

Module-theoretic Characterizations of Strongly t -linked Extensions

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ABSTRACT. In this paper, we introduce and study the concept of “strongly t -linked extensions”, which is a stronger version of t -linked extensions of integral domains. We show that for an extension of Prüfer v -multiplication domains, this concept is equivalent to that of “ w -faithfully flat”.

1. Introduction

Let R be an integral domain with quotient field K . Then for any nonzero (fractional) ideal I set $I^{-1} := \{x \in K \mid xI \subseteq R\}$ and an ideal J of R is called a GV -ideal, denoted by $J \in GV(R)$, if J is a finitely generated ideal of R with $J^{-1} = R$.

Let R be a subring of the integral domain T . Following [7], we say that T is t -linked over R if $J \in GV(R)$ implies $JT \in GV(T)$. As pointed out in [1], an extension $R \subseteq T$ of Krull domains is t -linked if and only if it satisfies Samuel’s PDE (Pas d’éclatement) or NBU (No blowing up) condition, i.e., for a height one prime $P \in \text{Spec}(T)$, the set of prime ideals of T , we have $\text{ht}(P \cap R) \leq 1$. Anderson *et al.* in [1] showed that if T is t -linked over R , then the map $[I] \mapsto [(IT)_t]$ gives a homomorphism $Cl_t(R) \rightarrow Cl_t(T)$ of the t -class groups. Recall from [8] that an integral domain R is called t -linkative if each overring T of R is t -linked over R , equivalently, if every (nonzero) ideal of R is w -ideal ([13]). Examples of t -linkative domains are Prüfer domains and domains with Krull dimension one ([7, Corollary 2.7]). In [13],

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module-theoretic characterizations of t -linked extensions and t -linkative domains are given. In [16], a stronger version of the PDE condition for an extension of Krull domains was introduced and studied. In this paper, we introduce and study the concept of “strongly t -linked extensions”, which is a stronger version of t -linked extensions of integral domains. In fact, this is a continuous work on the project of studying some properties over Prüfer v -multiplication domains ([12, 14]).

We first introduce some definitions and notations. Let R be an integral domain with quotient field K . Let I be a nonzero fractional ideal I of R . Then $I_v := (I^{-1})^{-1}$, $I_t := \bigcup \{J_v \mid J \subseteq I \text{ is a nonzero finitely generated ideal}\}$, and $I_w := \{x \in K \mid Jx \subseteq I \text{ for some } J \in GV(R)\}$. We say that I is a t -ideal (resp., w -ideal) if $I = I_t$ (resp., $I = I_w$). A fractional ideal I of R is said to be t -invertible (resp., w -invertible) if $(II^{-1})_t = R$ (resp., $(II^{-1})_w = R$). It is known that a fractional ideal I is t -invertible if and only if I is w -invertible. We say that a fractional ideal I of R is of w -finite type if $I_w = J_w$ for some finitely generated ideal J of R . A maximal t -ideal (resp., w -ideal) is an ideal of R maximal among proper integral t -ideals (resp., w -ideals) of R . Let $t\text{-Max}(R)$ (resp., $w\text{-Max}(R)$) be the set of maximal t -ideals (resp., w -ideals). Then it is easy to see that $t\text{-Max}(R) = w\text{-Max}(R)$; if R is not a field, then $t\text{-Max}(R) \neq \emptyset$. An integral domain R is a *Prüfer v -multiplication domain* (PvMD) if every nonzero finitely generated ideal of R is t -invertible. It is well known that an integral domain R is a PvMD if and only if $R_{\mathfrak{p}}$ is a valuation domain for any prime t -ideal \mathfrak{p} of R ; if a domain R is a PvMD, then $t = w$; if T is t -linked over a PvMD R , then T is w -flat over R (The definition of w -flatness will be reviewed later). Let M be a module over the Prüfer domain R . Then it is well known that M is torsion-free if and only if M is flat. From this result it also follows that a finitely generated module over a valuation domain is torsion-free if and only if it is free, since a finitely generated module over a local ring is free or, equivalently, projective, if and only if it is flat.

