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By Mark Kisin

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Abstract

We prove that, under some mild conditions, a two dimensional p-adic Galois representation which is residually modular and potentially Barsotti-Tate at p is modular. This provides a more conceptual way of establishing the Shimura-Taniyama-Weil conjecture, especially for elliptic curves which acquire good reduction over a wildly ramified extension of $\mathbb{Q}_3$. The main ingredient is a new technique for analyzing flat deformation rings. It involves resolving them by spaces which parametrize finite flat group scheme models of Galois representations.

Introduction

One of the most important advances in number theory in recent years was the proof of modularity of elliptic curves of $\mathbb{Q}$. After the breakthrough of Wiles and Taylor-Wiles [Wi], [TW], proving the modularity of semi-stable elliptic curves, the general case was completed in a series of papers culminating in the work of Breuil-Conrad-Diamond-Taylor [BCDT] treating elliptic curves which attain good reduction over a wildly ramified extension of $\mathbb{Q}_3$. These results were proved by establishing the modularity of certain 2-dimensional global p-adic Galois represen-tations which are potentially Barsotti-Tate at p. That is, after restriction to some finite extension $K/\mathbb{Q}_p$, the representation is required to be the generic fiber of a p-divisible group over the corresponding ring of integers $\mathbb{O}_K$. Unfortunately, when $K/\mathbb{Q}_p$ is wildly ramified such results were known only in certain very special cases (which, luckily, suffice for the application to elliptic curves) and even then only as a result of rather lengthy calculations. The purpose of this paper is to introduce a new technique for dealing with the difficulties which arise in the presence of wild ramification. This has the benefit of yielding a more general modularity result and a more conceptual proof. To be precise, we prove the following:

THEOREM. *Let $p > 2$ be a prime, and S a finite set of primes containing p and the infinite prime. We denote by $G_{\mathbb{Q},S}$ the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathbb{O}$, and residue field $\mathbb{F}$. Let*

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{O})$$

be a continuous representation whose determinant is the cyclotomic character times a finite character. Suppose that

(1) *The composite $\bar{\rho} : G_{\mathbb{Q},S} \xrightarrow{\rho} \mathrm{GL}_2(\mathbb{O}) \to \mathrm{GL}_2(\mathbb{F})$ is absolutely irreducible when restricted to $\mathbb{Q}(\sqrt{(-1)^{p-1/2}p})$,*

(2) *$\bar{\rho}$ is modular,*

(3) *ρ is potentially Barsotti-Tate at p.*

Then ρ is modular.

The proof of the theorem is based on a modularity result over totally real fields. The basic idea is a modification of the patching argument of Taylor-Wiles or, more precisely, of a variant introduced by Diamond [Di2] and Fujiwara [Fu]. There one proves an equality $R \xrightarrow{\sim} \mathbb{T}$ between a deformation ring and a Hecke ring by showing instead that an auxiliary surjection $R_\infty \to \mathbb{T}_\infty$, obtained by patching deformation rings and Hecke rings at auxiliary levels, is an isomorphism. To do this one shows that $\mathbb{T}_\infty$ has a faithful module which is finite flat over a power series

ring $\mathbb{O}[\![x_1, \ldots, x_r]\!]$ for some $r \in \mathbb{N}^+$, while R_∞ is a quotient of another such ring $\mathbb{O}[\![y_1, \ldots, y_r]\!]$. Comparing dimensions then shows that $R_\infty \xrightarrow{\sim} \mathbb{T}_\infty$.

Here we show instead that R_∞ is a power series ring over a certain local deformation ring $R_p^{\mathbf{v}, \psi}$ (see Section (3.5) for the notation) in $r - [F : \mathbb{Q}]$ variables, where F denotes the totally real field. Note that in the cases previously considered by Taylor-Wiles and others the corresponding local deformation ring was always a power series ring (cf. [CDT], [BCDT], [Sav]), and then this condition is equivalent to the previous one. In order to complete the argument we need to show that $R_p^{\mathbf{v}, \psi}$ has dimension $[F : \mathbb{Q}] + 1$ and that it is a domain. The former statement is not too hard to show. Unfortunately the latter statement is not always true, but we are able to give a precise description of the components of $R_p^{\mathbf{v}, \psi}$, and this suffices for (some) applications.

Our method for analyzing local flat deformation rings involves the construction of certain auxiliary schemes which we term "moduli of finite flat group schemes". To explain this, suppose $K/\mathbb{Q}_p$ is a finite extension with absolute Galois group G_K, and that $p > 2$. Let $\mathbb{F}/\mathbb{F}_p$ be a finite extension, and $V_{\mathbb{F}}$ an $\mathbb{F}$-vector space of dimension $d < \infty$, equipped with a continuous G_K-action. Suppose that $V_{\mathbb{F}}$ arises as the generic fiber of a finite flat group scheme over $\mathbb{O}_K$ and (for simplicity) that $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$. We are interested in understanding the universal flat deformation ring $R_{V_{\mathbb{F}}}^{\mathrm{fl}}$. This is a complete local, Noetherian ring, (pro)-representing the functor which to a local Artinian $\mathrm{W}(\mathbb{F})$-algebra A, with residue field $\mathbb{F}$, assigns the set of deformations V_A of $V_{\mathbb{F}}$ to a finite free A-module with a continuous action of G_K, such that V_A arises as the generic fiber of a finite flat group scheme over $\mathbb{O}_K$. When $K/\mathbb{Q}_p$ is ramified, and especially in the presence of wild ramification, one expects that $R_{V_{\mathbb{F}}}^{\mathrm{fl}}$ is highly singular. One reason for the intractability of these rings is that the condition on V_A involves the existence of an auxiliary object - the finite flat group scheme - which is in general not unique. To overcome this problem we study directly the finite flat group schemes which give rise to $V_{\mathbb{F}}$ and its deformations. This leads to the construction of a certain projective $R_{V_{\mathbb{F}}}^{\mathrm{fl}}$-scheme

$$\Theta_{V_{\mathbb{F}}} : \mathscr{GR}_{V_{\mathbb{F}}} \to \mathrm{Spec}\, R_{V_{\mathbb{F}}}^{\mathrm{fl}}.$$

The construction of $\mathscr{GR}_{V_{\mathbb{F}}}$ does not work directly with finite flat group schemes. Instead we use (a variant of) a closely related category of modules defined by Breuil [Br5]. Although $\mathscr{GR}_{V_{\mathbb{F}}}$ is not smooth, its local structure can be described in terms of the local models of Shimura varieties studied by Pappas-Rapoport [PR]. In the case when $\dim_{\mathbb{F}} V_{\mathbb{F}} = 2$ the relevant local models are those for Hilbert modular varieties studied by Deligne-Pappas [DeP]. Thus $\mathscr{GR}_{V_{\mathbb{F}}}$ may be thought of as a resolution of $\mathrm{Spec}\, R_{V_{\mathbb{F}}}^{\mathrm{fl}}$.

It turns out that the map $\Theta_{V_{\mathbb{F}}}$ becomes an isomorphism after p is inverted. Thus the analysis of the components of $\mathrm{Spec}\, R_{V_{\mathbb{F}}}^{\mathrm{fl}}$ is reduced (modulo p-torsion which

we ignore) to those of $\mathscr{GR}_{V_{\mathbb{F}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The singularities of the scheme $\mathscr{GR}_{V_{\mathbb{F}}}$ are sufficiently well behaved that the components of $\mathscr{GR}_{V_{\mathbb{F}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ can often be related to those of $\mathscr{GR}_{V_{\mathbb{F}},0}$, the fiber of $\mathscr{GR}_{V_{\mathbb{F}}}$ over the closed point of $R_{V_{\mathbb{F}}}^{\mathrm{fl}}$. When $d = 2$, we can compute the components of the projective $\mathbb{F}$-scheme $\mathscr{GR}_{V_{\mathbb{F}},0}$ in certain cases. This involves a rather ad hoc calculation inside an affine Grassmannian, and is at present the least conceptual part of the theory. To explain the outcome of this computation note that if $E/\mathbb{Q}_p$ is a finite extension and $x \in \operatorname{Spec} R_{V_{\mathbb{F}}}^{\mathrm{fl}}(E)$ then there is a corresponding representation of G_K on an E-vector space V_x of dimension d. It is not hard to check that the p-adic Hodge type, as well as the dimensions of the maximal unramified quotient and maximal multiplicative subspace of V_x are invariants of the component of $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\mathrm{fl}}$ containing the image of x. One can define corresponding invariants for the components of $\mathscr{GR}_{V_{\mathbb{F}},0}$. When our calculations are successful we show that two points on $\mathscr{GR}_{V_{\mathbb{F}},0}$ with the same invariants are connected by a chain of rational curves, and hence that a component is characterized by these invariants. We conjecture that this is always the case.

When the ramification degree of K satisfies $e(K/\mathbb{Q}_p) < p-1$, then the map $\Theta_{V_{\mathbb{F}}}$ is an isomorphism. In this case we find that $R_{V_{\mathbb{F}}}^{\mathrm{fl}}$ is itself isomorphic to a complete local ring on a Hilbert modular variety. From this optic Ramakrishna's result that this ring is formally smooth when $K = \mathbb{Q}_p$ is related to the fact that modular curves of level prime to p are smooth, even at supersingular points!

We have assumed above that $\operatorname{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$. However, we carry through the theory and the applications without this assumption by using framed deformation rings and the language of groupoids. One of the consequences of this is that the theorem above yields new results even if ρ is ordinary, since we do not assume that the restriction of $\bar{\rho}$ to the decomposition group at p has distinct diagonal characters as in [SW2]. Indeed, this restriction may even have trivial image!

Our motivation for developing this local theory was the global applications to modularity. However it yields results on finite flat group schemes which seem of independent interest. Let us mention two of these (see Sections (2.1) and (2.2)). From now on we no longer assume $\operatorname{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$. The first result describes the set of models of $V_{\mathbb{F}}$. More precisely, by a *finite flat model* of $V_{\mathbb{F}}$ we mean a finite flat group scheme $\mathscr{G}$, equipped with an action of $\mathbb{F}$, and an isomorphism of $\mathbb{F}[G_K]$-modules $\mathscr{G}(\bar{K}) \xrightarrow{\sim} V_{\mathbb{F}}$, where $\bar{K}$ is an algebraic closure of K. Then we have

THEOREM. *There exists a projective $\mathbb{F}$-scheme $\mathscr{GR}_{V_{\mathbb{F}},0}$ such that for any finite extension $\mathbb{F}'/\mathbb{F}$ the set of finite flat models of $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ is in bijection with $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$.*

Of course when $e(K/\mathbb{Q}_p) < p-1$, Raynaud's results [Ra] imply that $\mathscr{GR}_{V_{\mathbb{F}},0}$ is a single point, but in general the theorem reveals a surprising structure on the set of finite flat models.

The second result gives an incredibly simple description of the category of p-divisible groups over $\mathbb{O}_K$, which had been conjectured by Breuil [Br5, 2.1.2]. To state it, we denote by k the residue field of K, and we write

$$\mathfrak{S} = \mathrm{W}(k)[\![u]\!].$$

Fix a uniformiser of $\mathbb{O}_K$ and let $E(u) \in \mathrm{W}(k)[u]$ denote its Eisenstein polynomial. $\mathfrak{S}$ is equipped with a Frobenius ϕ which is the canonical Frobenius on $\mathrm{W}(k)$ and takes u to u^p. We denote by $(\mathrm{Mod\,FI}/\mathfrak{S})_{\mathbb{Z}_p}$ the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a ϕ-semilinear map $\phi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the image of $1 \otimes \phi_{\mathfrak{M}} : \phi^*\mathfrak{M} \to \mathfrak{M}$ contains $E(u)\mathfrak{M}$. Then

THEOREM. *The category* $(\mathrm{Mod\,FI}/\mathfrak{S})_{\mathbb{Z}_p}$ *is equivalent to the category of p-divisible groups over* $\mathbb{O}_K$.

Let us mention a final local application. As remarked above, the usual method of Taylor-Wiles requires a suitable local deformation ring to be a power series ring. In [BCDT] the authors considered local deformations rings $R_{V,\mathbb{O}}^D$, attached to a representation D of the inertia group at p (a p-type). These correspond to certain deformations which come from finite flat group schemes when restricted to a particular extension $K/\mathbb{Q}_p$. The techniques of *loc. cit.* succeed in establishing modularity when $R_{V,\mathbb{O}}^D$ is a power series ring. The authors prove this is so in particular cases when $p = 3$, and they give a conjectural description in terms of the p-type of precisely when this should occur. Later Breuil and Mézard [BrM, 2.3.1.1] generalized this conjecture by predicting the Hilbert-Samuel multiplicity of $R_{V,\mathbb{O}}^D$ in terms of an invariant μ_{aut} defined in terms of D.

It turns out that one can *use* the modularity result stated above, together with the techniques of Section 3.4 to relate the Hilbert-Samuel multiplicity of $R_{V,\mathbb{O}}^D$ to that of a certain patched Hecke ring, and then to μ_{aut}. (The construction is explained in [Ki2, §2.2], where it is used to deduce modularity results from cases of the Breuil-Mézard conjecture proved by a different method.) This gives a proof of the Breuil-Mézard conjecture when $k = 2$. We hope to report on this in a future paper.

1. p-adic Hodge theory with coefficients

(1.1) *Breuil modules.* We begin by recalling some definitions from the papers [Br3], [Br4] and [Br5].

We fix a finite extension k of $\mathbb{F}_p$, and we assume that $p \neq 2$. Write $W = W(k)$ for the ring of Witt vectors of k, $K_0 = W[1/p]$ and K for a finite totally ramified extension of K_0. Let $\mathbb{O}_K$ be the ring of integers in K and $\pi \in \mathbb{O}_K$ a fixed uniformiser. We fix an algebraic closure $\bar{K}$ of K, and write $\mathbb{O}_{\bar{K}}$ for its ring of integers. We set $G_K = \mathrm{Gal}(\bar{K}/K)$.

(1.1.1) Let $E(u)$ be the minimal polynomial of π over K_0, and consider the surjective map $s : W[u] \overset{u \mapsto \pi}{\to} \mathbb{O}_K$. We denote by S the p-adic completion of the divided power envelope of $W[u]$ with respect to $\ker(s)$. Let $\mathrm{Fil}^1 S \subset S$ be the p-adic completion of the ideal generated by the divided powers $\gamma_i(E(u)) = E(u)^i/i!$. The map s extends uniquely to a surjection $S \to \mathbb{O}_K$. There is a unique map $\phi : W[u] \to W[u]$ which extends the Frobenius on W and satisfies $\phi(u) = u^p$. Since $E(u) = u^e$ modulo p, where $e = [K : K_0]$, the ideal $(p, E(u))$ is stable by ϕ, and ϕ has a unique continuous extension to S, which we again denote by ϕ. One checks that $\phi(\mathrm{Fil}^1 S) \subset pS$, and we write $\phi_1 = p^{-1}\phi|_{\mathrm{Fil}^1 S}$, and $c = \phi_1(E(u))$. Note that c is a unit in S. For n a positive integer, we write $S_n = S/p^n S$.

