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# MODULES WITH SEMI-LOCAL ENDOMORPHISM RING

DOLORS HERBERA AND AHMAD SHAMSUDDIN

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ABSTRACT. We use the concept of dual Goldie dimension and a characterization of semi-local rings due to Camps and Dicks (1993) to find some classes of modules with semi-local endomorphism ring. We deduce that linearly compact modules have semi-local endomorphism ring, cancel from direct sums and satisfy the $n$ th root uniqueness property. We also deduce that modules over commutative rings satisfying $AB5^*$ also cancel from direct sums and satisfy the $n$ th root uniqueness property.

Let $R$ be an associative ring with 1 and let $M$ be a right unital $R$-module. A finite set $A_1, \dots, A_n$ of proper submodules of $M$ is said to be *coindependent* if for each $i$, $1 \le i \le n$, $A_i + \bigcap_{j \ne i} A_j = M$, and a family of submodules of $M$ is said to be *coindependent* if each of its finite subfamilies is coindependent. The module $M$ is said to have *finite dual Goldie dimension* if every coindependent family of submodules of $M$ is finite. It can be shown that, in this case, there is a maximal coindependent family of submodules of $M$. If this set is finite, then its cardinality (denoted by $\operatorname{codim}(M)$) is uniquely determined and is called the *dual Goldie dimension* of $M$. If this set is infinite we set $\operatorname{codim}(M) = \infty$ and say that $M$ *has infinite dual Goldie dimension.* A module with dual Goldie dimension 1 is said to be *hollow*, and a cyclic hollow module is said to be *local*. We have

$$\begin{gathered} \operatorname{codim}(M_1 \oplus M_2) = \operatorname{codim}(M_1) + \operatorname{codim}(M_2), \\ \operatorname{codim}(M/N) \le \operatorname{codim}(M) \text{ for every submodule } N \text{ of } M, \\ \operatorname{codim}(M/N) = \operatorname{codim}(M) \text{ if } N \text{ is a small submodule of } M, \\ \operatorname{codim}(M) = 0 \quad \text{if and only if } M = 0; \end{gathered}$$

refer to [10] and [20] for details concerning the dual Goldie dimension.

A ring $R$ with Jacobson radical $J(R)$ is said to be *semi-local* if $R/J(R)$ is a semi-simple ring. Semi-local rings are characterized as those rings with finite dual Goldie dimension. Note that for a semi-local ring $R$,

$$\operatorname{codim}(R_R) = \text{length of the right } R\text{-module } R/J$$

and so $\operatorname{codim}(R_R) = \operatorname{codim}({}_RR)$; this common value is denoted by $\operatorname{codim}(R)$.

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Denote by $\dim(M)$ the *Goldie dimension* of $M$ and by $\mathrm{U}(R)$ the group of units in the ring $R$. Camps and Dicks recently proved the following characterization of semi-local rings.

**Theorem 1** (Camps and Dicks [3, Theorem 1(e)]). *A ring $R$ is semi-local if and only if there exist an integer $n$ and a function $d: R \to \{0, \dots, n\}$ satisfying the conditions:*

(1) *for any $r, s \in R$, $d(r - rsr) = d(r) + d(1 - rs)$,*
(2) *$d(r) = 0$ if and only if $r \in \mathrm{U}(R)$.*

*Moreover, it follows in this situation that* $\operatorname{codim}(R) = \dim(R/J(R)) \leq n$. □

Recall that a ring $R$ has 1 *in its stable range* if whenever the equation $ax + b = 1$ has a solution for $x$ in $R$, there exists $c \in R$ such that $a + bc \in U(R)$. A right $R$-module $M$ *cancels from direct sums* if for any right $R$-modules $A$ and $B$, $M \oplus A \cong M \oplus B$ implies $A \cong B$.

By a result of Evans [6, Theorem 2], if 1 is in the stable range of the endomorphism ring of a module $M$, then $M$ cancels from direct sums. Bass in [2] proves that a semi-local ring has 1 in its stable range, and hence a module whose endomorphism ring is semi-local cancels from direct sums.

It has been recently proved by Facchini, Herbera, Levy and Vamos [7] that if $M$ and $N$ are modules for which the endomorphism rings $\operatorname{End} M$ and $\operatorname{End} N$ are semi-local, then $M^n \cong N^n$ for $n \in \mathbb{N}$ implies that $M \cong N$. This latter property is called the *$n$ th root uniqueness property.*

We summarize these results in the following theorem.