Let M be a module over an integral domain R . Following [13] and [19], M is said to be *GV-torsion-free* (or *co-semi-divisorial*) if $\{x \in M \mid (ann_R(x))_w = R\} = 0$; equivalently, if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, we have that $x = 0$. M is called *GV-torsion* (or *w -null*) if $\{x \in M \mid (ann_R(x))_w = R\} = M$. We call an R -module M *semi-divisorial* (or a *w -module*) if it is torsion-free and $M = W_R(M)$, where the *w -envelope* of M is defined as $W_R(M) = \bigcap_{P \in w\text{-Max}(R)} M_P$,

where the intersection is taken within $K \otimes_R M$. In particular, the domain R itself is semi-divisorial as an R -module. Any R -linear map $u : M \rightarrow N$ between torsion-free R -modules induces a map $W_R(u) : W_R(M) \rightarrow W_R(N)$, i.e., W_R may be viewed as a covariant functor on torsion-free R -modules. Let M, N be semi-divisorial modules over R . Suppose that $f : M \rightarrow N$ is an R -homomorphism and $f_P : M_P \rightarrow N_P$ is an isomorphism for all $P \in w\text{-Max}(R)$. Then it is easy to see that f is an isomorphism.

Let R be a PvMD. Then for any $\mathfrak{p} \in w\text{-Spec}(R)$ the ring $R_{\mathfrak{p}}$ is a valuation domain, hence, an $R_{\mathfrak{p}}$ -module is $R_{\mathfrak{p}}$ -flat if and only if it is torsion-free. Since $R_{\mathfrak{p}}$ is a flat R -module, any $R_{\mathfrak{p}}$ -module that is $R_{\mathfrak{p}}$ -flat is R -flat. Hence any semi-divisorial R -module M is an intersection in $K \otimes_R M$ of flat R -modules $M_{\mathfrak{p}}$.

Any undefined terminology is standard, as in [9] or [10].

2. Main results

We begin this section by listing some characterizations of t -linked extensions of integral domains in the literature.

If R is an integral domain, we set $R\langle X \rangle := R[X]_{N_t}$, where $N_t := \{f \in R[X] \mid c(f)_t = R\}$, a multiplicative set in $R[X]$ ($c(f)$ is the ideal of R generated by the coefficients of $f \in R[X]$). $R\langle X \rangle$ is called the t -Nagata ring of R .

Theorem 2.1. *Let $R \subseteq T$ be an extension of domains. Then the following conditions are equivalent.*

- (1) T is t -linked over R .
- (2) If I is a (finitely generated) ideal of R with $I_t = R$, then $(IT)_t = T$.
- (3) If Q is a prime t -ideal of T with $Q \cap R \neq 0$, then $(Q \cap R)_t \subsetneq R$.
- (4) If Q is a maximal t -ideal of T with $Q \cap R \neq 0$, then $(Q \cap R)_t \subsetneq R$.
- (5) If I and J are t -invertible ideals of R with $I_t = J_t$, then $(IT)_t = (JT)_t$.
- (6) If I is a t -invertible ideal of R , then $(IT)_t = (I_t T)_t$.
- (7) $I_w \subseteq (IT)_w$, for any ideal I of R .
- (8) $A \cap R$ is a w -ideal of R for any w -ideal A of T .
- (9) $(IT)_w \cap R$ is a w -ideal of R for any ideal I of R .
- (10) $(IT)_w \cap R$ is a w -ideal of R for any finitely generated ideal I of R .
- (11) $P \cap R$ is a (prime) w -ideal of R for any prime w -ideal P of T .
- (12) $T = T\langle X \rangle \cap qf(T)$, where X is an indeterminate over T .
- (13) T is semi-divisorial as an R -module.
- (14) Every GV -torsion-free T -module is a GV -torsion-free R -module.
- (15) $M \otimes_R T$ is a GV -torsion T -module for any GV -torsion R -module M .

Proof. The proof of [7, Proposition 2.1] shows that (1)-(3) are equivalent. For the equivalences of (2), (4), (5), and (6), See [1, Proposition 2.1]. It was shown in [17, Proposition 1.2] that (1) and (7)-(11) are equivalent. (1) \Leftrightarrow (12). See the proof of [4, Lemma 3.2]. (1) \Leftrightarrow (13) \Leftrightarrow (14). See [13, Theorem 9.10]. (14) \Leftrightarrow (15). See [23, Lemma 1.1(2)]. \square

Corollary 2.2. *Let $R \subseteq T$ be a t -linked extension of domains and let M be an R -module. If $M \otimes_R T$ is a GV -torsion-free T -module, then M is a GV -torsion-free R -module.*

Proof. This follows from [23, Lemma 1.1(1)] and Theorem 2.1. \square

Consider an inclusion of domains $i : R \hookrightarrow T$. Taking intersections with A , this inclusion induces a continuous map (for the Zariski topology)

$${}^a i : \text{Spec}(T) \rightarrow \text{Spec}(R), Q \mapsto Q \cap R,$$

which does, in general, not restrict to a map $w\text{-Spec}(T) \rightarrow w\text{-Spec}(R)$. If it does, i.e., if $Q \cap R \in w\text{-Spec}(R)$, for all $Q \in w\text{-Spec}(T)$, then we say that i is a *t-linked extension*.

