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MODULI OF BRIDGELAND SEMISTABLE OBJECTS ON $\mathbf{P}^2$

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1. Introduction

Let X be a smooth projective surface and $D^b(X)$ the bounded derived category of coherent sheaves on X. We study Bridgeland stability conditions σ on $D^b(X)$. We show that if a stability condition σ has a certain property, the moduli space of σ-(semi)stable objects in $D^b(X)$ coincides with a certain moduli space of Gieseker-(semi)stable coherent sheaves on X. On the other hand, when X has a full strong exceptional collection, we define the notion of σ being "algebraic", and we show that for any algebraic stability condition σ_{alg}, the moduli space of σ_{alg}-(semi)stable objects in $D^b(X)$ coincides with a certain moduli space of modules over a finite dimensional $\mathbf{C}$-algebra. Using these observations, we construct moduli spaces of Gieseker-(semi)stable coherent sheaves on $\mathbf{P}^2$ as moduli spaces of certain modules (Theorem 5.1). This gives a new proof (§5.3) of Le Potier's result [P] and establishes some related results (§6).

1.1. Bridgeland stability conditions

The notion of stability conditions on a triangulated category $\mathscr{T}$ was introduced in [Br1] to give the mathematical framework for the Douglas's work on Π-stability. Roughly speaking, it consists of data $\sigma = (Z, \mathscr{A})$, where Z is a group homomorphism from the Grothendieck group $K(\mathscr{T})$ to the complex number field $\mathbf{C}$, $\mathscr{A}$ is a full abelian subcategory of $\mathscr{T}$ and these data should have some properties (see Definition 2.3). Then Bridgeland [Br1] showed that the set of some good stability conditions has a structure of a complex manifold. This set is denoted by $\mathrm{Stab}(X)$ when $\mathscr{T} = D^b(X)$. An element σ of $\mathrm{Stab}(X)$ is called a Bridgeland stability condition on X. For a full abelian subcategory $\mathscr{A} \subset \mathscr{T}$, $\mathrm{Stab}(\mathscr{A})$ denotes the subset of $\mathrm{Stab}(X)$ consisting of all stability conditions of the form $\sigma = (Z, \mathscr{A})$.

Let $K(X)$ be the Grothendieck group of X. For $\alpha \in K(X)$, the Chern character of α is the element $\mathrm{ch}(\alpha) := (\mathrm{rk}(\alpha), c_1(\alpha), \mathrm{ch}_2(\alpha))$ of the lattice $\mathscr{N}(X) := \mathbf{Z} \oplus \mathrm{NS}(X) \oplus \frac{1}{2}\mathbf{Z}$. For $\sigma = (Z, \mathscr{A}) \in \mathrm{Stab}(X)$, we consider the moduli functor $\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$ of σ-(semi)stable objects E in $\mathscr{A}$ with $\mathrm{ch}(E) = \mathrm{ch}(\alpha)$.

1.2. Geometric Bridgeland stability conditions

For $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$ such that ω is in the ample cone $\mathrm{Amp}(X)$, we consider a pair $\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathscr{A}_{(\beta,\omega)})$ as in [ABL], where $Z_{(\beta,\omega)} : K(X) \to \mathbf{C}$

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is a group homomorphism and $\mathscr{A}_{(\beta,\omega)}$ is a full abelian subcategory of $D^b(X)$ defined from β and ω (see Definition 3.3 for details). It is shown in [ABL] that $\sigma_{(\beta,\omega)}$ is a Bridgeland stability condition if $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{Q}$. For general $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$, we do not know whether $\sigma_{(\beta,\omega)}$ belongs to $\mathrm{Stab}(X)$ or not (cf. §3.2).

Let $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ be the universal cover of the group $\mathrm{GL}^+(2,\mathbf{R}) := \{T \in \mathrm{GL}(2,\mathbf{R}) \mid \det T > 0\}$. The group $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ acts on $\mathrm{Stab}(X)$ in a natural way (cf. §2.3). Two stability conditions σ and σ' are said to be $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$-equivalent if σ and σ' are in a single orbit of this action. In such cases σ and σ' correspond to isomorphic moduli functors of semistable objects. $\sigma \in \mathrm{Stab}(X)$ is said to be geometric if σ is $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$-equivalent to $\sigma_{(\beta,\omega)}$ for some $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$ with $\omega \in \mathrm{Amp}(X)$. We have a criterion due to [Br2] for $\sigma \in \mathrm{Stab}(X)$ to be geometric (Proposition 3.6).

On the other hand, for an integral ample divisor ω and $\beta \in \mathrm{NS}(X) \otimes \mathbf{Q}$, we consider (β,ω)-twisted Gieseker-stability of torsion free sheaves on X, which was introduced in [MW] generalizing the Gieseker-stability. For $\alpha \in K(X)$, we assume $\mathrm{rk}(\alpha) > 0$ and consider the moduli functor $\mathscr{M}_X(\mathrm{ch}(\alpha),\beta,\omega)$ of (β,ω)-semistable sheaves E with $\mathrm{ch}(E) = \mathrm{ch}(\alpha)$. There is a scheme $M_X(\mathrm{ch}(\alpha),\beta,\omega)$ which corepresents $\mathscr{M}_X(\mathrm{ch}(\alpha),\beta,\omega)$ [MW], and is called the moduli space (cf. Definition 2.6).

One of our main results is the following.

THEOREM 1.1. *Let ω be an integral ample divisor, $\beta \in \mathrm{NS}(X) \otimes \mathbf{Q}$ and $\alpha \in K(X)$ with $\mathrm{rk}(\alpha) > 0$. Take a real number t with $0 < t \le 1$ and assume that $\sigma_{(\beta,t\omega)} \in \mathrm{Stab}(X)$. If $0 < c_1(\alpha)\cdot\omega - \mathrm{rk}(\alpha)\beta\cdot\omega \le \min\left\{t, \frac{1}{\mathrm{rk}(\alpha)}\right\}$ then the moduli space $M_X\left(\mathrm{ch}(\alpha), \beta - \frac{1}{2}K_X, \omega\right)$ corepresents the moduli functor $\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma_{(\beta,t\omega)})$.*

A proof of Theorem 1.1 will be given in §3.3. Similar results are obtained by [Br2] and [To] when X is a K3 surface, but our choices of ω and β are different from theirs.

1.3. Algebraic Bridgeland stability conditions

For a finite dimensional $\mathbf{C}$-algebra B, mod-B denotes the abelian category of finitely generated right B-modules and $K(B)$ denotes the Grothendieck group. For any B-module N, we denote by $[N]$ the image of N by the map mod-$B \to K(B)$. King [K] introduced the notion of θ_B-stability of B-modules, where θ_B is a group homomorphism $\theta_B : K(B) \to \mathbf{R}$. It is shown in [K] that the moduli space $M_B(\alpha_B, \theta_B)$ of θ_B-semistable B-modules N with $[N] = \alpha_B$ exists, for any $\alpha_B \in K(B)$ and $\theta_B \in \alpha_B^{\perp} := \{\theta_B \in \mathrm{Hom}_{\mathbf{Z}}(K(B),\mathbf{R}) \mid \theta_B(\alpha_B) = 0\}$.

When X has a full strong exceptional collection $\mathfrak{E} = (E_0,\dots,E_n)$ in $D^b(X)$ (cf. §4.2), we put $\mathscr{E} = \bigoplus_i E_i$ and consider the finite dimensional $\mathbf{C}$-algebra $B_{\mathscr{E}} = \mathrm{End}_X(\mathscr{E})$. Then by Bondal's Theorem [Bo], the functor $\mathbf{R}\,\mathrm{Hom}_X(\mathscr{E},\cdot)$ gives an equivalence of triangulated categories $\Phi_{\mathscr{E}} : D^b(X) \cong D^b(B_{\mathscr{E}})$, where $D^b(B_{\mathscr{E}})$ is the bounded derived category of mod-$B_{\mathscr{E}}$. $\Phi_{\mathscr{E}}$ induces an isomorphism of the

Grothendieck groups $\varphi_{\mathscr{E}} : K(X) \cong K(B_{\mathscr{E}})$. Let $\mathscr{A}_{\mathscr{E}}$ be the full abelian subcategory of $D^b(X)$ corresponding to mod-$B_{\mathscr{E}} \subset D^b(B_{\mathscr{E}})$ by $\Phi_{\mathscr{E}}$. $\sigma \in \mathrm{Stab}(X)$ is called an algebraic Bridgeland stability condition associated to $\mathfrak{E} = (E_0, \dots, E_n)$ if σ is $\widetilde{\mathrm{GL}}^+(2, \mathbf{R})$-equivalent to $(Z, \mathscr{A}_{\mathscr{E}})$ for some $Z : K(X) \to \mathbf{C}$.

For any $\sigma = (Z, \mathscr{A}_{\mathscr{E}}) \in \mathrm{Stab}(\mathscr{A}_{\mathscr{E}})$ and $\alpha \in K(X)$, we associate the group homomorphism $\theta_Z^\alpha : K(B_{\mathscr{E}}) \to \mathbf{R}$ defined by

$$\theta_Z^\alpha(\beta) = \begin{vmatrix} \mathrm{Re}\, Z(\varphi_{\mathscr{E}}^{-1}(\beta)) & \mathrm{Re}\, Z(\alpha) \\ \mathrm{Im}\, Z(\varphi_{\mathscr{E}}^{-1}(\beta)) & \mathrm{Im}\, Z(\alpha) \end{vmatrix}$$

for $\beta \in K(B_{\mathscr{E}})$. Clearly $\theta_Z^\alpha \in \varphi_{\mathscr{E}}(\alpha)^\perp$, so we have the moduli space $M_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta_Z^\alpha)$.

Proposition 1.2. *The moduli space $M_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta_Z^\alpha)$ of $B_{\mathscr{E}}$-modules corepresents the moduli functor $\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$ for any $\alpha \in K(X)$ and $\sigma = (Z, \mathscr{A}_{\mathscr{E}}) \in \mathrm{Stab}(\mathscr{A}_{\mathscr{E}})$.*

A proof of Proposition 1.2 will be given in §4.2.

1.4. Application in the case $X = \mathbf{P}^2$

We prove that there exist Bridgeland stability conditions on $\mathbf{P}^2$ which are both geometric and algebraic by using the criterion Proposition 3.6.

The Neron-Severi group $\mathrm{NS}(\mathbf{P}^2)$ of $\mathbf{P}^2$ is generated by the hyperplane class H. Hence when $X = \mathbf{P}^2$ the twisted Gieseker-stability coincides with the classical one defined by H. We sometimes identify $\mathrm{NS}(\mathbf{P}^2)$ with $\mathbf{Z}$ by the map $\beta \mapsto \beta \cdot H$. For $\alpha \in K(\mathbf{P}^2)$ with $\mathrm{rk}(\alpha) > 0$, we consider the moduli space $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H)$ and $\sigma_{(bH, tH)}$ for $b, t > 0$.

On the other hand, for each $k \in \mathbf{Z}$ there exist full strong exceptional collections on $\mathbf{P}^2$

$$\mathfrak{E}_k := (\mathscr{O}_{\mathbf{P}^2}(k+1), \Omega^1_{\mathbf{P}^2}(k+3), \mathscr{O}_{\mathbf{P}^2}(k+2)) \quad \text{and}$$
$$\mathfrak{E}'_k := (\mathscr{O}_{\mathbf{P}^2}(k), \mathscr{O}_{\mathbf{P}^2}(k+1), \mathscr{O}_{\mathbf{P}^2}(k+2)).$$

We put $\mathscr{E}_k := \mathscr{O}_{\mathbf{P}^2}(k+1) \oplus \Omega^1_{\mathbf{P}^2}(k+3) \oplus \mathscr{O}_{\mathbf{P}^2}(k+2)$ and $\mathscr{E}'_k := \mathscr{O}_{\mathbf{P}^2}(k) \oplus \mathscr{O}_{\mathbf{P}^2}(k+1) \oplus \mathscr{O}_{\mathbf{P}^2}(k+2)$. Up to natural isomorphism, $\mathrm{End}_{\mathbf{P}^2}(\mathscr{E}_k)$ and $\mathrm{End}_{\mathbf{P}^2}(\mathscr{E}'_k)$ do not depend on k, hence we identify and denote them by B and B' respectively. Using the notation in §1.3, we put

$$\Phi_k := \Phi_{\mathscr{E}_k} : D^b(\mathbf{P}^2) \cong D^b(B), \quad \Phi'_k := \Phi_{\mathscr{E}'_k} : D^b(\mathbf{P}^2) \cong D^b(B'),$$

induced isomorphisms $\varphi_k := \varphi_{\mathscr{E}_k} : K(\mathbf{P}^2) \cong K(B)$, $\varphi'_k := \varphi_{\mathscr{E}'_k} : K(\mathbf{P}^2) \cong K(B')$ and hearts of induced bounded t-structures $\mathscr{A}_k := \mathscr{A}_{\mathscr{E}_k} \subset D^b(\mathbf{P}^2)$, $\mathscr{A}'_k := \mathscr{A}_{\mathscr{E}'_k} \subset D^b(\mathbf{P}^2)$.

For $\alpha \in K(\mathbf{P}^2)$ and $\theta \in \alpha^\perp := \{\theta \in \mathrm{Hom}_{\mathbf{Z}}(K(\mathbf{P}^2), \mathbf{R}) \mid \theta(\alpha) = 0\}$, we put

$$\theta_k := \theta \circ \varphi_k^{-1} : K(B) \to \mathbf{R}, \quad \theta'_k := \theta \circ \varphi_k'^{-1} : K(B') \to \mathbf{R}.$$

There exists $\theta \in \alpha^\perp$ such that $\Phi'_1 \circ \Phi_0^{-1}$ and $\Phi_1 \circ \Phi_1'^{-1}$ induce the following isomorphisms (Proposition 5.4)

$$M_B(-\varphi_0(\alpha), \theta_0) \cong M_{B'}(-\varphi'_1(\alpha), \theta'_1) \cong M_B(-\varphi_1(\alpha), \theta_1). \tag{1}$$

We find algebraic Bridgeland stability conditions $\sigma^b = (Z^b, \mathscr{A}_1) \in \mathrm{Stab}(\mathscr{A}_1)$ parametrized by real numbers b with $0 < b < 1$ such that for each b there exist an element $g \in \widetilde{\mathrm{GL}}^+(2, \mathbf{R})$ and $t > 0$ satisfying

$$\sigma^b g = \sigma_{(bH, tH)}, \tag{2}$$

where g and $t > 0$ may depend on b. Then $M_B(-\varphi_1(\alpha), \theta^{\alpha}_{Z^b})$ corepresents the moduli functors $\mathscr{M}_{D^b(\mathbf{P}^2)}(-\mathrm{ch}(\alpha), \sigma^b)$ by Proposition 1.2. Furthermore by (2) and Theorem 1.1, $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H)$ also corepresents the same moduli functor for suitable choice of b. From these facts and isomorphisms (1), we have our main results (see §5.1 for the choice of $\theta \in \alpha^{\perp}$). We denote by $\cdot\,[1]$ the shift functor $D^b(\mathbf{P}^2) \to D^b(\mathbf{P}^2) : E \mapsto E[1]$.

Main Theorem 1.3. *For $\alpha \in K(\mathbf{P}^2)$ with $c_1(\alpha) = sH$, assume $0 < s \le \mathrm{rk}(\alpha)$ and $\mathrm{ch}_2(\alpha) < \frac{1}{2}$. Then there exists $\theta \in \alpha^{\perp}$ such that $\Phi_1(\cdot\,[1])$, $\Phi_1'(\cdot\,[1])$ and $\Phi_0(\cdot\,[1])$ induce the following isomorphisms.*
(i) $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_1(\alpha), \theta_1) : E \mapsto \Phi_1(E[1])$
(ii) $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_{B'}(-\varphi_1'(\alpha), \theta_1') : E \mapsto \Phi_1'(E[1])$
(iii) $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_0(\alpha), \theta_0) : E \mapsto \Phi_0(E[1])$.
These isomorphisms keep open subsets consisting of stable objects.

We remark that if we assume $0 < s \le \mathrm{rk}(\alpha)$ and $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \ne \emptyset$ in Main Theorem 1.3, then we have

$$\dim M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) = s^2 - \mathrm{rk}(\alpha)^2 + 1 - 2\,\mathrm{rk}(\alpha)\,\mathrm{ch}_2(\alpha) \ge 0.$$

Hence we have $\mathrm{ch}_2(\alpha) \le \frac{1}{2}$, and $\mathrm{ch}_2(\alpha) = \frac{1}{2}$ if and only if $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) = \{\mathscr{O}_{\mathbf{P}^2}(1)\}$. In this case, similar isomorphisms hold via $\Phi_1(\cdot\,[1])$ in (i), Φ_1' in (ii) and Φ_0 in (iii) respectively. A proof of Main Theorem 1.3 will be given in §5.

(ii) is obtained by Le Potier [P] (cf. [KW, §4] and [P2, Theorem 14.7.1]) by a different method.

1.5. Wall-crossing phenomena

In §6 we consider the case $\mathrm{rk}(\alpha) = 1$, $c_1(\alpha) = H$ and $\mathrm{ch}_2(\alpha) = \frac{1}{2} - n$ with $n \ge 1$. By Main Theorem 1.3 we have

$$M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_0(\alpha), \theta_0) \cong M_B(-\varphi_1(\alpha), \theta_1)$$

for some $\theta \in \alpha^{\perp}$. We study how $M_B(-\varphi_k(\alpha), \theta_k^{\dagger})$ changes when $\theta_k^{\dagger} \in \varphi_k(\alpha)^{\perp}$ varies for $k = 0, 1$, where $\varphi_k(\alpha)^{\perp} := \{\theta_k \in \mathrm{Hom}_{\mathbf{Z}}(K(B), \mathbf{R}) \mid \theta_k(\varphi_k(\alpha)) = 0\}$. We define a wall-and-chamber structure on $\varphi_k(\alpha)^{\perp}$ as follows (cf. §5.1). Within $\varphi_k(\alpha)^{\perp}$, there are finitely many rays corresponding to certain B-modules. In our case, a ray may be called a wall, since $\varphi_k(\alpha)^{\perp} \cong \mathbf{R}^2$. Let W_k be the union of such rays. A connected component of the complement of W_k is called a chamber. The moduli space $M_B(-\varphi_k(\alpha), \theta_k^{\dagger})$ does not change when $\theta_k^{\dagger}$ moves in a chamber.

If two chambers $\hat{C}_{\varphi_k(\alpha)}$ and $\bar{C}_{\varphi_k(\alpha)}$ on $\varphi_k(\alpha)^{\perp}$ are adjacent to each other having a common wall w_k, then for $\hat{\theta}_k \in \hat{C}_{\varphi_k(\alpha)}$, $\bar{\theta}_k \in \bar{C}_{\varphi_k(\alpha)}$ and $\tilde{\theta}_k \in w_k$ we have a diagram:

$$\begin{array}{ccc} M_B(-\varphi_k(\alpha), \bar{\theta}_k) & \overset{\kappa}{\longleftarrow - - - - - -} & M_B(-\varphi_k(\alpha), \hat{\theta}_k). \\ & \searrow^{f''} \quad {}^{f'}\swarrow & \\ & M_B(-\varphi_k(\alpha), \tilde{\theta}_k) & \end{array} \tag{3}$$

Further, if both $M_B(-\varphi_k(\alpha), \hat{\theta}_k)$ and $M_B(-\varphi_k(\alpha), \bar{\theta}_k)$ are non-empty, then we see that f', f'' are birational morphisms by general theory of Thaddeus [Th].

Within $\varphi_k(\alpha)^{\perp}$, we have a chamber $C^{\mathbf{P}^2}_{\varphi_k(\alpha)}$ such that $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_k(\alpha), \theta_k)$ for any $\theta_k \in C^{\mathbf{P}^2}_{\varphi_k(\alpha)}$. In the case $\mathrm{rk}(\alpha) = 1$, $c_1(\alpha) = 1$ and $\mathrm{ch}_2(\alpha) = \frac{1}{2} - n$, diagrams (3) with $k = 0, 1$ give the two birational transformations of the Hilbert schemes $(\mathbf{P}^2)^{[n]}$ (Theorem 6.5). In the case $\mathrm{rk}(\alpha) = r$, $c_1(\alpha) = 1$, $\mathrm{ch}_2(\alpha) = \frac{1}{2} - n$ with arbitrary $r > 0$, we will describe these diagrams more explicitly in [O].

Similar phenomena as in (3), sometimes called Wall-crossing phenomena, occur by variation of polarizations on some surfaces X in case of Gieseker-stability. However the polarization is essentially unique in our case $X = \mathbf{P}^2$ since $\mathrm{Pic}\,\mathbf{P}^2 \cong \mathbf{Z}H$. So our phenomena are of different nature. We expect that Bridgeland theory is useful to study such phenomena systematically.