**Theorem 2.** *Let $R$ be a ring and $M$ a right $R$-module with semi-local endomorphism ring. Then* 1 *is in the stable range of the endomorphism ring of $M$, $M$ cancels from direct sums and $M$ satisfies the $n$ th root uniqueness property.* □

In this paper we use Theorem 1 and the concept of dual Goldie dimension to find classes of modules whose endomorphism rings are semi-local. Our main result is Theorem 3 which contains the result of Camps and Dicks [3, Theorem 5] and has consequences for quasi-projective modules (Corollary 4), linearly compact modules (Corollary 5) and modules satisfying $AB5^*$ and for which the number of non-isomorphic simple subfactors is finite (Corollary 7). In general the endomorphism ring of a right $R$-module $M$ satisfying $AB5^*$ is not semi-local, but we can show that if $R$ is commutative, the endomorphism ring of $M$ is a product of semi-local rings, hence the conclusions of Theorem 2 are still valid for $AB5^*$ modules over commutative rings (Corollary 9).

Example 10 (1) shows that any ring that can be embedded in a local ring can be realized as the endomorphism ring of a local module over some ring; this contrasts with the situation for quasi-projective modules (see Corollary 4), or with the situation for commutative or right noetherian rings (see the remarks preceding Corollary 4).

All our results seem to indicate that there is a close relation between having semi-local endomorphism ring and having finite dual Goldie dimension. However Example 10 (2) shows that there exist cyclic modules with semi-local endomorphism ring whose dual Goldie dimension is not finite.

We denote the endomorphism ring of the right $R$-module $M$ by $\operatorname{End}_R(M) = \operatorname{End}(M)$.

**Theorem 3.** *Let $R$ be a ring and $M$ a right $R$-module.*

(1) (Camps and Dicks [3, Theorem 5]) *If $M$ has finite Goldie dimension and every injective endomorphism of $M$ is bijective, then the endomorphism ring of $M$ is semi-local and*

$$\operatorname{codim}(\operatorname{End}(M)) = \dim(\operatorname{End}(M)/J(\operatorname{End} M)) \leq \dim(M).$$

(2) *If $M$ has finite dual Goldie dimension and every surjective endomorphism of $M$ is bijective, then the endomorphism ring of $M$ is semi-local and*

$$\operatorname{codim}(\operatorname{End}(M)) = \dim(\operatorname{End}(M)/J(\operatorname{End} M)) \leq \operatorname{codim}(M).$$

(3) *If $M$ has finite dual Goldie dimension and finite Goldie dimension, then the endomorphism ring of $M$ is semi-local and*

$$\dim(\operatorname{End}(M)/J(\operatorname{End} M)) \leq \dim(M) + \operatorname{codim}(M).$$

*Proof.* If $f$ and $g$ are endomorphisms of $M$, then

$$\ker(f - fgf) = \ker(f) \oplus \ker(1 - gf),$$

for it is clear that $\ker(f) \cap \ker(1 - gf) = 0$ and for any $x \in \ker(f - fgf)$, $x = gf(x) + (1 - gf)(x)$ where $gf(x) \in \ker(1 - gf)$ and $(1 - gf)(x) \in \ker(f)$.

Dually,

$$\operatorname{coker}(f - fgf) \cong \operatorname{coker}(f) \oplus \operatorname{coker}(1 - fg)$$

which holds because

$$M = \operatorname{im}(fg) + \operatorname{im}(1 - fg) = \operatorname{im}(f) + \operatorname{im}(1 - fg)$$

and

$$\operatorname{im}(f - fgf) = \operatorname{im}(f) \cap \operatorname{im}(1 - fg).$$

The endomorphism $f$ induces isomorphisms between $\ker(1 - gf)$ and $\ker(1 - fg)$, and between $\operatorname{coker}(1 - gf)$ and $\operatorname{coker}(1 - fg)$.