Let $'(\mathrm{Mod}/S)$ denote the category whose objects are triples $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \phi_1)$, consisting of

(1) an S-module $\mathcal{M}$.

(2) an S-submodule $\mathrm{Fil}^1\mathcal{M} \subset \mathcal{M}$ containing $\mathrm{Fil}^1 S \cdot \mathcal{M}$.

(3) a ϕ-semi-linear map $\phi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ such that for all $s \in \mathrm{Fil}^1 S$ and $x \in \mathcal{M}$ we have $\phi_1(sx) = c^{-1}\phi_1(s)\phi_1(E(u)x)$

Morphisms are given by maps of S-modules respecting Fil^1's and ϕ_1. There is a natural structure of exact category on $'(\mathrm{Mod}/S)$. A sequence is short exact if it is short exact as a sequence of S-modules, and induces a short exact sequence on Fil^1's.

We denote by $(\mathrm{Mod}\,\mathrm{FI}/S)$ the full subcategory of $'(\mathrm{Mod}/S)$ consisting of objects such that

(1) As an S-module $\mathcal{M}$ is isomorphic to $\bigoplus_{i \in I} S/p^{n_i}S$ for some finite set I, and n_i a positive integer.

(2) $\phi_1(\mathrm{Fil}^1\mathcal{M})$ generates $\mathcal{M}$ over S.

Finally we denote by (Mod/S) the smallest full subcategory of $'(\mathrm{Mod}/S)$ which contains the objects of $(\mathrm{Mod}\,\mathrm{FI}/S)$ which are killed by p, and is stable by extension. (Mod/S) contains $(\mathrm{Mod}\,\mathrm{FI}/S)$ as a full subcategory, and the objects killed by p in these two categories coincide.

(1.1.2) Let $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)$ denote the category of finite flat group schemes over $\mathrm{Spec}\,\mathbb{O}_K$ of p-power order. We denote by $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}}$ the full subcategory of $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)$ consisting of objects G such that for all positive integers n, $G[p^n]$ (considered as an fppf sheaf) is again represented by a finite flat group scheme.

We will make extensive use of the following result:

THEOREM (1.1.3) (Breuil). *There exist quasi-inverse anti-equivalences of categories*

$$\mathrm{Gr} : (\mathrm{Mod}/S) \to (p\text{-}\mathrm{Gr}/\mathbb{O}_K) \text{ and } \mathrm{Mod} : (p\text{-}\mathrm{Gr}/\mathbb{O}_K) \to (\mathrm{Mod}/S).$$

These restrict to anti-equivalences

$$\mathrm{Gr} : (\mathrm{Mod}\,\mathrm{FI}/S) \to (p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}} \text{ and } \mathrm{Mod} : (p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}} \to (\mathrm{Mod}\,\mathrm{FI}/S).$$

These functors preserve short exact sequences.

Proof. This is in [Br3, 4.2.1.6, 4.2.2.5]. □

(1.1.4) We denote by

$$D : (p\text{-}\mathrm{Gr}/\mathbb{O}_K) \to (p\text{-}\mathrm{Gr}/\mathbb{O}_K)$$

the involution given by Cartier duality.

It will be more convenient to work with the functors obtained from those in (1.1.3) by composing with D. For this we need the following:

LEMMA (1.1.5). *D induces an involution*

$$D : (p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}} \to (p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}}.$$

Proof. Let G be in $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}}$. Consider the exact sequence of fppf sheaves

$$(1.1.6) \qquad\qquad 0 \to G[p^n] \to G \xrightarrow{p^n} G \to G/p^n G \to 0$$

Since $G[p^n]$ is in $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)$, $p^n G = G/G[p^n] \subset G$ is in $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)$, and hence is a closed subgroup of G. Thus $G/p^n G$ is also in $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)$. This shows that (1.1.6) splits into two short exact sequences in $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)$. Since D is exact we see that $D(G)[p^n]$ may be identified with $D(G/p^n G)$, so that $D(G)$ is in $(p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}}$. □

COROLLARY (1.1.7). *Composing D with* Gr *and* Mod *induces equivalences of categories*

$$\mathrm{Mod}_D := \mathrm{Mod} \circ D : (p\text{-}\mathrm{Gr}/\mathbb{O}_K) \to (\mathrm{Mod}/S)$$

and

$$\mathrm{Gr}_D := D \circ \mathrm{Gr} : (\mathrm{Mod}/S) \to (p\text{-}\mathrm{Gr}/\mathbb{O}_K).$$

These induce equivalences of categories $\mathrm{Gr}_D : (\mathrm{Mod}\,\mathrm{FI}/S) \to (p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}}$ *and* $\mathrm{Mod}_D : (p\text{-}\mathrm{Gr}/\mathbb{O}_K)^{\mathrm{fl}} \to (\mathrm{Mod}\,\mathrm{FI}/S).$ □

(1.1.8) We recall the definitions of [Br3] and [Br4], as well as a slight variant. Let $\mathfrak{S} = W[\![u]\!]$ and $\mathfrak{S}_n = W_n[\![u]\!]$. The ring $\mathfrak{S}$ is equipped with a Frobenius endomorphism ϕ given by $u \mapsto u^p$, and the natural Frobenius on W.

Denote by $'(\mathrm{Mod}/\mathfrak{S})$ the category of $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a ϕ-semi-linear map $\phi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\phi^*(\mathfrak{M}) := \mathfrak{S} \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$, the $\mathfrak{S}$-linear map induced by ϕ, is killed by $E(u)$. We give $'(\mathrm{Mod}/\mathfrak{S})$ the structure of an exact category induced by that on the abelian category of $\mathfrak{S}$-modules.

We denote by $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})$ the full subcategory of $'(\mathrm{Mod}/\mathfrak{S})$ consisting of those $\mathfrak{M}$ such that as an $\mathfrak{S}$-module $\mathfrak{M}$ is isomorphic to $\bigoplus_{i \in I} \mathfrak{S}_{n_i}$, where I is a finite set of positive integers, and n_i is a non-negative integer.

Finally we denote by $(\mathrm{Mod}/\mathfrak{S})$ the smallest full subcategory of $'(\mathrm{Mod}/\mathfrak{S})$ which contains the objects of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})$ which are killed by p, and is stable under extensions. $(\mathrm{Mod}/\mathfrak{S})$ contains $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})$ and the objects killed by p in these two categories coincide.

LEMMA (1.1.9). *For any $\mathfrak{M}$ in* $(\mathrm{Mod}/\mathfrak{S})$, *the map* $1 \otimes \phi : \phi^*\mathfrak{M} \to \mathfrak{M}$ *is injective.*

Proof. By dévissage it suffices to consider the case when $\mathfrak{M}$ is killed by p. Then $\mathfrak{M}$ is a free $\mathfrak{S}/p\mathfrak{S}$-module of some rank $r \in \mathbb{N}$, and the determinant of $1 \otimes \phi$ (in any choice of bases) divides u^{er} because the image of $1 \otimes \phi$ contains $u^e\mathfrak{M}$. Since $u^{er} \in \mathfrak{S}/p\mathfrak{S}$ is not a zero divisor, this implies the claimed injectivity. $\square$

(1.1.10) We have a functor $(\mathrm{Mod}/\mathfrak{S}) \to (\mathrm{Mod}/S)$ given as follows (cf. [Br5, §2.2] and the proof of [Br4, 3.3.2]): We have a map of W-algebras $\mathfrak{S} \to S$ given by $u \mapsto u$, so we regard S as an $\mathfrak{S}$-algebra. We will denote by ϕ the map $\mathfrak{S} \to S$ obtained by composing this map with ϕ on $\mathfrak{S}$. Given $\mathfrak{M}$ in $(\mathrm{Mod}/\mathfrak{S})$, set $\mathcal{M} = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$.

One has the map $1 \otimes \phi : S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \to S \otimes_{\mathfrak{S}} \mathfrak{M}$. Note that $\mathrm{Tor}^1_{\mathfrak{S}}(S/\mathrm{Fil}^1 S, \mathfrak{M}) = 0$, since $\mathfrak{M}$ is a successive extension of free $\mathfrak{S}/p\mathfrak{S}$-modules, so that $\mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M}$ is a submodule of $S \otimes_{\mathfrak{S}} \mathfrak{M}$. Set

$$\mathrm{Fil}^1 \mathcal{M} = \{ y \in \mathcal{M} : (1 \otimes \phi)(y) \in \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M} \}$$

and define $\phi_1 : \mathrm{Fil}^1 \mathcal{M} \to \mathcal{M}$ as the composite

$$\mathrm{Fil}^1 \mathcal{M} \xrightarrow{1 \otimes \phi} \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\phi_1 \otimes 1} S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}.$$

This gives $\mathcal{M}$ the structure of an object of $'(\mathrm{Mod}/S)$.

We have the following result (cf. [Br5, 3.1.3]):

PROPOSITION (1.1.11). *The functor* $(\mathrm{Mod}/\mathfrak{S}) \to {}'(\mathrm{Mod}/S)$ *above*, *induces exact and fully faithful functors*

$$(\mathrm{Mod}/\mathfrak{S}) \to (\mathrm{Mod}/S) \ \text{and} \ (\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S}) \to (\mathrm{Mod}\,\mathrm{FI}/S).$$

These functors are equivalences of categories between the full subcategories consisting of objects killed by p.

Proof. We first check that the functor of (1.1.10) is exact. Let

$$0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$$

be a short exact sequence in (Mod/$\mathfrak{S}$). Since $\mathfrak{M}''$ is a successive extension of free $\mathfrak{S}/p\mathfrak{S}$-modules, and S is p-torsion free, one sees that $\mathrm{Tor}^1_{\mathfrak{S}}(S, \mathfrak{M}) = 0$ where we regard S as a $\mathfrak{S}$-module via ϕ. It follows easily that the functor of (1.1.10) gives rise to an exact sequence of S-modules

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0,$$

and a left exact sequence

$$0 \to \mathrm{Fil}^1 \mathcal{M}' \to \mathrm{Fil}^1 \mathcal{M} \to \mathrm{Fil}^1 \mathcal{M}''.$$

To check that the final map is a surjection, choose $x \in \mathrm{Fil}^1 \mathcal{M}''$. Since $\mathrm{Fil}^1 S + \mathfrak{S} = S$ and $\mathrm{Fil}^1 S \cdot \mathcal{M} \subset \mathrm{Fil}^1 \mathcal{M}$, we may alter x by an element of $\mathrm{Fil}^1 S \cdot \mathcal{M}''$, and assume that x is the image of some $\tilde{x} \in \phi^*(\mathfrak{M}'')$. Now, by definition

$$\mathrm{Fil}^1 \mathcal{M}'' = \ker(\mathcal{M}'' = S \otimes_{\phi, \mathfrak{S}} \mathfrak{M}'' \xrightarrow{1 \otimes \phi} S \otimes_{\mathfrak{S}} \mathfrak{M}'' / \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M}'' \xrightarrow{\sim} \mathfrak{M}'' / E(u) \cdot \mathfrak{M}'').$$

Hence $\tilde{x} \in (1 \otimes \phi)^{-1}(E(u) \cdot \mathfrak{M}'')$. Since $E(u) \cdot \mathfrak{M}$ surjects onto $E(u) \cdot \mathfrak{M}''$, (1.1.9) implies that there exists $\tilde{y} \in \phi^*(\mathfrak{M})$ which maps to $\tilde{x}$ and is such that $(1 \otimes \phi)(\tilde{y}) \in E(u) \cdot \mathfrak{M}$. Then the image of $\tilde{y}$ in $\mathcal{M}$ is contained in $\mathrm{Fil}^1 \mathcal{M}$ and maps to x. This proves the exactness.

Next we note that if $\mathfrak{M}$ is in (Mod/$\mathfrak{S}$) then the image of ϕ_1 generates $\mathcal{M}$ as an S-module, because $1 \otimes \phi(\mathrm{Fil}^1 \mathcal{M})$ contains $E(u) \otimes \mathfrak{M}$, so that $\phi_1(\mathrm{Fil}^1 \mathcal{M})$ contains $c \otimes \mathfrak{M}$, where $c \in S^{\times}$ is the element defined in (1.1.1). Together with the exactness proved above, this shows that we have functors

$$(\mathrm{Mod}/\mathfrak{S}) \to (\mathrm{Mod}/S) \quad \text{and} \quad (\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S}) \to (\mathrm{Mod}\,\mathrm{FI}/S).$$

The proof of [Br4, 3.3.2] shows that our functors induce equivalences between the full subcategories consisting of objects killed by p. The proposition now follows by a dévissage argument as in [FoL, p. 584]. This uses the fact that short exact sequences in (Mod/$\mathfrak{S}$) and (Mod/S) give rise to exact sequences of Hom's and Ext^1's in the usual way. This property holds in any exact category in the sense of [Qu, §2]. □

(1.1.12) We now explain the connection between the categories introduced above and Galois representations (cf. [Fo, A.3] and [Br1, 4.2]).

Let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ where the transition maps are given by Frobenius. By the universal property of the Witt vectors $\mathrm{W}(R)$ of R, there is a unique surjective map

$\theta : W(R) \to \widehat{\mathbb{O}_{\bar{K}}}$ to the p-adic completion $\widehat{\mathbb{O}_{\bar{K}}}$, which lifts the projection $R \to \mathbb{O}_{\bar{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$, with respect to $\ker(\theta)$.

Let $\sqrt[p^n]{\pi} \in \bar{K}$ be a root of π, such that $(\sqrt[p^{n+1}]{\pi})^p = \sqrt[p^n]{\pi}$. Write $\underline{\pi} = (\sqrt[p^n]{\pi})_{n \geq 0} \in R$, and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative. We embed the W-algebra $W[u] \subset S$ into A_{cris} by $u \mapsto [\underline{\pi}]$. Since $\theta([\underline{\pi}]) = \pi$ this embedding extends to an embedding $S \hookrightarrow A_{\mathrm{cris}}$, and $\theta|_S$ is the map $S \to \mathbb{O}_K$ constructed in (1.1.1). This embedding is compatible with Frobenius endomorphisms, and identifies $\mathfrak{S} \subset S$ with a subring of $W(R)$.