Convention

Throughout this paper we work over $\mathbf{C}$. Any scheme is of finite type over $\mathbf{C}$. For a scheme Y, we denote by $\mathrm{Coh}(Y)$ the abelian category of coherent sheaves on Y and by $D^b(Y)$ (respectively, $D^-(Y)$) the bounded (respectively, bounded above) derived category of $\mathrm{Coh}(Y)$. For $E \in \mathrm{Coh}(Y)$, by $\dim E$ we denote the dimension of the support of E. For a ring B, by mod-B we denote the abelian category of finitely generated right B-modules. We denote by $D^b(B)$ (respectively, $D^-(B)$) the bounded (respectively, bounded above) derived category of mod-B. For an abelian category $\mathscr{A}$ and a triangulated category $\mathscr{T}$, their Grothendieck groups are denoted by $K(\mathscr{A})$ and $K(\mathscr{T})$. For any object E of $\mathscr{A}$ (resp. $\mathscr{T}$) we denote by $[E]$ the image of E by the map $\mathscr{A} \to K(\mathscr{A})$ (resp. $\mathscr{T} \to K(\mathscr{T})$). When $\mathscr{A} = $ mod-B and $\mathscr{T} = D^b(Y)$, we simply write them $K(B)$ and $K(Y)$. For objects E, F, G of $\mathscr{T}$, the distinguished triangle $E \to F \to G \to E[1]$ is denoted by:

$$\begin{array}{ccc} E & \longrightarrow & F \\ {}_{[1]}\nwarrow & & \swarrow \\ & G & \end{array}$$

For objects $F_0, \ldots, F_n$ in $\mathscr{T}$ we denote by $\langle F_0, \ldots, F_n \rangle$ the smallest full subcategory of $\mathscr{T}$ containing $F_0, \ldots, F_n$, which is closed under extensions.

2. Generalities on Bridgeland stability conditions

Here we collect some basic definitions and results of Bridgeland stability conditions on triangulated categories in [Br1], [Br2].

2.1. Bridgeland stability conditions on triangulated categories

Let $\mathscr{A}$ be an abelian category.

Definition 2.1. A stability function on $\mathscr{A}$ is a group homomorphism $Z : K(\mathscr{A}) \to \mathbf{C}$ such that $Z(E) \in \mathbf{R}_{>0} \exp(\sqrt{-1}\pi\phi(E))$ with $0 < \phi(E) \leq 1$ for any nonzero object E of $\mathscr{A}$. The real number $\phi(E) \in (0,1]$ is called the phase of the object E. A nonzero object E of $\mathscr{A}$ is said to be Z-(semi)stable if for every proper subobject $0 \neq F \subsetneq E$ we have $\phi(F) < \phi(E)$ (resp. $\leq$).

If we define the slope of E by

$$\mu_\sigma(E) := -\frac{\mathrm{Re}(Z(E))}{\mathrm{Im}(Z(E))},$$

which possibly be infinity, then a nonzero object E of $\mathscr{A}$ is Z-(semi)stable if and only if $\mu_\sigma(F) < \mu_\sigma(E)$ (resp. $\leq$) for any subobject $0 \neq F \subsetneq E$ in $\mathscr{A}$.

The stability function Z is said to have the Harder-Narasimhan property if every nonzero object $E \in \mathscr{A}$ has a finite filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are Z-semistable objects of $\mathscr{A}$ with

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

Let $\mathscr{T}$ be a triangulated category. We recall the definition of a t-structure and its heart (cf. [Br1]).

Definition 2.2. A t-structure on $\mathscr{T}$ is a full subcategory $\mathscr{T}^{\leq 0}$ of $\mathscr{T}$ satisfying the following properties.

(1) $\mathscr{T}^{\leq 0}[1] \subset \mathscr{T}^{\leq 0}$.

(2) If one defines $\mathscr{T}^{\geq 1} := \{F \in \mathscr{T} \mid \mathrm{Hom}_{\mathscr{T}}(G,F) = 0 \text{ for any } G \in \mathscr{T}^{\leq 0}\}$, then for any object $E \in \mathscr{T}$ there is a distinguished triangle

$$G \to E \to F \to G[1]$$

with $G \in \mathscr{T}^{\leq 0}$ and $F \in \mathscr{T}^{\geq 1}$.

We define $\mathscr{T}^{\leq -i} := \mathscr{T}^{\leq 0}[i]$ and $\mathscr{T}^{\geq -i} := \mathscr{T}^{\geq 1}[i+1]$. Then the heart of the t-structure is defined to be the full subcategory $\mathscr{A} := \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$. It was proved in [BBD] that $\mathscr{A}$ is an abelian category, with the short exact sequences in $\mathscr{A}$ being precisely the triangles in $\mathscr{T}$ all of whose vertices are objects of $\mathscr{A}$. A t-structure $\mathscr{T}^{\leq 0} \subset \mathscr{T}$ is said to be bounded if

$$\mathscr{T} = \bigcup_{i,j \in \mathbf{Z}} \mathscr{T}^{\leq i} \cap \mathscr{T}^{\geq j}.$$

If $\mathscr{A}$ is the heart of a bounded t-structure on $\mathscr{T}$, then we have $K(\mathscr{A}) \cong K(\mathscr{T})$.

Definition 2.3. A Bridgeland stability condition σ on a triangulated category $\mathscr{T}$ is a pair $(Z, \mathscr{A})$ of a group homomorphism $Z : K(\mathscr{T}) \to \mathbf{C}$ and the heart $\mathscr{A}$ of a bounded t-structure on $\mathscr{T}$ such that Z is a stability function on $\mathscr{A}$ having the Harder-Narasimhan property.

For each $n \in \mathbf{Z}$ and $\phi' \in (0, 1]$, we define a full subcategory $\mathscr{P}(n + \phi')$ of $\mathscr{T}$ by

$$\mathscr{P}(n + \phi') := \{E \in \mathscr{T} \mid E[-n] \in \mathscr{A} \text{ is } Z\text{-semistable and } \phi(E[-n]) = \phi'\}.$$

For any $\phi \in \mathbf{R}$, a nonzero object E of $\mathscr{P}(\phi)$ is said to be σ-semistable and ϕ is called the phase of E. $E \in \mathscr{P}(\phi)$ is said to be σ-stable if $\phi = n + \phi'$ with $n \in \mathbf{Z}$ and $\phi' \in (0, 1]$, and $E[-n] \in \mathscr{A}$ is Z-stable. It is easy to see that each subcategory $\mathscr{P}(\phi)$ of $\mathscr{T}$ is an abelian category (cf. [Br1, Lemma 5.2]). $E \in \mathscr{P}(\phi)$ is σ-stable if and only if E is a simple object in $\mathscr{P}(\phi)$. For any interval $I \subset \mathbf{R}$, $\mathscr{P}(I)$ is defined by $\mathscr{P}(I) := \langle \{\mathscr{P}(\phi) \mid \phi \in I\} \rangle$. In particular the Harder-Narasimhan property implies that $\mathscr{P}((0, 1]) = \mathscr{A}$.

Proposition 2.4. (1) *The pair* $(Z, \mathscr{P})$ *of the group homomorphism* $Z : K(\mathscr{T}) \to \mathbf{C}$ *and the family* $\mathscr{P} = \{\mathscr{P}(\phi) \mid \phi \in \mathbf{R}\}$ *of full subcategories of* $\mathscr{T}$ *has the following property.*

(a) $\mathscr{P}(\phi)$ *is a full additive subcategory of* $\mathscr{T}$.

(b) $\mathscr{P}(\phi + 1) = \mathscr{P}(\phi)[1]$.

(c) *If* $\phi_1 > \phi_2$ *and* $E_i \in \mathscr{P}(\phi_i)$, *then* $\mathrm{Hom}_{\mathscr{T}}(E_1, E_2) = 0$.

(d) $Z(E) \in \mathbf{R}_{>0} \exp(\sqrt{-1}\pi\phi)$ *for any nonzero object* E *of* $\mathscr{P}(\phi)$.

(e) *For a nonzero object* $E \in \mathscr{T}$, *we have a collection of triangles*

$$\begin{array}{ccccccccccc} 0 = E_0 & \longrightarrow & & E_1 & \longrightarrow & & E_2 & \longrightarrow \cdots \longrightarrow & & & E_n = E \\ & \nwarrow_{[1]} & \swarrow & & \nwarrow_{[1]} & \swarrow & & & \nwarrow_{[1]} & \swarrow & \\ & & F_1 & & & F_2 & & & & F_n & \end{array}$$

such that $F_j \in \mathscr{P}(\phi_j)$ *with* $\phi_1 > \phi_2 > \cdots > \phi_n$.

(2) *Giving a stability condition* $\sigma = (Z, \mathscr{A})$ *on* $\mathscr{T}$ *is equivalent to giving a pair* $(Z, \mathscr{P})$ *with the above properties.*

Proof. See [Br1, Definition 5.1 and Proposition 5.3]. Originally the pair $(Z, \mathscr{P})$ is called the stability condition σ in [Br1]. □

The filtration in (e) of Proposition 2.4 is called the Harder-Narasimhan filtration of E and the objects F_j are called σ-semistable factors of E. We can easily check that the Harder-Narasimhan filtration is unique up to isomorphism. For a Bridgeland stability condition $\sigma = (Z, \mathscr{A})$ (or $(Z, \mathscr{P})$), Z, $\mathscr{A}$ and $\mathscr{P}$ is denoted by Z_σ, $\mathscr{A}_\sigma$ and $\mathscr{P}_\sigma$.

2.2. Bridgeland stability conditions on smooth projective surfaces

Let X be a smooth complex projective surface. The Chern character of an object E of $D^b(X)$ is the element $\mathrm{ch}(E) := (\mathrm{rk}(E), c_1(E), \mathrm{ch}_2(E))$ of the lattice $\mathcal{N}(X) := \mathbf{Z} \oplus \mathrm{NS}(X) \oplus \frac{1}{2}\mathbf{Z}$. We define the Euler form on the Grothendieck group $K(X)$ of X by

$$\chi(E,F) := \Sigma_i (-1)^i \dim_{\mathbf{C}} \mathrm{Hom}_{D^b(X)}(E, F[i]). \tag{4}$$

Let $K(X)^{\perp} = \{\alpha \in K(X) \mid \chi(\alpha,\beta) = 0 \text{ for each } \beta \in K(X)\}$ and $K(X)/K(X)^{\perp}$ is called the *numerical Grothendieck group* of $D^b(X)$.

By the Riemann-Roch theorem the Chern character gives an inclusion $K(X)/K(X)^{\perp} \to \mathcal{N}(X)$. Furthermore we define a symmetric bilinear form $(\cdot,\cdot)_M$ on $\mathcal{N}(X)$, called Mukai pairing, by the following formula

$$((r_1, D_1, s_1), (r_2, D_2, s_2))_M := D_1 \cdot D_2 - r_1 s_2 - r_2 s_1. \tag{5}$$

This bilinear form makes $\mathcal{N}(X)$ a lattice of signature $(2,\rho)$ by the Hodge Index Theorem, where $\rho \geq 1$ is the Picard number of X.

A Bridgeland stability condition $\sigma = (Z, \mathcal{A})$ is said to be *numerical* if there is a vector $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbf{C}$ such that

$$Z(E) = (\pi(\sigma), \mathrm{ch}(E))_M \tag{6}$$

for any $[E] \in K(X)$. σ is said to be *local finite* if it satisfies some technical conditions [Br1, Definition 5.7].

The set of all the numerical local finite Bridgeland stability conditions on $D^b(X)$ is denoted by $\mathrm{Stab}(X)$. It is shown in [Br1, Section 6] that $\mathrm{Stab}(X)$ has a natural structure as a complex manifold. The map

$$\pi : \mathrm{Stab}(X) \to \mathcal{N}(X) \otimes \mathbf{C}, \tag{7}$$

defined by (6), is holomorphic.

For the fixed heart $\mathcal{A}$ of a bounded t-structure on $D^b(X)$, we write

$$\mathrm{Stab}(\mathcal{A}) := \{\sigma \in \mathrm{Stab}(X) \mid \mathcal{A}_\sigma = \mathcal{A}\}.$$

2.3. $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ action on $\mathrm{Stab}(X)$

Let $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ be the universal cover of $\mathrm{GL}^+(2,\mathbf{R}) = \{T \in \mathrm{GL}(2,\mathbf{R}) \mid \det T > 0\}$. The group $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ can be viewed as the set of pairs (T,f) where $T \in \mathrm{GL}^+(2,\mathbf{R})$ and f is the automorphism of $\mathbf{R} \cong \widetilde{S^1}$ such that f covers the automorphism $\overline{T}$ of $S^1 \cong (\mathbf{R}^2 \backslash 0)/\mathbf{R}_{>0}$ induced by T.

The topological space $\mathrm{Stab}(X)$ carries the right action of the group $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ [Br1, Lemma 8.2] as follows. Given $\sigma \in \mathrm{Stab}(X)$ and $g = (T,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbf{R})$, a new stability condition σg is defined to be the pair $(Z_{\sigma g}, \mathcal{P}_{\sigma g})$ where $Z_{\sigma g} := T^{-1} \circ Z_\sigma$ and $\mathcal{P}_{\sigma g}(\phi) := \mathcal{P}_\sigma(f(\phi))$ for $\phi \in \mathbf{R}$, where we identify $\mathbf{C}$ with $\mathbf{R}^2$ by

$$x + \sqrt{-1}y \mapsto \begin{pmatrix} x \\ y \end{pmatrix}.$$

It is easy to check that the pair $(Z_{\sigma g}, \mathcal{P}_{\sigma g})$ satisfies the properties of Proposition 2.4 (1). Hence by Proposition 2.4 (2), we have $\sigma g = (Z_{\sigma g}, \mathcal{P}_{\sigma g}) \in \mathrm{Stab}(X)$. We remark that the sets of the (semi)stable objects of σ and σg are the same, but the phases have been relabelled. For our purpose, it is convenient to introduce the following definition.

Definition 2.5. Two stability conditions $\sigma, \sigma' \in \mathrm{Stab}(X)$ are said to be $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$-equivalent to each other if σ and σ' are in a single $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ orbit.

For any element $T \in \mathrm{GL}^+(2,\mathbf{R})$, the right $\mathrm{GL}^+(2,\mathbf{R})$ action on $\mathcal{N}(X) \otimes \mathbf{C}$ is defined by $\mathrm{id}_{\mathcal{N}(X)} \otimes T^{-1}$. Hence the $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ acts on $\mathcal{N}(X) \otimes \mathbf{C}$ via the covering map

$$\widetilde{\mathrm{GL}}^+(2,\mathbf{R}) \to \mathrm{GL}^+(2,\mathbf{R}) : (T, f) \mapsto T.$$

The map $\pi : \mathrm{Stab}(X) \to \mathcal{N}(X) \otimes \mathbf{C}$ is equivariant for these $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ actions.

2.4. Moduli functors of Bridgeland semistable objects

For $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$ and $\alpha \in K(X)$, we define a moduli functor

$$\mathcal{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma) : (\mathrm{scheme}/\mathbf{C}) \to (\mathrm{sets}) : S \mapsto \mathcal{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)(S)$$

as follows, where (scheme/$\mathbf{C}$) is the category of schemes of finite type over $\mathbf{C}$ and (sets) is the category of sets. For a scheme S, the set $\mathcal{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)(S)$ consists of isomorphism classes of $E \in D^b(X \times S)$ such that for every closed point $s \in S$ the restriction to the fiber

$$E_s := \mathbf{L}\iota^*_{X \times \{s\}} E$$

is a σ-semistable object in $\mathcal{A}$ with $\mathrm{ch}(E_s) = \mathrm{ch}(\alpha) \in \mathcal{N}(X)$, where $\iota_{X \times \{s\}}$ is the embedding

$$\iota_{X \times \{s\}} : X \times \{s\} \to X \times S.$$

Note that by definition each object E_s belongs to $\mathcal{A} \subset D^b(X)$ for every closed point $s \in S$, so $\mathrm{ch}(E_s) \in \mathcal{N}(X)$ is well-defined. Let $\mathcal{M}^s_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$ be the subfunctor of $\mathcal{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$ corresponding to σ-stable objects of $\mathcal{A}$.

Since the action of $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ does not change the set of (semi)stable objects, for any $g \in \widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ there exists an integer n such that the shift functor $[n]$ gives an isomorphism

$$\mathcal{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma) \cong \mathcal{M}_{D^b(X)}((-1)^n \mathrm{ch}(\alpha), \sigma g) : E \mapsto E[n]. \tag{8}$$

Here we recall the definition of a moduli space. For a scheme Z, we denote by $\underline{Z}$ the functor

$$\underline{Z} : (\mathrm{scheme}/\mathbf{C}) \to (\mathrm{sets}) : S \mapsto \mathrm{Hom}(S, Z).$$

The Yoneda lemma tells us that every natural transformation $\underline{Y} \to \underline{Z}$ is of the form $\underline{f}$ for some morphism $f : Y \to Z$ of schemes, where $\underline{f}$ sends $t \in \underline{Y}(T)$ to $\underline{f}(t) = f \circ t \in \underline{Z}(T)$ for any scheme T. A functor (scheme/$\mathbf{C}$) $\to$ (sets) isomorphic to $\underline{Z}$ is said to be represented by Z.

In the terminology introduced by Simpson [S, Section 1], a *moduli space* is a scheme which 'corepresents' a moduli functor.

Definition 2.6. Let $\mathcal{M} : (\text{scheme}/\mathbf{C}) \to (\text{sets})$ be a functor, M a scheme and $\psi : \mathcal{M} \to \underline{M}$ a natural transformation. We say that M corepresents $\mathcal{M}$ if for each scheme Y and each natural transformation $h : \mathcal{M} \to \underline{Y}$, there exists a unique morphism $\sigma : M \to Y$ such that $h = \underline{\sigma} \circ \psi$:

$$\begin{array}{ccc} \mathcal{M} & & \\ {\scriptstyle\psi}\downarrow & \searrow^{h} & \\ \underline{M} & \xrightarrow[\underline{\sigma}]{} & \underline{Y} \end{array}$$

This characterizes M up to a unique isomorphism. If M represents $\mathcal{M}$ we say that M is a fine moduli space.

For any functor $\mathcal{M} : (\text{scheme}/\mathbf{C}) \to (\text{sets})$, we consider the sheafication of $\mathcal{M}$

$$^{sh}\mathcal{M} : (\text{scheme}/\mathbf{C}) \to (\text{sets})$$

with respect to the Zariski topology. For a scheme S, $^{sh}\mathcal{M}(S)$ is defined as follows. For an open cover $\mathcal{U} = \{U_i\}$ of S, $S = \bigcup U_i$, let $\mathcal{M}_{\mathcal{U}} := \{(E_i) \in \prod \mathcal{M}(U_i) \mid E_i|_{U_i \cap U_j} = E_j|_{U_i \cap U_j}\}$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we have a natural map $\mathcal{M}_{\mathcal{U}} \to \mathcal{M}_{\mathcal{V}}$. The set of open covers forms a direct system with respect to the preorder defined by refinement. We define a functor $\mathcal{M}'$ by

$$\mathcal{M}' : (\text{scheme}/\mathbf{C}) \to \text{Sets} : S \mapsto \mathcal{M}'(S) := \varinjlim_{\mathcal{U}} \mathcal{M}_{\mathcal{U}}. \tag{9}$$

Then $^{sh}\mathcal{M}(S)$ is defined by $^{sh}\mathcal{M} := (\mathcal{M}')'$. Actually, the limit can be computed over affine coverings only, because every covering $\mathcal{U}$ has a refinement which is affine. Since any scheme Y satisfies $\underline{Y} \cong {}^{sh}\underline{Y}$, we have

$$\text{Hom}(\mathcal{M}, \underline{Y}) \cong \text{Hom}({}^{sh}\mathcal{M}, \underline{Y}). \tag{10}$$

In particular, a scheme M corepresents $\mathcal{M}$ if and only if M corepresents $^{sh}\mathcal{M}$.

3. Geometric Bridgeland stability conditions

Let X be a smooth projective surface. In this section, we introduce the notion of geometric Bridgeland stability conditions on $D^b(X)$ and see that if $\sigma \in \text{Stab}(X)$ is geometric, then under suitable assumptions the above functor $\mathcal{M}^s_{D^b(X)}(\text{ch}(\alpha), \sigma)$ (resp. $\mathcal{M}_{D^b(X)}(\text{ch}(\alpha), \sigma)$) is corepresented by a certain moduli space of Gieseker-(semi)stable coherent sheaves on X.

3.1. Twisted Gieseker-stability and μ-stability

We recall the notion of twisted Gieseker-stability and μ-stability. For details, we can consult [HL], [MW]. Take $\gamma, \omega \in \text{NS}(X) \otimes \mathbf{R}$, and suppose that ω is in the ample cone

$$\text{Amp}(X) = \{\omega \in \text{NS}(X) \otimes \mathbf{R} \mid \omega^2 > 0 \text{ and } \omega \cdot C > 0 \text{ for any curve } C \subset X\}.$$

For a coherent sheaf E with $\mathrm{rk}(E) \neq 0$, define $\mu_\omega(E)$ and $\nu_\gamma(E)$ by

$$\mu_\omega(E) := \frac{c_1(E)\cdot\omega}{\mathrm{rk}(E)}, \quad \nu_\gamma(E) := \frac{\mathrm{ch}_2(E)}{\mathrm{rk}(E)} - \frac{c_1(E)\cdot K_X}{2\,\mathrm{rk}(E)} - \frac{c_1(E)\cdot\gamma}{\mathrm{rk}(E)}. \tag{11}$$

Definition 3.1. Let E be a torsion free sheaf.