To prove (1) let $n = \dim(M)$, define $d_1 : \operatorname{End}(M) \to \{0, \dots, n\}$ by $d_1(f) = \dim \ker(f)$ and set $d = d_1$. To prove (2) let $m = \operatorname{codim}(M)$, define $d_2 : \operatorname{End}(M) \to \{0, \dots, m\}$ by $d_2(f) = \operatorname{codim} \operatorname{coker}(f)$ and set $d = d_2$. To prove (3) set $d = d_1 + d_2 : \operatorname{End}(M) \to \{0, \dots, n + m\}$. In each of the three cases $d$ satisfies the conditions of Theorem 1 and the result is now clear. □

Camps and Dicks use Theorem 3 (1) to prove that artinian modules have semi-local endomorphism rings [3, Corollary 6] since for an artinian module any injective endomorphism is bijective.

Following Goodearl [9] we say that a ring $R$ is *right repetitive* if for any elements $a$, $b \in R$ the right ideal $I = \sum_{i \geq 0} a^i bR$ is finitely generated. Right repetitive rings include commutative rings, matrices over commutative rings and right noetherian rings. Goodearl in [9] shows that $M_n(R)$ is right repetitive for any $n \geq 1$ if and only if any surjective endomorphism of a finitely generated module $M$ is an isomorphism. Thus if $M$ is a finitely generated module with finite dual Goldie dimension over a right repetitive ring whose matrices are also right repetitive, then $\operatorname{End} M$ is semi-local, and if further $M$ is hollow, then $\operatorname{End} M$ is local. Example 10 (1) shows that this result is not true for an arbitrary ring.

It is well known that a quasi-injective module $M$ has finite Goldie dimension if and only if $\operatorname{End} M$ is a semi-perfect ring. Theorem 3 (2) gives an "almost" dual result for quasi-projective modules.

**Corollary 4.** *Let $R$ be a ring and $P$ a right quasi-projective module.*

(1) *If $P$ has finite dual Goldie dimension, then* $\operatorname{End}(P)$ *is semi-local.*
(2) (Ware, [21]) *If $P$ has small radical, then* $\operatorname{End}(P)$ *is local if and only if $P$ is a local module.*
(3) *If $P$ has small radical, then* $\operatorname{End}(P)$ *is semi-local if and only if $P$ has finite dual Goldie dimension.*

*Proof*. To prove (1) observe that if $P$ is a quasi-projective module and $f: P \to P$ is a surjective endomorphism, then $P \cong X \oplus f(P) \cong X \oplus P$ and hence $P \cong X^n \oplus P$ for all $n \geq 1$. If $P$ has finite dual Goldie dimension $k$, then it cannot be a direct sum of more than $k$ proper summands. Thus $X = 0$ and we conclude that $f$ is an isomorphism. Now Theorem 3 (2) implies that $\operatorname{End}(P)$ is semi-local.

If $P_R$ is local, then Theorem 3 (2) implies immediately that $\operatorname{End} P_R$ is local. A slight modification in the proof of Proposition 17.19 of [1] yields the converse of this statement. This proves (2).

To prove (3) we only need to show that if $P$ is a quasi-projective module with small radical and whose endomorphism ring is semi-local, then $P$ has finite dual Goldie dimension. Observe that $\overline{P} = P/J(P)$ is quasi-projective as a module over $\overline{R} = R/J(R)$ and $\operatorname{End}(\overline{P}_{\overline{R}}) \cong \operatorname{End}(P)/J(\operatorname{End} P)$ (cf. [21, Proposition 1.1]) is semi-simple. Hence there exist primitive orthogonal idempotents $e_1, \dots, e_n$ in $\operatorname{End}(\overline{P}_{\overline{R}})$ such that $1 = e_1 + \cdots + e_n$ and $\operatorname{End} e_i\overline{P} \cong e_i \operatorname{End}(\overline{P})e_i$ is a division ring. Hence $\overline{P} = e_1\overline{P} \oplus \cdots \oplus e_n\overline{P}$. It follows from (2) that $(e_i\overline{P})_{\overline{R}}$ is local and hence simple because it has zero radical. This shows that $\overline{P}_{\overline{R}}$ is semi-simple and so $\operatorname{codim}(P) = \operatorname{codim}(\overline{P}_{\overline{R}}) = n < \infty$. □

A right $R$-module $M$ is said to be *linearly compact* (in the discrete topology), if any system of finitely solvable congruences

$$x \equiv x_i \quad \operatorname{mod} N_i, \quad i \in I, \quad N_i \subseteq M, \quad x_i \in M,$$

is solvable. Artinian modules are linearly compact but the importance of linearly compact modules comes from the fact proved by Müller in [18] (see also [22, Corollary 4.2]) that when a ring $R$ has a right Morita duality then the reflexive modules are exactly the right linearly compact ones.

