(\mathrm{rad}^\infty(\mathrm{mod}\ C))^2$ and hence in

$(\mathrm{rad}^{\infty}(\mathrm{mod} A))^2$, which contradicts our assumption. We show, in a similar way, that also every non-trivial connected component of $\mathcal{C}_r$ contains oriented cycles.

We shall show now that there are at most finitely many τ-orbits in $\mathcal{C}$ which are not τ_A-periodic. Let $\mathcal{E}$ be a non-trivial connected component of $\mathcal{C}_l$. If $\mathcal{E}$ contains a periodic module, then either $\mathcal{E}$ is a stable tube or it is of the form $\mathbb{Z}Q/G$, where Q is a Dynkin quiver and G is a group of automorphisms of $\mathbb{Z}Q$ [**HPR**]. In this case, all τ_A-orbits of $\mathcal{E}$ are periodic. On the other hand, if $\mathcal{E}$ has no periodic module, then, by [**L**](2.3), $\mathcal{E}$ has only finitely many τ_A-orbits. Therefore, we infer that there is at most a finite number of non-periodic τ_A-orbits in $\mathcal{C}$ (see [**BC**](4.2)), and our claim is proved.

Let us observe now that a stable tube $\mathcal{T}$ in Γ_A has no module from $\mathcal{M}$, because all modules in $\mathcal{T}$ are starting vertices of infinite sectional paths. Hence, there exists a τ-orbit $\mathcal{O}$ in $\mathcal{C}$ containing infinitely many modules from $\mathcal{M}$. Without loss of generality, we can assume that, for some $Y \in \mathcal{O}$, $\Omega = \{\tau_A^i Y, i \geq 0\}$ contains infinitely many modules of $\mathcal{M}$. Obviously then there is a connected component $\mathcal{F}$ of $\mathcal{C}_l$ that contains all but finitely many modules of Ω. Since $\mathcal{F}$ has no periodic modules but contains oriented cycles, it follows from [**L**](2.3) that there exists an infinite sectional path

$$\cdots \to \tau_A^{2t} X_1 \to \tau_A^t X_s \to \cdots \to \tau_A^t X_1 \to X_s \to \cdots \to X_1,$$

in $\mathcal{F}$, where $t > s$, at least one of the modules X_j is not stable, and $\{X_1, ..., X_s\}$ is a complete set of representatives of τ_A-orbits in $\mathcal{F}$.

It follows that $\mathcal{M}$ contains a module of the form $\tau_A^i X_j$, for some $i \geq t$ and $1 \leq j \leq s$. Observe that there exists an infinite sectional path starting in $\tau_A^i X_j$, which is a contradiction to the fact that this module belongs to $\mathcal{M}$.

This completes the proof of the theorem.

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