(i) E is said to be (γ,ω)-semistable if for every proper nonzero subsheaf F of E we have

$$(\mu_\omega(F), \nu_\gamma(F)) \leq (\mu_\omega(E), \nu_\gamma(E)) \tag{12}$$

in the lexicographic order, namely $\mu_\omega(F) < \mu_\omega(E)$ or $\mu_\omega(F) = \mu_\omega(E)$, $\nu_\gamma(F) \leq \nu_\gamma(E)$. E is said to be (γ,ω)-stable if $(\mu_\omega(F), \nu_\gamma(F)) < (\mu_\omega(E), \nu_\gamma(E))$ for any such F.

(ii) E is said to be μ_ω-semistable if $\mu_\omega(F) \leq \mu_\omega(E)$ for any such F. E is said to be μ_ω-stable if in addition $\mu_\omega(F) < \mu_\omega(E)$ for any F with $\mathrm{rk}\,F < \mathrm{rk}\,E$.

(γ,ω)-stability is called twisted Gieseker-stability in [To]. Correspondingly to these semistability notions, every torsion free sheaf E on X has a unique Harder-Narasimhan filtration (cf. [J, Example 4.16 and 4.17]). If

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

is the Harder-Narasimhan filtration with respect to μ_ω-semistability, we define $\mu_{\omega\text{-min}}(E) := \mu_\omega(E_n/E_{n-1})$ and $\mu_{\omega\text{-max}}(E) := \mu_\omega(E_1)$.

Theorem 3.2 (Bogomolov-Gieseker Inequality). *Let X be a smooth projective surface and ω an ample divisor on X. If E is a μ_ω-semistable torsion free sheaf on X, then*

$$c_1^2(E) - 2\,\mathrm{rk}(E)\,\mathrm{ch}_2(E) \geq 0.$$

Proof. See [HL, Theorem 3.4.1]. □

We take $\alpha \in K(X)$ with $\mathrm{rk}(\alpha) > 0$ and consider the moduli functor $\mathscr{M}_X(\mathrm{ch}(\alpha), \gamma, \omega)$ of (γ,ω)-semistable torsion free sheaves E with $\mathrm{ch}(E) = \mathrm{ch}(\alpha) \in \mathrm{NS}(X)$. Let $\mathscr{M}_X^s(\mathrm{ch}(\alpha),\gamma,\omega)$ be the subfunctor of $\mathscr{M}_X(\mathrm{ch}(\alpha),\gamma,\omega)$ corresponding to (γ,ω)-stable ones.

We denote by $M_X(\mathrm{ch}(\alpha),\gamma,\omega)$ the moduli space of (γ,ω)-semistable torsion-free sheaves if it exists. When ω is an integral ample divisor and $\gamma \in \mathrm{NS}(X)\otimes\mathbf{Q}$, the moduli space $M_X(\mathrm{ch}(\alpha),\gamma,\omega)$ exists [MW, Theorem 5.7]. Furthermore if $\gamma = 0$, we write $M_X(\mathrm{ch}(\alpha),\omega)$ instead of $M_X(\mathrm{ch}(\alpha),0,\omega)$ for the sake of simplicity. In this case there is an open subset $M_X^s(\mathrm{ch}(\alpha),\omega)$ of $M_X(\mathrm{ch}(\alpha),\omega)$ that corepresents the functor $\mathscr{M}_X^s(\mathrm{ch}(\alpha),\omega)$ [HL, Theorem 4.3.4].

3.2. Geometric Bridgeland stability conditions

We construct some Bridgeland stability conditions on $D^b(X)$ following [ABL]. For every coherent sheaf E on X, we denote the torsion part of E

by E_{tor} and the torsion free part of E by $E_{\mathrm{fr}} = E/E_{\mathrm{tor}}$. Suppose that $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$ with $\omega \in \mathrm{Amp}(X)$, then we define two full subcategories $\mathfrak{T}$ and $\mathfrak{F}$ of $\mathrm{Coh}(X)$ as follows;

$$\mathrm{ob}(\mathfrak{T}) = \{\text{torsion sheaves}\} \cup \{E \mid E_{\mathrm{fr}} \neq 0 \text{ and } \mu_{\omega\text{-min}}(E_{\mathrm{fr}}) > \beta \cdot \omega\}$$
$$\mathrm{ob}(\mathfrak{F}) = \{E \mid E_{\mathrm{tor}} = 0 \text{ and } \mu_{\omega\text{-max}}(E) \leq \beta \cdot \omega\}.$$

We define a pair $\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathscr{A}_{(\beta,\omega)})$ of the heart $\mathscr{A}_{(\beta,\omega)}$ of a bounded t-structure on $D^b(X)$ and a stability function $Z_{(\beta,\omega)}$ on $\mathscr{A}_{(\beta,\omega)}$ in the following way.

Definition 3.3. A full subcategory $\mathscr{A}_{(\beta,\omega)}$ of $D^b(X)$ is defined as follows;

$$\mathscr{A}_{(\beta,\omega)} := \{E \in D^b(X) \mid \mathscr{H}^i(E) = 0 \text{ for all } i \neq 0, 1 \text{ and}$$
$$\mathscr{H}^0(E) \in \mathfrak{T} \text{ and } \mathscr{H}^{-1}(E) \in \mathfrak{F}\}.$$

The group homomorphism $Z_{(\beta,\omega)}$ is defined by $Z_{(\beta,\omega)}(\alpha) := (\exp(\beta + \sqrt{-1}\omega), \mathrm{ch}(\alpha))_M$, where

$$\exp(\beta + \sqrt{-1}\omega) = \left(1, \beta + \sqrt{-1}\omega, \frac{1}{2}(\beta^2 - \omega^2) + \sqrt{-1}(\beta \cdot \omega)\right) \in \mathscr{N}(X)$$

and $(\cdot,\cdot)_M$ is the Mukai pairing defined in §2.2.

From the general theory called tilting we see that $\mathscr{A}_{(\beta,\omega)}$ is the heart of a bounded t-structure on $D^b(X)$ (for example, see [Br1, §3]). By definition, for $\alpha \in K(X)$ with $\mathrm{ch}(\alpha) = (r, c_1, \mathrm{ch}_2)$ we have

$$Z_{(\beta,\omega)}(\alpha) = -\mathrm{ch}_2 + c_1 \cdot \beta + \frac{r}{2}(\omega^2 - \beta^2) + \sqrt{-1}\omega \cdot (c_1 - r\beta). \tag{13}$$

Furthermore if $r \neq 0$, we can write

$$Z_{(\beta,\omega)}(\alpha) = \frac{1}{2r}((c_1^2 - 2r\,\mathrm{ch}_2) + r^2\omega^2 - (c_1 - r\beta)^2) + \sqrt{-1}\omega(c_1 - r\beta). \tag{14}$$

Our $\sigma_{(\beta,\omega)}$ is slightly different from that in [Br2], [To].

Proposition 3.4 [ABL, Corollary 2.1]. *For each pair $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{Q}$ with $\omega \in \mathrm{Amp}(X)$, $\sigma_{(\beta,\omega)}$ is a Bridgeland stability condition on $D^b(X)$.*

For general $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$, we do not know whether $\sigma_{(\beta,\omega)}$ belongs to $\mathrm{Stab}(X)$ or not since we do not know if $Z_{(\beta,\omega)}$ has the Harder-Narasimhan property. If $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{Q}$ it directly follows from [Br2, Proposition 7.1]. However we consider the following definition.

Definition 3.5. $\sigma \in \mathrm{Stab}(X)$ is called geometric if σ is $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$-equivalent to $\sigma_{(\beta,\omega)}$ for some $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$ with $\omega \in \mathrm{Amp}(X)$.

We have the following criterion due to [Br2] for $\sigma \in \mathrm{Stab}(X)$ to be geometric. It reduces the proof of Theorem 5.1 to easy calculations (§5.2).

Proposition 3.6. *$\sigma \in \mathrm{Stab}(X)$ is geometric if and only if*

1. *For all $x \in X$, the structure sheaves $\mathcal{O}_x$ are σ-stable of the same phase.*
2. *There exist $T \in \mathrm{GL}^+(2,\mathbf{R})$ and $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$ such that $\omega^2 > 0$ and*
$$\pi(\sigma)T = \exp(\beta + \sqrt{-1}\omega),$$
where $\pi : \mathrm{Stab}(X) \to \mathcal{N}(X)$ is defined by (7) and $\mathrm{GL}^+(2,\mathbf{R})$ action on $\mathcal{N}(X) \otimes \mathbf{C}$ is defined in §2.3.

Proof. From [Br2, Lemma 10.1 and Proposition 10.3] the assertion holds because [Br2, Lemma 6.3 and Lemma 10.1] hold for an arbitrary smooth projective surface. However we give the proof of this proposition for the reader's convenience.

The only if part is easy. By [Br2, Lemma 6.3], for any closed point $x \in X$ the structure sheaf $\mathcal{O}_x$ is a simple object of the abelian category $\mathcal{A}_{(\beta,\omega)}$, hence $\sigma_{(\beta,\omega)}$-stable for any $\beta, \omega \in \mathrm{NS}(X)$ with $\omega \in \mathrm{Amp}(X)$. Since $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ action does not change stable objects, $\mathcal{O}_x$ is also σ-stable. Furthermore since the map π is equivariant for $\widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ actions, σ also satisfies condition 2 (cf. §2.3).

Now we consider the if part. We show that $\sigma g = \sigma_{(\beta,\omega)}$ for some $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbf{R})$, where β, ω and T are as in the condition 2. We may assume $\pi(\sigma) = \exp(\beta + \sqrt{-1}\omega)$ for some $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$ with $\omega^2 > 0$. The kernel of the homomorphism $\widetilde{\mathrm{GL}}^+(2,\mathbf{R}) \to \mathrm{GL}^+(2,\mathbf{R})$ acts on $\mathrm{Stab}(X)$ by even shifts, so we may assume furthermore that $\mathcal{O}_x \in \mathcal{P}_\sigma(1)$ for all $x \in X$.

We show that ω is ample. It is enough to show that $C \cdot \omega > 0$ for any curve $C \subset X$. The condition 1 and [Br2, Lemma 10.1(c)] show that the torsion sheaf $\mathcal{O}_C$ lies in the subcategory $\mathcal{P}_\sigma((0,1])$. If $Z_\sigma(\mathcal{O}_C)$ lies on the real axis it follows that $\mathcal{O}_C \in \mathcal{P}_\sigma(1)$ which is impossible by [Br2, Lemma 10.1(b)]. Thus $\mathrm{Im}\, Z_\sigma(\mathcal{O}_C) = C \cdot \omega > 0$.

The same argument of STEP 2 in [Br2, Proposition 10.3] holds and we see that $\mathcal{P}_\sigma((0,1]) = \mathcal{A}_{(\beta,\omega)}$. □

3.3. Moduli spaces corepresenting $\mathcal{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma_{(\beta,\omega)})$ and $\mathcal{M}^s_{D^b(X)}(\mathrm{ch}(\alpha), \sigma_{(\beta,\omega)})$

In this subsection we fix $\alpha \in K(X)$ with $\mathrm{ch}(\alpha) = (r, c_1, \mathrm{ch}_2) \in \mathcal{N}(X)$, $r > 0$ and $\beta \in \mathrm{NS}(X) \otimes \mathbf{R}$, $\omega \in \mathrm{NS}(X)$ with ω ample. We put

$$\varepsilon := \mathrm{Im}\, Z_{(\beta,\omega)}(\alpha) = c_1 \cdot \omega - r\beta \cdot \omega \in \mathbf{R} \tag{15}$$

and $\gamma := \beta - \frac{1}{2}K_X \in \mathrm{NS}(X) \otimes \mathbf{R}$. We take $0 < t \le 1$ and assume that $\sigma_{(\beta,t\omega)} = (Z_{(\beta,t\omega)}, \mathcal{A}_{(\beta,t\omega)})$ satisfies the Harder-Narasimhan property, that is, $\sigma_{(\beta,t\omega)} \in$

$\mathrm{Stab}(X)$. We will show that if $\varepsilon > 0$ is small enough and the moduli space $M_X(\mathrm{ch}(\alpha), \gamma, \omega)$ exists, then it corepresents the moduli functor $\mathcal{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma_{(\beta, t\omega)})$.

LEMMA 3.7. *For any $\sigma_{(\beta,t\omega)}$-semistable object $E \in \mathcal{A}_{(\beta,t\omega)}$ with $[E] = \alpha$, the following hold.*

(1) *Assume that $0 < \varepsilon \le t$ and $\mathrm{Re}\, Z_{(\beta,t\omega)}(\alpha) \ge 0$. Then E is a torsion free sheaf.*

(2) *Furthermore assume that $\varepsilon \le \dfrac{1}{r}$. Then E is a μ_ω-semistable torsion free sheaf.*

Proof. (1) For a contradiction we assume that $\mathcal{H}^{-1}(E) \ne 0$ and take $\mathrm{ch}(\mathcal{H}^{-1}(E)) = (r', c_1', \mathrm{ch}_2') \in \mathcal{N}(X)$. Then there exists an exact sequence in $\mathcal{A}_{(\beta,t\omega)}$,

$$0 \to \mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E) \to 0 \tag{16}$$

and we have

$$Z_{(\beta,t\omega)}(E) = Z_{(\beta,t\omega)}(\mathcal{H}^0(E)) + Z_{(\beta,t\omega)}(\mathcal{H}^{-1}(E)[1]).$$

Since $\mathrm{Im}\, Z_{(\beta,t\omega)}(\mathcal{H}^0(E)) > 0$ and $\mathrm{Im}\, Z_{(\beta,t\omega)}(\mathcal{H}^{-1}(E)[1]) \ge 0$, we get

$$0 \le t\omega \cdot (-c_1' + r'\beta) = \mathrm{Im}\, Z_{(\beta,t\omega)}(\mathcal{H}^{-1}(E)[1]) < \mathrm{Im}\, Z_{(\beta,t\omega)}(E) = t\varepsilon.$$

By the Hodge Index Theorem, we have

$$(-c_1' + r'\beta)^2 < \frac{\varepsilon^2}{\omega^2} \le t^2. \tag{17}$$

Here we assume that $\mathcal{H}^{-1}(E)$ is μ_ω-semistable. Then by Theorem 3.2 we have $-(c_1'^2 - 2r'\,\mathrm{ch}_2') \le 0$. It follows from (14), (17) and $r'^2\omega^2 \in \mathbf{Z}_{>0}$ that

$$\begin{aligned}\mathrm{Re}\, Z_{(\beta,t\omega)}(\mathcal{H}^{-1}(E)[1]) &= \frac{1}{2r'}(-(c_1'^2 - 2r'\,\mathrm{ch}_2') - r'^2t^2\omega^2 + (c_1' - r'\beta)^2) \\ &< \frac{1}{2r'}(-r'^2\omega^2 + 1)t^2 \le 0.\end{aligned}$$

In the general case, $\mathcal{H}^{-1}(E)$ factors into μ_ω-semistable sheaves and we also get the inequality

$$\mathrm{Re}\, Z_{(\beta,t\omega)}(\mathcal{H}^{-1}(E)[1]) < 0.$$

Hence we have $0 < \mu_{\sigma_{(\beta,t\omega)}}(\mathcal{H}^{-1}(E)[1])$.

On the other hand by the assumption that $\mathrm{Re}\, Z_{(\beta,t\omega)}(E) \ge 0$, we have $\mu_{\sigma_{(\beta,t\omega)}}(E) \le 0$. Thus we have $\mu_{\sigma_{(\beta,t\omega)}}(E) < \mu_{\sigma_{(\beta,t\omega)}}(\mathcal{H}^{-1}(E)[1])$. This contradicts the fact that E is $\sigma_{(\beta,t\omega)}$-semistable since $\mathcal{H}^{-1}(E)[1]$ is a subobject of E in $\mathcal{A}_{(\beta,t\omega)}$ by (16). Thus $\mathcal{H}^{-1}(E) = 0$ and E is a sheaf.

Next we show that E is torsion free. We assume that E has a torsion $E_{\mathrm{tor}} \ne 0$. In the case $\dim E_{\mathrm{tor}} = 1$, we have $m := \omega \cdot c_1(E_{\mathrm{tor}}) \ge 1$. Since

$E \in \mathcal{A}_{(\beta, t\omega)}$ we get $t\omega \cdot \beta < \mu_{t\omega}(E_{\text{fr}}) = \dfrac{tc_1 \cdot \omega - mt}{r}$. However by (15), $t\omega \cdot \beta = \dfrac{tc_1 \cdot \omega - t\varepsilon}{r}$. This implies that $\varepsilon > m \geq 1$. This contradicts the assumption that $\varepsilon \leq t \leq 1$. In the case $\dim E_{\text{tor}} = 0$, we get a nonzero subobject E_{tor} of E in $\mathcal{A}_{(\beta, t\omega)}$. However the slope $\mu_{\sigma_{(\beta, t\omega)}}(E_{\text{tor}})$ is infinity and greater than $\mu_{\sigma_{(\beta, t\omega)}}(E)$. This contradicts the fact that E is $\sigma_{(\beta, t\omega)}$-semistable.

(2) By (1), E is a torsion free sheaf. For a contradiction we assume that E is not μ_ω-semistable. Then there exists an exact sequence in $\mathrm{Coh}(X)$

$$0 \to E'' \to E \to E' \to 0.$$

Here E' is a μ_ω-semistable factor of E with the smallest slope $\mu_\omega(E')$. Since $E \in \mathcal{A}_{(\beta, t\omega)}$, we have $t\omega \cdot \beta < \mu_{t\omega\text{-min}}(E) = \mu_{t\omega}(E')$. Hence

$$\mu_{t\omega}(E) - \mu_{t\omega}(E') < \mu_{t\omega}(E) - t\omega \cdot \beta = t\varepsilon/r.$$

On the other hand, since $\mu_\omega(E) - \mu_\omega(E') > 0$ and $\mathrm{rk}(E')c_1 \cdot \omega - rc_1(E') \cdot \omega$ is an integer, we have

$$\mu_\omega(E) - \mu_\omega(E') = \frac{\mathrm{rk}(E')c_1 \cdot \omega - rc_1(E') \cdot \omega}{r\,\mathrm{rk}(E')} > 1/r^2.$$

Hence we get $\varepsilon/r > \mu_\omega(E) - \mu_\omega(E') > 1/r^2$ and this contradicts the assumption that $\varepsilon \leq \dfrac{1}{r}$. Thus E is μ_ω-semistable. □

Next we consider the relationship between $\sigma_{(\beta, t\omega)}$ and the (γ, ω)-stability, where $\gamma = \beta - \frac{1}{2}K_X$. By (13) the slope $\mu_{\sigma_{(\beta, t\omega)}}(E)$ is written as

$$\mu_{\sigma_{(\beta, t\omega)}}(E) = \frac{\nu_\gamma(E) - \frac{1}{2}(t^2\omega^2 - \beta^2)}{t\mu_\omega(E) - t\beta \cdot \omega} \tag{18}$$

for any coherent sheaf $E \in \mathrm{Coh}(X)$ with $\mathrm{rk}(E) \neq 0$.

Theorem 3.8. *Assume that* $0 < \varepsilon \leq \min\left\{t, \dfrac{1}{r}\right\}$ *and* $\mathrm{Re}\, Z_{(\beta, t\omega)}(\alpha) \geq 0$. *Then for* $E \in \mathcal{A}_{(\beta, t\omega)}$ *with* $[E] = \alpha$, E *is* $\sigma_{(\beta, t\omega)}$-*(semi)stable if and only if* E *is a* (γ, ω)-*(semi)stable torsion free sheaf.*

Proof. $\Rightarrow$) From Lemma 3.7, E is a μ_ω-semistable torsion free sheaf. Hence to see that E is (γ, ω)-(semi)stable it is enough to show that for any subsheaf $F \subset E$ with E/F torsion free and $\mu_\omega(F) = \mu_\omega(E)$, the inequality $\nu_\gamma(F) < \nu_\gamma(E)$, (resp. $\leq$) holds. Since E is μ_ω-semistable and $\mu_\omega(F) = \mu_\omega(E/F) = \mu_\omega(E)$, both F and E/F are μ_ω-semistable and belong to $\mathcal{A}_{(\beta, t\omega)}$. Hence the exact sequence in $\mathrm{Coh}(X)$

$$0 \to F \to E \to E/F \to 0$$

is also exact in $\mathcal{A}_{(\beta, t\omega)}$.