It was shown in [17, Proposition 1.1] that for an extension $R \subseteq T$ of domains, if P is a prime ideal of T such that $P \cap R$ is a w -ideal of R , then $P_w \neq T_w$. As a corollary, for a prime ideal \mathfrak{p} of R , \mathfrak{p} is a w -ideal if and only if $\mathfrak{p}_w \neq R$. It is easy to see that if a domain T is semi-divisorial over a domain R and I is a w -ideal of T . Then I is semi-divisorial over R (cf. [17, Remark 1]).

Let $R \subseteq T$ be an extension of rings. Suppose T is a flat R -module. Then it is known that if P is a prime ideal of T and write $\mathfrak{p} = P \cap R$, then T_P is a faithfully flat $R_{\mathfrak{p}}$ -module. Note that if R is a valuation domain, then any nonzero (prime) ideal of R is w -ideal. Recall that for two local rings (R, \mathfrak{m}_R) and (T, \mathfrak{m}_T) , a homomorphism $\psi : R \rightarrow T$ is called a *local homomorphism* if $\psi(\mathfrak{m}_R) \subseteq \mathfrak{m}_T$.

Theorem 2.3. *Let $i : R \hookrightarrow T$ be an extension of PvMDs. Then i is a t-linked extension if and only if the $R_{\mathfrak{p}}$ -module T_P is (faithfully) flat, for every $P \in w\text{-Spec}(T)$ and $\mathfrak{p} = P \cap R$.*

Proof. (\Rightarrow) If i is a t -linked extension, then $\mathfrak{p} := P \cap R \in w\text{-Spec}(R)$, for all $P \in w\text{-Spec}(T)$. Thus $R_{\mathfrak{p}}$ is a valuation domain. Since T_P is torsion-free over $R_{\mathfrak{p}}$, it is flat over $R_{\mathfrak{p}}$.

(\Leftarrow) If $P \in w\text{-Spec}(T)$ and if T_P is flat over $R_{\mathfrak{p}}$ with $\mathfrak{p} = P \cap R$, then T_P is faithfully flat over $R_{\mathfrak{p}}$ since $R_{\mathfrak{p}} \rightarrow T_P$ is local. Therefore T_P is semi-divisorial over $R_{\mathfrak{p}}$ and $\mathfrak{p}_P T_P \neq T_P$, and so $(T_P)_{w_{\mathfrak{p}}} = T_P$ as an $R_{\mathfrak{p}}$ -module (i.e., $w_{\mathfrak{p}}$ is the w -operation on $R_{\mathfrak{p}}$) and $\mathfrak{p}_P T_P \subseteq P_P$. Note that P_P is a prime w -ideal of T_P , since $R_{\mathfrak{p}}$ is a valuation domain. We will show that \mathfrak{p} is a prime w -ideal of R . Suppose that $\mathfrak{p}_w = R$. Then $(\mathfrak{p}_P)_{w_{\mathfrak{p}}} = R_{\mathfrak{p}}$. Thus we have $P_P = (P_P)_{w_{\mathfrak{p}}} \supseteq (\mathfrak{p}_P T_P)_{w_{\mathfrak{p}}} = ((\mathfrak{p}_P)_{w_{\mathfrak{p}}} T_P)_{w_{\mathfrak{p}}} = (T_P)_{w_{\mathfrak{p}}} = T_P$ (the first equality follows from the remark just above, while the second equality follows from [20, Proposition 2.8]) as (torsion-free) $R_{\mathfrak{p}}$ -modules, which is a contradiction. Therefore, $\mathfrak{p}_w \neq R$. Thus by [21, Proposition 1.1] \mathfrak{p} is a prime w -ideal of R . \square

The following result provides the first link between the notion of the t -linked extension and that of a semi-divisorial module.