Denote by $\mathbb{O}_{\mathscr{E}}$ the p-adic completion of $\mathfrak{S}[1/u]$. Then $\mathbb{O}_{\mathscr{E}}$ is a discrete valuation ring with residue field the Laurent series ring $k((u))$. We write $\mathscr{E}$ for the field of fractions of $\mathbb{O}_{\mathscr{E}}$. If $\mathrm{Fr}R$ denotes the field of fractions of R, then the inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathscr{E} \hookrightarrow W(\mathrm{Fr}R)[1/p]$. Let

$$\mathscr{E}^{\mathrm{ur}} \subset W(\mathrm{Fr}R)[1/p]$$

denote the maximal unramified extension of $\mathscr{E}$ contained in $W(\mathrm{Fr}R)[1/p]$, and $\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}} \subset W(\mathrm{Fr}R)$ its ring of integers. The field $\mathrm{Fr}R$ is algebraically closed [Fo, A.3.1.6], so the residue field $\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}/p\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}$ is a separable closure of $k((u))$. We denote by $\mathbb{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}}$ the p-adic completion of $\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}$, and by $\widehat{\mathscr{E}^{\mathrm{ur}}}$ its field of fractions. We set $\mathfrak{S}^{\mathrm{ur}} = \mathbb{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}} \cap W(R) \subset W(\mathrm{Fr}R)$.

Let $K_\infty = \cup_{n \geq 1} K(\sqrt[p^n]{\pi})$, and write $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$. We will denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_\infty})$ the category of continuous representations of G_{K_∞} on finite $\mathbb{Z}_p$-modules. G_{K_∞} fixes $\mathbb{O}_{\mathscr{E}} \subset W(\mathrm{Fr}R)$, and hence $\mathbb{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}}$ is stable under the action of G_{K_∞}.

We denote by $\Phi\mathbf{M}_{\mathbb{O}_{\mathscr{E}}}$ the category of finite $\mathbb{O}_{\mathscr{E}}$-modules M, equipped with the ϕ-semi-linear map $M \to M$, such that the induced $\mathbb{O}_{\mathscr{E}}$-linear map $\phi^*(M) \to M$ is an isomorphism.

The argument of [Br1, 2.1.1] shows that K_∞/K is a strictly APF extension in the sense of [Win]. Then [Fo, A. 1.2.6], and the constructions of [Fo, A. 3] imply that the functor

$$T : \Phi\mathbf{M}_{\mathbb{O}_{\mathscr{E}}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_{K_\infty}); \ M \mapsto (\mathbb{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}} \otimes_{\mathbb{O}_{\mathscr{E}}} M)^{\phi=1}$$

is an equivalence of abelian categories (cf. also [Br4, 3.3]). A quasi-inverse is given by

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_\infty}) \to \Phi\mathbf{M}_{\mathbb{O}_{\mathscr{E}}}; \ V \mapsto (\mathbb{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}.$$

Note that $E(u) \in \mathbb{O}_{\mathscr{E}}$ is a unit, since its image in the residue field of $\mathbb{O}_{\mathscr{E}}$ is u^e. Thus, if $\mathfrak{M}$ is in (Mod FI/$\mathfrak{S}$), then $\mathbb{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ has a natural structure of an object of $\Phi\mathbf{M}_{\mathbb{O}_{\mathscr{E}}}$.

PROPOSITION (1.1.13). *Let $\mathfrak{M}$ be in $(\mathrm{Mod}/\mathfrak{S})$, and $\mathcal{M}$ in (Mod/S), the image of $\mathfrak{M}$ under the functor of (1.1.11). There is a canonical isomorphism of G_{K_∞}-representations*

$$T\left(\mathbb{O}_{\mathscr{E}} \underset{\mathfrak{S}}{\otimes} \mathfrak{M}\right)(1) \xrightarrow{\sim} \mathrm{Gr}_D(\mathcal{M})(\mathbb{O}_{\bar{K}})|_{G_{K_\infty}},$$

where, as usual (1), *denotes the Tate twist.*

Proof. Since $\mathrm{Gr}_D(\mathcal{M})(\mathbb{O}_{\bar{K}}) = \mathrm{Hom}(\mathrm{Gr}(\mathcal{M})(\mathbb{O}_{\bar{K}}), \mathbb{Q}_p/\mathbb{Z}_p(1))$, it suffices by [Fo, A.1.2.7] to construct a canonical G_{K_∞}-invariant isomorphism

$$\mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, \mathscr{E}^{\mathrm{ur}}/\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}) \xrightarrow{\sim} \mathrm{Gr}(\mathcal{M})(\mathbb{O}_{\bar{K}}).$$

For this we follow the method of [Br4, 3.3.2]. By [Fo, B.1.8.4], the natural map

$$\mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, \mathscr{E}^{\mathrm{ur}}/\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}})$$

is an isomorphism. In particular [Fo, A.§1.2] implies that the left-hand side is exact in $\mathfrak{M}$. On the other hand by [Br2, 2.3.11] and the description of $\mathrm{Gr}(\mathcal{M})(\mathbb{O}_{\bar{K}})$ given in [Br3, 5.3.1], we have an isomorphism of G_{K_∞}-modules

$$\mathrm{Gr}(\mathcal{M})(\mathbb{O}_{\bar{K}}) \xrightarrow{\sim} \mathrm{Hom}_{S,\phi_1,\mathrm{Fil}^1}(\mathcal{M}, A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}).$$

Here the filtration on $A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}$ is induced by the usual filtration on A_{cris}. Since $\phi(\mathrm{Fil}^1 A_{\mathrm{cris}}) \subset p A_{\mathrm{cris}}$, A_{cris} and $A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}$ are equipped with the structure of an object of $'(\mathrm{Mod}/S)$. Thus, it suffices to show that the natural map

$$(1.1.14) \quad \mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}) \to \mathrm{Hom}_{S,\phi_1,\mathrm{Fil}^1}(\mathcal{M}, A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}})$$

obtained by sending $f : \mathfrak{M} \to \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}}$ to

$$\mathcal{M} = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \overset{1\otimes f}{\to} S \otimes_{\phi,\mathfrak{S}} \mathfrak{S}^{\mathrm{ur}}[1/p]/\mathfrak{S}^{\mathrm{ur}} \overset{1\otimes \phi}{\to} A_{\mathrm{cris}}[1/p]/A_{\mathrm{cris}}$$

is an isomorphism. We have already seen that the left-hand side of (1.1.14) is exact in $\mathfrak{M}$, while the right side is exact in $\mathcal{M}$ since Gr is exact. Thus, by dévissage, it suffices to consider the case when $p \cdot \mathfrak{M} = 0$, and this follows from [Br4, 3.3.2] and its proof. □

LEMMA (1.1.15). *Let $\mathfrak{M}$ and $\mathcal{M}$ be as in (1.1.13). Then $\mathrm{Gr}_D(\mathcal{M})$ is étale (resp. multiplicative), if and only if the map $1 \otimes \phi : \phi^* \mathfrak{M} \to \mathfrak{M}$ has image equal to $E(u) \cdot \mathfrak{M}$ (resp. is an isomorphism).*

Proof. By dévissage, it suffices to consider the case when $\mathfrak{M}$ and $\mathcal{M}$ are killed by p. Using [BCDT, 5.1.3], one sees that $\mathrm{Gr}_D(\mathcal{M})$ is étale (resp. multiplicative) if and only if $\mathrm{Fil}^1 \mathcal{M} = \mathcal{M}$ (resp. $\mathrm{Fil}^1 \mathcal{M} = \mathrm{Fil}^1 S \cdot \mathcal{M}$). (Note that the result of *loc. cit.* describes the Dieudonné module of $\mathrm{Gr}(\mathcal{M})$.)

Now from the definition of $\mathrm{Fil}^1\mathcal{M}$ we have an embedding

$$\mathcal{M}/\mathrm{Fil}^1\mathcal{M} \overset{1\otimes\phi}{\hookrightarrow} S\otimes_{\mathfrak{S}}\mathfrak{M}/(\mathrm{Fil}^1 S\otimes_{\mathfrak{S}}\mathfrak{M}) \overset{\sim}{\longrightarrow} \mathfrak{M}/E(u)\mathfrak{M}.$$

The image of this embedding is the image of the composite $\phi^*(\mathfrak{M}) \to \mathfrak{M} \to \mathfrak{M}/E(u)\mathfrak{M}$. From this we see that $\mathrm{Fil}^1\mathcal{M} = \mathcal{M}$ if and only if $1\otimes\phi$ has image $E(u)\cdot\mathfrak{M}$. Similarly, using the above embedding, and the fact that $\mathcal{M}/\mathrm{Fil}^1 S\cdot\mathcal{M}\overset{\sim}{\longrightarrow}\phi^*(\mathfrak{M})/E(u)\phi^*(\mathfrak{M})$, one sees that $\mathrm{Fil}^1\mathcal{M} = \mathrm{Fil}^1 S\cdot\mathcal{M}$ if and only if $1\otimes\phi$ is an isomorphism. $\square$

(1.1.16) We call an object $\mathfrak{M}$ of $(\mathrm{Mod}/\mathfrak{S})$ étale if the image of $1\otimes\phi$ is equal to $E(u)\mathfrak{M}$, and *multiplicative* if $1\otimes\phi$ is an isomorphism.

(1.2) *Breuil modules with coefficients.* For a $\mathbb{Z}_p$-algebra A, we set $S_A = S\otimes_{\mathbb{Z}_p} A$ and $\mathfrak{S}_A = \mathfrak{S}\otimes_{\mathbb{Z}_p} A$. We denote by $'(\mathrm{Mod}/S)_A$ the category consisting of pairs $(\mathcal{M}, \iota)$ where $\mathcal{M}$ is in $'(\mathrm{Mod}/S)$ and $\iota: A \to \mathrm{End}(\mathcal{M})$ is a $\mathbb{Z}_p$-algebra map. We define similarly the category $'(\mathrm{Mod}/\mathfrak{S})_A$.

(1.2.1) We denote by $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_A$ the full subcategory of $'(\mathrm{Mod}/\mathfrak{S})_A$ consisting of those objects $\mathfrak{M}$ such that $\mathfrak{M}$ is a finite projective $\mathfrak{S}_A$-module.

We denote by $(\mathrm{Mod}\,\mathrm{FI}/S)_A$ the full subcategory of $'(\mathrm{Mod}/S)_A$ consisting of those objects $\mathcal{M}$ such that

(1) $\mathcal{M}$ is a finite projective S_A-module.

(2) The quotient $\mathcal{M}/\mathrm{Fil}^1\mathcal{M}$ is a finite projective A-module.

(3) The image of ϕ_1 generates $\mathcal{M}$ as an S-module.

We will only make use of the category $(\mathrm{Mod}\,\mathrm{FI}/S)_A$ when A is a finite $\mathbb{Z}_p$-algebra. However, we will make crucial use of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_A$ in more general situations. Note that if A is finite then an object of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_A$ (resp. $(\mathrm{Mod}\,\mathrm{FI}/S)_A$) may be viewed as an object of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})$ (resp. $(\mathrm{Mod}\,\mathrm{FI}/S)$).

LEMMA (1.2.2). *Suppose that $\mathfrak{M}$ is in $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_A$. Then*:

(1) *The natural map $1\otimes\phi: \phi^*(\mathfrak{M}) \to \mathfrak{M}$ is injective.*

(2) *The cokernel of the map $1\otimes\phi$ is a finite projective A-module.*

(3) *$(1\otimes\phi)(\phi^*(\mathfrak{M}))/E(u)\mathfrak{M}$ is a finite projective A-module.*

(4) *Suppose that $|A| < \infty$. Then locally on $\mathrm{Spec}\,A$, $\mathfrak{M}$ is a finite free $\mathfrak{S}_A$-module.*

Proof. We may assume that A is a finitely generated $\mathbb{Z}_p$-algebra, and that $\mathfrak{M}$ has constant $\mathfrak{S}\otimes_{\mathbb{Z}_p} A$-rank r. The map $1\otimes\phi$ is a map between two free $\mathfrak{S}\otimes_{\mathbb{Z}_p} A$-modules of the same rank. Since its image contains $E(u)\mathfrak{M}$, its determinant in any choice of bases divides $E(u)^r$. Since $\mathfrak{S}/E(u)^r$ is $\mathbb{Z}_p$-flat, being an extension of copies of $\mathcal{O}_K$, $E(u)^r$ is not a zero divisor in $\mathfrak{S}\otimes_{\mathbb{Z}_p} A$, and (1) follows. The same

argument also shows that $1 \otimes \phi$ remains injective after applying $\otimes_A A/I$ for any ideal I of A. Since $\mathfrak{M}$ is A-flat, this shows that the cokernel of $1 \otimes \phi$ is A-flat. Since it is a quotient of the finite $\mathbb{O}_K \otimes_{\mathbb{Z}_p} A$-module $\mathfrak{M}/E(u)\mathfrak{M}$, it must be finite and hence projective over A. This proves (2).

For (3) note that we have an exact sequence

$$0 \to (1 \otimes \phi)(\phi^*(\mathfrak{M}))/E(u)\mathfrak{M} \to \mathfrak{M}/E(u)\mathfrak{M} \to \mathfrak{M}/(1 \otimes \phi)(\phi^*(\mathfrak{M})) \to 0.$$

The term on the right is A-projective, and the middle term is $\mathbb{O}_K \otimes_{\mathbb{Z}_p} A$-projective, and hence A-projective. (3) follows.