Since E is $\sigma_{(\beta,t\omega)}$-(semi)stable, we have $\mu_{\sigma_{(\beta,t\omega)}}(F) < \mu_{\sigma_{(\beta,t\omega)}}(E)$, (resp. $\leq$). By equation (18) we have the desired inequality $\nu_\gamma(F) < \nu_\gamma(E)$, (resp. $\leq$).

$\Leftarrow$) We take an arbitrary exact sequence in $\mathscr{A}_{(\beta,t\omega)}$

$$0 \to K \to E \to Q \to 0 \tag{19}$$

with $K \neq 0$ and $Q \neq 0$. We will show the inequality

$$\mu_{\sigma_{(\beta,t\omega)}}(\mathscr{H}^{-i}(Q)[i]) > \mu_{\sigma_{(\beta,t\omega)}}(E), \quad (\text{resp. } \geq) \tag{20}$$

if $\mathscr{H}^{-i}(Q) \neq 0$ for $i = 0, 1$. Then since $Z_{(\beta,t\omega)}(Q) = Z_{(\beta,t\omega)}(\mathscr{H}^0(Q)) + Z_{(\beta,t\omega)}(\mathscr{H}^{-1}(Q)[1])$, we have the desired inequality

$$\mu_{\sigma_{(\beta,t\omega)}}(Q) > \mu_{\sigma_{(\beta,t\omega)}}(E), \quad (\text{resp. } \geq),$$

showing that E is $\sigma_{(\beta,t\omega)}$-(semi)stable.

First we assume $\mathscr{H}^{-1}(Q) \neq 0$ and show (20). In fact we see that the inequality is always strict. The fact that E is a torsion free sheaf implies that K is also a torsion free sheaf. Hence we have $\operatorname{Im} Z_{(\beta,t\omega)}(K) > 0$. Since

$$\operatorname{Im} Z_{(\beta,t\omega)}(E) = \operatorname{Im} Z_{(\beta,t\omega)}(K) + \operatorname{Im} Z_{(\beta,t\omega)}(\mathscr{H}^0(Q)) + \operatorname{Im} Z_{(\beta,t\omega)}(\mathscr{H}^{-1}(Q)[1]),$$

we see that $0 \leq \operatorname{Im} Z_{(\beta,t\omega)}(\mathscr{H}^{-1}(Q)[1]) < \operatorname{Im} Z_{(\beta,t\omega)}(E) = t\varepsilon$. The same argument as in the proof of Lemma 3.7 (1) shows the strict inequality $\operatorname{Re} Z_{(\beta,t\omega)}(\mathscr{H}^{-1}(Q)[1]) < 0$. Hence by the assumption that $\operatorname{Re} Z_{(\beta,t\omega)}(E) \geq 0$ we have the strict inequality

$$\mu_{\sigma_{(\beta,t\omega)}}(E) < \mu_{\sigma_{(\beta,t\omega)}}(\mathscr{H}^{-1}(Q)[1]).$$

Next we assume $\mathscr{H}^0(Q) \neq 0$. We take the cohomology long exact sequence of (19) in $\operatorname{Coh}(X)$;

$$0 \to \mathscr{H}^{-1}(Q) \to K \to E \to \mathscr{H}^0(Q) \to 0.$$

We take $I := \operatorname{im}(K \to E)$. Since the fact that $K, Q \in \mathscr{A}_{(\beta,t\omega)}$ implies $\mu_{t\omega}(K) > \mu_{t\omega}(\mathscr{H}^{-1}(Q))$, we have $K \not\cong \mathscr{H}^{-1}(Q)$. Hence I is not equal to 0 and is torsion free.

If the strict inequality

$$\mu_\omega(I) < \mu_\omega(E) \tag{21}$$

holds we show a contradiction in the following way. We can write

$$\mu_{t\omega}(E) - \mu_{t\omega}(I) = \frac{t(r(I)c_1 \cdot \omega - rc_1(I) \cdot \omega)}{rr(I)}.$$

By (21) we have $(r(I)c_1 \cdot \omega - rc_1(I) \cdot \omega) \in \mathbf{Z}_{>0}$. Hence we get

$$\mu_{t\omega}(E) - \mu_{t\omega}(I) \geq \frac{t}{r^2}. \tag{22}$$

On the other hand since $K \to I$ is surjective, we have the following inequalities

$$\beta \cdot t\omega < \mu_{t\omega\text{-}\min}(K) \leq \mu_{t\omega}(I).$$

Hence we get

$$\mu_{t\omega}(E) - \mu_{t\omega}(I) < \frac{c_1 \cdot t\omega}{r} - \beta \cdot t\omega = \frac{t\varepsilon}{r} \tag{23}$$

by (15). Combining (22) and (23) with the assumption that $\varepsilon \leq \frac{1}{r}$, we get a contradiction.

In the case $r(I) = r$ and $\dim \mathscr{H}^0(Q) = 1$ we have $\mu_\omega(I) < \mu_\omega(E)$. Hence we may assume that $0 < \mathrm{rk}(I) < \mathrm{rk}(E)$ holds or that $\mathrm{rk}(I) = \mathrm{rk}(E)$ and $\dim(\mathscr{H}^0(Q)) = 0$ holds. In the latter case, we see that the slope $\mu_{\sigma_{(\beta, t\omega)}}(\mathscr{H}^0(Q))$ is infinity and the desired inequality $\mu_{\sigma_{(\beta, t\omega)}}(E) < \mu_{\sigma_{(\beta, t\omega)}}(\mathscr{H}^0(Q))$ holds.

We assume that $\mathrm{rk}(I) < \mathrm{rk}(E)$. Since E is (γ, ω)-(semi)stable,

$$(\mu_\omega(E), \nu_\gamma(E)) < (\mu_\omega(\mathscr{H}^0(Q)), \nu_\gamma(\mathscr{H}^0(Q))), \quad (\text{resp. } \leq).$$

Then since $\mu_\omega(I) = \mu_\omega(E)$ by the above argument, we have

$$\mu_\omega(E) = \mu_\omega(\mathscr{H}^0(Q)) \quad \text{and} \quad \nu_\gamma(E) < \nu_\gamma(\mathscr{H}^0(Q)), \quad (\text{resp. } \leq).$$

Hence by (18) we get the desired inequality $\mu_{\sigma_{(\beta, t\omega)}}(E) < \mu_{\sigma_{(\beta, t\omega)}}(\mathscr{H}^0(Q))$, (resp. $\leq$). □

Here we assume that β belongs to $\mathrm{NS}(X) \otimes \mathbf{Q}$, or that $\gamma = \beta - \frac{1}{2}K_X$ is proportional to ω in $\mathrm{NS}(X) \otimes \mathbf{R}$. In the latter case we have $\mathscr{M}_X(\mathrm{ch}(\alpha), \gamma, \omega) = \mathscr{M}_X(\mathrm{ch}(\alpha), 0, \omega)$ by (11) and (12). We recall that ω is an integral divisor. Hence in both cases we have moduli spaces $M_X(\mathrm{ch}(\alpha), \gamma, \omega)$ of $\mathscr{M}_X(\mathrm{ch}(\alpha), \gamma, \omega)$ by [MW, Theorem 5.7].

Corollary 3.9. *Under the assumptions in the above theorem the moduli space* $M_X(\mathrm{ch}(\alpha), \gamma, \omega)$ *of* (γ, ω)*-semistable sheaves corepresents the moduli functor* $\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma_{(\beta, t\omega)})$. *In the case where* γ *is proportional to* ω, *or* $\gamma = 0$, *the open subset* $M_X^s(\mathrm{ch}(\alpha), \omega) \subset M_X(\mathrm{ch}(\alpha), \omega)$ *corepresents the functor* $\mathscr{M}^s_{D^b(X)}(\mathrm{ch}(\alpha), \sigma_{(\beta, t\omega)})$.

Proof. This follows directly from Theorem 3.8 and [Hu, Lemma 3.31]. □

By this corollary we get Theorem 1.1 in the introduction.

4. Algebraic Bridgeland stability conditions

4.1. Moduli functors of representations of algebras

For a finite dimensional $\mathbf{C}$-algebra B, we consider the abelian category mod-B of finitely generated right B-modules and introduce the notion of θ_B-stability of B-modules and families of B-modules over schemes following [K].

Definition 4.1. Let $\theta_B : K(B) \to \mathbf{R}$ be an additive function on the Grothendieck group $K(B)$. An object $N \in$ mod-B is called θ_B-semistable if $\theta_B(N) = 0$

and every subobject $N' \subset N$ satisfies $\theta_B(N') \geq 0$. Such an N is called θ_B-stable if the only subobjects N' with $\theta_B(N') = 0$ are N and 0.

For $S \in (\text{scheme}/\mathbf{C})$, define $\mathrm{Coh}_B(S)$ to be the category with objects (F, ρ) for F a coherent sheaf on S and $\rho : B \to \mathrm{Hom}_S(F, F)$ a $\mathbf{C}$-linear homomorphism with $\rho(ab) = \rho(b) \circ \rho(a)$ for each $a, b \in B$, and morphisms $\eta : (F, \rho) \to (F', \rho')$ to be morphisms of sheaves $\eta : F \to F'$ with $\eta \circ \rho(a) = \rho'(a) \circ \eta$ in $\mathrm{Hom}_S(F, F')$ for all $a \in B$. It is easy to show $\mathrm{Coh}_B(S)$ is an abelian category. Let $\mathrm{Vec}_B(S)$ be the full subcategory of $\mathrm{Coh}_B(S)$ consisting of objects $(E, \rho) \in \mathrm{Coh}_B(S)$ where E is locally free.

Definition 4.2. [K, Definition 5.1] Objects of $\mathrm{Vec}_B(S)$ are called families of B-modules over S.

For $\alpha_B \in K(B)$ and an additive function $\theta_B : K(B) \to \mathbf{R}$ as in Definition 4.1, let $\mathscr{M}_B(\alpha_B, \theta_B)$ be the moduli functor which sends $S \in (\text{scheme}/\mathbf{C})$ to the set $\mathscr{M}_B(\alpha_B, \theta_B)(S)$ consisting of isomorphism classes of families of θ_B-semistable right B-modules N with $[N] = \alpha_B$. Let $\mathscr{M}_B^s(\alpha_B, \theta_B)$ be the subfunctor of $\mathscr{M}_B(\alpha_B, \theta_B)$ corresponding to θ_B-stable right B-modules. There exist moduli spaces $M_B^s(\alpha_B, \theta_B) \subset M_B(\alpha_B, \theta_B)$ of $\mathscr{M}_B^s(\alpha_B, \theta_B)$ and $\mathscr{M}_B(\alpha_B, \theta_B)$ [K, Proposition 5.2].

Here we recall the definition of the S-equivalence. Since any object of mod-B is finite dimensional $\mathbf{C}$-vector space, any θ_B-semistable B-module N has a filtration, called *Jordan-Hölder filtration*,

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

such that N_i/N_{i-1} is θ_B-stable for any i. The *grading* $Gr_{\theta_B}(N) := \bigoplus_i N_i/N_{i-1}$ does not depend on a choice of a Jordan-Hölder filtration up to isomorphism (for example, see [HL, Proposition 1.5.2]). θ_B-semistable B-modules N and N' are said to be *S-equivalent* if $Gr_{\theta_B}(N) \cong Gr_{\theta_B}(N')$.

Proposition 4.3 (cf. [K, Proposition 3.2]). *For B-modules N and N' with $[N] = [N'] = \alpha_B \in K(B)$, N and N' define the same point of $M_B(\alpha_B, \theta_B)$ if and only if they are S-equivalent to each other.*

4.2. Algebraic Bridgeland stability conditions

Let X be a smooth projective surface. An object $E \in D^b(X)$ is said to be *exceptional* if

$$\mathrm{Hom}^k_{D^b(X)}(E, E) = \begin{cases} \mathbf{C} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

An *exceptional collection* in $D^b(X)$ is a sequence of exceptional objects $\mathfrak{E} = (E_0, \ldots, E_n)$ of $D^b(X)$ such that

$$n \geq i > j \geq 1 \Rightarrow \mathrm{Hom}^k_{D^b(X)}(E_i, E_j) = 0 \quad \text{for all } k \in \mathbf{Z}.$$

The exceptional collection $\mathfrak{E}$ is said to be full if $E_0, \dots, E_n$ generates $D^b(X)$, namely the smallest full triangulated subcategory containing $E_0, \dots, E_n$ coincides with $D^b(X)$. The exceptional collection $\mathfrak{E}$ is said to be strong if for all $1 \le i, j \le n$ one has

$$\mathrm{Hom}^k_{D^b(X)}(E_i, E_j) = 0 \quad \text{for } k \neq 0.$$

We assume that $D^b(X)$ has a full strong exceptional collection $\mathfrak{E} = (E_0, \dots, E_n)$ on $D^b(X)$. We put $\mathscr{E} := E_0 \oplus \cdots \oplus E_n$, $B_{\mathscr{E}} := \mathrm{End}_X(\mathscr{E})$. By Bondal's theorem [Bo] we have an equivalence

$$\Phi_{\mathscr{E}} : D^b(X) \cong D^b(B_{\mathscr{E}}) : E \mapsto \mathbf{R}\,\mathrm{Hom}_X(\mathscr{E}, E).$$

We obtain the heart $\mathscr{A}_{\mathscr{E}} \subset D^b(X)$ by pulling back mod-$B_{\mathscr{E}}$ via the equivalence $\Phi_{\mathscr{E}}$. The equivalence $\Phi_{\mathscr{E}}$ induces an isomorphism $\varphi_{\mathscr{E}} : K(X) \cong K(B_{\mathscr{E}})$ of the Grothendieck groups.

For a stability function Z on $\mathscr{A}_{\mathscr{E}}$ and $\alpha \in K(X)$, we define $\theta^\alpha_Z : K(B_{\mathscr{E}}) \to \mathbf{R}$ by

$$\theta^\alpha_Z(\beta) := \begin{vmatrix} \mathrm{Re}\, Z(\varphi_{\mathscr{E}}^{-1}(\beta)) & \mathrm{Re}\, Z(\alpha) \\ \mathrm{Im}\, Z(\varphi_{\mathscr{E}}^{-1}(\beta)) & \mathrm{Im}\, Z(\alpha) \end{vmatrix} \tag{24}$$

for any $\beta \in K(B_{\mathscr{E}})$. Then for an object $E \in \mathscr{A}_{\mathscr{E}}$ with $[E] = \alpha \in K(X)$, E is Z-(semi)stable if and only if $\Phi_{\mathscr{E}}(E)$ is θ^α_Z-(semi)stable. We also notice that by the existence of full exceptional collection, $K(X)$ is isomorphic to the numerical Grothendieck group $K(X)/K(X)^\perp$. Hence for $E \in D^b(X)$ the class $[E]$ is equal to α in $K(X)$ if and only if $\mathrm{ch}(E) = \mathrm{ch}(\alpha)$.

Proposition 4.4. *The moduli space $M_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta^\alpha_Z)$ (resp. $M^s_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta^\alpha_Z)$) corepresents the moduli functor $\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$ (resp. $\mathscr{M}^s_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$) for any $\alpha \in K(X)$, $\sigma = (Z, \mathscr{A}_{\mathscr{E}}) \in \mathrm{Stab}(\mathscr{A}_{\mathscr{E}})$.*

Proof. We only give a proof for the moduli functor $\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$, since a similar argument also holds for the other moduli functor $\mathscr{M}^s_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$ corresponding to stable objects. We show that

$${}^{sh}\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma) \cong {}^{sh}\mathscr{M}_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta^\alpha_Z). \tag{25}$$

Then, since $M_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta^\alpha_Z)$ corepresents $\mathscr{M}_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta^\alpha_Z)$, the assertion holds by (10). By the remark after (9), to establish (25) it is enough to give a functorial isomorphism

$$\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)(S) \cong \mathscr{M}_{B_{\mathscr{E}}}(\varphi_{\mathscr{E}}(\alpha), \theta^\alpha_Z)(S), \tag{26}$$

for every affine scheme $S = \mathrm{Spec}\, R$. We consider $X_S := X \times S$, projections p and q from X_S to X and S, the pull back $\mathscr{E}_S := p^*\mathscr{E}$ of $\mathscr{E}$ and R-algebra $B_{\mathscr{E}_S} := \mathrm{Hom}_{X_S}(\mathscr{E}_S, \mathscr{E}_S)$. Since $B_{\mathscr{E}_S} \cong R \otimes B_{\mathscr{E}}$, we have mod-$B_{\mathscr{E}_S} \cong \mathrm{Coh}_{B_{\mathscr{E}}}(S)$. From [TU, Lemma 8] we see that via the above identification $\Phi_{\mathscr{E}_S}(\cdot) := \mathbf{R}\,\mathrm{Hom}_{X_S}(\mathscr{E}_S, \cdot)$ gives equivalences

$$D^b(X_S) \cong D^b(\mathrm{Coh}_{B_{\mathscr{E}}}(S)), \quad D^-(X_S) \cong D^-(\mathrm{Coh}_{B_{\mathscr{E}}}(S)).$$

These equivalences are compatible with pull backs, that is, the following diagram is commutative

$$\begin{array}{ccc} D^-(X_S) & \xrightarrow{\Phi_{\mathscr{E}_S}} & D^-(\mathrm{Coh}_{B_{\mathscr{E}}}(S)) \\ {\scriptstyle \mathbf{L}f^*}\downarrow & & \downarrow{\scriptstyle \mathbf{L}f^*} \\ D^-(X_{S'}) & \xrightarrow{\Phi_{\mathscr{E}_{S'}}} & D^-(\mathrm{Coh}_{B_{\mathscr{E}}}(S')) \end{array}$$

for every morphism $f : S' \to S$ of affine schemes. In the following we show that this equivalence $\Phi_{\mathscr{E}_S}$ defines an isomorphism (26).

For any S-valued point E of $\mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)$, by the above diagram the fact that $E \in \mathscr{M}_{D^b(X)}(\mathrm{ch}(\alpha), \sigma)(S)$ implies that $\mathbf{L}\iota_s^* \Phi_{\mathscr{E}_S}(E) \in D^-(\mathrm{Coh}_{B_{\mathscr{E}}}(\{s\})) \cong D^-(B_{\mathscr{E}})$ is a θ_Z^{α}-semistable $B_{\mathscr{E}}$-module for any closed point $s \in S$, where $\iota_s : \{s\} \to S$ is the embedding. By the standard argument using the spectral sequence (for example, [Hu, Lemma 3.31]), we see that $\Phi_{\mathscr{E}_S}(E)$ belongs to $\mathrm{Vec}_{B_{\mathscr{E}}}(S) \subset \mathrm{Coh}_{B_{\mathscr{E}}}(S)$. Hence $\Phi_{\mathscr{E}_S}$ defines a desired map. We see that this map is an isomorphism since $\Phi_{\mathscr{E}_S}^{-1}$ gives the inverse map by a similar argument. □

By this proposition we get Proposition 1.2 in the introduction.

Definition 4.5. $\sigma \in \mathrm{Stab}(X)$ is called an algebraic Bridgeland stability condition associated to the full strong exceptional collection $\mathfrak{E} = (E_0, \dots, E_n)$ if σ is $\widetilde{\mathrm{GL}}^+(2, \mathbf{R})$-equivalent to $(Z, \mathscr{A}_{\mathscr{E}})$ for some $Z : K(X) \to \mathbf{C}$, where $\mathscr{E} = E_0 \oplus \cdots \oplus E_n$.

4.3. Full strong exceptional collections on $\mathbf{P}^2$

In the rest of the paper, we assume that $X = \mathbf{P}^2$ and H is the hyperplane class on $\mathbf{P}^2$. We put $\mathscr{O}_{\mathbf{P}^2}(1) := \mathscr{O}_{\mathbf{P}^2}(H)$ and denote the homogeneous coordinates of $\mathbf{P}^2$ by $[z_0 : z_1 : z_2]$. We introduce two types of full strong exceptional collections $\mathfrak{E}_k$ and $\mathfrak{E}'_k$ on $\mathbf{P}^2$ for each $k \in \mathbf{Z}$ as follows,

$$\mathfrak{E}_k := (\mathscr{O}_{\mathbf{P}^2}(k+1), \Omega^1_{\mathbf{P}^2}(k+3), \mathscr{O}_{\mathbf{P}^2}(k+2)),$$
$$\mathfrak{E}'_k := (\mathscr{O}_{\mathbf{P}^2}(k), \mathscr{O}_{\mathbf{P}^2}(k+1), \mathscr{O}_{\mathbf{P}^2}(k+2)).$$

We put

$$\mathscr{E}_k := \mathscr{O}_{\mathbf{P}^2}(k+1) \oplus \Omega^1_{\mathbf{P}^2}(k+3) \oplus \mathscr{O}_{\mathbf{P}^2}(k+2),$$
$$\mathscr{E}'_k := \mathscr{O}_{\mathbf{P}^2}(k) \oplus \mathscr{O}_{\mathbf{P}^2}(k+1) \oplus \mathscr{O}_{\mathbf{P}^2}(k+2)$$

and $B := \mathrm{End}_{\mathbf{P}^2}(\mathscr{E}_k)$, $B' := \mathrm{End}_{\mathbf{P}^2}(\mathscr{E}'_k)$, which do not depend on k up to natural isomorphism. Using the notation in §4.2, we define functors

$$\Phi_k := \Phi_{\mathscr{E}_k} : D^b(\mathbf{P}^2) \cong D^b(B), \quad \Phi'_k := \Phi_{\mathscr{E}'_k} : D^b(\mathbf{P}^2) \cong D^b(B'),$$

induced isomorphisms $\varphi_k := \varphi_{\mathscr{E}_k} : K(\mathbf{P}^2) \cong K(B)$, $\varphi'_k := \varphi_{\mathscr{E}'_k} : K(\mathbf{P}^2) \cong K(B')$ and full subcategories $\mathscr{A}_k := \mathscr{A}_{\mathscr{E}_k}$, $\mathscr{A}'_k := \mathscr{A}_{\mathscr{E}'_k}$ of $D^b(\mathbf{P}^2)$.