Proposition 2.4([13, Corollary 9.11]). *Let $R \subseteq T$ be a t-linked extension of integral domains. If M is a semi-divisorial T -module, then M is also semi-divisorial as an R -module.*

If the map ${}^a i : w\text{-Spec}(T) \rightarrow w\text{-Spec}(R)$ is surjective, i.e., if for every $P \in w\text{-Spec}(R)$ there exists some $Q \in w\text{-Spec}(T)$ with the property that $Q \cap R = P$, then we will say that i is a *strongly t-linked extension* or that T is *strongly t-linked* over

R . Thus it is clear to see that a t -linked extension $R \subseteq T$ of domains is a strongly t -linked extension if and only if the pair (R, T) satisfies “lying over” property for prime w -ideals of R and T .

Following [22], an ideal J of a commutative ring R is called a *Glaz-Vasconcelos ideal* or a *GV-ideal*, denoted by $J \in GV(R)$, if J is finitely generated and the natural homomorphism $\alpha : R \rightarrow \text{Hom}_R(J, R)$, defined by $\alpha(r)(a) = ra, \forall r \in R, \forall a \in J$, is an isomorphism. An R -module M is said to be *GV-torsion-free* if whenever $Jx = 0$, for some $J \in GV(R)$ and $x \in M$, then $x = 0$.

Now we extend this concept to any module. Let R be a commutative ring with identity and let M be an R -module. Define $r(M) := \{x \in M \mid (\text{ann}_R(x))_w = R\}$. Then $r(M)$ is a submodule of M . It is easy to see that M is GV-torsion-free if and only if $r(M) = 0$ and that $M/r(M)$ is GV-torsion-free. Define $M_w := \{x \in E(M) \mid Jx \subseteq M/r(M) \text{ for some } J \in GV(R)\}$, where $E(M)$ denotes the *injective envelope* (or *injective hull*) of M . Then it is also easy to see that $W(M) = M_w$ for any torsion-free R -module M . An R -module M is said to be a *w-module* if $M_w = M$. Let M and N be any modules over any commutative ring R . Then we define the *w-tensor product* of M and N as follows: $M \hat{\otimes} N := (M \otimes_R N)_w$.

The following definitions and proposition are easily derived from [3, 1.4, 1.5 Proposition, 1.6 Proposition]: Let M be an R -module. Then it is clear that the functor $M \hat{\otimes} -$ is right exact. We call M a *w-flat* R -module if $M \hat{\otimes} -$ is exact. Then M is *w-flat* if and only if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$ -module for every $\mathfrak{p} \in w\text{-Max}(R)$. M is said to be *w-faithfully flat* if it satisfies one of the equivalent conditions of the following proposition.

Proposition 2.5. *Let M be an R -module. Then the following statements are equivalent:*

- (1) *For all GV-torsion-free semi-divisorial R -modules A, B , and C , the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $M \hat{\otimes} A \xrightarrow{1_M \hat{\otimes} f} M \hat{\otimes} B \xrightarrow{1_M \hat{\otimes} g} M \hat{\otimes} C$ is exact.*
- (2) *M is w-flat and for all GV-torsion-free semi-divisorial R -module N we have $M \hat{\otimes} N = 0$ if and only if $N = 0$.*
- (3) *M is w-flat and for all $\mathfrak{p} \in w\text{-Max}(R)$ we have $(M/\mathfrak{p}M)_w \neq 0$.*
- (4) *For all $\mathfrak{p} \in w\text{-Max}(R)$ the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is faithfully flat.*

It is well known that for an extension $R \subseteq T$ of integral domains having the same quotient field, if T is faithfully flat over R , then $R = T$. The following result is the w -theoretic analogue of this result.

Corollary 2.6. *Let $R \subseteq T$ be an extension of integral domains having the same quotient field. If T is w-faithfully flat over R , then $R = T$.*

Proof. Let $\mathfrak{p} \in w\text{-Max}(R)$. Then $T_{R \setminus \mathfrak{p}}$ is $R_{\mathfrak{p}}$ -faithfully flat by Proposition 2.5.