Carl Faith made the conjecture that a linearly compact module should have semi-local endomorphism ring. Since a linearly compact module has both finite dual Goldie dimension (by Zelinsky [23, Proposition 6]) and finite Goldie dimension (by Sandomierski [19, Lemma 2.3] or [22, Propositions 3.4 and 3.3]), Theorem 3 (3) settles the conjecture of Faith in the affirmative.

**Corollary 5.** *Let $R$ be a ring and $M$ a linearly compact right $R$-module. Then the endomorphism ring of $M$ is semi-local.* □

Right linearly compact rings are semi-perfect ([19, Proposition 2.6 corollary] or [22, Corollary 3.14]), and since any linearly compact module over a commutative ring is pure-injective, it has semi-perfect endomorphism ring (cf. [12,

p. 174 and Corollary 8.27]). However in [4, Theorem 3.5] Camps and Menal give an example of a cyclic indecomposable artinian module whose endomorphism ring is semi-local but not local, thus in general it is not true that the endomorphism ring of a linearly compact module is semi-perfect.

We say that a module $M$ *satisfies* $AB5^*$ if

$$\bigcap_{i\in I}(N+M_i)=N+\bigcap_{i\in I}M_i$$

for all submodules $N$ and inverse systems of submodules $\{M_i\}_{i\in I}$ of $M$.

Leptin proved that linearly compact modules satisfy $AB5^*$ ([14, Satz 1] or [22, Corollary 3.9]), but in general a module satisfying $AB5^*$ need not have finite Goldie or dual Goldie dimension (consider for example the $\mathbb{Z}$-module $M=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$, where $\mathbb{Z}$ denotes the ring of integers and $P$ is an infinite set of different primes).

**Lemma 6.** *Let $R$ be a ring and $M$ a right $R$-module satisfying $AB5^*$. Then the following statements are equivalent:*

(1) *Any quotient of $M$ has finite Goldie dimension.*
(2) *Any submodule of $M$ has finite dual Goldie dimension.*

*Proof.* To prove that (1) implies (2), let $\{A_i\}_{i\in\mathbb{N}}$ be an infinite countable coindependent family of submodules of $M$. Set $P_i=\{J\subseteq\mathbb{N}\mid J \text{ is finite and } i\notin J\}$, for any $J\in P_i$ set $M_J=\bigcap_{j\in J}A_j$. Now $\{M_J\}$ is an inverse subsystem of submodules of $M$. Applying $AB5^*$ we have

$$\bigcap_{J\in P_i}(A_i+M_J)=A_i+\bigcap_{J\in P_i}M_J.$$

By the definition of coindependence $A_i+M_J=M$, thus for any $i\in\mathbb{N}$, $A_i+\bigcap_{J\in P_i}M_J=A_i+\bigcap_{j\neq i}A_j=M$. This proves that the image of the natural morphism $M\longrightarrow\prod_{i\in\mathbb{N}}M/A_i$ contains an infinite direct sum, which contradicts (1). Hence $M$ has no infinite coindependent families of submodules. Since submodules of modules with $AB5^*$ also have this property, the result follows.

It is very easy to see that (2) always implies (1). □

If $M$ is a right $R$-module, we denote by $\mathscr{S}(M)$ the set of non-isomorphic simple images of submodules of $M$.

**Corollary 7.** *If $R$ is a ring and $M$ a right $R$-module satisfying $AB5^*$ such that $\mathscr{S}(M)$ is finite, then* $\operatorname{End}(M)$ *is semi-local.*

*Proof.* Lemonnier in [13, Lemme 2] proves that if $M$ is a right $R$-module satisfying $AB5^*$ such that $\mathscr{S}(M)$ is finite, then any quotient of $M$ has finite Goldie dimension. Now the result follows from Lemma 6 and Theorem 3(3). □

The result of Corollary 7 does not include all linearly compact modules, since there exist examples of linearly compact modules such that $\mathscr{S}(M)$ is not finite—see [8, Examples 3 and 4].