To explain finite dimensional algebras B and B' we introduce some notations. For any $l \in \mathbf{Z}$, we denote by z_i the morphism $\mathcal{O}_{\mathbf{P}^2}(l) \to \mathcal{O}_{\mathbf{P}^2}(l+1)$ defined by multiplication of z_i for $i = 0, 1, 2$. We put $V := \mathbf{C}e_0 \oplus \mathbf{C}e_1 \oplus \mathbf{C}e_2$ and denote i-th projection and i-th embedding by $e_i^* : V \to \mathbf{C}$ and $e_i : \mathbf{C} \to V$ for $i = 0, 1, 2$. We consider the exact sequence for each $k \in \mathbf{Z}$

$$0 \to \Omega^1_{\mathbf{P}^2}(k+3) \xrightarrow{\iota} \mathcal{O}_{\mathbf{P}^2}(k+2) \otimes V \xrightarrow{j} \mathcal{O}_{\mathbf{P}^2}(k+3) \to 0, \tag{27}$$

where we put $j := z_0 \otimes e_0^* + z_1 \otimes e_1^* + z_2 \otimes e_2^*$ and identify $\Omega^1_{\mathbf{P}^2}(k+3)$ with $\ker j$. We define morphisms $p_i : \Omega^1_{\mathbf{P}^2}(k+3) \to \mathcal{O}_{\mathbf{P}^2}(k+2)$ by $p_i := (\mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(k+2)} \otimes e_i^*) \circ \iota$ and $q_i : \mathcal{O}_{\mathbf{P}^2}(k+1) \to \Omega^1_{\mathbf{P}^2}(k+3)$ by $q_i := z_{i+2} \otimes e_{i+1} - z_{i+1} \otimes e_{i+2}$ for $i \in \mathbf{Z}/3\mathbf{Z}$.

We introduce the following quiver Q with 3 vertices $\{v_0, v_1, v_2\}$ and 6 arrows $\{\gamma_0, \gamma_1, \gamma_2, \delta_0, \delta_1, \delta_2\}$

$$\overset{v_0}{\bullet} \xleftarrow{\quad \gamma_i \quad} \overset{v_1}{\bullet} \xleftarrow{\quad \delta_j \quad} \overset{v_2}{\bullet} \qquad (i, j = 0, 1, 2)$$

and consider ideals J and J' of the path algebra $\mathbf{C}Q$ defined as follows. J and J' are two-sided ideals generated by $\{\gamma_i\delta_j + \gamma_j\delta_i \mid i, j = 0, 1, 2\}$ and $\{\gamma_i\delta_j - \gamma_j\delta_i \mid i, j = 0, 1, 2\}$, respectively. We have isomorphisms

$$\rho : \mathbf{C}Q/J \cong B : \gamma_i, \delta_j \mapsto p_i, q_j, \quad \rho' : \mathbf{C}Q/J' \cong B' : \gamma_i, \delta_j \mapsto z_i, z_j. \tag{28}$$

These isomorphisms ρ and ρ' map vertices $v_0, v_1, v_2 \in \mathbf{C}Q/J$ (resp. $\mathbf{C}Q/J'$) to idempotent elements

$$\rho(v_0) = \mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(k+2)}, \quad \rho(v_1) = \mathrm{id}_{\Omega^1_{\mathbf{P}^2}(k+3)}, \quad \rho(v_2) = \mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(k+1)} \in B$$
$$(\text{resp. } \rho'(v_0) = \mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(k+2)}, \rho'(v_1) = \mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(k+1)}, \rho'(v_2) = \mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(k)} \in B').$$

They also map $\gamma_i, \delta_j \in \mathbf{C}Q/J$ (resp. $\mathbf{C}Q/J'$) to

$$\rho(\gamma_i) = p_i, \quad \rho(\delta_j) = q_j \in B \quad (\text{resp. } \rho'(\gamma_i) = z_i, \rho'(\delta_j) = z_j \in B')$$

for $i, j = 0, 1, 2$. We identify B and B' with $\mathbf{C}Q/J$ and $\mathbf{C}Q/J'$ via isomorphisms ρ and ρ'.

For any finitely generated right B-module N, we consider the right action on N of a path p of Q as a pull back by p and denote it by p^*. Notice that vertices $v_i's$ are regarded as paths with the length 0. We have the decomposition $N = Nv_0^* \oplus Nv_1^* \oplus Nv_2^*$ as a vector space. This gives the dimension vector $\underline{\dim}(N) = (\dim_{\mathbf{C}} Nv_0^*, \dim_{\mathbf{C}} Nv_1^*, \dim_{\mathbf{C}} Nv_2^*)$ of N and an isomorphism $\underline{\dim} : K(B) \cong \mathbf{Z}^{\oplus 3}$. The B-module structure of N is written as;

$$Nv_0^* \xrightarrow{\gamma_i^*} Nv_1^* \xrightarrow{\delta_j^*} Nv_2^* \quad (i, j = 0, 1, 2).$$

We sometimes use notation $\gamma_i^*|_N$ and $\delta_j^*|_N$ to avoid confusion. We define B-modules $\mathbf{C}v_i$ for $i = 0, 1, 2$ as follows. As vector spaces $\mathbf{C}v_i = \mathbf{C}$ and can be decomposed by $(\mathbf{C}v_i)v_i^* = \mathbf{C}$, $(\mathbf{C}v_i)v_j^* = 0$ for $j \neq i$. Actions of B are defined in obvious way. They are simple objects of mod-B and we have

$$\text{mod-}B = \langle \mathbf{C}v_0, \mathbf{C}v_1, \mathbf{C}v_2 \rangle \tag{29}$$

as a full subcategory of $D^b(B)$. Similar results hold for B' and we use similar notations for B'.

Since $\mathcal{O}_{\mathbf{P}^2}(k-1)[2]$, $\mathcal{O}_{\mathbf{P}2}(k)[1]$ and $\mathcal{O}_{\mathbf{P}^2}(k+1)$ correspond to B-modules $\mathbf{C}v_0$, $\mathbf{C}v_1$ and $\mathbf{C}v_2$ via Φ_k, we have

$$\mathcal{A}_k = \langle \mathcal{O}_{\mathbf{P}^2}(k-1)[2], \mathcal{O}_{\mathbf{P}2}(k)[1], \mathcal{O}_{\mathbf{P}^2}(k+1)\rangle.$$

Similarly we have

$$\mathcal{A}_k' = \langle \mathcal{O}_{\mathbf{P}^2}(k-1)[2], \Omega^1_{\mathbf{P}2}(k+1)[1], \mathcal{O}_{\mathbf{P}^2}(k)\rangle.$$

On the other hand, $\mathcal{O}_{\mathbf{P}^2}(k+1)$, $\Omega^1_{\mathbf{P}^2}(k+3)$ and $\mathcal{O}_{\mathbf{P}^2}(k+2)$ correspond to B-modules B, v_1B and v_2B via Φ_k. Similarly $\mathcal{O}_{\mathbf{P}^2}(k)$, $\mathcal{O}_{\mathbf{P}^2}(k+1)$ and $\mathcal{O}_{\mathbf{P}^2}(k+2)$ correspond to B'-modules B', v_1B' and v_2B' via Φ_k'. They are projective modules and we can compute Ext groups by using them. Hence we get the following lemma.

Lemma 4.6. *For bounded complexes E, F of coherent sheaves on $\mathbf{P}^2$, the following hold for each $k \in \mathbf{Z}$.*

(1) *By E^i, we denote each term of complex E. We assume that* (i) *E^i is a direct sum of $\mathcal{O}_{\mathbf{P}^2}(k+1)$, $\Omega^1_{\mathbf{P}^2}(k+3)$ and $\mathcal{O}_{\mathbf{P}^2}(k+2)$ for any $i \in \mathbf{Z}$ and F belongs to $\mathcal{A}_k$, or that* (ii) *E^i is a direct sum of $\mathcal{O}_{\mathbf{P}^2}(k)$, $\mathcal{O}_{\mathbf{P}^2}(k+1)$ and $\mathcal{O}_{\mathbf{P}^2}(k+2)$ for any $i \in \mathbf{Z}$ and F belongs to $\mathcal{A}_k'$. Then the complex $\mathbf{R}\operatorname{Hom}_{\mathbf{P}^2}(E,F)$ is quasi-isomorphic to the following complex*

$$\cdots \to \operatorname{Hom}_{D^b(\mathbf{P}^2)}(E^{-i},F) \xrightarrow{d^i} \operatorname{Hom}_{D^b(\mathbf{P}^2)}(E^{-i-1},F) \to \cdots, \tag{30}$$

where $\operatorname{Hom}_{D^b(\mathbf{P}^2)}(E^{-i},F)$ lies on degree i and d^i is defined by

$$d^i(f) := f \circ d_E^{-i-1} : E^{-i-1} \to F \quad \text{for } f \in \operatorname{Hom}_{\mathbf{P}^2}(E^{-i},F).$$

In particular, we have $\operatorname{Hom}_{D^b(\mathbf{P}^2)}(E,F[i]) \cong \ker d^i/\operatorname{im} d^{i-1}$

(2) *If E belongs to $\mathcal{A}_k$ (resp. $\mathcal{A}_k'$), then we have the following isomorphism in $D^b(\mathbf{P}^2)$*

$$E \cong (\mathcal{O}_{\mathbf{P}^2}(k-1)^{\oplus a_0} \to \mathcal{O}_{\mathbf{P}^2}(k)^{\oplus a_1} \to \mathcal{O}_{\mathbf{P}^2}(k+1)^{\oplus a_2}),$$
$$(\text{resp. } E \cong (\mathcal{O}_{\mathbf{P}^2}(k-1)^{\oplus a_0} \to \Omega^1_{\mathbf{P}^2}(k+1)^{\oplus a_1} \to \mathcal{O}_{\mathbf{P}^2}(k)^{\oplus a_2})),$$

where $(a_0,a_1,a_2) \in \mathbf{Z}^3_{\geq 0}$ and $\mathcal{O}_{\mathbf{P}^2}(k+1)^{\oplus a_2}$ (resp. $\mathcal{O}_{\mathbf{P}^2}(k)^{\oplus a_2}$) lies on degree 0.

Proof. (1) We only prove (i). We put $N := \Phi_k(E)$, $M := \Phi_k(F)$. Then by the assumption the each term N^i of the complex N is a direct sum of B, v_1B and v_2B for any i. Hence N^i is a projective module. Furthermore since the fact $F \in \mathcal{A}_k$ implies that M is a B-module, $\mathbf{R}\operatorname{Hom}_{\mathbf{P}^2}(E,F) \cong \mathbf{R}\operatorname{Hom}_B(N,M)$ is quasi-isomorphic to the following complex

$$\cdots \to \operatorname{Hom}_B(N^{-i},M) \xrightarrow{d^i} \operatorname{Hom}_B(N^{-i-1},M) \to \cdots.$$

Via Φ_k this complex coincides with (30).

(2) For any object $E \in \mathscr{A}_k$ we consider the B-module $N = \Phi_k(E)$. If we put $\underline{\dim}(N) = (a_0, a_1, a_2)$, then N can be obtained by extensions

$$0 \to (\mathbf{C}v_1)^{\oplus a_1} \to N' \to (\mathbf{C}v_0)^{\oplus a_0} \to 0, \tag{31}$$

$$0 \to (\mathbf{C}v_2)^{\oplus a_2} \to N \to N' \to 0. \tag{32}$$

Since $\Phi_k(\mathscr{O}_{\mathbf{P}^2}(k-1)[1]) = \mathbf{C}v_0[-1]$ and $\Phi_k(\mathscr{O}_{\mathbf{P}^2}(k)[1]) = \mathbf{C}v_1$, we have a homomorphism

$$f : \mathscr{O}_{\mathbf{P}^2}(k-1)^{\oplus a_0} \to \mathscr{O}_{\mathbf{P}^2}(k)^{\oplus a_1}$$

in $\mathrm{Coh}(\mathbf{P}^2)$ such that $\Phi_k(C(f)[1]) \cong N'$, where $C(f)$ is the mapping cone of f. From (32) E can be obtained as a mapping cone of a certain homomorphism in $\mathrm{Hom}_{D^b(\mathbf{P}^2)}(C(f), \mathscr{O}_{\mathbf{P}^2}(k+1)^{\oplus a_2})$, since $\Phi_k(\mathscr{O}_{\mathbf{P}^2}(k+1)) = \mathbf{C}v_2$. By (1) this homomorphism is identified with a homomorphism

$$g : \mathscr{O}_{\mathbf{P}^2}(k)^{\oplus a_1} \to \mathscr{O}_{\mathbf{P}^2}(k+1)^{\oplus a_2}$$

in $\mathrm{Coh}(\mathbf{P}^2)$ satisfying $g \circ f = 0$. Thus E is isomorphic to the following complex

$$(\mathscr{O}_{\mathbf{P}^2}(k-1)^{\oplus a_0} \xrightarrow{f} \mathscr{O}_{\mathbf{P}^2}(k)^{\oplus a_1} \xrightarrow{g} \mathscr{O}_{\mathbf{P}^2}(k+1)^{\oplus a_2}),$$

where $\mathscr{O}_{\mathbf{P}^2}(k+1)^{\oplus a_2}$ lies on degree 0. $\square$

The vector $(a_0, a_1, a_2) \in \mathbf{Z}^3_{\geq 0}$ in Lemma 4.6 (2) coincides with $\underline{\dim}(\Phi_k(E))$ and is explicitly computed from $\mathrm{ch}(E) = (r, sH, \mathrm{ch}_2)$. For example, we assume that E belongs to $\mathscr{A}_1$. Since

$$\mathrm{ch}(\mathscr{O}_{\mathbf{P}^2}[2]) = (1,0,0),\ \mathrm{ch}(\mathscr{O}_{\mathbf{P}^2}(1)[1]) = -\left(1, H, \frac{1}{2}\right),\ \mathrm{ch}(\mathscr{O}_{\mathbf{P}^2}(2)) = (1, 2H, 2), \tag{33}$$

we have $(a_0, a_1, a_2) = r(1,0,0) - \frac{s}{2}(3,4,1) + \mathrm{ch}_2(1,2,1)$.

5. Proof of Main Theorem 1.3

In this section we fix $\alpha \in K(\mathbf{P}^2)$ with $\mathrm{ch}(\alpha) = (r, sH, \mathrm{ch}_2)$ and $0 < s \leq r$. In the sequel, we sometimes identify $\mathrm{NS}(\mathbf{P}^2)$ with $\mathbf{Z}$ by the isomorphism $\mathrm{NS}(\mathbf{P}^2) \cong \mathbf{Z} : \beta \mapsto \beta \cdot H$.

5.1. Wall-and-chamber structure

We consider the full strong exceptional collection $\mathfrak{E}_1 = (\mathscr{O}_{\mathbf{P}^2}(2), \Omega^1_{\mathbf{P}^2}(4), \mathscr{O}_{\mathbf{P}^2}(3))$ on $\mathbf{P}^2$, the equivalence $\Phi_1(\cdot) = \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathscr{E}_1, \cdot) : D^b(\mathbf{P}^2) \cong D^b(B)$ and the induced isomorphism $\varphi_1 : K(\mathbf{P}^2) \cong K(B)$, where $\mathscr{E}_1 = \mathscr{O}_{\mathbf{P}^2}(2) \oplus \Omega^1_{\mathbf{P}^2}(4) \oplus \mathscr{O}_{\mathbf{P}^2}(3)$ and $B = \mathrm{End}_{\mathbf{P}^2}(\mathscr{E}_1)$. We consider the plane $\varphi_1(\alpha)^{\perp} := \{\theta_1 \in \mathrm{Hom}_{\mathbf{Z}}(K(B), \mathbf{R}) \mid \theta_1(\varphi_1(\alpha)) = 0\}$ and define a subset $W_1 \subset \varphi_1(\alpha)^{\perp}$ as follows. A subset W_1 consists of elements $\theta_1 \in \varphi_1(\alpha)^{\perp}$ satisfying that there exists a θ_1-semistable B-module N with $[N] = \varphi_1(\alpha)$ such that N has a proper nonzero submodule $N' \subset N$ with $\theta_1(N') = 0$ and $[N'] \notin \mathbf{Q}_{>0}\varphi_1(\alpha)$ in $K(B)$. The subset W_1 is a union of finitely many rays in $\varphi_1(\alpha)^{\perp}$. These rays are called walls and the connected components of $\varphi_1(\alpha)^{\perp}\backslash W_1$ are called chambers.

We take a line l_1 in $\varphi_1(\alpha)^{\perp}$ defined by $l_1 := \{\theta_1 \in \varphi_1(\alpha)^{\perp} \mid \theta_1(\varphi_1(\mathcal{O}_x)) = 0\}$, where $\mathcal{O}_x$ is the structure sheaf of a point $x \in \mathbf{P}^2$. We take a chamber $C^{\mathbf{P}^2}_{\varphi_1(\alpha)} \subset \varphi_1(\alpha)^{\perp}$, if any, such that the closure intersects with l_1 and there exists an element $\theta_1 \in C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ satisfying the inequality $\theta_1(\varphi_1(\mathcal{O}_x)) > 0$ and $M_B(-\varphi_1(\alpha), \theta_1) \neq \emptyset$. These conditions characterize $C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ uniquely.

We have the following theorem, which gives a proof of (i) in Main Theorem 1.3. The proof of Theorem 5.1 in the next subsection shows that if there is not such a chamber $C^{\mathbf{P}^2}_{\varphi_1(\alpha)} \subset \varphi_1(\alpha)^{\perp}$, then $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) = \emptyset$.

Theorem 5.1. *The map $E \mapsto \Phi_1(E[1])$ gives an isomorphism*

$$M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_1(\alpha), \theta_1)$$

for any $\theta_1 \in C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$. This isomorphism keeps open subsets consisting of stable objects.

Here we remark that if we assume $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \neq \emptyset$, then $\dim M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) = s^2 - r^2 + 1 - 2r\,\mathrm{ch}_2 \geq 0$. Hence we have $\mathrm{ch}_2 \leq \frac{1}{2}$. We see that $\mathrm{ch}_2 = \frac{1}{2}$ if and only if $\mathrm{ch}(\alpha) = \left(1, 1, \frac{1}{2}\right)$.

5.2. Proof of Theorem 5.1

We will find Bridgeland stability conditions σ in $\mathrm{Stab}(\mathcal{A}_1) \cap \{\sigma_{(bH, tH)} \in \mathrm{Stab}(\mathbf{P}^2) \mid t > 0\}\,\widetilde{\mathrm{GL}}^+(2, \mathbf{R})$ for suitable $b \in \mathbf{R}$ and obtain Theorem 5.1.

We put $\mathbf{H} = \{r \exp(\sqrt{-1}\pi\phi) \mid r > 0 \text{ and } 0 < \phi \leq 1\}$ the strict upper half-plane and $F_0 = \mathcal{O}_{\mathbf{P}^2}[2]$, $F_1 = \mathcal{O}_{\mathbf{P}^2}(1)[1]$ and $F_2 = \mathcal{O}_{\mathbf{P}^2}(2)$. The full subcategory $\mathcal{A}_1$ of $D^b(\mathbf{P}^2)$ is generated by F_0, F_1 and F_2,

$$\mathcal{A}_1 = \langle \mathcal{O}_{\mathbf{P}^2}[2], \mathcal{O}_{\mathbf{P}^2}(1)[1], \mathcal{O}_{\mathbf{P}^2}(2) \rangle. \tag{34}$$

Since $K(\mathbf{P}^2) = \mathbf{Z}[F_0] \oplus \mathbf{Z}[F_1] \oplus \mathbf{Z}[F_2]$, a stability function Z on $\mathcal{A}_1$ is identified with the element $(Z(F_0), Z(F_1), Z(F_2))$ of $\mathbf{H}^3$. Furthermore since the category $\mathcal{A}_1 \cong \text{mod-}B$ has finite length, all stability functions on $\mathcal{A}_1$ satisfy the Harder-Narasimhan property. Hence $\mathrm{Stab}(\mathcal{A}_1) \cong \mathbf{H}^3$.