Since $T_{R \setminus \mathfrak{p}}$ and $R_{\mathfrak{p}}$ have the same quotient field, we have that $T_{R \setminus \mathfrak{p}} = R_{\mathfrak{p}}$. Hence $R = \bigcap_{\mathfrak{p} \in w\text{-Max}(R)} R_{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in w\text{-Max}(R)} T_{R \setminus \mathfrak{p}} \supseteq T \supseteq R$, and thus $R = T$. \square

To address the question of “what is a w -faithfully flat ideal of an integral domain R ?”, we need the following lemma.

Lemma 2.7 ([19, Theorem 2.6.22]). *Let (R, \mathfrak{m}) be a local ring and let I be a faithfully flat ideal of R . Then I is a principal ideal, which is also free as an R -module. More precisely, if $a \in I \setminus \mathfrak{m}I$, then $I = (a)$.*

A nonzero ideal I of R is called a w -cancellation ideal if $(IA)_w = (IB)_w$ for nonzero ideals A and B of R implies $A_w = B_w$. In [6, Corollary 2.4], it was shown that I is a w -cancellation ideal if and only if I is w -locally principal. The following result is the w -theoretic analogue of [19, Theorem 2.6.23].

Theorem 2.8. *Let R be a domain and let I be a nonzero ideal of R . Then I is w -faithfully flat if and only if I is w -locally principal.*

Proof. Suppose that I is w -faithfully flat and let \mathfrak{m} be a maximal w -ideal of R . Then by Proposition 2.5, $I_{\mathfrak{m}}$ is a faithfully flat ideal of $R_{\mathfrak{m}}$. Hence $I_{\mathfrak{m}}$ is a principal ideal by Lemma 2.7.

Conversely, suppose that $I_{\mathfrak{m}}$ is a principal ideal of $R_{\mathfrak{m}}$ for any maximal w -ideal \mathfrak{m} of R . Then I is w -flat by the remark before Proposition 2.5. Clearly, $I_{\mathfrak{m}} \neq \mathfrak{m}I_{\mathfrak{m}}$, and hence $(I/\mathfrak{m}I)_w \neq 0$. Therefore by Proposition 2.5, I is w -faithfully flat. \square

Now we give an example of a w -faithfully flat, but not w -invertible ideal.

Example 2.9. It is known that a nonzero ideal I of R is w -invertible if and only if I is of w -finite type and I is w -locally principal. Consider $R := \mathbb{Z} + X\mathbb{Q}[[X]]$. Then it is known that R is a two-dimensional Prüfer domain. Let I be an ideal of R generated by the set $\{\frac{1}{p}X\}$, where p ranges over the set of prime numbers in \mathbb{Z} . Then it is shown in [19, Example 8.6.25] that I is a faithfully flat ideal, which is not finitely generated. Since R is a Prüfer domain, R is t -linkative, i.e., every (nonzero) ideal of R is a w -ideal. Therefore by Theorem 2.8, I is a not w -invertible but w -locally principal ideal of R .

Lemma 2.10. *Let $R \subseteq T$ be an extension of domains. If T is a w -faithfully flat R -module, then $(IT)_w \cap R = I_w$ for any ideal I of R .*

Proof. Let \mathfrak{p} be a maximal w -ideal of R . Then we have that $(IT \cap R)_{\mathfrak{p}} = I_{\mathfrak{p}}T_{\mathfrak{p}} \cap R_{\mathfrak{p}} = I_{\mathfrak{p}}$ (the second equality follows from the fact that $T_{\mathfrak{p}}$ is faithfully flat as an $R_{\mathfrak{p}}$ -module). Thus $(IT)_w \cap R = (IT \cap R)_w = I_w$. \square

Lemma 2.11. *Let $R \subseteq T$ be an extension of domains and let $\mathfrak{p} \in w\text{-Spec}(R)$. Then there is a $P \in w\text{-Spec}(T)$ lying over \mathfrak{p} if and only if $(\mathfrak{p}T)_w \cap R = \mathfrak{p}$.*

Proof. Suppose that there is a $P \in w\text{-Spec}(T)$ lying over \mathfrak{p} . Then we have that $\mathfrak{p} \subseteq (\mathfrak{p}T)_w \cap R \subseteq P_w \cap R = P \cap R = \mathfrak{p}$. Therefore $(\mathfrak{p}T)_w \cap R = \mathfrak{p}$.