The next result enables us to show that over a commutative ring a module satisfying $AB5^*$ satisfies the conclusions of Theorem 2.

If $M$ is a right $R$-module and $A$ is a subset of $M$, put $r_R(A)=\{r\in R\mid Ar=0\}$.

**Lemma 8.** *Let $R$ be a commutative ring and $M$ an $R$-module such that for any $x \in M$, $R/r_R(x)$ is a semi-perfect ring. Then $M = \bigoplus_{i\in I} M_i$ where $M_i$ is a module over a local ring $R_i$ and $\operatorname{End}_R(M) \cong \prod_{i\in I} \operatorname{End}_{R_i}(M_i)$.*

*Proof.* For any $S_i \in \mathscr{S}(M)$ consider $E(S_i)$, the injective hull of the simple module $S_i$, and set

$$M_i = \{ x \in M \mid \operatorname{Hom}(xR, E(S_j)) = 0 \text{ for all } j \neq i \}.$$

It is easy to see that

$$M_i = \{ x \in M \mid R/r_R(x) \text{ is a local ring with simple module } S_i \} \cup \{0\}.$$

We prove first that $M_i$ is a submodule of $M$. Since $E(S_i)$ is injective it is clear that $xr \in M_i$ whenever $x \in M_i$ and $r \in R$. Let $x$ and $y$ be non-zero elements of $M_i$ and let $f\colon (x+y)R \longrightarrow E(S_j)$, $j \neq i$, be any morphism. Then $f((x+y)R)r_R(x) = 0$ and so $\operatorname{im} f$ is an $R/r_R(x)$-module. But $x \in M_i$ and by the definition of $M_i$, $R/r_R(x)$ is a local ring with simple module $S_i$, thus $\operatorname{im} f = 0$ and we conclude that $x + y \in M_i$.

It is clear that $\{M_i\}$ form a family of independent submodules of $M$ such that $\operatorname{Hom}(M_i, M_j) = 0$ for $i \neq j$, and since for any $x \in M$, $xR \cong R/r_R(xR)$ is a commutative semi-perfect ring, we deduce that $M = \bigoplus M_i$.

Consider $S_i \in \mathscr{S}(M)$, $S_i \cong R/P_i$, for a suitable maximal ideal $P_i$ of $R$. To finish the proof, we show that $M_i$ is an $R_{P_i}$-module and $\operatorname{End}_R(M_i) = \operatorname{End}_{R_{P_i}}(M_i)$. The definition of $M_i$ implies that $r_R(x) \subseteq P_i$ for any $0 \neq x \in M_i$, hence for any $a \in R \setminus P_i$ multiplication by $a$ induces an injective $R/r_R(x)$-endomorphism $f$ of $xR$, and since $a$ is a unit in $R/r_R(x)$, $f$ is also surjective. We conclude that $M_i$ is an $R_{P_i}$-module, and as it is clear that $\operatorname{End}_R(M_i) = \operatorname{End}_{R_{P_i}}(M_i)$, the proof of the lemma is complete. □

**Corollary 9.** *Let $R$ be a commutative ring and $M$ a module satisfying $AB5^*$. Then $\operatorname{End}(M)$ is a product of semi-local rings, 1 is in the stable range of $\operatorname{End}(M)$, and $M$ cancels from direct sums and satisfies the $n$ th root uniqueness property.*

*Proof.* If $M$ is a module satisfying $AB5^*$, then by [13, Proposition 4] for any $x \in M$, $xR \cong R/r_R(x)$ is a semi-perfect ring. Apply Lemma 8 and Corollary 7 to conclude that $\operatorname{End}(M)$ is a product of semi-local rings. Thus by Theorem 2, 1 is in the stable range of $\operatorname{End}(M)$, and by [6, Theorem 2] $M$ cancels from direct sums.