For $\sigma = (Z, \mathcal{A}_1) \in \mathrm{Stab}(\mathcal{A}_1)$, we put $Z(F_i) = x_i + \sqrt{-1}y_i \in \mathbf{H}^3$ and consider the conditions for σ to be geometric. In the next lemmas we consider the condition 1 of Proposition 3.6. For any point $x \in \mathbf{P}^2$ we take a resolution of $\mathcal{O}_x$

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 2} \to \mathcal{O}_{\mathbf{P}^2}(2) \to \mathcal{O}_x \to 0. \tag{35}$$

Hence from (34) we have $\mathcal{O}_x \in \mathcal{A}_1$ and $[\mathcal{O}_x] = [F_0] + 2[F_1] + [F_2] \in K(\mathbf{P}^2)$.

Lemma 5.2. *For any subobject E of $\mathcal{O}_x$ in $\mathcal{A}_1$, the class $[E]$ in $K(\mathbf{P}^2)$ is equal to $[F_2]$, $[F_1] + [F_2]$ or $2[F_1] + [F_2]$.*

Proof. If the conclusion is not true, we can find a subobject $\mathcal{F}[i] \subset \mathcal{O}_x$ in $\mathcal{A}_1$ with $\mathcal{F}$ a nonzero sheaf on $\mathbf{P}^2$ and $i = 1$ or 2; for example, if E is a subobject of $\mathcal{O}_x$ in $\mathcal{A}_1$ and $[E] = [F_0] + [F_1] + [F_2]$ in $K(\mathbf{P}^2)$, then by Lemma 4.6 (2), E is written as

$$E = (\mathcal{O}_{\mathbf{P}^2} \xrightarrow{f} \mathcal{O}_{\mathbf{P}^2}(1) \xrightarrow{g} \mathcal{O}_{\mathbf{P}^2}(2)).$$

If $g = 0$ and $f \neq 0$, then $E = \mathcal{O}_\ell(1)[1] \oplus \mathcal{O}_{\mathbf{P}^2}(2)$, where ℓ is a line on $\mathbf{P}^2$ determined by $\mathcal{O}_\ell(1) = \operatorname{coker} f$. If $g = f = 0$, then $E = \mathcal{O}_{\mathbf{P}^2}[2] \oplus \mathcal{O}_{\mathbf{P}^2}(1)[1] \oplus \mathcal{O}_{\mathbf{P}^2}(2)$. If $g \neq 0$, then we have a distinguished triangle

$$\mathcal{O}_{\ell'}(2) \to E \to \mathcal{O}_{\mathbf{P}^2}[2] \to \mathcal{O}_{\ell'}(2)[1]$$

for a line ℓ' on $\mathbf{P}^2$ determined by $\mathcal{O}_{\ell'}(2) = \operatorname{coker} g$. The fact that $\operatorname{Hom}_{D^b(\mathbf{P}^2)}(\mathcal{O}_{\mathbf{P}^2}[2], \mathcal{O}_{\ell'}(2)[1]) = 0$ implies $E = \mathcal{O}_{\mathbf{P}^2}[2] \oplus \mathcal{O}_{\ell'}(2)$.

However the fact that $\operatorname{Hom}_{D^b(\mathbf{P}^2)}(\mathcal{F}[i], \mathcal{O}_x) = 0$ for $i \geq 1$ contradicts the fact that $\mathcal{F}[i]$ is a nonzero subobject of $\mathcal{O}_x$ in $\mathcal{A}_1$. □

LEMMA 5.3. *For* $\sigma = (Z, \mathcal{A}_1) \in \operatorname{Stab}(\mathcal{A}_1)$, $\mathcal{O}_x$ *is* σ*-stable for each* $x \in \mathbf{P}^2$ *if and only if* (a), (b) *and* (c) *hold*;

$$\text{(a)} \begin{vmatrix} x_2 & x_0 + 2x_1 + x_2 \\ y_2 & y_0 + 2y_1 + y_2 \end{vmatrix} > 0, \quad \text{(b)} \begin{vmatrix} x_1 + x_2 & x_0 + 2x_1 + x_2 \\ y_1 + y_2 & y_0 + 2y_1 + y_2 \end{vmatrix} > 0,$$

$$\text{(c)} \begin{vmatrix} 2x_1 + x_2 & x_0 + 2x_1 + x_2 \\ 2y_1 + y_2 & y_0 + 2y_1 + y_2 \end{vmatrix} > 0.$$

Proof. By Lemma 5.2, it is enough to show $\phi(\beta) < \phi(\mathcal{O}_x)$ for each $\beta = [F_2]$, $[F_1] + [F_2]$, $2[F_1] + [F_2]$, where $\phi(\beta)$ is the phase of $Z(\beta) \in \mathbf{C}$. It is equivalent to

$$\begin{vmatrix} \operatorname{Re} Z(\beta) & \operatorname{Re} Z(\mathcal{O}_x) \\ \operatorname{Im} Z(\beta) & \operatorname{Im} Z(\mathcal{O}_x) \end{vmatrix} > 0,$$

which is equivalent to (a), (b) and (c) for the case $\beta = [F_2]$, $[F_1] + [F_2]$ and $2[F_1] + [F_2]$ respectively. Hence the assertion follows. □

By Lemma 5.3 and some easy calculations, we can find Bridgeland stability conditions $\sigma^b = (Z^b, \mathcal{A}_1)$ with $0 < b < 1$ which satisfy the conditions 1 and 2 in Proposition 3.6 as follows. We put $x_0 := -b$, $x_1 := -1 + b$, $x_2 := -3b + 3$ and $y_0 = y_1 = 0$, $y_2 = 1$, that is,

$$Z^b(F_0) := -b, \quad Z^b(F_1) := -1 + b, \quad Z^b(F_2) := -3b + 3 + \sqrt{-1}. \tag{36}$$

$\sigma^b = (Z^b, \mathcal{A}_1) \in \operatorname{Stab}(\mathbf{P}^2)$ satisfies the conditions (a), (b) and (c) in Lemma 5.3. The vector $\pi(\sigma^b)$ is written as

$$\pi(\sigma^b) = u + \sqrt{-1}v \in \mathcal{N}(\mathbf{P}^2) \otimes \mathbf{C}$$

with $u = \left(2b - 1, \left(b + \frac{1}{2}\right)H, b\right)$, $v = \left(-1, -\frac{1}{2}H, 0\right) \in \mathcal{N}(\mathbf{P}^2)$. If we put

$$T^{-1} := \begin{pmatrix} b - \frac{1}{2} & 2b^2 - 2b - \frac{1}{2} \\ \sqrt{b - b^2} & (2b - 1)\sqrt{b - b^2} \end{pmatrix} \in \mathrm{GL}^+(2, \mathbf{R}),$$

then $\pi(\sigma^b)T = \exp(bH + \sqrt{-1}\sqrt{b-b^2}H)$;

$$\begin{pmatrix} b-\frac{1}{2} & 2b^2-2b-\frac{1}{2} \\ \sqrt{b-b^2} & (2b-1)\sqrt{b-b^2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & bH & b^2-\frac{1}{2}b \\ 0 & \sqrt{b-b^2}H & b\sqrt{b-b^2} \end{pmatrix}.$$

Hence σ^b also satisfies the condition 2 of Proposition 3.6 and $\sigma^b \in \mathrm{Stab}(\mathbf{P}^2)$ is geometric. The proof of Proposition 3.6 implies that there exists a lift $g \in \widetilde{\mathrm{GL}}^+(2,\mathbf{R})$ of $T \in \mathrm{GL}^+(2,\mathbf{R})$ such that $\pi(\sigma^b g) = \pi(\sigma^b)T$ and

$$\sigma^b g = \sigma_{(bH,tH)}, \tag{37}$$

where we put $t = \sqrt{b-b^2}$. We fix $\alpha \in K(\mathbf{P}^2)$ with $\mathrm{ch}(\alpha) = (r, sH, \mathrm{ch}_2)$, $0 < s \le r$. By the remark after Main Theorem 5.1 we may assume that $\mathrm{ch}_2 \le \frac{1}{2}$. We choose $0 < b < \dfrac{s}{r}$ such that $\alpha \in K(\mathbf{P}^2)$ and $\sigma_{(bH,tH)} = (Z_{(bH,tH)}, \mathscr{A}_{(bH,tH)})$ satisfy the conditions in Theorem 3.8;

$$0 < \varepsilon = \mathrm{Im}\, Z_{(bH,tH)}(\alpha) = s - rb \le \min\left\{t = \sqrt{b-b^2}, \frac{1}{r}\right\} \tag{38}$$

$$\text{and} \quad \mathrm{Re}\, Z_{(bH,tH)}(\alpha) = -\mathrm{ch}_2 + r/2(b-2b^2) + sb \ge 0.$$

In the following we assume that $s/r - b > 0$ is small enough such that these inequalities are satisfied. Then by Corollary 3.9 we have

$$\mathscr{M}_{D^b(\mathbf{P}^2)}(\mathrm{ch}(\alpha), \sigma_{(bH,tH)}) \cong \mathscr{M}_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H). \tag{39}$$

Since $\sigma^b g = \sigma_{(bH,tH)}$, by (8) we see that the shift functor $\cdot\,[n]$ gives an isomorphism

$$\mathscr{M}_{D^b(\mathbf{P}^2)}(\mathrm{ch}(\alpha), \sigma_{(bH,tH)}) \cong \mathscr{M}_{D^b(\mathbf{P}^2)}((-1)^n\, \mathrm{ch}(\alpha), \sigma^b) : E \mapsto E[n] \tag{40}$$

for some $n \in \mathbf{Z}$. We show that $n = 1$. First notice that $\alpha = a_0[F_0] + a_1[F_1] + a_2[F_2] \in K(\mathbf{P}^2)$, where $(a_0, a_1, a_2) \in \mathbf{Z}^3$ is defined by

$$a_0 := r - \frac{3}{2}s + \mathrm{ch}_2$$
$$a_1 := -2s + 2\,\mathrm{ch}_2$$
$$a_2 := -\frac{s}{2} + \mathrm{ch}_2.$$

For every $\mathbf{C}$-valued point E of $\mathscr{M}_{D^b(\mathbf{P}^2)}(\mathrm{ch}(\alpha), \sigma_{(bH,tH)})$, by Lemma 4.6 (2), $E[n]$ is written as

$$E[n] \cong (\mathscr{O}_{\mathbf{P}^2}^{(-1)^n a_0} \to \mathscr{O}_{\mathbf{P}^2}(1)^{(-1)^n a_1} \to \mathscr{O}_{\mathbf{P}^2}(2)^{(-1)^n a_2}) \in \mathscr{A}_1, \tag{41}$$

where $\mathscr{O}_{\mathbf{P}^2}(2)^{(-1)^n a_2}$ lies on degree 0. The conditions that $0 < s \le r$ and $\mathrm{ch}_2 \le \frac{1}{2}$ imply that $a_2 \le 0$ and that $a_2 = 0$ if and only if $\mathrm{ch}(\alpha) = (1, 1, \frac{1}{2})$. In the case $a_2 < 0$, the form (41) of $E[n]$ implies $n = 1$ since E is a sheaf. In the case $a_2 = 0$, we have $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) = \{\mathscr{O}_{\mathbf{P}^2}(1)\}$. Since $\mathscr{O}_{\mathbf{P}^2}(1)[1] \in \mathscr{A}_1$, we also have $n = 1$.

On the other hand we define $\theta^{\alpha}_{Z^b} : K(B) \to \mathbf{R}$ by (24) using $\varphi_1 : K(\mathbf{P}^2) \cong K(B)$. Then by Proposition 4.4 the moduli functor $\mathscr{M}_{D^b(\mathbf{P}^2)}(-\mathrm{ch}(\alpha), \sigma^b)$ is co-represented by the moduli scheme $M_B(-\varphi_1(\alpha), \theta^{\alpha}_{Z^b})$. Combining this with the above isomorphisms (39) and (40) with $n = 1$ we have an isomorphism

$$M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_1(\alpha), \theta^{\alpha}_{Z^b}) : E \mapsto \Phi_1(E[1]). \tag{42}$$

Isomorphisms (39) and (40) hold for moduli functors corresponding to stable objects. Hence the isomorphism (42) keeps open subsets of stable objects.

Finally we see that if $s/r - b > 0$ is small enough, this $\theta^{\alpha}_{Z^b}$ belongs to $C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ in the Main Theorem as follows. The above isomorphism (42) implies that if $s/r - b > 0$ is small enough, $\theta^{\alpha}_{Z^b}$ belongs to the same chamber $C_{\varphi_1(\alpha)}$. This chamber $C_{\varphi_1(\alpha)}$ satisfies the desired conditions. In fact we have $\theta^{\alpha}_{Z^b}(\varphi_1(\mathscr{O}_x)) > 0$ for $b < s/r$ and $\theta^{\alpha}_{Z^{s/r}}(\varphi_1(\mathscr{O}_x)) = 0$, furthermore $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \neq \emptyset$ implies $M_B(-\varphi_1(\alpha), \theta_1) \neq \emptyset$ for $\theta_1 \in C_{\varphi_1(\alpha)}$ because of the isomorphism (42). This completes the proof of Main Theorem 5.1.

5.3. Comparison with Le Potier's result

In the sequel we show that our Theorem 5.1 implies Main Theorem 1.3 (ii), (iii), in particular, Le Potier's result. In addition to $\mathfrak{E}_1$, we consider the following full strong exceptional collections on $\mathbf{P}^2$

$$\mathfrak{E}'_1 = (\mathscr{O}_{\mathbf{P}^2}(1), \mathscr{O}_{\mathbf{P}^2}(2), \mathscr{O}_{\mathbf{P}^2}(3)), \quad \mathfrak{E}_0 = (\mathscr{O}_{\mathbf{P}^2}(1), \Omega^1_{\mathbf{P}^2}(3), \mathscr{O}_{\mathbf{P}^2}(2)),$$

the equivalences $\Phi'_1(\cdot) = \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathscr{E}'_1, \cdot)$, $\Phi_0(\cdot) = \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathscr{E}_0, \cdot)$ between $D^b(\mathbf{P}^2)$ and $D^b(B')$, $D^b(B)$ and the induced isomorphisms $\varphi'_1 : K(\mathbf{P}^2) \cong K(B')$, $\varphi_0 : K(\mathbf{P}^2) \cong K(B)$, where $\mathscr{E}'_1 = \mathscr{O}_{\mathbf{P}^2}(1) \oplus \mathscr{O}_{\mathbf{P}^2}(2) \oplus \mathscr{O}_{\mathbf{P}^2}(3)$, $\mathscr{E}_0 = \mathscr{O}_{\mathbf{P}^2}(1) \oplus \Omega^1_{\mathbf{P}^2}(3) \oplus \mathscr{O}_{\mathbf{P}^2}(2)$ and $B' = \mathrm{End}_{\mathbf{P}^2}(\mathscr{E}'_1)$, $B = \mathrm{End}_{\mathbf{P}^2}(\mathscr{E}_0)$. We also recall from §4.3 that

$$\mathscr{A}'_1 = \langle \mathscr{O}_{\mathbf{P}^2}[2], \Omega^1_{\mathbf{P}2}(2)[1], \mathscr{O}_{\mathbf{P}^2}(1) \rangle, \quad \mathscr{A}_0 = \langle \mathscr{O}_{\mathbf{P}^2}(-1)[2], \mathscr{O}_{\mathbf{P}2}[1], \mathscr{O}_{\mathbf{P}^2}(1) \rangle. \tag{43}$$

We remark that $\mathscr{A}'_1$ is the left tilt of $\mathscr{A}_1 = \langle \mathscr{O}_{\mathbf{P}^2}[2], \mathscr{O}_{\mathbf{P}^2}(1)[1], \mathscr{O}_{\mathbf{P}^2}(2) \rangle$ at $\mathscr{O}_{\mathbf{P}^2}(1)[1]$ and $\mathscr{A}_0$ is the left tilt of $\mathscr{A}'_1$ at $\mathscr{O}_{\mathbf{P}^2}[2]$. See [Br3] for this terminology and relationship between tilting and exceptional collections although we do not use this fact.

For $\theta \in \mathrm{Hom}_{\mathbf{Z}}(K(\mathbf{P}^2), \mathbf{R})$, we put $\theta_k := \theta \circ \varphi_k^{-1} \in \mathrm{Hom}_{\mathbf{Z}}(K(B), \mathbf{R})$ for $k = 0, 1$ and $\theta'_1 := \theta \circ \varphi_1'^{-1} \in \mathrm{Hom}_{\mathbf{Z}}(K(B'), \mathbf{R})$. We put

$$\begin{aligned} (\theta^0_k, \theta^1_k, \theta^2_k) &:= (\theta_k(\mathbf{C}v_0), \theta_k(\mathbf{C}v_1), \theta_k(\mathbf{C}v_2)) \quad \text{for } k = 0, 1, \\ (\theta'^0_1, \theta'^1_1, \theta'^2_1) &:= (\theta'_1(\mathbf{C}v_0), \theta'_1(\mathbf{C}v_1), \theta'_1(\mathbf{C}v_2)). \end{aligned} \tag{44}$$

For any B-module N and B'-module M, we have

$$\theta_k(N) = \theta^0_k \dim_{\mathbf{C}}(Nv^*_0) + \theta^1_k \dim_{\mathbf{C}}(Nv^*_1) + \theta^2_k \dim_{\mathbf{C}}(Nv^*_2) \quad \text{for } k = 0, 1,$$

$$\theta'_1(M) = \theta'^0_1 \dim_{\mathbf{C}}(Mv^*_0) + \theta'^1_1 \dim_{\mathbf{C}}(Mv^*_1) + \theta'^2_1 \dim_{\mathbf{C}}(Mv^*_2).$$

By abbreviation we denote this by $\theta_k = (\theta_k^0, \theta_k^1, \theta_k^2)$ and $\theta_1' = (\theta_1'^0, \theta_1'^1, \theta_1'^2)$. It is also convenient to write the following equality

$$(45)\quad \begin{aligned} (\theta_k^0, \theta_k^1, \theta_k^2) &= (\theta(\mathcal{O}_{\mathbf{P}^2}(k-1)[2]), \theta(\mathcal{O}_{\mathbf{P}^2}(k)[1]), \theta(\mathcal{O}_{\mathbf{P}^2}(k+1))) \quad \text{for } k = 0, 1, \\ (\theta_1'^0, \theta_1'^1, \theta_1'^2) &= (\theta(\mathcal{O}_{\mathbf{P}^2}[2]), \theta(\Omega_{\mathbf{P}^2}(2)[1]), \theta(\mathcal{O}_{\mathbf{P}^2}(1))). \end{aligned}$$

Proposition 5.4. *Let* $\theta : K(\mathbf{P}^2) \to \mathbf{R}$ *be an additive function with* $\theta_1 = (\theta_1^0, \theta_1^1, \theta_1^2)$ *and* $\alpha \in K(\mathbf{P}^2)$ *with* $\theta(\alpha) = 0$. *If* $\theta_1^0, \theta_1^1 < 0$, *then equivalences* $\Phi_1' \circ \Phi_1^{-1} : D^b(B) \cong D^b(B')$ *and* $\Phi_0 \circ \Phi_1'^{-1} : D^b(B') \cong D^b(B)$ *between derived categories induce the isomorphisms*

$$M_B(\varphi_1(\alpha), \theta_1) \cong M_{B'}(\varphi_1'(\alpha), \theta_1') \cong M_B(\varphi_0(\alpha), \theta_0).$$

These isomorphisms keep open subsets of stable modules.

We only show the first isomorphism using the assumption that $\theta_1^1 < 0$. The other assumption that $\theta_1^0 < 0$ is used for the second isomorphism.

Step 1. The assumption $\theta_1^1 < 0$ implies that $\Phi_1' \circ \Phi_1^{-1}(N) \in \text{mod-}B'$ for any $N \in M_B(\varphi_1(\alpha), \theta_1)$.