Conversely, suppose that $(\mathfrak{p}T)_w \cap R = \mathfrak{p}$. Set $S := R \setminus \mathfrak{p}$. Then $(\mathfrak{p}T)_w \cap S = \emptyset$. Thus there is a prime w -ideal P of T such that $(\mathfrak{p}T)_w \subseteq P$ and $P \cap S = \emptyset$. Hence $\mathfrak{p} \subseteq P \cap R$. Let $x \in P \cap R$. Since $P \cap S = \emptyset$, $x \notin S$, and so $x \in \mathfrak{p}$. Therefore $P \cap R = \mathfrak{p}$. \square

Lemma 2.12([19, Theorem 5.2.17]). *Let (R, \mathfrak{m}_R) and (T, \mathfrak{m}_T) be local rings and let $f : R \rightarrow T$ be a homomorphism with $f^{-1}(\mathfrak{m}_T) = \mathfrak{m}_R$. Suppose T is a flat R -module.*

- (1) *T is faithfully flat and f is monomorphic.*
- (2) *The map ${}^a f : \text{Spec}(T) \rightarrow \text{Spec}(R)$ defined by ${}^a f(P) = f^{-1}(P)$ is a surjection.*

Let T be an R -algebra and M be a T -module. Then it is known that if T is flat over R and M is flat over T , then M is flat over R . Let $R \subseteq T$ be a t -linked extension. We say that (R, T) satisfies w -GD if (R, T) satisfies “going down” in the sense that for $\mathfrak{p}, \mathfrak{q} \in w\text{-Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ and let $P \in w\text{-Spec}(T)$ with $P \cap R = \mathfrak{p}$, there exists a $Q \subseteq P$ such that $Q \cap R = \mathfrak{q}$. We also define $w\text{-dim}(R) := \sup\{ht(\mathfrak{p}) \mid \mathfrak{p} \in w\text{-Max}(R)\}$.

Theorem 2.13. *Let $R \subseteq T$ be an extension of rings.*

- (1) *If T is a w -flat R -module, then (R, T) satisfies w -GD.*
- (2) *If T is a w -faithfully flat R -module, then $w\text{-dim}(R) \leq w\text{-dim}(T)$.*

Proof. (1) Let $\mathfrak{p}, \mathfrak{q} \in w\text{-Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$ and let $P \in w\text{-Spec}(T)$ with $P \cap R = \mathfrak{p}$. Consider the extension $R_{\mathfrak{p}} \subseteq T_{\mathfrak{p}}$. Since $T_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in w\text{-Spec}(R)$ and $T_{\mathfrak{p}}$ is a quotient ring of $R_{\mathfrak{p}}$, it follows from the above remark that $T_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$. Thus by Lemma 2.12, there is a $Q \in \text{Spec}(T)$ with $Q \subseteq P$ and $QT_{\mathfrak{p}} \cap R_{\mathfrak{p}} = \mathfrak{q}R_{\mathfrak{p}}$. Note that $Q \in w\text{-Spec}(T)$ since $Q \subseteq P$. It follows that $Q \cap R = \mathfrak{q}$.

(2) Let \mathfrak{p} be a maximal w -ideal of R and let $\mathfrak{p}_s \subset \cdots \subset \mathfrak{p}_1 \subset \mathfrak{p}$ be a chain of prime (w -)ideals of R . By Lemma 2.10 and Lemma 2.11, there is a maximal w -ideal P of T such that $P \cap R = \mathfrak{p}$. By (1), (R, T) satisfies the w -GD. Hence there is a chain of prime (w -)ideals $P_s \subset \cdots \subset P_1 \subset P$ such that $P_i \cap R = \mathfrak{p}_i$. It follows that $ht(\mathfrak{p}) \leq ht(P)$. Therefore $w\text{-dim}(R) \leq w\text{-dim}(T)$. \square

The following result is a connection between strong t -linkedness and w -faithful flatness among extensions of PvMDs.

Theorem 2.14. *Let $R \subseteq T$ be a t -linked extension of PvMDs. Then T is a strongly t -linked extension over R if and only if T is a w -faithfully flat R -module.*

Proof. Assume that T is a strongly t -linked extension over R and let $\mathfrak{p} \in w\text{-Spec}(R)$. Then $T_{\mathfrak{p}}$ is torsion-free over the valuation domain $R_{\mathfrak{p}}$, and hence it is flat over $R_{\mathfrak{p}}$. To prove that $T_{\mathfrak{p}}$ is faithfully flat, we have to show that $IT_{\mathfrak{p}} \neq T_{\mathfrak{p}}$ for every maximal ideal I of $R_{\mathfrak{p}}$. Note that $\mathfrak{p}_{\mathfrak{p}}$ is a unique maximal of $R_{\mathfrak{p}}$. We thus have to show that $\mathfrak{p}T_{\mathfrak{p}} \neq T_{\mathfrak{p}}$. By assumption, there is a prime ideal $P \in w\text{-Spec}(T)$ with $P \cap R = \mathfrak{p}$. Then clearly $\mathfrak{p}T_{\mathfrak{p}} \subseteq PT_{\mathfrak{p}} \neq T_{\mathfrak{p}}$ (as $P \cap (R \setminus \mathfrak{p}) = \emptyset$).