Theorem 2 implies that $M$ is a direct sum of modules that cancel from direct sums and satisfy the $n$ th root uniqueness property, so $M$ itself satisfies the $n$ th root uniqueness property. □

*Remark.* It is easy to see that the rings such that every right ideal and every left ideal is an annihilator satisfy $AB5^*$ (on both sides). These rings were studied by Hajarnavis and Norton in [11]. Lemonnier's results in [13] give alternative and shorter proofs to Theorems 3.9 and 5.3 in the Hajarnavis and Norton paper, who also show that if $R$ is a ring such that any right and left ideal is an annihilator, then $R/\bigcap_{n=1}^{\infty} J(R)^n$ is a noetherian ring; it is easy to see that their proof also works for rings satisfying $AB5^*$. Müller in [17] or [22, Lemma 17.1] proves that if $R$ is a right linearly compact ring, then $R/\bigcap_{n=1}^{\infty} J(R)^n$ is a right noetherian ring and in his proof only right $AB5^*$ is used. In [15] Menini proved that a two-sided noetherian and right linearly compact ring satisfies that $\bigcap_{n=1}^{\infty} J(R)^n = 0$

(see also [22, Corollary 17.5]). Again in Menini's proof the only property used of right linear compactness is right $AB5^*$.

Thus if $R$ is a ring satisfying right $AB5^*$, then:

(1) (Müller [17]) *$R/\bigcap_{n=1}^{\infty} J(R)^n$ is a noetherian ring.*

(2) (Menini [15]) *If $R$ is right and left noetherian, then $\bigcap_{n=1}^{\infty} J(R)^n = 0$.*

In [16, Question 11, p. 106] Mohamed and Müller ask for examples of local modules whose endomorphism ring is not local. In [4, Theorem 3.5] Camps and Menal construct examples of indecomposable artinian cyclic modules $M$ whose endomorphism ring is semi-local but not local. It is easy to see that in some of these examples $M$ is also a local module. The next example, patterned after Camps and Menal techniques, shows that any ring that can be embedded in a local ring can be realized as the endomorphism ring of a local module.

Until now all the examples we have given of modules with semi-local endomorphism ring (except perhaps injective modules with finite Goldie dimension) have finite dual Goldie dimension. It is clear that if $R$ is commutative any cyclic module with semi-local endomorphism ring should have finite dual Goldie dimension but, as the next example shows, this is not true over arbitrary rings.

**Example 10.** (1) *Let $R$ be a ring that can be embedded in a local ring $S$. Then $R$ can be realized as the endomorphism ring of a local module.*

(2) *There exist cyclic modules with infinite dual Goldie dimension whose endomorphism ring is semi-local.*

*Proof*. Let $R \subseteq S$ be an embedding of rings, and consider the $(S, R)$-bimodule $M = \mathrm{Hom}_R({}_RS, {}_RS/R)$ and the sub-bimodule $N = \{f \in M \mid f(R) = 0\}$. Let $T$ be the ring $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ and consider the right ideal $I = \begin{pmatrix} 0 & N \\ 0 & R \end{pmatrix}$ of $T$. The idealizer of $I$ is $I' = \begin{pmatrix} R & N \\ 0 & R \end{pmatrix}$ because an element $\begin{pmatrix} s & f \\ 0 & r \end{pmatrix} \in I'$ if and only if $sN \subseteq N$ and $fR \subseteq N$ which implies that $s \in R$ and $f \in N$. Thus $\mathrm{End}_T(T/I) = I'/I = R$.

To prove (1) assume that $S$ is a local ring. The proper right ideals of $T$ containing $I$ are of the form $\begin{pmatrix} J & K \\ 0 & R \end{pmatrix}$, where $J$ is a right ideal of $S$ different from $S$, and $K$ is a sub-bimodule of $M$ containing $N$. Since $J$ is a small submodule of $S$, every proper submodule of $T/I$ is small. Hence $T/I$ is a local right $T$-module with endomorphism ring $R$.

To prove (2) assume that $R$ is semi-local and $S$ is not, thus $S$ has an infinite co-independent family $\{A_i\}_{i\in\mathbb{N}}$ of right ideals. The right ideals of $T$, $\{\begin{pmatrix} A_i & M \\ 0 & R \end{pmatrix}\}_{i\in\mathbb{N}}$, will give an infinite family of coindependent submodules of $T/I$. Thus $T/I$ has infinite dual Goldie dimension but its endomorphism ring is the semi-local ring $R$. □

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Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903
*Current address*: Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellatera, Barcelona, Spain
*E-mail address*: `dolors@mat.uab.es`

Department of Mathematics, American University of Beirut, Beirut, Lebanon
*E-mail address*: `ahmad@layla.aub.ac.lb`