Proof. We take $E \in \mathscr{A}_1$ such that $\Phi_1(E) = N$. Then the decomposition of $N = \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathscr{E}_1, E)$ is given by

$$(46)\quad \begin{aligned} Nv_0^* &= \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(3), E) \\ Nv_1^* &= \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\Omega^1_{\mathbf{P}^2}(4), E) \\ Nv_2^* &= \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(2), E), \end{aligned}$$

and $\gamma_i^*|_N = p_i^*$, $\delta_j^*|_N = q_j^*$ from (28). On the other hand, we have

$$(47)\quad \begin{aligned} \Phi_1' \circ \Phi_1^{-1}(N) &= \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathscr{E}_1', E) \\ &= \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(3), E) \oplus \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(2), E) \\ &\quad \oplus \mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(1), E). \end{aligned}$$

The fact that $N \in \text{mod-}B$ and (46) implies

$$\mathbf{R}^i\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(3), E) = \mathbf{R}^i\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(2), E) = 0$$

for $i \neq 0$. From the exact sequence

$$(48)\quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(1) \xrightarrow{\Sigma z_i \otimes e_i} \mathcal{O}_{\mathbf{P}^2}(2) \otimes V \xrightarrow{q_i \otimes e_i^*} \Omega^1_{\mathbf{P}^2}(4) \longrightarrow 0,$$

we have an isomorphism of complexes in $D^b(\mathbf{P}^2)$

$$(49)\quad \mathcal{O}_{\mathbf{P}^2}(1) \cong (\mathcal{O}_{\mathbf{P}^2}(2) \otimes V \xrightarrow{\Sigma q_i \otimes e_i^*} \Omega^1_{\mathbf{P}^2}(4)),$$

where $\mathcal{O}_{\mathbf{P}^2}(2) \otimes V$ lies on degree 0. By applying Lemma 4.6 (1) to (49) and $E \in \mathcal{A}_1$, we have an isomorphism in $D^b(\mathbf{C})$

$$\mathbf{R}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(1), E) \cong (Nv_1^* \xrightarrow{\delta_V^*} (Nv_2^*) \otimes V), \tag{50}$$

where $(Nv_2^*) \otimes V$ lies on degree 0 and $\delta_V^* = \delta_0^* \otimes e_0 + \delta_1^* \otimes e_1 + \delta_2^* \otimes e_2$. Hence $\Phi_1' \circ \Phi_1^{-1}(N)$ belongs to mod-B' if and only if

$$\ker \delta_V^* = \mathbf{R}^{-1}\,\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(1), E) = 0.$$

However if $\ker \delta_V^* \neq 0$, we can view $\ker \delta_V^*$ as a submodule N' of N with $N'v_0^* = N'v_2^* = 0$ and $N'v_1^* = \ker \delta_V^*$. This contradicts θ_1-semistability of N since $\theta_1(\ker \delta_V^*) = \theta_1^1 \cdot \dim_{\mathbf{C}}(\ker \delta_V^*) < 0$. □

Step 2. For any $N \in M_B(\varphi_1(\alpha), \theta_1)$, θ_1-(semi)stability of N implies θ_1'-(semi)stability of $M := \Phi_1' \circ \Phi_1^{-1}(N) \in$ mod-B'.

Proof. We recall that $v_i \in \mathbf{C}Q/J'$ correspond to $\mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(3-i)} \in B'$ for $i = 0, 1, 2$ via the isomorphism (28). Hence by (46), (47) and (50) we have

$$Mv_0^* = Nv_0^*, \quad Mv_1^* = Nv_2^*, \quad Mv_2^* = \mathrm{coker}\,\delta_V^*. \tag{51}$$

Since $z_i = p_{i+2} \circ q_{i+1} \in \mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(2), \mathcal{O}_{\mathbf{P}^2}(3))$, $\gamma_i^*|_M : Mv_0^* \to Mv_1^*$ is defined by

$$\gamma_i^*|_M := \delta_{i+1}^*|_N \circ \gamma_{i+2}^*|_N : Nv_0^* \to Nv_2^*.$$

Via the isomorphism (49), homomorphisms $z_i : \mathcal{O}_{\mathbf{P}^2}(1) \to \mathcal{O}_{\mathbf{P}^2}(2)$ correspond to homotopy classes of homomorphisms $\mathrm{id}_{\mathcal{O}_{\mathbf{P}^2}(2)} \otimes e_i^* : \mathcal{O}_{\mathbf{P}^2}(2) \otimes V \to \mathcal{O}_{\mathbf{P}^2}(2)$ in

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathbf{P}^2)}(\mathcal{O}_{\mathbf{P}^2}(1), \mathcal{O}_{\mathbf{P}^2}(2)) &\cong \mathrm{coker}(\mathrm{Hom}_{\mathbf{P}^2}(\Omega^1_{\mathbf{P}^2}(4), \mathcal{O}_{\mathbf{P}^2}(2))\\ &\qquad \to \mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}_{\mathbf{P}^2}(2) \otimes V, \mathcal{O}_{\mathbf{P}^2}(2))\end{aligned}$$

for $i = 0, 1, 2$. Hence $\delta_j^*|_M : Mv_1^* \to Mv_2^*$ is defined by

$$\delta_j^*|_M : Nv_2^* \xrightarrow{\mathrm{id}_{Nv_2^*} \otimes e_j} (Nv_2^*) \otimes V \longrightarrow \mathrm{coker}\,\delta_V^*,$$

where $(Nv_2^*) \otimes V \to \mathrm{coker}\,\delta_V^*$ is a natural surjection.

Conversely from this description we see easily that the above B-module N is reconstructed from the B'-module $M = \Phi_1' \circ \Phi_1^{-1}(N)$ as follows. We define

$$\delta^{*V} := \Sigma_i(\delta_i^*|_M) \otimes e_i^* : (Mv_1^*) \otimes V \to Mv_2^*. \tag{52}$$

We put

$$Nv_0^* := Mv_0^*, \quad Nv_1^* := \ker \delta^{*V}, \quad Nv_2^* := Mv_1^* \tag{53}$$

and define $\gamma_i^*|_N : Nv_0^* \to Nv_1^*$ and $\delta_j^*|_N : Nv_1^* \to Nv_2^*$ by

$$\begin{aligned}&\gamma_i^*|_N := (\gamma_{i+1}^*|_M) \otimes e_{i+2} - (\gamma_{i+2}^*|_M) \otimes e_{i+1} : Mv_0^* \to \ker \delta^{*V},\\ &\delta_j^*|_N : \ker \delta^{*V} \subset (Mv_1^*) \otimes V \xrightarrow{\mathrm{id}_{Mv_1^*} \otimes e_j^*} Mv_1^*.\end{aligned} \tag{54}$$

Imitating this, for any B'-submodule M' of M we construct an B-submodule N' of N by (52), (53) and (54) with Mv_i^* and Nv_j^* replaced by $M'v_i^*$ and $N'v_j^*$. However in this case

$$\delta^{*V} : (M'v_1^*) \otimes V \to M'v_2^*$$

is not necessarily surjective. Hence we have

$$\dim_{\mathbf{C}}(N'v_1^*) = \dim_{\mathbf{C}} \ker(\delta^{*V}|_{(M'v_1^*)\otimes V}) \geq 3 \dim_{\mathbf{C}}(M'v_1^*) - \dim_{\mathbf{C}}(M'v_2^*).$$

Hence the assumption that $\theta_1^1 < 0$ and the following equality by (45)

$$(\theta_1^0, \theta_1^1, \theta_1^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{pmatrix} = (\theta_1'^0, \theta_1'^1, \theta_1'^2)$$

implies $\theta_1(N') \leq \theta_1'(M')$. Thus θ_1-(semi)stability of N implies θ_1'-(semi)stability of M and we have

$$\Phi_1' \circ \Phi_1^{-1}(M_B(\varphi_1(\alpha), \theta_1)) \subset M_{B'}(\varphi_1'(\alpha), \theta_1').$$

The proof of the opposite inclusion is similar and we leave it to the readers. □

If we assume $\mathrm{ch}_2 < \frac{1}{2}$, the chamber $C_{\varphi_1(\alpha)}^{\mathbf{P}^2} \subset \varphi_1(\alpha)^{\perp}$ defined in Section 5.1 intersect with the region defined by the inequalities $\theta_1^0, \theta_1^1 < 0$. Hence from the above proposition and Theorem 5.1 we have isomorphisms

$$M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_{B'}(-\varphi_1'(\alpha), \theta_1') : E \mapsto \Phi_1'(E[1]) \tag{55}$$

$$M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_0(\alpha), \theta_0) : E \mapsto \Phi_0(E[1]) \tag{56}$$

for $\alpha \in K(\mathbf{P}^2)$ with $0 < c_1(\alpha) \leq \mathrm{rk}(\alpha)$, $\mathrm{ch}_2 < \frac{1}{2}$ and $\theta : K(\mathbf{P}^2) \to \mathbf{R}$ satisfying $\theta_1 \in C_{\varphi_1(\alpha)}^{\mathbf{P}^2}$ with $\theta_1^0, \theta_1^1 < 0$. This completes the proof of Main Theorem 1.3. (55) was obtained by Le Potier [P].

6. Computations of the wall-crossing

In this section, we identify the Hilbert schemes of points on $\mathbf{P}^2$

$$(\mathbf{P}^2)^{[n]} := \{\mathscr{I} \subset \mathcal{O}_{\mathbf{P}^2} \mid \mathrm{Length}(\mathcal{O}_{\mathbf{P}^2}/\mathscr{I}) = n\}$$

with the moduli spaces $M_B(-\varphi_0(\alpha), \theta_0) \cong M_B(-\varphi_1(\alpha), \theta_1)$ by Theorem 5.1 and Proposition 5.4, where $\alpha \in K(\mathbf{P}^2)$ with $\mathrm{ch}(\alpha) = \left(1, 1, \frac{1}{2} - n\right)$, $\theta_1 \in C_{\varphi_1(\alpha)}^{\mathbf{P}^2}$ and $\theta_0 = \theta_1 \circ \varphi_1 \circ \varphi_0^{-1}$. We study the wall-crossing phenomena of the Hilbert schemes of points on $\mathbf{P}^2$ via this identification.

6.1. Geometry of Hilbert schemes of points on $\mathbf{P}^2$

We recall the geometry of Hilbert schemes of points on $\mathbf{P}^2$ (cf. [LQZ]). Let ℓ be a line in $\mathbf{P}^2$, and $x_1, \ldots, x_{n-1} \in \mathbf{P}^2$ be distinct fixed points in ℓ. Let

$$M_2(x_1) = \{\xi \in (\mathbf{P}^2)^{[2]} \mid \mathrm{Supp}(\xi) = x_1\}$$

be the punctual Hilbert scheme parameterizing length-2 0-dimensional subschemes supported at x_1. It is known that $M_2(x_1) \cong \mathbf{P}^1$. Let $N_1((\mathbf{P}^2)^{[n]})$ be the $\mathbf{R}$-vector space of numerical equivalence classes of one-cycles on $(\mathbf{P}^2)^{[n]}$. We define two curves β_n and ζ_ℓ in $(\mathbf{P}^2)^{[n]}$ as elements in $N_1((\mathbf{P}^2)^{[n]})$ by the following formula

$$\beta_n := \{\xi + x_2 + \cdots + x_{n-1} \in (\mathbf{P}^2)^{[n]} \mid \xi \in M_2(x_1)\}$$
(57)
$$\zeta_\ell := \{x + x_1 + \cdots + x_{n-1} \in (\mathbf{P}^2)^{[n]} \mid x \in \ell\}.$$

The definition of β_n and ζ_ℓ does not depend on the choice of a line ℓ on $\mathbf{P}^2$ and points $x_1, \ldots, x_{n-1}$ on ℓ (cf. [LQZ, Theorem 3.2 and Theorem 5.1]). We define a cone $\mathrm{NE}((\mathbf{P}^2)^{[n]})$ in $N_1((\mathbf{P}^2)^{[n]})$ by

$$\mathrm{NE}((\mathbf{P}^2)^{[n]}) := \{\Sigma a_i[C_i] \mid C_i \subset (\mathbf{P}^2)^{[n]} \text{ an irreducible curve, } a_i \geq 0\}$$

and $\overline{\mathrm{NE}}((\mathbf{P}^2)^{[n]})$ to be its closure.

THEOREM 6.1 [LQZ, Theorem 4.1]. *$\overline{\mathrm{NE}}((\mathbf{P}^2)^{[n]})$ is spanned by β_n and ζ_ℓ.*

Let $S^n(\mathbf{P}^2)$ be the nth symmetric product of $\mathbf{P}^2$, that is, $S^n(\mathbf{P}^2) := (\mathbf{P}^2)^n/\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group of degree n. The Hilbert-Chow morphism $\pi : (\mathbf{P}^2)^{[n]} \to S^n(\mathbf{P}^2)$ is defined by $\pi(\mathscr{I}) = \mathrm{Supp}(\mathscr{O}_{\mathbf{P}^2}/\mathscr{I}) \in S^n(\mathbf{P}^2)$ for every $\mathscr{I} \in (\mathbf{P}^2)^{[n]}$. The morphism π is the contraction of the extremal ray $\mathbf{R}_{>0}\beta_n$.

Denote by $\psi : (\mathbf{P}^2)^{[n]} \to Z$ the contraction morphism of the extremal ray $\mathbf{R}_{>0}\zeta_\ell$. In the case $n = 2$, $\psi : (\mathbf{P}^2)^{[2]} \to Z$ coincide with the morphism $\mathrm{Hilb}^2(\mathbf{P}((T_{(\mathbf{P}^2)^*})^*)) \to (\mathbf{P}^2)^*$ up to isomorphism, where $\mathrm{Hilb}^2(\mathbf{P}((T_{(\mathbf{P}^2)^*})^*))$ is the relative Hilbert scheme. In the case $n = 3$, $\psi : (\mathbf{P}^2)^{[3]} \to Z$ is a divisorial contraction. In the case $n \geq 4$, $\psi : (\mathbf{P}^2)^{[n]} \to Z$ is a flipping contraction.

6.2. Wall-Crossing of the Hilbert schemes of points on $\mathbf{P}^2$

We take $\alpha \in K(\mathbf{P}^2)$ with $\mathrm{ch}(\alpha) = \left(r, 1, \frac{1}{2} - n\right)$ and assume that $n \geq 1$. By (33), we have $\underline{\dim}(-\varphi_1(\alpha)) = (n - r + 1, 2n + 1, n)$. For $b \in \mathbf{R}$ with $0 < b < \dfrac{1}{r}$ we put $t = \sqrt{b - b^2}$. From (40) and Proposition 4.4, we have isomorphisms

$$\text{(58)} \qquad {}^{sh}\mathscr{M}_{D^b(\mathbf{P}^2)}(\mathrm{ch}(\alpha), \sigma_{(bH, tH)}) \cong {}^{sh}\mathscr{M}_{D^b(\mathbf{P}^2)}(-\mathrm{ch}(\alpha), \sigma^b) : E \mapsto E[1]$$

$$\text{(59)} \qquad {}^{sh}\mathscr{M}_{D^b(\mathbf{P}^2)}(-\mathrm{ch}(\alpha), \sigma^b) \cong {}^{sh}\mathscr{M}_B(-\varphi_1(\alpha), \theta^\alpha_{Z^b}) : E[1] \mapsto \Phi_1(E[1]),$$

where σ^b is defined by (36) and $\theta^\alpha_{Z^b}$ is defined by (24) using $\varphi_1 : K(\mathbf{P}^2) \cong K(B)$. We recall that from §5.2, if $\dfrac{1}{r} - b_0 > 0$ is small enough, then $M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H)$ corepresents ${}^{sh}\mathscr{M}_{D^b(\mathbf{P}^2)}(\mathrm{ch}(\alpha), \sigma_{(b_0 H, t_0 H)})$, where $t_0 := \sqrt{b_0 - b_0^2}$. We have $\theta^\alpha_{Z^{b_0}} \in C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ and the isomorphism

$$M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_1(\alpha), \theta^\alpha_{Z^{b_0}})$$

in Theorem 5.1. In fact the following lemma holds.

LEMMA 6.2. *We have* $\mathbf{R}_{>0}\theta^{\alpha}_{Z^0} + \mathbf{R}_{>0}\theta^{\alpha}_{Z^{1/r}} \subset C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$, *that is, the moduli functor* $\mathscr{M}_{D^b(\mathbf{P}^2)}(\mathrm{ch}(\alpha), \sigma_{(bH,tH)})$ *does not change as b moves in the interval* $\left(0, \dfrac{1}{r}\right)$.

Proof. We assume that there exists a $\mathbf{C}$-valued point E of $\mathscr{M}_{D^b(\mathbf{P}^2)}(\mathrm{ch}(\alpha), \sigma_{(b_0H,t_0H)})$ such that E is not $\sigma_{(b_1H,t_1H)}$-semistable for some $b_1 \in \left(0, \dfrac{1}{r}\right)$, where we put $t_1 := \sqrt{b_1 - b_1^2}$. From (58) and (59), $\sigma_{(bH,tH)}$-semistability for E and $\theta^{\alpha}_{Z^b}$-semistability for $\Phi_1(E[1])$ are equivalent for $b \in \left(0, \dfrac{1}{r}\right)$. Using the notation (44) in §5.3, $\theta^{\alpha}_{Z^b}$ is computed from (36) and (45) as follows:

$$\theta^{\alpha}_{Z^b} = (1-b)(0,-n,2n+1) + b(-n,0,n+1-r) \in \mathrm{Hom}_{\mathbf{Z}}(K(B),\mathbf{R}) \cong \mathbf{R}^3.$$

If we fix any $\beta \in K(B)$, then $\theta^{\alpha}_{Z^b}(\beta)$ is a monotonic function for b. Hence we may assume that such a real number b_1 is small enough.

We take the $\sigma_{(b_1H,t_1H)}$-semistable factor G of E with the smallest slope $\mu_{\sigma_{(b_1H,t_1H)}}(G)$ and the exact sequence in $\mathscr{A}_{(b_1H,t_1H)}$

$$0 \to F \to E \to G \to 0, \tag{60}$$

where F is a nonzero object of $\mathscr{A}_{(b_1H,t_1H)}$. From (60) we see that F is a sheaf since E is a sheaf and $\mathscr{H}^i(G) = 0$ for $i \neq 0, -1$. From (58) we have $E[1] \in \mathscr{A}_1$. By the uniqueness of Harder-Narasimhan filtration we see that $G[1]$ and $F[1]$ also belong to $\mathscr{A}_1$. Hence from the exact sequence (60), we see that dimension vectors of B-modules $\Phi_1(G[1])$ and $\Phi_1(F[1])$ are bounded from above by $\underline{\dim}(-\varphi_1(\alpha))$. In particular there exists a bound of $\mathrm{rk}(F)$ and $\mathrm{rk}(G)$ independent of the choice of E and b_1. The inequality $0 < \mathrm{Im}\, Z_{(b_1H,t_1H)}(F) = t_1(c_1(F) - r(F)b_1) < \mathrm{Im}\, Z_{(b_1H,t_1H)}(E)$ implies that $0 < c_1(F) \leq c_1(E) = 1$ since we can take arbitrary small $b_1 > 0$ and $\mathrm{rk}(F)$ is bounded from above. So we have $c_1(F) = 1$ and $c_1(G) = c_1(E) - c_1(F) = 0$.

We put $I := \mathrm{im}(F \to E)$. Since $F \to I$ is surjective we have $0 < \mu_{H\text{-min}}(F) \leq \mu(I)$. Furthermore since E is Gieseker-semistable, we have $\mu(I) \leq \mu(E) = \dfrac{1}{r}$. Hence $\mathrm{rk}(I) = r$, $c_1(I) = 1$ and $\mathscr{H}^0(G)$ is a 0-dimensional sheaf. Since $G[1] \in \mathscr{A}_1$, by Lemma 4.6 (2) we have an isomorphism

$$G[1] \cong (\mathscr{O}_{\mathbf{P}^2}^{\oplus a_0} \to \mathscr{O}_{\mathbf{P}^2}(1)^{\oplus a_1} \to \mathscr{O}_{\mathbf{P}^2}(2)^{\oplus a_2}),$$

where $(a_0, a_1, a_2) = -r(G)(1,0,0) - \mathrm{ch}_2(G)(1,2,1) \in \mathbf{Z}^3_{\geq 0}$. Hence $\mathrm{ch}_2(G)$ must be non-positive and $\mathrm{ch}_2(G) = 0$ if and only if $G[1] \cong \mathscr{O}_{\mathbf{P}^2}^{\oplus a_0}[2]$. In this case, we have $\theta^{\alpha}_{Z^{b_1}}(\Phi_1(G[1])) = -nb_1a_0 < 0$ and $\Phi_1(G[1])$ does not break $\theta^{\alpha}_{Z^{b_1}}$-semistability of $\Phi_1(E[1])$. This contradicts the choice of G. We have $\mathrm{ch}_2(\mathscr{H}^{-1}(G)) = -\mathrm{ch}_2(G) + \mathrm{ch}_2(\mathscr{H}^0(G)) > 0$. On the other hand, we have $c_1(\mathscr{H}^{-1}(G)) = -c_1(G) + c_1(\mathscr{H}^0(G)) = 0$ and from $G \in \mathscr{A}_{(b_1H,t_1H)}$ we have $\mu_{H\text{-max}}(\mathscr{H}^{-1}(G)) \leq 0$ for small enough $b_1 > 0$. Hence $\mathscr{H}^{-1}(G)$ is μ_H-semistable and satisfy the inequality $-2r(\mathscr{H}^{-1}(G))\,\mathrm{ch}_2(\mathscr{H}^{-1}(G)) \geq 0$ by Theorem 3.2. This is a contradiction. $\square$

In the following we consider the case $r = 1$. We fix $\alpha \in K(\mathbf{P}^2)$ with $\mathrm{ch}(\alpha) = \left(1, 1, \frac{1}{2} - n\right)$, $n \geq 1$ and $\theta_1 \in C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$. Tensoring by $\mathscr{O}_{\mathbf{P}^2}(1) = \mathscr{O}_{\mathbf{P}^2}(H)$ does not change Gieseker-semistability of torsion free sheaves on $\mathbf{P}^2$ and induces an automorphism of $K(\mathbf{P}^2)$ sending $\hat{\alpha}$ with $\mathrm{ch}(\hat{\alpha}) = (1, 0, -n)$ to α. Since by definition $(\mathbf{P}^2)^{[n]} = M_{\mathbf{P}^2}(\mathrm{ch}(\hat{\alpha}), H)$, we have an isomorphism

$$(\mathbf{P}^2)^{[n]} \cong M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) : \mathscr{I} \mapsto \mathscr{I}(1).$$

On the other hand, by Theorem 5.1 and Proposition 5.4, we have isomorphisms

$$\Phi_k(\cdot\,[1]) : M_{\mathbf{P}^2}(\mathrm{ch}(\alpha), H) \cong M_B(-\varphi_k(\alpha), \theta_k)$$

for $k = 0, 1$, where $\theta_0 = \theta_1 \circ \varphi_1 \circ \varphi_0^{-1}$. In what follows, we often use these identifications

$$(\mathbf{P}^2)^{[n]} \cong M_B(-\varphi_k(\alpha), \theta_k) : \mathscr{I} \mapsto \Phi_k(\mathscr{I}(1)[1]), \quad \text{and} \quad \Phi_k : \mathscr{A}_k \cong \text{mod-}B.$$

For any 0-dimensional subscheme Z of $\mathbf{P}^2$, $\mathscr{I}_Z$ denotes the ideal of Z, that is, the structure sheaf $\mathscr{O}_Z$ is defined by $\mathscr{O}_Z := \mathscr{O}_{\mathbf{P}^2}/\mathscr{I}_Z$. If the length of Z is n, then $\mathscr{I}_Z$ is an element of $(\mathbf{P}^2)^{[n]}$.