Conversely, assume that for every $\mathfrak{p} \in w\text{-Spec}(R)$, the $R_{\mathfrak{p}}$ -module $T_{\mathfrak{p}}$ is faithfully flat. Consider, for the moment, a fixed $\mathfrak{p} \in w\text{-Spec}(R)$ and the induced map $R_{\mathfrak{p}} \hookrightarrow T_{\mathfrak{p}}$, which is faithfully flat. Since $T_{\mathfrak{p}}$ is faithfully flat over $R_{\mathfrak{p}}$, the extension $R_{\mathfrak{p}} \hookrightarrow T_{\mathfrak{p}}$ is t -linked, equivalently $T_{\mathfrak{p}}$ is semi-divisorial over $R_{\mathfrak{p}}$, and so $(T_{\mathfrak{p}})_w = T_{\mathfrak{p}}$. Again since $T_{\mathfrak{p}}$ is faithfully flat over $R_{\mathfrak{p}}$, there exists $Q' \in \text{Max}(T_{\mathfrak{p}})$ with $Q' \cap R_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$ ([2, I, 3.5, Proposition 9]). We will show that $Q' \in w\text{-Spec}(T_{\mathfrak{p}})$. Since $R_{\mathfrak{p}}$ is a valuation domain, $\mathfrak{p}_{\mathfrak{p}}$ is a w -prime ideal of $R_{\mathfrak{p}}$. Then by [17, Proposition 1.1], $(Q')_w \neq (T_{\mathfrak{p}})_w = T_{\mathfrak{p}}$. Thus by [21, Proposition 1.1], we have $Q' \in w\text{-Spec}(T_{\mathfrak{p}})$. Set $Q = Q' \cap T$. Then $Q \in w\text{-Spec}(T)$, since $T \hookrightarrow T_{\mathfrak{p}}$ is t -linked, with $Q \cap R = \mathfrak{p}$. \square

The following result follows immediately from Proposition 2.5 and Theorem 2.14.

Corollary 2.15. *Let T be an extension of a PvMD R . Then the following assertions are equivalent:*

- (1) T is strongly t -linked over R .
- (2) For any GV-torsion-free semi-divisorial R -module M , we have that $M \hat{\otimes}_R T = 0$ if and only if $M = 0$.

In particular, it follows from Corollary 2.15 that the functor $T \hat{\otimes}_R -$ is left exact on GV-torsion-free semi-divisorial R -modules.

Example 2.16. We provide examples of strongly t -linked extensions of domains.

- (1) Any faithfully flat extension $i : R \hookrightarrow T$ of PvMDs is strongly t -linked. Indeed, the fact that i is a t -linked extension follows from the flatness of i and it is trivial to see that for any $\mathfrak{p} \in w\text{-Spec}(R)$ the induced map $i_{\mathfrak{p}} : R_{\mathfrak{p}} \hookrightarrow T_{\mathfrak{p}}$ is faithfully flat.
- (2) If $i : R \hookrightarrow T$ is an extension of PvMDs which makes T into a semi-divisorial R -lattice, then T is strongly t -linked over R . Indeed, since T is semi-divisorial over R , i is a t -linked extension. On the other hand, for every $\mathfrak{p} \in w\text{-Spec}(R)$, the $R_{\mathfrak{p}}$ -module $T_{\mathfrak{p}}$ is free of finite rank, hence certainly faithfully flat.
- (3) As in [18], we say that an element $u \in K$ is w -integral over R if $uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of R . Set $R^w := \{x \in K \mid x \text{ is } w\text{-integral over } R\}$. It is known that R^w is an integrally closed overring of R (see [18, section 3]); R^w is called the w -integral closure of R . The w -integral closure R^w of an integral domain R is strongly t -linked over R . Indeed, this follows from [5, Lemma 1.2 and Corollary 1.4].