Finally, to show (4), we may replace A by a residue field at a prime ideal, and assume that A is a finite field. Then $\mathfrak{S}_A$ has finitely many maximal ideals. If $\mathfrak{p}$ is such an ideal, denote by $r_{\mathfrak{p}}$ the rank of $\mathfrak{M}$ at $\mathfrak{p}$. We extend ϕ to an A-linear map $\phi : \mathfrak{S}_A \to \mathfrak{S}_A$. Since $1 \otimes \phi$ identifies $\phi^*(\mathfrak{M})$ with a submodule of $\mathfrak{M}$, by (1), we have $r_{\mathfrak{p}} \leqslant r_{\phi(\mathfrak{p})}$. But ϕ permutes the maximal ideals of $\mathfrak{S}_A$ transitively, and so all the $r_{\mathfrak{p}}$ must be equal. It follows that $\mathfrak{M}$ is a free $\mathfrak{S}_A$-module. $\square$

LEMMA (1.2.3). *Let $A \to A'$ be a map of $\mathbb{Z}_p$-algebras. Then there are natural functors*

$$\otimes_A A' : (\mathrm{Mod}\ \mathrm{FI}/S)_A \to (\mathrm{Mod}\ \mathrm{FI}/S)_{A'}$$

and

$$\otimes_A A' : (\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_A \to (\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_{A'}.$$

Proof. If $(\mathcal{M}, \mathrm{Fil}^1, \phi_1)$ is in $(\mathrm{Mod}\ \mathrm{FI}/S)_A$, then $(\mathcal{M} \otimes_A A', \mathrm{Fil}^1 \mathcal{M} \otimes_A A', \phi_1 \otimes 1)$ is in $(\mathrm{Mod}\ \mathrm{FI}/S)_{A'}$. Note that $\mathrm{Fil}^1 \mathcal{M} \otimes_A A' \subset \mathcal{M} \otimes_A A'$ as $\mathcal{M}/\mathrm{Fil}^1 \mathcal{M}$ is A-projective. A similar remark applies to $(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_A$. $\square$

LEMMA (1.2.4). *When $|A| < \infty$, the functor of (1.1.10) induces an exact and fully faithful functor*

$$(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_A \to (\mathrm{Mod}\ \mathrm{FI}/S)_A.$$

Proof. If $\mathfrak{M}$ is in $(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_A$, then $\mathcal{M} = S \otimes_{\phi, \mathfrak{S}} \mathfrak{M}$ has a natural structure of an object of $(\mathrm{Mod}\ \mathrm{FI}/S)$ by (1.1.10), and it is equipped with an A-action by functoriality. To check that $\mathcal{M}/\mathrm{Fil}^1 \mathcal{M}$ is a projective A-module, note that, as in the proof of (1.1.15), we have an embedding

$$\mathcal{M}/\mathrm{Fil}^1 \mathcal{M} \overset{1 \otimes \phi}{\hookrightarrow} S \otimes_{\mathfrak{S}} \mathfrak{M}/(\mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M}) \overset{\sim}{\to} \mathfrak{M}/E(u)\mathfrak{M}$$

whose image is the image of the composite $\phi^*(\mathfrak{M}) \to \mathfrak{M} \to \mathfrak{M}/E(u)\mathfrak{M}$, which is A-projective by (1.2.2)(3). This proves the existence of the required functor. The exactness and full faithfulness follow from the corresponding properties in (1.1.11). $\square$

LEMMA (1.2.5). *If $|A| < \infty$ and $p \cdot A = 0$, then the functor of (1.2.4) is an equivalence.*

Proof. Suppose $\mathcal{M}$ is in $(\mathrm{Mod}\,\mathrm{FI}/S)_A$. By (1.1.11), $\mathcal{M}$ viewed as an object of $(\mathrm{Mod}\,\mathrm{FI}/S)$ arises from an object $\mathfrak{M}$ of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})$, and $\mathfrak{M}$ is equipped with an action of A by functoriality. Suppose that $\mathcal{M}$ is a free S_A-module of rank d. Since

$$W \otimes_{\mathfrak{S}} \mathfrak{M} \underset{\phi \otimes 1}{\overset{\sim}{\longrightarrow}} W \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \overset{\sim}{\longrightarrow} W \otimes_S \mathcal{M},$$

where we view W as an S-algebra via $u \mapsto 0$, $W \otimes_{\mathfrak{S}} \mathfrak{M}$ is free of rank d over $k \otimes_{\mathbb{F}_p} A$. Hence there exists a surjection $(\mathfrak{S}_A)^d \to \mathfrak{M}$. On the other hand, this map must be an isomorphism, since both sides are free $\mathfrak{S}/p$-modules, and they have the same rank, as can be checked after applying $S \otimes_{\phi,\mathfrak{S}}$. $\qquad\square$

(1.2.6) If $|A| < \infty$, then we denote by $\mathrm{Rep}'_A(G_{K_\infty})$ the category of continuous representations of G_{K_∞} on finite A-modules. We denote by $\mathrm{Rep}_A(G_{K_\infty})$ the full subcategory of $\mathrm{Rep}'_A(G_{K_\infty})$ consisting of those objects which are free as A-modules.

We denote by $\Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}},A}$ the category consisting of an object of $\Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}}}$ equipped with an action of A.

LEMMA (1.2.7). (1) *The functor T of (1.1.12) induces an equivalence of abelian categories*

$$T_A : \Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}},A} \to \mathrm{Rep}'_A(G_{K_\infty}).$$

(2) *If $A \to A'$ is a finite map then there is a functor*

$$\otimes_A A' : \Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}},A} \to \Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}},A'}; \quad M \mapsto M \otimes_A A'.$$

(3) *For M in $\Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}},A}$ there is a natural isomorphism*

$$T_A(M) \otimes_A A' \overset{\sim}{\longrightarrow} T_{A'}(M \otimes_A A').$$

(4) *If M is in $\Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}},A}$, then $T_A(M)$ is free over A of rank $r \in \mathbb{N}^+$ if and only if M is a free $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$-module of rank r.*

Proof. (1) follows immediately from the fact that T is an equivalence. (2) is proved as in (1.2.3). For (3) note that for any finite A-module N we have a natural map

(1.2.8) $T_A(M) \otimes_A N \to T_A(M \otimes_A N).$

Since (1.2.8) is evidently an isomorphism when N is free over A, and both sides are right exact in N, it must be an isomorphism in general, as can be seen by taking a presentation of N, and using the 5-lemma. In particular,

$$T_A(M) \otimes_A A' \overset{\sim}{\longrightarrow} T_A(M \otimes_A A') = T_{A'}(M \otimes_A A').$$

For (4), we immediately reduce to the case where A is an Artinian local ring. Let $\mathfrak{m}_A$ denote the maximal ideal of A. The isomorphism (1.2.8), and the exactness of T_A imply that M is A-flat if and only if $T_A(M)$ is A-flat. Thus, it suffices to show that if M is A-flat, then it is finite free over $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$, since

the A-rank of $T_A(M)$ and the $\mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A$-rank of M must then be equal, as the $A/\mathfrak{m}_A$-rank of $T_{A/\mathfrak{m}_A}(M \otimes_A A/\mathfrak{m}_A)$ and the $\mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A/\mathfrak{m}_A$-rank of M are equal by [Fo, A. 1.2.4(i)].

Now if M is A-flat, then M is free over $\mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A$ if and only if $M \otimes_A A/\mathfrak{m}_A$ is free over $\mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A/\mathfrak{m}_A$; so we may assume that A is a finite field. Then $\mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A$ is a product of fields, so that M is automatically projective over $\mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A$. Now an argument using Frobenius, as in the proof of (1.2.2)(4), shows that M is free over $\mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A$. $\qquad\square$

LEMMA (1.2.9). *Suppose that A is a local $\mathbb{Z}_p$-algebra with $|A| < \infty$.*

(1) *The functor T_A of (1.2.7) induces a functor*

$$T_{\mathfrak{S},A} : (\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_A \to \mathrm{Rep}_A(G_{K_\infty}); \quad \mathfrak{M} \mapsto T_A(\mathbb{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}).$$

(2) *If $A \to A'$ is a finite map, then for $\mathfrak{M}$ in $(\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_A$ there is a natural isomorphism $T_{\mathfrak{S},A}(\mathfrak{M}) \otimes_A A' \xrightarrow{\sim} T_{\mathfrak{S},A'}(\mathfrak{M} \otimes_A A')$.*

Proof. This follows from (1.2.7). $\qquad\square$

(1.2.10) Now let A be any $\mathbb{Z}_p$-algebra. Following (1.1.16) we call an object $\mathfrak{M}_A$ of $(\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_A$ étale (resp. multiplicative) if $\phi^*\mathfrak{M}_A \to \mathfrak{M}_A$ has image equal to $E(u)\mathfrak{M}_A$ (resp. is an isomorphism).

Given $\mathfrak{M}_A$ in $(\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_A$, we define its dual $\mathfrak{M}_A^*$ as follows: As an $\mathfrak{S}_A$-module, $\mathfrak{M}_A^* = \mathrm{Hom}_{\mathfrak{S}_A}(\mathfrak{M}_A, \mathfrak{S}_A)$, and the map $1 \otimes \phi : \phi^*(\mathfrak{M}_A^*) \to \mathfrak{M}_A^*$ is defined by

$$\phi^*(\mathfrak{M}_A^*) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_A}(\phi^*(\mathfrak{M}_A), \phi^*(\mathfrak{S}_A)) \to \mathrm{Hom}_{\mathfrak{S}_A}(E(u)\mathfrak{M}_A, \phi^*(\mathfrak{S}_A))$$
$$\xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_A}(E(u)\mathfrak{M}_A, \mathfrak{S}_A) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}_A}(\mathfrak{M}_A, \mathfrak{S}_A),$$

where the second map is induced by regarding $\phi^*(\mathfrak{M}_A)$ as an $\mathfrak{S}_A$-submodule of $\mathfrak{M}_A$ via the injective map $1 \otimes \phi$, and restricting maps to $E(u)\mathfrak{M}_A \subset \phi^*(\mathfrak{M}_A)$, the third map is induced by the isomorphism $1 \otimes \phi : \phi^*(\mathfrak{S}_A) \xrightarrow{\sim} \mathfrak{S}_A$, and the final map by the isomorphism

$$\mathfrak{M}_A \xrightarrow{\cdot E(u)} E(u)\mathfrak{M}_A.$$

One sees easily that $\mathfrak{M}_A$ is étale if and only if $\mathfrak{M}_A^*$ is multiplicative, and that the usual double duality isomorphism $\mathfrak{M}_A^{**} \xrightarrow{\sim} \mathfrak{M}_A$ is an isomorphism in $(\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_A$.

PROPOSITION (1.2.11). *Suppose that $|A| < \infty$, and let $\mathfrak{M}_A$ be in $(\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_A$. There exists a maximal multiplicative subobject $\mathfrak{M}_A^m \subset \mathfrak{M}_A$, and a maximal étale quotient $\mathfrak{M}_A^{\mathrm{ét}}$ of $\mathfrak{M}_A$. (That is any other étale quotient of $\mathfrak{M}_A$ is a quotient of $\mathfrak{M}_A^{\mathrm{ét}}$.) These satisfy the following properties*:

(1) *Both the quotient $\mathfrak{M}_A/\mathfrak{M}_A^m$ and the kernel of $\mathfrak{M}_A \to \mathfrak{M}_A^{\mathrm{ét}}$ are objects of $(\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_A$.*

(2) *For any finite A-algebra B,*

$$(\mathfrak{M}_A \otimes_A B)^m = \mathfrak{M}_A^m \otimes_A B \ and \ (\mathfrak{M}_A \otimes_A B)^{\text{ét}} = \mathfrak{M}_A^{\text{ét}} \otimes_A B.$$

(3) *There are natural isomorphisms*

$$(\mathfrak{M}_A^*)^m \xrightarrow{\sim} (\mathfrak{M}_A^{\text{ét}})^* \ and \ (\mathfrak{M}_A^*)^{\text{ét}} \xrightarrow{\sim} (\mathfrak{M}_A^m)^*.$$

Proof. Since $\phi^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ is injective by (1.2.2)(1), and ϕ on $\mathfrak{S}$ is a flat map, the maps $1 \otimes \phi^r : (\phi^*)^r(\mathfrak{M}_A) \to \mathfrak{M}_A$ are also injective, for $r \in \mathbb{N}^+$. From now on we regard $(\phi^*)^r(\mathfrak{M}_A)$ as a submodule of $\mathfrak{M}_A$ via $1 \otimes \phi^r$.

We set $\mathfrak{M}_A^m = \bigcap_{r=1}^\infty (\phi^*)^r \mathfrak{M}_A$. It will be clear that $\mathfrak{M}_A^m \subset \mathfrak{M}_A$ is the maximal multiplicative submodule, once we check that it is a projective $\mathfrak{S}_A$-module. Since the map $\phi^*(\mathfrak{M}_A^m) \to \mathfrak{M}_A^m$ is a bijection, the induced map $\phi^*(\mathfrak{M}_A^m/u\mathfrak{M}_A^m) \to \mathfrak{M}_A^m/u\mathfrak{M}_A^m$ is surjective, and hence bijective. It follows that an element x of $\mathfrak{M}_A^m$ is divisible by u if and only if $\phi^i(x)$ is divisible by u for i sufficiently large. This implies that $\mathfrak{M}_A/\mathfrak{M}_A^m$ is u-torsion free, for if $y \in \mathfrak{M}_A$ and $uy \in \mathfrak{M}_A^m$, then the sequence $\{\phi^i(uy)\}_{i \geq 0}$ goes u-adically to 0 in $\mathfrak{M}_A$ and hence also in $\mathfrak{M}_A^m$ by the Artin-Rees lemma. Hence uy is divisible by u in $\mathfrak{M}_A^m$, and $y \in \mathfrak{M}_A^m$.

Since A is Artinian, there is a canonical decomposition

$$\mathfrak{M}_A/u\mathfrak{M}_A = (\mathfrak{M}_A/u\mathfrak{M}_A)_{\text{nil}} \oplus (\mathfrak{M}_A/u\mathfrak{M}_A)_{\text{unit}}$$

where ϕ is nilpotent on the first factor, and bijective on the second. By what we have just seen $\mathfrak{M}_A^m/u\mathfrak{M}_A^m$ may be identified with an $\mathfrak{S}_A$-submodule of $(\mathfrak{M}_A/u\mathfrak{M}_A)_{\text{unit}} = \bigcap_{r=1}^\infty \phi^r(\mathfrak{M}_A/u\mathfrak{M}_A)$. In fact this inclusion is an equality. To see this suppose that $x_0 \in \bigcap_{r=1}^\infty \phi^r(\mathfrak{M}_A/u\mathfrak{M}_A)$, and for $r \geq 1$, choose $x_r \in \mathfrak{M}_A/u\mathfrak{M}_A$ such that $\phi^r(x_r) = x_0$. Let $\tilde{x}_r \in \mathfrak{M}_A$ be a lift of x_r. Then one checks easily that the sequence $\phi^r(\tilde{x}_r)$ converges to an element of $\mathfrak{M}_A^m$ which maps to x_0.

In particular we see that $\mathfrak{M}_A^m/u\mathfrak{M}_A^m$ is a direct summand in $\mathfrak{M}_A/u\mathfrak{M}_A$ and hence is a projective $\mathfrak{S}_A/u\mathfrak{S}_A$-module. After replacing A by a localization at a maximal ideal, we may assume that $\mathfrak{M}_A/u\mathfrak{M}_A$ is a free A-module, and then an argument as in (1.2.2) shows that it is free over $\mathfrak{S}_A/u\mathfrak{S}_A$. Lifting an isomorphism $(\mathfrak{S}_A/u\mathfrak{S}_A)^d \xrightarrow{\sim} \mathfrak{M}_A^m/u\mathfrak{M}_A^m$, to a surjection $\mathfrak{S}_A^d \to \mathfrak{M}_A^m$, and applying Nakayama's lemma to the kernel of this last map, we see that $\mathfrak{S}_A^d \xrightarrow{\sim} \mathfrak{M}_A^m$.