We recall that

$$\text{(61)} \quad \mathscr{A}_1 = \langle \mathscr{O}_{\mathbf{P}^2}[2], \mathscr{O}_{\mathbf{P}^2}(1)[1], \mathscr{O}_{\mathbf{P}^2}(2) \rangle, \quad \mathscr{A}_0 = \langle \mathscr{O}_{\mathbf{P}^2}(-1)[2], \mathscr{O}_{\mathbf{P}^2}[1], \mathscr{O}_{\mathbf{P}^2}(1) \rangle,$$
$$\underline{\dim}(-\varphi_1(\alpha)) = (n, 2n+1, n), \quad \underline{\dim}(-\varphi_0(\alpha)) = (n, 2n, n-1).$$

For $b \in \mathbf{R}$, we put

$$\text{(62)} \quad \theta(b)_1 := (1-b)(0, -n, 2n+1) + b(-n, 0, n) \in \mathrm{Hom}_{\mathbf{Z}}(K(B), \mathbf{R})$$

$$\text{(63)} \quad \theta(b)_0 := (1-b)(-n+1, 0, n) + b(-2n, n, 0) \in \mathrm{Hom}_{\mathbf{Z}}(K(B), \mathbf{R}).$$

If $0 < b < 1$, by (36) and (45) we have $\theta(b)_1 = \theta^{\alpha}_{Z^b}$ and $\theta(b)_0 = \theta^{\alpha}_{Z^b} \circ \varphi_1 \circ \varphi_0^{-1}$. By Lemma 6.2, we have $\mathbf{R}_{>0}\theta(0)_1 + \mathbf{R}_{>0}\theta(1)_1 \subset C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ in $\varphi_1(\alpha)^{\perp}$. We define a wall-and-chamber structure on $\varphi_0(\alpha)^{\perp}$ as in §5.1 and take the chamber $C^{\mathbf{P}^2}_{\varphi_0(\alpha)}$ on $\varphi_0(\alpha)^{\perp}$ containing $\mathbf{R}_{>0}\theta(0)_0 + \mathbf{R}_{>0}\theta(1)_0$.

Lemma 6.3. *The following hold.*

(1) $\mathbf{R}_{>0}\theta(0)_1 + \mathbf{R}_{>0}\theta(1)_1 = C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ *for* $n \geq 1$.

(2) $\mathbf{R}_{>0}\theta(0)_0 + \mathbf{R}_{>0}\theta(1)_0 = C^{\mathbf{P}^2}_{\varphi_0(\alpha)}$ *for* $n \geq 2$.

Proof. It is enough to show that $\theta(0)_k$ and $\theta(1)_k$ lie on walls on $\varphi_k(\alpha)^{\perp}$ for $k = 0, 1$.

(1) Any B-module N with $[N] = \varphi_1(\alpha)$ has a surjection $N \to \mathbf{C}v_0$ and $\theta(0)_1(\mathbf{C}v_0) = 0$. Thus $\theta(0)_1$ lies on a wall on $\varphi_1(\alpha)^{\perp}$. We take any element $\mathscr{I}_Z \in (\mathbf{P}^2)^{[n]}$. We have an exact sequence

$$\text{(64)} \quad 0 \to \mathscr{I}_Z \to \mathscr{O}_{\mathbf{P}^2} \to \mathscr{O}_Z \to 0.$$

$\mathcal{O}_Z$ can be obtained by extensions of $\{\mathcal{O}_x \mid x \in \mathrm{Supp}(Z)\}$. Since $\mathcal{O}_x$ belongs to $\mathscr{A}_1$ by (35), we have $\mathcal{O}_Z \in \mathscr{A}_1$. From (64), tensoring by $\mathcal{O}_{\mathbf{P}^2}(1)$ we have an exact sequence in $\mathscr{A}_1$

$$0 \to \mathcal{O}_Z \to \mathscr{I}_Z(1)[1] \to \mathcal{O}_{\mathbf{P}^2}(1)[1] \to 0.$$

Furthermore we have $\theta(1)_1(\Phi_1(\mathcal{O}_Z)) = 0$, since $\underline{\dim}(\Phi_1(\mathcal{O}_x)) = (1,2,1)$ and $\theta(1)_1(\Phi_1(\mathcal{O}_x)) = 0$ for any closed point $x \in \mathbf{P}^2$ by (62). Thus $\theta(1)_1$ also lies on a wall on $\varphi_1(\alpha)^\perp$.

(2) Any B-module N with $[N] = \varphi_0(\alpha)$ has a submodule $\mathbf{C}v_2$. Since $\theta(1)_0(\mathbf{C}v_2) = 0$, $\theta(1)_0$ lies on a wall on $\varphi_0(\alpha)^\perp$. On the other hand, for any line ℓ on $\mathbf{P}^2$ we take an element $\mathscr{I}_Z$ of ζ_ℓ. Since Z is a closed subscheme of ℓ by the definition (57), we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}_Z & \longrightarrow & \mathcal{O}_{\mathbf{P}^2} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^2}(-1) & \longrightarrow & \mathcal{O}_{\mathbf{P}^2} & \longrightarrow & \mathcal{O}_\ell & \longrightarrow & 0. \end{array}$$

Hence tensoring by $\mathcal{O}_{\mathbf{P}^2}(1)$, we get an exact sequence in $\mathrm{Coh}(\mathbf{P}^2)$

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to \mathscr{I}_Z(1) \to \mathcal{O}_\ell(-n+1) \to 0,$$

where $\mathcal{O}_\ell(-n+1) = \ker(\mathcal{O}_\ell(1) \to \mathcal{O}_Z)$. This gives a distinguished triangle in $D^b(\mathbf{P}^2)$

$$\mathcal{O}_{\mathbf{P}^2}[1] \to \mathscr{I}_Z(1)[1] \to \mathcal{O}_\ell(-n+1)[1] \to \mathcal{O}_{\mathbf{P}^2}[2]. \tag{65}$$

We show that this gives an exact sequence in $\mathscr{A}_0$. It is enough to show that $\mathcal{O}_\ell(-n+1)[1] \in \mathscr{A}_0$. An exact sequence in $\mathrm{Coh}(\mathbf{P}^2)$

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-1) \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{O}_\ell \to 0$$

implies that $\mathcal{O}_\ell[1] \in \mathscr{A}_0$ from (61). For an integer $m > 0$ and a closed point x in ℓ, we consider an exact sequence in $\mathrm{Coh}(\mathbf{P}^2)$

$$0 \to \mathcal{O}_\ell(-m) \to \mathcal{O}_\ell(-m+1) \to \mathcal{O}_x \to 0.$$

This gives a distinguished triangle in $D^b(\mathbf{P}^2)$

$$\mathcal{O}_x \to \mathcal{O}_\ell(-m)[1] \to \mathcal{O}_\ell(-m+1)[1] \to \mathcal{O}_x[1].$$

Since $\mathcal{O}_x$ belongs to $\mathscr{A}_0$ as in Lemma 5.3, by induction on m we have $\mathcal{O}_\ell(-m)[1] \in \mathscr{A}_0$ for any $m \geq 0$. Since $\theta(0)_0(\varphi(\mathcal{O}_{\mathbf{P}^2}[1])) = 0$, $\mathscr{I}_Z(1)[1]$ and the subobject $\mathcal{O}_{\mathbf{P}^2}[1]$ define a wall $\mathbf{R}_{\geq 0}\theta(0)_0$ on $\varphi_0(\alpha)^\perp$. □

We take the chamber $C^+_{\varphi_1(\alpha)} \neq C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ in $\varphi_1(\alpha)^\perp$ sharing the wall $\mathbf{R}_{\geq 0}\theta(1)_1$ with $C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$. Similarly we take the chamber $C^-_{\varphi_0(\alpha)} \neq C^{\mathbf{P}^2}_{\varphi_0(\alpha)}$ in $\varphi_0(\alpha)^\perp$ sharing the wall $\mathbf{R}_{\geq 0}\theta(0)_0$ with $C^{\mathbf{P}^2}_{\varphi_0(\alpha)}$. We take a real number $0 < \varepsilon < 1$ small enough such that $\theta(1-\varepsilon)_1 \in C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$, $\theta(1+\varepsilon)_1 \in C^+_{\varphi_1(\alpha)}$ and $\theta(\varepsilon)_0 \in C^{\mathbf{P}^2}_{\varphi_0(\alpha)}$, $\theta(-\varepsilon)_0 \in C^-_{\varphi_0(\alpha)}$.

LEMMA 6.4. *The following hold.*
(1) $M_B(-\varphi_1(\alpha), \theta(1+\varepsilon)_1) \neq \emptyset$ *for* $n \geq 1$.
(2) $M_B(-\varphi_0(\alpha), \theta(-\varepsilon)_0) \neq \emptyset$ *for* $n \geq 3$.

Proof. (1) For any $N \in M_B(-\varphi_1(\alpha), \theta(1-\varepsilon)_1)$, we show that the dual vector space $N^* := \mathrm{Hom}_{\mathbf{C}}(N, \mathbf{C})$ has a natural B-module structure and belongs to $M_B(-\varphi_1(\alpha), \theta(1+\varepsilon)_1)$ as follows. We put $N^* v_i^* := \mathrm{Hom}_{\mathbf{C}}(N v_{2-i}^*, \mathbf{C})$ and define $\gamma_i^*|_{N^*}$ and $\delta_j^*|_{N^*}$ by pull backs of $\delta_i^*|_N$ and $\gamma_j^*|_N$, respectively. Any surjection $N^* \to (N')^*$ corresponds to a submodule N' of N and

$$\underline{\dim}((N')^*) = (\dim_{\mathbf{C}} N' v_2^*, \dim_{\mathbf{C}} N' v_1^*, \dim_{\mathbf{C}} N' v_0^*). \tag{66}$$

On the other hand, from (62) we have

$$\theta(1+\varepsilon)_1 = \varepsilon(-2n-1, n, 0) + \frac{n-(n+1)\varepsilon}{n}(-n, 0, n) \in \mathrm{Hom}_{\mathbf{Z}}(K(B), \mathbf{R}). \tag{67}$$

By (66) and (67), we have the following equality

$$\theta(1+\varepsilon)_1((N')^*) = -\left(\varepsilon\theta(0)_1 + \frac{n-(n+1)\varepsilon}{n}\theta(1)_1\right)(N'). \tag{68}$$

Since by Lemma 6.3, we see that $\theta(1-\varepsilon)_1$ and $\varepsilon\theta(0)_1 + \frac{n-(n+1)\varepsilon}{n}\theta(1)_1$ belong to the same chamber $C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ for ε small enough, the right hand side of (68) is non-positive for any submodule N' of $N \in M_B(-\varphi_1(\alpha), \theta(1-\varepsilon)_1)$. We have $\theta(1+\varepsilon)_1((N')^*) \leq 0$ for any surjection $N^* \to (N')^*$. Thus N^* belongs to $M_B(-\varphi_1(\alpha), \theta(1+\varepsilon)_1)$.

(2) For $n \geq 3$ we take an element $\mathscr{I}_Z \in (\mathbf{P}^2)^{[n]}$ such that $\mathrm{Supp}(\mathscr{O}_{\mathbf{P}^2}/\mathscr{I}_Z)$ is not contained in any line ℓ on $\mathbf{P}^2$. Hence we have $\mathrm{Hom}_{\mathbf{P}^2}(\mathscr{O}_{\mathbf{P}^2}, \mathscr{I}_Z(1)) = 0$. Below we show that this implies that the B-module $M := \Phi_0(\mathscr{I}_Z(1)[1]) \in M_B(-\varphi_0(\alpha), \theta(\varepsilon)_0)$ is also $\theta(-\varepsilon)_0$-semistable. For any B-submodule $M' \subset M$, if $\theta(0)_0(M') > 0$ then by taking ε small enough we have $\theta(-\varepsilon)_0(M') > 0$ and M' does not break $\theta(-\varepsilon)_0$-semistability of M. If $\theta(0)_0(M') = 0$, then from (63) $\underline{\dim}\, M' = (n, *, n-1)$ or $(0, *, 0)$. However the latter case contradicts the fact that $\mathrm{Hom}_B(\mathbf{C}v_1, M) \cong \mathrm{Hom}_{\mathbf{P}^2}(\mathscr{O}_{\mathbf{P}^2}, \mathscr{I}_Z(1)) = 0$. Hence we have $\underline{\dim}\, M' = (n, l, n-1)$ with $0 \leq l \leq 2n$ and $\theta(-\varepsilon)_0(M') \geq 0$. Thus M is $\theta(-\varepsilon)_0$-semistable. □

For $\theta_k \in C^{\mathbf{P}^2}_{\varphi_k(\alpha)}$, we have natural morphisms

$$(\mathbf{P}^2)^{[n]} \cong M_B(-\varphi_k(\alpha), \theta_k) \to M_B(-\varphi_k(\alpha), \theta(k)_k) \tag{69}$$

for $k = 0, 1$, since $\mathbf{R}_{\geq 0}\theta(1)_1$ and $\mathbf{R}_{\geq 0}\theta(0)_0$ are walls of the chamber $C^{\mathbf{P}^2}_{\varphi_1(\alpha)}$ and $C^{\mathbf{P}^2}_{\varphi_0(\alpha)}$, respectively. We study the Stein factorization $\pi_k' : (\mathbf{P}^2)^{[n]} \to Y_k$ of the above morphism (69) for each $k = 0, 1$. Since by Lemma 6.4, for $n \geq 3$ our situations satisfy the assumptions in [Th, Theorem (3.3)], we see that π_1' and π_0' are birational morphisms and have the following diagram:

$$
\begin{array}{ccccc}
M_B(-\varphi_0(\alpha), \theta(-\varepsilon)_0) & \overset{\kappa}{\dashleftarrow} & (\mathbf{P}^2)^{[n]} & & \\
& \searrow & \swarrow_{\pi_0'} & \searrow^{\pi_1'} & \\
& & Y_0 & & Y_1.
\end{array}
\tag{70}
$$

Theorem 6.5. *The following hold.*

(1) *There exists an isomorphism* $Y_1 \cong S^n(\mathbf{P}^2)$ *and via this isomorphism, the morphism* π_1' *coincide with the Hilbert-Chow morphism* π.

(2) *For* $n \geq 3$, *the morphism* π_0' *is the contraction morphism of the extremal ray* $\mathbf{R}_{>0}\zeta_\ell$. *Hence* π_0' *coincide with* ψ *defined in* §6.1 *up to isomorphism.*

Proof. (1) We take two elements $\mathscr{I}_Z, \mathscr{I}_{Z'} \in (\mathbf{P}^2)^{[n]}$. We show that if $\mathrm{Supp}(Z) = \mathrm{Supp}(Z')$, then $\Phi_1(\mathscr{I}_Z(1)[1])$ and $\Phi_1(\mathscr{I}_{Z'}(1)[1])$ are S-equivalent $\theta(1)_1$-semistable B-modules. By Proposition 4.3 this implies that π_1' contracts the curve β_n to one point. This shows that the morphism π_1' coincides with the Hilbert-Chow morphism π via an isomorphism $Y_1 \cong S^n(\mathbf{P}^2)$, since the Picard number of $(\mathbf{P}^2)^{[n]}$ is two ($n \geq 2$).

We put $\mathrm{Supp}(\mathscr{O}_Z) = \mathrm{Supp}(\mathscr{O}_{Z'}) = \{x_1, \dots, x_n\}$ and consider a filtration of $\mathscr{I}_Z(1)[1]$ in $\mathscr{A}_1$. We put $Z_0 := Z \in (\mathbf{P}^2)^{[n]}$ and inductively define $Z_{i+1} \in (\mathbf{P}^2)^{[n-i-1]}$ from Z_i by the following exact sequence in $\mathrm{Coh}(\mathbf{P}^2)$

$$0 \to \mathscr{O}_{Z_{i+1}} \to \mathscr{O}_{Z_i} \to \mathscr{O}_{x_{i+1}} \to 0 \tag{71}$$

for $i = 0, \dots, n-2$. We have $\mathscr{O}_{Z_{n-1}} = \mathscr{O}_{x_n}$ and $\mathscr{O}_{x_i} \in \mathscr{A}_1$ for any i by (35). By (71) we have $\mathscr{O}_{Z_i} \in \mathscr{A}_1$ for $i = 0, \dots, n-1$. Hence (71) is also exact in $\mathscr{A}_1$. On the other hand, from the exact sequence in $\mathrm{Coh}(\mathbf{P}^2)$

$$0 \to \mathscr{I}_Z \to \mathscr{O}_{\mathbf{P}^2} \to \mathscr{O}_Z \to 0 \tag{72}$$

we have an exact sequence in $\mathscr{A}_1$

$$0 \to \mathscr{O}_Z \to \mathscr{I}_Z(1)[1] \to \mathscr{O}_{\mathbf{P}^2}(1)[1] \to 0. \tag{73}$$

Since $\underline{\dim}(\Phi_1(\mathscr{O}_{\mathbf{P}^2}(1)[1])) = (0,1,0)$ and $\underline{\dim}(\Phi_1(\mathscr{O}_x)) = (1,2,1)$ for any closed point $x \in \mathbf{P}^2$, we have $\theta(1)_1(\Phi_1(\mathscr{O}_{\mathbf{P}^2}(1)[1])) = \theta(1)_1(\Phi_1(\mathscr{O}_x)) = 0$ from (62). Furthermore from (71) we have $\theta(1)_1(\Phi_1(\mathscr{O}_{Z_i})) = 0$ for any i. Hence (71) and (73) give a Jordan-Hölder filtration of $\Phi_1(\mathscr{I}_Z(1)[1])$ with $\theta(1)_1$-stable quotients $\{\Phi_1(\mathscr{O}_{\mathbf{P}^2}(1)[1]), \Phi_1(\mathscr{O}_{x_1}), \dots, \Phi_1(\mathscr{O}_{x_n})\}$. This set only depends on $\mathrm{Supp}(Z)$. Thus $\Phi_1(\mathscr{I}_Z(1)[1])$ and $\Phi_1(\mathscr{I}_{Z'}(1)[1])$ represent the same S-equivalence class of $\theta(1)_1$-semistable B-modules.

(2) For a line ℓ, we take an element $\mathscr{I}_Z$ of ζ_ℓ. As in Lemma 6.3, we get an exact sequence in $\mathscr{A}_0$

$$0 \to \mathscr{O}_{\mathbf{P}^2}[1] \to \mathscr{I}_Z(1)[1] \to \mathscr{O}_\ell(-n+1)[1] \to 0$$

and $\theta(0)_0(\Phi_0(\mathcal{O}_{\mathbf{P}^2}[1])) = \theta(0)_0(\Phi_0(\mathcal{O}_\ell(-n+1)[1])) = 0$. Hence by a similar argument as in the proof of (1), we see that π_0' contracts the curve ζ_ℓ on $(\mathbf{P}^2)^{[n]}$ to one point. □

If $n \geq 4$, the morphism ψ is small and induces a flip in the sense of [Th]. For general $r > 0$ it will be shown in [O] that κ in the above diagram (70) is the Mori flip for $n \gg 0$ and described by stratified Grassmann bundles.

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