The w -tensor product $M \hat{\otimes}_R N$ of two R -modules M and N can be redefined as the image under the reflector W of their ordinary tensor product $M \otimes_R N$. If M and N are torsion-free, then $M \hat{\otimes}_R N = W_R(MN)$, where MN is the image of $M \otimes_R N$ in $K \otimes_R (M \otimes_R N) \cong (K \otimes_R M) \otimes_K (K \otimes_R N)$. The w -tensor product behaves in many ways as the ordinary tensor product of R -modules.

For (m, n) in $M \times N$ let $\alpha(m, n) = 1 \otimes m \otimes n \in K \otimes_R (M \times_R N)$. View α as a map to $M \hat{\otimes}_R N$. Let β be the map sending (m, n) to $m \otimes n$ in $M \otimes_R N$. Because α is bilinear, there is a commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\alpha} & M \hat{\otimes}_R N \\ & \searrow \beta & \uparrow \exists! \gamma \\ & & M \otimes_R N \end{array}$$

An integral domain R is said to be of *w-finite character* if every nonzero nonunit of R belongs to at most finitely many maximal w -ideals of R . The proof of the following result is easy, and so we omit it.

Lemma 2.17. *Let S be a multiplicative set of a domain of w -finite character and M an R -module. If M is semi-divisorial over R , then M_S is semi-divisorial over R_S .*

The following proposition summarizes some basic properties of the w -tensor product.

Proposition 2.18. *Let L, M, M_i be torsion-free R -modules. Then:*

- (1) *Given an R -homomorphism $f : M \rightarrow N$, there exists a unique R -homomorphism $W_R(f) : W_R(M) \rightarrow W_R(N)$ which on M restricts to f . For $g : L \rightarrow M$ we have $W_R(fg) = W_R(f)W_R(g)$.*
- (2) *$M \hat{\otimes}_R N$ is semi-divisorial.*
- (3) *$M \hat{\otimes}_R N$ has the universal mapping property in the following sense. If L is semi-divisorial and $\delta : M \times N \rightarrow L$ is R -bilinear, then there exists a unique R -homomorphism $\lambda : M \hat{\otimes}_R N \rightarrow L$ satisfying $\lambda\alpha = \delta$ (with α as above).*
- (4) *If M and N are R -lattices, so is $M \hat{\otimes}_R N$.*
- (5) *If M is R -flat, then it is semi-divisorial. If in addition R is of w -finite character and N is semi-divisorial, then the map γ above is an isomorphism.*
- (6) *If B is an R -algebra and M is a B -module, then there is a B -module structure on $M \hat{\otimes}_R N$ which makes γ a B -module homomorphism.*
- (7) *$R \hat{\otimes}_R M = W_R(M)$.*
- (8) *$L \hat{\otimes}_R (M \hat{\otimes}_R N) \cong (L \hat{\otimes}_R M) \hat{\otimes}_R N$.*
- (9) *$L \hat{\otimes}_R (\bigoplus_i M_i) \cong \bigoplus_i (L \hat{\otimes}_R M_i)$.*
- (10) *$M \hat{\otimes}_R N \cong N \hat{\otimes}_R M$.*
- (11) *Let S be a multiplicative subset of R , a domain of w -finite character. Then $(M \hat{\otimes}_R N)_S \cong M_S \hat{\otimes}_{R_S} N_S$.*

Proof. (2) is clear. (1) and (3) follow from the fact that W_R is a reflector functor. (4) follows from the well-known fact that if M and N are R -lattices, so is MN .

The first assertion of (5) is given in [11]. The second assertion of (5) follows from [15, Corollary 2 to Proposition 1]. (6) follows from (2) and (3). Assertions (7) to (10) are easy to prove.

To prove (11), we will establish that there are maps in both directions between $(M \hat{\otimes}_R N)_S$ and $M_S \hat{\otimes}_{R_S} N_S$ whose composites are clearly the identity maps. First note that each of the modules involved is semi-divisorial over R_S . For $(M \hat{\otimes}_R N)_S$ this is true by (2) and Lemma 2.17. For $M_S \hat{\otimes}_{R_S} N_S$ we need only invoke (2) with R replaced by R_S . The existence of the maps we want is now easily established using (3) and properties of the localizing functor $(\)_S$. \square

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