To check that the formation of $\mathfrak{M}_A^m$ is compatible with extension of scalars, write $\mathfrak{M}_B = \mathfrak{M}_A \otimes_A B$. Since $\mathfrak{M}_A/\mathfrak{M}_A^m$ is u-torsion free, and its reduction modulo u is a projective $\mathfrak{S}_A/u\mathfrak{S}_A$-module, $\mathfrak{M}_A/\mathfrak{M}_A^m$ is flat and hence projective over $\mathfrak{S}_A$. Now the reductions of $\mathfrak{M}_A/\mathfrak{M}_A^m \otimes_A B$ and $\mathfrak{M}_B/\mathfrak{M}_B^m$ modulo u may both be identified with $(\mathfrak{M}_A/u\mathfrak{M}_A)_{\text{nil}} \otimes_A B \xrightarrow{\sim} (\mathfrak{M}_B/u\mathfrak{M}_B)_{\text{nil}}$, and hence have the same B-rank. Hence the natural surjection $\mathfrak{M}_A/\mathfrak{M}_A^m \otimes_A B \to \mathfrak{M}_B/\mathfrak{M}_B^m$ is an isomorphism, and $\mathfrak{M}_A^m \otimes_A B \xrightarrow{\sim} \mathfrak{M}_B^m$.

Finally, we define $\mathfrak{M}_A^{\text{ét}}$ by $\mathfrak{M}_A^{\text{ét}} = (\mathfrak{M}_A^{*,m})^*$. Then $\mathfrak{M}_A^{\text{ét}}$ is the maximal étale quotient of $\mathfrak{M}_A$, since an object of $(\text{Mod FI}/\mathfrak{S})_A$ is étale if and only if its dual is multiplicative. We have already seen (1) and the first isomorphism in (2). The second isomorphism in (2) follows from duality. Similarly, the isomorphisms in (3) follow using the definition of $\mathfrak{M}_A^{\text{ét}}$, and duality. $\qquad\square$

(1.3) *Weakly admissible modules with coefficients.* We will need the analogues of some of the above results for weakly admissible modules. The proofs are usually easier, since we now work in a situation where p is invertible.

(1.3.1) Let A be a finite, local $\mathbb{Q}_p$-algebra. We denote by $(\text{Mod}/K_0)_A$ the category of weakly admissible φ-modules equipped with an action of A, and by $(\text{Mod}/K_0)_{A\text{-fr}}$ the full subcategory of $(\text{Mod}/K_0)_A$ consisting of those weakly admissible modules $\mathcal{M}$ such that $\text{gr}^\bullet \mathcal{M}_K$ is a projective A-module, where $\mathcal{M}_K = \mathcal{M} \otimes_{K_0} K$.

LEMMA (1.3.2). *If $\mathcal{M}$ is in $(\text{Mod}/K_0)_{A\text{-fr}}$, then $\text{gr}^\bullet \mathcal{M}_K$ is a finite projective $A \otimes_{\mathbb{Q}_p} K$-module, and $\mathcal{M}$ is free over $A \otimes_{\mathbb{Q}_p} K_0$.*

Proof. The first claim follows from the isomorphism of functors

$$\text{Hom}_{A \otimes_{\mathbb{Q}_p} K}(\mathcal{M}_K, \_) \xrightarrow{\sim} \text{Hom}_{K \otimes_{\mathbb{Q}_p} K}(K, \text{Hom}_A(\mathcal{M}_K, \_))$$

where K is a $K \otimes_{\mathbb{Q}_p} K$-algebra via the multiplication map. In fact the right-hand side is a composite of two exact functors, since K is a projective $K \otimes_{\mathbb{Q}_p} K$-module.

By descent this implies that $\mathcal{M}$ is finite projective over $A \otimes_{\mathbb{Q}_p} K_0$. That it is free follows by an argument using Frobenius as in the proof of (1.2.7). $\qquad\square$

(1.3.3) We denote by Rep^{cris} the category of crystalline G_K-representations, and for A as above, we denote by $\text{Rep}_A^{\text{cris}}$ the category of G_K-representations on finite free A-modules, which are crystalline when considered as representations on a $\mathbb{Q}_p$-vector space. By the main result of [CF], we have an exact equivalence of abelian categories

$$D_{\text{cris}} : \text{Rep}^{\text{cris}} \xrightarrow{\sim} (\text{Mod}/K_0)_{\mathbb{Q}_p}; \quad V \mapsto (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

with a quasi-inverse given by

$$V_{\text{cris}} : (\text{Mod}/K_0)_{\mathbb{Q}_p} \xrightarrow{\sim} \text{Rep}^{\text{cris}}; \quad M \mapsto \text{Fil}^0(B_{\text{cris}} \otimes_{\mathbb{Q}_p} M)^{\varphi=1}.$$

PROPOSITION (1.3.4). *The functors D_{cris} and V_{cris} induce exact equivalences of categories*

$$D_{\text{cris},A} : \text{Rep}_A^{\text{cris}} \xrightarrow{\sim} (\text{Mod}/K_0)_{A\text{-fr}} \text{ and } V_{\text{cris}} : (\text{Mod}/K_0)_{A\text{-fr}} \xrightarrow{\sim} \text{Rep}_A^{\text{cris}}.$$

Proof. For any finite A-module N and $\mathcal{M}$ in $(\text{Mod}/K_0)_A$, $\mathcal{M} \otimes_A N$ has a natural structure of object in $(\text{Mod}/K_0)_A$. This can be seen by writing a presentation for N,

and keeping in mind that $(\mathrm{Mod}/K_0)_A$ is abelian, and hence, in particular, admits cokernels. Now the same argument as in the proof of (1.2.7) shows that there is a natural isomorphism $V_{\mathrm{cris}}(\mathcal{M}) \otimes_A N \xrightarrow{\sim} V_{\mathrm{cris}}(\mathcal{M} \otimes_A N)$. As in the proof of (1.2.2)(4), this implies that, if $\mathcal{M}$ is in $(\mathrm{Mod}/K_0)_{A\text{-fr}}$ then $V_{\mathrm{cris}}(\mathcal{M})$ is a free A-module.

Conversely, if V is in $\mathrm{Rep}_A^{\mathrm{cris}}$, an analogous argument shows that $D_{\mathrm{cris}}(V) \otimes_A N \xrightarrow{\sim} D_{\mathrm{cris}}(V \otimes_A N)$. Since the category of crystalline representations is stable under quotients $V \otimes_A N$ is crystalline, so that $D_{\mathrm{cris}}(V) \otimes_A N$ is naturally an object of $(\mathrm{Mod}/K_0)_A$. Since the functor $\mathcal{M} \mapsto \mathrm{gr}^\bullet \mathcal{M}_K$ on the category of weakly admissible modules is exact (because morphisms between weakly admissible modules are strict for filtrations),

$$(\mathrm{gr}^\bullet D_{\mathrm{cris}}(V)_K) \otimes_A N \xrightarrow{\sim} \mathrm{gr}^\bullet(D_{\mathrm{cris}}(V)_K \otimes_A N) \xrightarrow{\sim} \mathrm{gr}^\bullet D_{\mathrm{cris}}(V \otimes_A N)_K.$$

Since the right-hand side is exact in N, this shows that $\mathrm{gr}^\bullet D_{\mathrm{cris}}(V)_K$ is a free A-module. $\qquad\square$

PROPOSITION (1.3.5). *If A' is a finite local A-algebra, then for V in $\mathrm{Rep}_A^{\mathrm{cris}}$ there is a natural isomorphism*

$$D_{\mathrm{cris},A}(V) \otimes_A A' \xrightarrow{\sim} D_{\mathrm{cris},A'}(V \otimes_A A'),$$

in $(\mathrm{Mod}/K_0)_{A\text{-fr}}$ and for $\mathcal{M}$ in $(\mathrm{Mod}/K_0)_{A\text{-fr}}$ there is a natural isomorphism

$$V_{\mathrm{cris},A}(\mathcal{M}) \otimes_A A' \xrightarrow{\sim} V_{\mathrm{cris},A'}(\mathcal{M} \otimes_A A')$$

in $\mathrm{Rep}_A^{\mathrm{cris}}$.

Proof. That the sources and targets of each of these two maps lie in the indicated category follows from (1.3.4). Now the proposition follows as in (1.2.7), by the observation already made in the proof of (1.3.4), that D_{cris} and V_{cris} commute with the functor $\otimes_A N$ for any finite A-module N. $\qquad\square$

2. **Moduli of finite flat groups schemes**

(2.1) *Definitions and first properties.* From now on we will freely use the language of groupoids. A summary of the definitions and properties we will need is given in the appendix. Readers who wish to avoid this, at least initially, may substitute the term "functor on" for "groupoid over". However, as mentioned in the appendix, the correct definition of fibers requires groupoids (either explicitly or implicitly).

Let $\mathbb{O}$ be a local $\mathbb{Z}_p$-algebra with maximal ideal $\mathfrak{m}_{\mathbb{O}}$. We will denote by $\mathfrak{AR}_{\mathbb{O}}$ the category of finite, local, Artinian $\mathbb{O}$-algebras A, with maximal ideal $\mathfrak{m}_A$, equipped with an isomorphism $A/\mathfrak{m}_A \xrightarrow{\sim} \mathbb{O}/\mathfrak{m}_{\mathbb{O}}$.

We denote by $\mathfrak{Aug}_{\mathbb{O}}$ the category consisting of pairs (A, I) where A is an $\mathbb{O}$-algebra, and $I \subset A$ is a nilpotent ideal with $\mathfrak{m}_{\mathbb{O}} A \subset I$. A map $(A, I) \to (B, J)$ is a map of rings $A \to B$ taking I into J.

Note that we do not assume that A is finite over $\mathbb{O}$. We remark that if $(A, \mathfrak{m}_A)$ is in $\mathfrak{AR}_{\mathbb{O}}$, then we may also regard it as an object in $\mathfrak{Aug}_{\mathbb{O}}$, so that $\mathfrak{AR}_{\mathbb{O}}$ is a full sub-category of $\mathfrak{Aug}_{\mathbb{O}}$.

A groupoid D over $\mathfrak{AR}_{\mathbb{O}}$ has a canonical extension to a groupoid over the category $\widehat{\mathfrak{AR}_{\mathbb{O}}}$ consisting of complete local $\mathbb{O}$-algebras with residue field $\mathbb{O}/\mathfrak{m}_{\mathbb{O}}$. This is explained in (A.7). We will again denote this extension by D (rather than by $\widehat{D}$ as in the appendix).

We fix a finite extension $\mathbb{F}$ of $\mathbb{F}_p$, and a continuous representation of G_K on an $\mathbb{F}$-vector space $V_{\mathbb{F}}$ of dimension $d \in \mathbb{N}^+$. We will assume that $V_{\mathbb{F}}$ is the generic fiber of a finite flat group scheme.

(2.1.1) As in (A.3), we define a groupoid $D_{V_{\mathbb{F}}}$ over $\mathfrak{AR}_{W(\mathbb{F})}$, by declaring the objects of $D_{V_{\mathbb{F}}}(A)$ to be finite free A-modules V_A equipped with a continuous G_K-action, and an $\mathbb{F}$-linear, G_K-equivariant isomorphism $\psi : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$. A morphism $(V_A, \psi) \to (V_{A'}, \psi')$ covering a given morphism $A \to A'$ is an equivalence class of isomorphisms $V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ which are A'-linear, G_K-equivariant, and respect the maps ψ and ψ'. Two isomorphisms are equivalent if they differ by a unit of A'. If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$ then this groupoid is pro-represented by a complete local $W(\mathbb{F})$-algebra $R_{V_{\mathbb{F}}}$.

We denote by $D_{V_{\mathbb{F}}}^{\mathrm{fl}}$ the full subcategory of $D_{V_{\mathbb{F}}}$ such that the objects of $D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A)$ consists of deformations V_A which are the generic fiber of a finite flat group scheme. By [Ram, §2] the morphism $D_{V_{\mathbb{F}}}^{\mathrm{fl}} \to D_{V_{\mathbb{F}}}$ is relatively representable. Hence, if $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$ then $D_{V_{\mathbb{F}}}^{\mathrm{fl}}$ is pro-represented by a quotient $R_{V_{\mathbb{F}}}^{\mathrm{fl}}$ of $R_{V_{\mathbb{F}}}$.

It will be convenient to extend $D_{V_{\mathbb{F}}}$ to a groupoid over $\mathfrak{Aug}_{W(\mathbb{F})}$. We will again denote the larger category by the same symbol. Given (A, I) in $\mathfrak{Aug}_{W(\mathbb{F})}$, denote by $\mathfrak{AR}_{W(\mathbb{F})}^{A,I}$ the category of rings A' in $\mathfrak{AR}_{W(\mathbb{F})}$ equipped with an injective map of $W(\mathbb{F})$-algebra map $A' \to A$, such that the radical of A' maps to I. The set of such rings forms a filtering direct limit: If A' and A'' are two such rings, then so is $\mathrm{Im}\,(A' \otimes_{W(\mathbb{F})} A'' \to A)$. We define

$$D_{V_{\mathbb{F}}}(A, I) = \varinjlim_{A' \in \mathfrak{AR}_{W(\mathbb{F})}^{A,I}} D_{V_{\mathbb{F}}}(A').$$

More precisely, if A' is in $\mathfrak{AR}_{W(\mathbb{F})}^{A,I}$, write $\mathfrak{AR}_{W(\mathbb{F}), \geq A'}^{A,I}$ for the full subcategory of $\mathfrak{AR}_{W(\mathbb{F})}^{A,I}$ consisting of rings which contain A'. We view $\mathfrak{AR}_{W(\mathbb{F}), \geq A'}^{A,I}$ as a category over $\mathfrak{AR}_{W(\mathbb{F})}$. Then

$$D_{V_{\mathbb{F}}}(A, I) = \varinjlim_{A' \in \mathfrak{AR}_{W(\mathbb{F})}^{A,I}} \mathrm{Hom}_{\mathfrak{AR}_{W(\mathbb{F})}}(\mathfrak{AR}_{W(\mathbb{F}), \geq A'}^{A,I}, D_{V_{\mathbb{F}}}).$$

Given a morphism $f : (A, I) \to (B, J)$, and objects $\eta \in D_{V_{\mathbb{F}}}(A, I)$ and $\xi \in D_{V_{\mathbb{F}}}(B, J)$, we define the set of maps $\eta \to \xi$ which cover f to be

$$\operatorname{Hom}_{D_{V_{\mathbb{F}}}}(\eta, \xi)_f = \varprojlim_{A'} \varinjlim_{B'} \operatorname{Hom}_{D_{V_{\mathbb{F}}}}(V_{A'}, V_{B'})_f$$

where A' runs over sufficiently large elements of $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}^{A, I}$, B' runs over sufficiently large elements of $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}^{B, J}$ which contain $f(A')$, $V_{A'} = \eta(A') \in D_{V_{\mathbb{F}}}(A')$, $V_{B'} = \xi(B') \in D_{V_{\mathbb{F}}}(B')$, and the subscript f in the right-hand side denotes maps in $D_{V_{\mathbb{F}}}$ which cover the induced map $A' \to B'$ in $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}$. Note that the transition maps in the inverse limit are all bijections.

We define $D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A, I)$ in an analogous way. For (A, I) in $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$ we will sometimes write $D_{V_{\mathbb{F}}}(A)$ and $D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A)$ for $D_{V_{\mathbb{F}}}(A, I)$ and $D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A, I)$, respectively, when this causes no confusion.

(2.1.2) Let $M_{\mathbb{F}}$ in $\Phi\mathbf{M}_{\mathcal{O}_{\mathscr{E}}, \mathbb{F}}$ be the pre-image of $V_{\mathbb{F}}(-1)$ under the equivalence $T_{\mathbb{F}}$ of (1.2.7). We define a groupoid $D_{M_{\mathbb{F}}}$ over $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}$ by declaring the objects of $D_{M_{\mathbb{F}}}(A)$ to be modules M_A in $\Phi\mathbf{M}_{\mathcal{O}_{\mathscr{E}}, A}$, which are free over $\mathcal{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A$, and equipped with an isomorphism $\psi : M_A \otimes_A \mathbb{F} \xrightarrow{\sim} M_{\mathbb{F}}$ in $\Phi\mathbf{M}_{\mathcal{O}_{\mathscr{E}}, \mathbb{F}}$. A morphism $(M_A, \psi) \to (M_{A'}, \psi')$ covering a given morphism $A \to A'$ in $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}$ is an equivalence class of isomorphisms $M_A \otimes_A A' \xrightarrow{\sim} M_{A'}$ in $\Phi\mathbf{M}_{\mathcal{O}_{\mathscr{E}}, A'}$, compatible with ψ and ψ', where two isomorphisms are equivalent of they differ by a unit of A'.

As in (2.1.1), we extend $D_{M_{\mathbb{F}}}$ to a groupoid over $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$, which we denote by the same symbol. If (A, I) is in $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$ then we have

$$D_{M_{\mathbb{F}}}(A) = D_{M_{\mathbb{F}}}(A, I) = \varinjlim_{A' \in \mathfrak{AR}_{\mathrm{W}(\mathbb{F})}^{A, I}} D_{M_{\mathbb{F}}}(A'),$$

and morphisms are defined as in (2.1.1). If $A' \in \mathfrak{AR}_{\mathrm{W}(\mathbb{F})}^{A, I}$ and $M_{A'} \in D_{M_{\mathbb{F}}}(A')$, we will slightly abuse notation, and regard $M_A = M_{A'} \otimes_{A'} A$ as the corresponding object of $D_{M_{\mathbb{F}}}(A)$. By (1.2.7), we have a morphism over $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$:

(2.1.3) $D_{V_{\mathbb{F}}} \to D_{M_{\mathbb{F}}}; \quad V_A \mapsto (\mathcal{O}_{\widehat{\mathscr{E}}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V_A(-1))^{G_{K_\infty}}.$

We now define a groupoid $D_{\mathfrak{S}, M_{\mathbb{F}}}$ over $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$ by declaring the objects of $D_{\mathfrak{S}, M_{\mathbb{F}}}(A, I)$ to be modules $\mathfrak{M}_A$ of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_A$ equipped with an isomorphism of $\mathcal{O}_{\mathscr{E}} \otimes_{\mathrm{W}(\mathbb{F}_p)} A/I$-modules

$$\psi : (\mathcal{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_A \otimes_A A/I) \xrightarrow{\sim} M_{\mathbb{F}} \otimes_{\mathbb{F}} A/I$$

which is compatible with the Frobenius endomorphism on both sides. A morphism $(\mathfrak{M}_A, \psi) \to (\mathfrak{M}_{A'}, \psi')$ covering a given morphism $(A, I) \to (A', I')$ in $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$ is an equivalence class of isomorphisms $\mathfrak{M}_A \otimes_A A' \xrightarrow{\sim} \mathfrak{M}_{A'}$ in $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_{A'}$ compatible with the maps ψ and ψ'. Again, two isomorphisms are equivalent if they differ by a unit in A'.

PROPOSITION (2.1.4). *There is a diagram of morphisms of categories over* $\mathfrak{Aug}_{W(\mathbb{F})}$ *which commutes up to equivalence*

$$
\begin{array}{ccc}
D^{\mathrm{fl}}_{V_{\mathbb{F}}} & \xrightarrow{\ (2.1.3)\ } & D_{M_{\mathbb{F}}} \\
& \Theta_{V_{\mathbb{F}}}\nwarrow & \uparrow \\
& & D_{\mathfrak{S},M_{\mathbb{F}}}
\end{array}
$$

where for (A, I) *in* $\mathfrak{Aug}_{W(\mathbb{F})}$ *the vertical functor on* (A, I)-*points is given by* $\mathfrak{M}_A \mapsto \mathbb{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_A$. *The horizontal functor, induced by* (2.1.3), *is fully faithful. In particular,* $\Theta_{V_{\mathbb{F}}}$ *is uniquely determined, up to equivalence, by requiring the commutativity of the diagram.*

Proof. Suppose (A, I) is in $\mathfrak{Aug}_{W(\mathbb{F})}$ and $\mathfrak{M}_A$ is in $D_{\mathfrak{S},M_{\mathbb{F}}}(A, I)$. Set $M_A = \mathbb{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_A$. Then M_A is equipped with a map $\phi^*(M_A) \to M_A$, which is an isomorphism as $\phi^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ has cokernel killed by $E(u)$. Moreover, by definition of $D_{\mathfrak{S},M_{\mathbb{F}}}$ there is a ϕ-equivariant isomorphism $M_A \otimes_A A/I \xrightarrow{\sim} M_{\mathbb{F}} \otimes_{\mathbb{F}} A/I$. In particular, this implies that M_A is a free $\mathbb{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$-module, since it is projective by construction.

Let A^+ denote the preimage of $\mathbb{F}$ under the projection $A \to A/I$, and let M_{A^+} denote the preimage of $M_{\mathbb{F}}$ under the composite

$$
M_A \to M_A \otimes_A A/I \xrightarrow{\sim} M_{\mathbb{F}} \otimes_{\mathbb{F}} A/I.
$$

Then M_{A^+} is a free $\mathbb{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A^+$-module equipped with an isomorphism

$$
\phi^*(M_{A^+}) \xrightarrow{\sim} M_{A^+}.
$$

A standard argument shows that there exists a finitely generated $W(\mathbb{F})$-subalgebra $A' \subset A^+$, and a free $\mathbb{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A'$-submodule $M_{A'} \subset M_{A^+}$ such that $M_A = M_{A'} \otimes_{A'} A$, and the above map induces an isomorphism $\phi^*(M_{A'}) \xrightarrow{\sim} M_{A'}$. But any finitely generated $W(\mathbb{F})$-subalgebra of A^+ is a local Artin ring. Hence A' is in $\mathfrak{AR}_{W(\mathbb{F})}$, and $M_{A'}$ is in $D_{M_{\mathbb{F}}}(A')$. Similarly for any finitely generated A'-subalgebra $A'' \subset A^+$ we obtain a module $M_{A''}$ in $D_{M_{\mathbb{F}}}(A'')$, and a canonical isomorphism $M_{A'} \otimes_{A'} A'' \xrightarrow{\sim} M_{A''}$. We define the vertical functor by sending $\mathfrak{M}_A$ to the image of $M_{A'}$ in $D_{M_{\mathbb{F}}}(A)$, or more precisely the image of the functor $\mathfrak{AR}^{A,I}_{W(\mathbb{F}),A'} \to D_{M_{\mathbb{F}}}$ given by $A'' \mapsto M_{A''}$. This gives a well defined morphism over $\mathfrak{AR}_{W(\mathbb{F})}$, because the construction of M_{A^+} is functorial.

To construct $\Theta_{V_{\mathbb{F}}}$, we retain the above notation. Set $\mathfrak{M}_{A'} = M_{A'} \cap \mathfrak{M}_A \subset M_A$. Then $\mathfrak{M}_{A'}$ is stable by ϕ. Moreover, using the exact sequence

$$
(2.1.5) \qquad 0 \to \mathfrak{M}_{A'} \to M_{A'} \oplus \mathfrak{M}_A \xrightarrow{(a,b) \mapsto b-a} M_A
$$

and the snake lemma, one sees that the cokernel of $\phi^*(\mathfrak{M}_{A'}) \to \mathfrak{M}_{A'}$ is killed by $E(u)$, as the analogous statements hold for $\mathfrak{M}_A$ and $M_{A'}$. (We do not claim that $\mathfrak{M}_{A'}$ is free over $\mathcal{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} A'$, and this is in general false). Thus, $\mathfrak{M}_{A'}$ has the structure of an object in $'(\mathrm{Mod}/\mathfrak{S})$, and we claim that it is actually in $(\mathrm{Mod}/\mathfrak{S})$. To see this, first note that, since $\mathfrak{M}_{A'}$ is a $\mathfrak{S}$-submodule of the finite $\mathcal{O}_{\mathscr{E}}$-module $M_{A'}$, and has no infinitely u-divisible elements, being contained in $\mathfrak{M}_A$, one easily checks that $\mathfrak{M}_{A'}$ is finitely generated over $\mathfrak{S}$. Next, for each non-negative integer i we set

$$\mathfrak{M}_{A'}^i = p^i M_{A'} \cap \mathfrak{M}_{A'} = p^i M_{A'} \cap \mathfrak{M}_A.$$

Then $\mathfrak{M}_{A'}^{i-1}/\mathfrak{M}_{A'}^i \subset p^{i-1} M_{A'}/p^i M_{A'}$ is a finitely generated $\mathfrak{S}/p\mathfrak{S}$-module, which is necessarily free, being u-torsion free. Moreover $\mathfrak{M}_{A'}^{i-1}/\mathfrak{M}_{A'}^i$ is stable under ϕ, and the cokernel of $\phi^*(\mathfrak{M}_{A'}^{i-1}/\mathfrak{M}_{A'}^i) \to \mathfrak{M}_{A'}^{i-1}/\mathfrak{M}_{A'}^i$ is killed by $E(u)$, since this is true for $1 \otimes \phi$ on $\mathfrak{M}_{A'}$. Hence $\mathfrak{M}_{A'}^{i-1}/\mathfrak{M}_{A'}^i$ is an object of $(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})$ killed by p, and $\mathfrak{M}_{A'}$ is a successive extension of such objects and hence in $(\mathrm{Mod}/\mathfrak{S})$.

Now let

$$V_{A'} = T_{A'}(M_{A'})(1) \xrightarrow{\sim} T_{A'}(\mathcal{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_{A'})(1).$$

By $(1.2.7)(4)$ $V_{A'}$ is a finite free A'-module, and by $(1.2.7)(3)$, it is equipped with a G_{K_∞}-equivariant isomorphism $V_{A'} \otimes_{A'} \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$. By $(1.1.13)$ the action of G_{K_∞} on $V_{A'}$ can be extended to an action of G_K in such a way that $V_{A'}$ becomes the set of $\mathcal{O}_{\bar{K}}$ points of a finite flat group scheme. Moreover, this extension is unique, by [Br4, 3.4.3], which asserts that the restriction functor from finite flat representations of G_K to representations of G_{K_∞} is fully faithful. The full faithfulness also implies that the G_K-action respects the A'-module structure on $V_{A'}$, and is compatible with the isomorphism $V_{A'} \otimes_{A'} \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$.

Now we define $\Theta_{V_{\mathbb{F}}}(M_A)$ to be the image of $V_{A'} \in D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A')$ in $D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A)$. One easily checks that this is independent of the choice of A' (use $(1.2.7)$), and gives a well defined morphism of categories over $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$. The full faithfulness of the horizontal functor, and hence the uniqueness of $\Theta_{V_{\mathbb{F}}}$, follows from the full faithfulness in [Br4, 3.4.3], already used above. $\square$

$(2.1.6)$ Let A be a ring in $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}$, with maximal ideal $\mathfrak{m}_A$, and fix an object $\xi = V_A \in D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A)$. Write

$$M_A = (\mathcal{O}_{\widehat{\mathscr{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V_A(-1))^{G_{K_\infty}}.$$

Since $D_{V_{\mathbb{F}}}^{\mathrm{fl}}$ is a groupoid over $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$, we can attach a category over $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$ to ξ, (which we again denote ξ; see $(A.5)$). Using this, and the morphism $\Theta_{V_{\mathbb{F}}}$ we may form the 2-fiber product

$$D_{\mathfrak{S}, M_{\mathbb{F}}, \xi} = \xi \times_{D_{V_{\mathbb{F}}}^{\mathrm{fl}}} D_{\mathfrak{S}, M_{\mathbb{F}}}.$$

We regard this product as a groupoid over $\mathfrak{Aug}_A$.

PROPOSITION (2.1.7). *With the notation above, there exists a projective A-scheme $\mathscr{GR}_{V_{\mathbb{F}},\xi}$ such that for any (B, I) in $\mathfrak{Aug}_A$, there is a canonical bijection*

$$(2.1.8) \qquad |D_{\mathfrak{S},M_{\mathbb{F}},\xi}|(B, I) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Spec} A}(\operatorname{Spec} B, \mathscr{GR}_{V_{\mathbb{F}},\xi}).$$

Proof. For (B, I) in $\mathfrak{Aug}_A$, write $M_B = M_A \otimes_A B$. By (2.1.4), (the class of) ξ is the unique class in $|D_{V_{\mathbb{F}}}^{\mathrm{fl}}|(A)$ mapping to $[M_A]$. It follows from the definitions, that $|D_{\mathfrak{S},M_{\mathbb{F}},\xi}|(B, I)$ may be identified with the set of $\mathfrak{S}_B$-submodules $\mathfrak{M}_B \subset M_B$ which are projective of rank d, span the $\mathfrak{S}_B[1/u]$-module M_B, are stable under the Frobenius endomorphism of M_B, and such that the induced map $1 \otimes \phi : \phi^*(\mathfrak{M}_B) \to \mathfrak{M}_B$ has cokernel killed by $E(u)$. Note that this is independent of I, as is the right-hand side of (2.1.8).

Now let $\hat{\mathfrak{S}}_B$ denote the u-adic completion of $\mathfrak{S}_B$, and write

$$\hat{M}_B = M_B \otimes_{\mathfrak{S}_B} \hat{\mathfrak{S}}_B.$$

The main result of [BL1] implies that the association $\mathfrak{M}_B \mapsto \mathfrak{M}_B \otimes_{\mathfrak{S}_B} \hat{\mathfrak{S}}_B$ induces a bijection between the set of finite projective $\hat{\mathfrak{S}}_B$-submodules $\hat{\mathfrak{M}}_B \subset \hat{M}_B$ of rank d which span $\hat{M}_B$, and the set of finite projective $\mathfrak{S}_B$-submodules of M_B of rank d which span M_B. It follows that the former functor is represented by the affine Grassmannian for $\operatorname{Res}_{W(k)/\mathbb{Z}_p} \operatorname{GL}_d$ over A. This is an Ind-projective scheme [Fa, p. 42] (cf. also [BL2, Thm. 2.5]). The condition that $1 \otimes \phi$ sends $\phi^*(\mathfrak{M}_B)$ into $\mathfrak{M}_B$ and has cokernel killed by $E(u)$ defines a subfunctor, represented by a closed subspace of the affine Grassmannian, which we denote by $\mathscr{GR}_{V_{\mathbb{F}},\xi}$.

Now fix a finite projective $\mathfrak{S}_A$-submodule $\mathfrak{N}_A \subset M_A$ of rank d, which spans M_A. A priori $\mathscr{GR}_{V_{\mathbb{F}},\xi}$ is Ind-projective. To show it is projective we have to show that there exists an integer i such that for any (B, I) in $\mathfrak{Aug}_A$, and any $\mathfrak{S}_B$ submodule $\mathfrak{M}_B \subset M_B$ corresponding to an elements of $D_{\mathfrak{S},M_{\mathbb{F}},\xi}(B, I)$, we have $u^i \mathfrak{N}_B \subset \mathfrak{M}_B \subset u^{-i} \mathfrak{N}_B$ (cf. the description in [Fa, §2]), where $\mathfrak{N}_B = \mathfrak{N}_A \otimes_A B$.[1]

To see this let r be the least integer such that $u^r \mathfrak{N}_B \subset (1 \otimes \phi) \phi^*(\mathfrak{N}_B) \subset u^{-r} \mathfrak{N}_B$, and i be the least integer such that $\mathfrak{N}_B \subset u^{-i} \mathfrak{M}_B$. By considering a matrix which transforms some $\mathfrak{S}_B$-basis of $\mathfrak{N}_B$ into a $\mathfrak{S}_B$-basis of $\mathfrak{M}_B$, we see that the least integer j such that $(1 \otimes \phi) \phi^*(\mathfrak{N}_B) \subset u^{-j} (1 \otimes \phi) \phi^*(\mathfrak{M}_B)$ is equal to ip. However

$$(1 \otimes \phi) \phi^*(\mathfrak{N}_B) \subset u^{-r} \mathfrak{N}_B \subset u^{-i-r} \mathfrak{M}_B$$
$$= E(u)^{-1} u^{-i-r} (E(u) \mathfrak{M}_B) \subset u^{-i-r-ek} (1 \otimes \phi) \phi^*(\mathfrak{M}_B),$$

where k is the least integer such that $p^k = 0$ in A. Hence $ip \leq ek + i + r$, and $i \leq (ek + r)/(p - 1)$.

---

[1] We thank Brian Conrad for pointing out that it is obviously not sufficient to show that $\mathfrak{M}'_B \subset u^{-i} \mathfrak{M}_B$ for $\mathfrak{M}'_B, \mathfrak{M}_B$ in $D_{\mathfrak{S},M_{\mathbb{F}},\xi}(B, I)$.

Similarly, if i is the least integer such that $\mathfrak{M}_B \subset u^{-i}\mathfrak{N}_B$, then

$$(1 \otimes \phi)\phi^*(\mathfrak{M}_B) \subset \mathfrak{M}_B \subset u^{-i}\mathfrak{N}_B \subset u^{-i-r}(1 \otimes \phi)\phi^*(\mathfrak{N}_B),$$

so that $ip \leq i + r$, and $i \leq r/(p-1)$. $\square$

REMARK (2.1.9). Had we defined $D_{\mathfrak{S},M_\mathbb{F}}$ over $W(\mathbb{F})$-schemes equipped with a nilpotent sheaf of ideals, rather than just on rings, then (2.1.7) would just say that the morphism $D_{\mathfrak{S},M_\mathbb{F}} \to D_{V_\mathbb{F}}$ is relatively representable.

Suppose that R is a complete local ring with residue field $\mathbb{F}$ and maximal ideal $\mathfrak{m}_R$. Fix a point $\xi \in D_{V_\mathbb{F}}^{\mathrm{fl}}(R)$, and a deformation V_R of $V_\mathbb{F}$ to R. For $i \geq 1$, let $\xi_i \in D_{V_\mathbb{F}}^{\mathrm{fl}}(R/\mathfrak{m}_R^i)$ be the image of ξ. Applying the construction of (2.1.6) with $A = R/\mathfrak{m}_R^i$, and $V_A = V_R \otimes_R R/\mathfrak{m}_R^i$, we obtain a groupoid $D_{\mathfrak{S},M_\mathbb{F},\xi_i}$ over $\mathfrak{Aug}_{R/\mathfrak{m}_R^i}$. We denote by $D_{\mathfrak{S},M_\mathbb{F},\xi}$ the category over $\mathfrak{Aug}_R$ whose fiber over a pair (B, I) is given by

$$D_{\mathfrak{S},M_\mathbb{F},\xi}(B, I) = \varprojlim_i D_{\mathfrak{S},M_\mathbb{F},\xi_i}(B, I)$$

where the right-hand side is defined for i sufficiently large since I is nilpotent, and where the morphisms are again defined by the obvious inverse limit. Note that in these inverse limits the transition maps are equivalences of categories for i large (depending on (B, I)). In fact we will make use only of the associated functor $|D_{\mathfrak{S},M_\mathbb{F},\xi}| = \varprojlim_i |D_{\mathfrak{S},M_\mathbb{F},\xi_i}|$.

PROPOSITION (2.1.10). *The functor $|D_{\mathfrak{S},M_\mathbb{F},\xi}|$ is represented by a projective R-scheme $\mathscr{GR}_{V_\mathbb{F},\xi}$, in the sense that for any (B, I) in $\mathfrak{Aug}_R$, there is a canonical isomorphism*

$$\mathrm{Hom}_R(\mathrm{Spec}\, B, \mathscr{GR}_{V_\mathbb{F},\xi}) \xrightarrow{\sim} |D_{\mathfrak{S},M_\mathbb{F},\xi}|(B, I).$$

Proof. Applying (2.1.7) with $A = R/\mathfrak{m}_R^i$, yields a projective $R/\mathfrak{m}_R^i$-scheme $\mathscr{GR}_{V_\mathbb{F},\xi,i}$ such that the reduction of $\mathscr{GR}_{V_\mathbb{F},\xi,i}$ modulo $\mathfrak{m}_R^{i-1}$ is canonically isomorphic to $\mathscr{GR}_{V_\mathbb{F},\xi,i-1}$. The inductive limit of the $\mathscr{GR}_{V_\mathbb{F},\xi,i}$ yields a formal scheme $\widehat{\mathscr{GR}}_{V_\mathbb{F},\xi}$. The proof of (2.1.7) shows that each $\mathscr{GR}_{V_\mathbb{F},\xi,i}$ is a closed subspace of the affine Grassmannian over R. Now the affine Grassmannian is equipped with a line bundle $\mathscr{L}$ whose restriction to any closed subscheme of finite type is very ample [Fa, p. 42-43]. Thus $\widehat{\mathscr{GR}}_{V_\mathbb{F},\xi}$ is equipped with a formal line bundle whose restriction to $\mathscr{GR}_{V_\mathbb{F},\xi,0}$ is very ample. It follows by formal GAGA [GrD, III, 5.4.5] that $\widehat{\mathscr{GR}}_{V_\mathbb{F},\xi}$ is the completion along $\mathfrak{m}_R$ of a projective R-scheme $\mathscr{GR}_{V_\mathbb{F},\xi}$. $\square$

COROLLARY (2.1.11). *If $\mathrm{End}_{\mathbb{F}[G_K]}V_\mathbb{F} = \mathbb{F}$, then $D_{V_\mathbb{F}}^{\mathrm{fl}}$ is (pro)-representable by a complete local $W(\mathbb{F})$-algebra $R_{V_\mathbb{F}}^{\mathrm{fl}}$, and there is a projective map of $W(\mathbb{F})$-schemes*

$$\Theta_{V_\mathbb{F}} : \mathscr{GR}_{V_\mathbb{F}} \to \mathrm{Spec}\, R_{V_\mathbb{F}}^{\mathrm{fl}}$$

which represents the morphism $\Theta_{V_{\mathbb{F}}}$ of (2.1.4), in the sense that for any (B, I) in $\mathfrak{Aug}_{W(\mathbb{F})}$ we have a canonical commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{W(\mathbb{F})}(\mathrm{Spec}\, B, \mathscr{GR}_{V_{\mathbb{F}}}) & \longrightarrow & \mathrm{Hom}_{W(\mathbb{F})}(\mathrm{Spec}\, B, \mathrm{Spec}\, R^{\mathrm{fl}}_{V_{\mathbb{F}}}) \\
\downarrow \sim & & \downarrow \sim \\
|D_{\mathfrak{S}, M_{\mathbb{F}}}|(B, I) & \xrightarrow{\ (2.1.4)\ } & |D^{\mathrm{fl}}_{V_{\mathbb{F}}}|(B, I)
\end{array}
$$

where the Homs in the top line denote maps of schemes such that the radical of $R^{\mathrm{fl}}_{V_{\mathbb{F}}}$ pulls back into the ideal I.

Proof. The pro-representability of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}$ was already remarked on in (2.1.1). The rest of the corollary follows by applying (2.1.10) to the tautological point $\xi^{\mathrm{univ}} \in D^{\mathrm{fl}}_{V_{\mathbb{F}}}(R^{\mathrm{fl}}_{V_{\mathbb{F}}})$. $\qquad\square$

(2.1.12) By a *finite flat model* of $V_{\mathbb{F}}$ we mean a finite flat group scheme $\mathscr{G}$ over $\mathbb{O}_K$, equipped with an action of $\mathbb{F}$, and an isomorphism $V_{\mathbb{F}} \xrightarrow{\sim} \mathscr{G}(\mathbb{O}_{\bar{K}})$ which respects the action of G_K and $\mathbb{F}$. We have

COROLLARY (2.1.13). *There exists a projective $\mathbb{F}$-scheme $\mathscr{GR}_{V_{\mathbb{F}},0}$ such that for any finite extension $\mathbb{F}'$ of $\mathbb{F}$, the set of (isomorphism classes of) finite flat models of $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ is in natural bijection with $\mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$.*

Proof. We take $\mathscr{GR}_{V_{\mathbb{F}},0}$ to be the projective $\mathbb{F}$-scheme obtained by taking $R = \mathbb{F}$, and $\xi = V_{\mathbb{F}} \in D^{\mathrm{fl}}_{V_{\mathbb{F}}}(\mathbb{F})$ in (2.1.10).

By definition, $\mathscr{GR}_{V_{\mathbb{F}}}(\mathbb{F}')$ is in bijection with the set of free $\mathfrak{S}_{\mathbb{F}'}$-submodules $\mathfrak{M}_{\mathbb{F}'} \subset M_{\mathbb{F}'}$ of rank d, which are stable under ϕ, span $M_{\mathbb{F}'}$ as a $\mathfrak{S}_{\mathbb{F}'}[1/u]$-module, and such that the cokernel of $\phi^*(\mathfrak{M}_{\mathbb{F}'}) \to \mathfrak{M}_{\mathbb{F}'}$ is killed by u^e. Given such an $\mathfrak{M}_{\mathbb{F}'}$, we obtain a finite flat $\mathbb{O}_K$-group scheme $\mathscr{G}$ by composing the functors of (1.1.3) and (1.1.11). It is equipped with an action of $\mathbb{F}'$ by functoriality. By (1.1.13) we have $V_{\mathbb{F}'} \xrightarrow{\sim} \mathscr{G}(\mathbb{O}_{\bar{K}})$ respecting the action of $\mathbb{F}'$ and G_{K_∞}. Hence by the full faithfulness result of [Br4] used above, this isomorphism respects the action of G_K.

Conversely, given a finite flat model $\mathscr{G}$ of $V_{\mathbb{F}'}$, we know that $\mathscr{G}$ arises from a module $\mathfrak{M}_{\mathbb{F}'}$ in $(\mathrm{Mod}\, \mathrm{FI}/\mathfrak{S})_{\mathbb{F}'}$, since (1.1.3) and (1.1.11) are equivalences on objects killed by p. We have

$$
T_{\mathbb{F}'}(M_{\mathbb{F}'})(1) \xrightarrow{\sim} V_{\mathbb{F}'} \xrightarrow{\sim} \mathscr{G}(\mathbb{O}_{\bar{K}}) \xrightarrow{\sim} T_{\mathbb{F}'}(\mathbb{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_{\mathbb{F}'})(1)
$$

by (1.1.13), so that $M_{\mathbb{F}'} \xrightarrow{\sim} \mathbb{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_{\mathbb{F}'}$. This isomorphism is compatible with the action of $\mathbb{F}'$ and ϕ, so that $\mathfrak{M}_{\mathbb{F}'}$ gives rise to an element of $|D_{\mathfrak{S}, M_{\mathbb{F}}, \xi}|(\mathbb{F}') = \mathscr{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$. $\qquad\square$

PROPOSITION (2.1.14). *Suppose that $e(K/\mathbb{Q}_p) < p - 1$. Then the map $\Theta_{V_{\mathbb{F}}}$ is an equivalence of categories.*

Proof. Let R and ξ be as in (2.1.9). By (2.1.13) and [Ra, 3.3.3], we see that the reduced fiber of $\mathscr{GR}_{V_{\mathbb{F}},\xi}$ over the closed point of R consists of a single point, corresponding to the unique finite flat model of $V_{\mathbb{F}}$. This implies that $\mathscr{GR}_{V_{\mathbb{F}},\xi} \to$ Spec R is a finite map, so it suffices to check that $\mathscr{GR}_{V_{\mathbb{F}},\xi} \otimes_R R/\mathfrak{m}_R$ is reduced.

Consider an $\mathbb{F}[\epsilon]$-valued point ($\epsilon^2 = 0$), y, of this scheme. By (1.1.11) and the construction of $\mathscr{GR}_{V_{\mathbb{F}},\xi}$, y corresponds to an extension of finite flat group schemes

$$0 \to \mathscr{G}_0 \to \mathscr{G} \to \mathscr{G}_0 \to 0,$$

such that the corresponding extension of $\mathbb{F}[G_K]$-modules splits. However, this implies that the above extension is split by [Ra, 3.3.6(2)], so that y factors through $\mathbb{F}[\epsilon] \to \mathbb{F}$. Since y was arbitrary, this implies that $\mathscr{GR}_{V_{\mathbb{F}},\xi} \otimes_R R/\mathfrak{m}_R$ is reduced. $\square$

(2.2) *Local analysis.* In this subsection we analyze the local structure of $\mathscr{GR}_{V_{\mathbb{F}}}$, and of certain of its subschemes. We begin by recalling some of the main results of [PR].

(2.2.1) Let Λ be a free $\mathbb{O}_K$-module of rank d, and K_0^{sep} an algebraic closure of K_0. For each K_0-algebra embedding $\varphi : K \to K_0^{\text{sep}}$ choose an integer $r_\varphi \in [0, d]$.

The reflex field $F \subset K_0^{\text{sep}}$ of $\mathbf{r} = (r_\varphi)_\varphi$ is a finite extension of K_0 corresponding to the subgroup of $\text{Gal}(K_0^{\text{sep}}/K_0)$ consisting of those automorphisms σ such that $r_{\sigma \circ \varphi} = r_\varphi$. Let $\mathbb{O}_F$ be the ring of integers of F, and $\mathbb{F}$ its residue field. For an $\mathbb{O}_F$-scheme T we write $M_{\mathbf{r}}(T)$ for the set of $\mathbb{O}_K \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_T$-submodules $L \subset \Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_T$ such that L is, locally on T, a direct summand as an $\mathbb{O}_T$-module, and for $a \in \mathbb{O}_K$,

$$\det_{\mathbb{O}_T}(a|L) = \prod_\varphi \varphi(a)^{r_\varphi}$$

as polynomial functions on $\mathbb{O}_K$ with values in $\Gamma(T, \mathbb{O}_T)$ in the sense of [Ko, §5]. Recall that this means that, if $X_0, X_1, \ldots, X_{e-1}$ are indeterminates, then there is an equality of polynomials with coefficients in $\Gamma(T, \mathbb{O}_T)$

$$\det_{\mathbb{O}_T}(X_0 + \pi X_1 + \ldots \pi^{e-1} X_{e-1}|L) = \prod_\varphi (X_0 + \varphi(\pi) X_1 + \ldots \varphi(\pi^{e-1}) X_{e-1})^{r_\varphi}.$$

Note that the right-hand side actually has coefficients in $\mathbb{O}_F$. The functor $M_{\mathbf{r}}$ is representable by a projective $\mathbb{O}_F$-scheme. Its closed fiber $\bar{M}_{\mathbf{r}} := M_{\mathbf{r}} \otimes_{\mathbb{O}_F} \mathbb{F}$ depends only on $r = \sum_\varphi r_\varphi$. More precisely, if T is an $\mathbb{F}$-scheme, then $\bar{M}_{\mathbf{r}}(T)$ consists of $\mathbb{O}_K \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_T$-submodules $L \subset \Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_T$ which are $\mathbb{O}_T$-direct summands, locally on T, and such that one has $\det(X - \pi \otimes 1|L) = X^r$ [PR, §2].

The reduced subscheme of $\bar{M}_{\mathbf{r}}$ has a stratification by locally closed subschemes $\bar{M}_{\mathbf{r},\mathbf{s}}$ where $\mathbf{s} = \{s_i\}_{i=1}^d$ is a partition of $r = \sum_\varphi r_\varphi$ satisfying $s_1 \geq s_2 \cdots \geq s_d$ and $s_i \in [0, e]$. If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, then a point in $\bar{M}_{\mathbf{r}}(\mathbb{F}')$ lies in $\bar{M}_{\mathbf{r},\mathbf{s}}(\mathbb{F}')$ if the nilpotent endomorphism $\pi \otimes 1$ acting on the corresponding $\mathbb{O}_K \otimes_{\mathbb{O}_{K_0}} \mathbb{F}'$-submodule $L \subset \Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{F}'$ has Jordan type $\mathbf{s}$. If $\mathbf{s}$ and $\mathbf{s}'$ are two such partitions,

then $\bar{M}_{\mathbf{r},\mathbf{s}'}$ is contained in the closure of $\bar{M}_{\mathbf{r},\mathbf{s}}$ if and only if $\mathbf{s}' \leqslant \mathbf{s}$, which means that $\sum_{i=1}^{j} s'_i \leqslant \sum_{i=1}^{j} s_i$ for $j \in [1, d]$ (see [PR, §3]).

In fact the $\bar{M}_{\mathbf{r},\mathbf{s}}$ can be identified with orbits of $\mathrm{GL}_d(\mathbb{F}[\![u]\!])$ on the affine Grassmannian for GL_d. Namely, if we regard $\mathbb{O}_K \otimes_{\mathbb{O}_{K_0}} \mathbb{F}$ as an $\mathbb{F}[\![u]\!]$-algebra via $u \mapsto \pi \otimes 1$, and we fix an isomorphism $\Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{F} = (\mathbb{F}[\![u]\!]/u^e)^d$, then one assigns to the submodule L the $\mathbb{F}'[\![u]\!]$-sublattice of $(\mathbb{F}'[\![u]\!])^d$ given by the pre-image of L under the composite

$$(\mathbb{F}'[\![u]\!])^d \to (\mathbb{F}'[\![u]\!]/u^e)^d \xrightarrow{\sim} \Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{F}'.$$

By [PR, 5.3, 5.8, 5.10], [We],

PROPOSITION (2.2.2). *Let $M_{\mathbf{r}}^{\mathrm{loc}}$ denote the scheme theoretic closure of $M_{\mathbf{r}} \otimes_{\mathbb{O}_F} F$ in $M_{\mathbf{r}}$, and set $\bar{M}_{\mathbf{r}}^{\mathrm{loc}} = M_{\mathbf{r}}^{\mathrm{loc}} \otimes_{\mathbb{O}_F} \mathbb{F}$. Then:*

(1) *$M_{\mathbf{r}}^{\mathrm{loc}}$ is normal and Cohen-Macaulay.*

(2) *$\bar{M}_{\mathbf{r}}^{\mathrm{loc}}$ is reduced and normal, with rational singularities. It is equal to the closure of $\bar{M}_{\mathbf{r},\check{\mathbf{r}}}$ in $\bar{M}_{\mathbf{r}}$, where $\check{\mathbf{r}}$ denotes the dual partition to $\mathbf{r}$.*

(3) *If any two of the integers r_φ differ by at most 1, then $M_{\mathbf{r}}^{\mathrm{loc}} = M_{\mathbf{r}}$ provided either $r = \sum_\varphi r_\varphi \leqslant e$, (i.e., $r_\varphi = 0$ or 1 for each φ) or $e \leqslant 2$.*

(2.2.3) In view of (2), when the r_φ differ by at most 1, the equality $M_{\mathbf{r}}^{\mathrm{loc}} = M_{\mathbf{r}}$ is equivalent to $\bar{M}_{\mathbf{r}}$ being reduced. In [PR] Pappas and Rapoport conjecture that this is always the case when the r_φ differ by at most 1, and they prove this in the cases mentioned in (3).

(2.2.4) Now for each $\mathbb{Q}_p$-algebra embedding $\psi : K \to K_0^{\mathrm{sep}}$ choose an integer $v_\psi \in [0, d]$. Denote by ψ_0 the restriction of ψ to K_0. For each $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$ we fix a lifting $\tilde{\sigma}$ of σ to K_0^{sep}. We set $\mathbf{v}_\sigma = (v_\psi)_{\tilde{\sigma}^{-1} \circ \psi}$, where in the indexing set, ψ runs over embeddings with $\psi_0 = \sigma$. Let $F \subset K_0^{\mathrm{sep}}$ be a finite Galois extension of $\mathbb{Q}_p$, which contains the reflex field of $\mathbf{v}_\sigma$ for each $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, and write $\mathbb{O}_F$ for its ring of integers, and $\mathbb{F}$ for its residue field.

Set $\mathbf{v} = (v_\psi)_\psi$. For an $\mathbb{O}_F$-scheme T, we denote by $M_{\mathbf{v}}(T)$ the set of $\mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_T$-submodules $L \subset \Lambda \otimes_{\mathbb{Z}_p} \mathbb{O}_T$ such that locally on T, L is a direct summand as an $\mathbb{O}_T$-module, and for $a \in \mathbb{O}_K$, we have

$$\det_{\mathbb{O}_T}(a|L) = \prod_\psi \psi(a)^{v_\psi}$$

as polynomial functions on $\mathbb{O}_K$.

We again write $M_{\mathbf{v}}$ for the pull-back of $M_{\mathbf{v}}$ (which is defined over the ring of integers of the reflex field of $\mathbf{v}$) to $\mathbb{O}_F$.

PROPOSITION (2.2.5). *Denote by $M_{\mathbf{v}_\sigma}^{\tilde{\sigma}}$ the pull-back of $M_{\mathbf{v}}$ by $\tilde{\sigma}|_{\mathbb{O}_F}$. There is a canonical isomorphism*

$$M_{\mathbf{v}} \overset{\sim}{\longrightarrow} \underset{\sigma}{\times} {}_{\mathbb{O}_F} M_{\mathbf{v}_\sigma}^{\tilde{\sigma}}.$$

Proof. For an $\mathbb{O}_F$-scheme T, write $T^{\tilde{\sigma}^{-1}}$ for the pull-back of T by $\tilde{\sigma}^{-1}$ on $\mathbb{O}_F$. We have a decomposition

$$(2.2.6) \qquad \mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_T \overset{\sim}{\longrightarrow} \bigoplus_\sigma \mathbb{O}_K \otimes_{\mathbb{O}_{K_0},\sigma} \mathbb{O}_T \overset{\sim}{\longrightarrow} \bigoplus_\sigma \mathbb{O}_K \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_{T^{\tilde{\sigma}^{-1}}},$$

where the first map is $\mathbb{O}_T$-linear, and the second map is $\tilde{\sigma}^{-1}|_{\mathbb{O}_F}$-semi-linear.

Thus, we have a corresponding decomposition $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{O}_T \overset{\sim}{\longrightarrow} \bigoplus_\sigma \Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$, and for $L \in M_{\mathbf{v}}(T)$ a decomposition $L = \bigoplus_\sigma L'_\sigma$, where $L'_\sigma \subset \Lambda \otimes_{\mathbb{Z}_p} \mathbb{O}_T$ is the preimage of an $\mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$-submodule $L_\sigma \subset \Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$, which is an $\mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$-module direct summand, locally on $T^{\tilde{\sigma}^{-1}}$.

For $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, let $\epsilon_\sigma \in \mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_F$ be the idempotent such that (2.2.6) identifies $\epsilon_\sigma \cdot \mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_T$ with the summand $\mathbb{O}_K \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$. For $i = 0, 1, \ldots, e-1$, let $X_{i,\sigma}$ be an indeterminate. Then the determinant condition on L means that

$$(2.2.7) \qquad \det_{\mathbb{O}_T} \left(\sum_{i,\sigma} \epsilon_\sigma \pi^i X_{i,\sigma} | L \right) = \prod_\psi \sum_{i,\sigma} (\psi(\epsilon_\sigma \pi^i) X_{i,\sigma})^{v_\psi}$$

where i runs over $0, 1, \ldots, e-1$, and σ over the elements of $\mathrm{Gal}(K_0/\mathbb{Q}_p)$. On the right-hand side of (2.2.7) we have again denoted by ψ the map $\mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_F \to \mathbb{O}_{K_0^{\mathrm{sep}}}$ induced by the inclusion $\psi : K \to K_0^{\mathrm{sep}}$. Now multiplication by ϵ_σ induces the identity on L'_σ and the zero map on $L'_{\sigma'}$ for $\sigma' \neq \sigma$. Similarly, $\psi(\epsilon_\sigma) = 1$, if $\psi_0 = \sigma$ and 0 otherwise. Hence, if for $\sigma' \neq \sigma$ we set $X_{i,\sigma'} = 0$ for $i \neq 0$, and $X_{0,\sigma'} = 1$, then (2.2.7) becomes

$$\det_{\mathbb{O}_T} \left(\sum_i \pi^i X_{i,\sigma} | L'_\sigma \right) = \prod_{\psi_0 = \sigma} \sum_i (\psi(\pi^i) X_{i,\sigma})^{v_\psi}$$

so that

$$\det_{\mathbb{O}_T} \left(\sum_i \pi^i X_{i,\sigma} | L_\sigma \right) = \prod_{\psi_0 = \sigma} \left(\sum_i \tilde{\sigma}^{-1} \circ \psi(\pi^i) X_{i,\sigma} \right)^{v_\psi}.$$

Thus $\{L_\sigma\}_\sigma \in \prod_\sigma M_{\mathbf{v}_\sigma}(T^{\tilde{\sigma}^{-1}}) = \prod_\sigma M_{\mathbf{v}_\sigma}^{\tilde{\sigma}}(T)$.

Conversely, given a collection $\{L_\sigma\}_\sigma \in \prod_\sigma M_{\mathbf{v}_\sigma}^{\tilde{\sigma}}(T)$, we can regard L_σ as an $\mathbb{O}_K \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$-submodule of $\Lambda \otimes_{\mathbb{O}_{K_0}} \mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$, which is a direct summand as an $\mathbb{O}_{T^{\tilde{\sigma}^{-1}}}$-module, locally on $T^{\tilde{\sigma}^{-1}}$. Then it is easy to invert the above construction to produce a submodule $L \subset \mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_T$ which corresponds to an element of $M_{\mathbf{v}}(T)$. $\square$

COROLLARY (2.2.8). *Let $M_{\mathbf{v}}^{\mathrm{loc}}$ denote the scheme theoretic closure of $M_{\mathbf{v}} \otimes_{\mathbb{O}_F} F$ in $M_{\mathbf{v}}$, and write $\bar{M}_{\mathbf{v}}^{\mathrm{loc}} = M_{\mathbf{v}}^{\mathrm{loc}} \otimes_{\mathbb{O}_F} \mathbb{F}$. Then*:

(1) $M_{\mathbf{v}}^{\mathrm{loc}}$ *is normal and Cohen-Macaulay.*

(2) $\bar{M}_{\mathbf{v}}^{\mathrm{loc}}$ *is reduced and normal with rational singularities. It is equal to the closure of $\underset{\sigma}{\times}_{\mathbb{O}_F} \bar{M}_{\mathbf{v}_\sigma, \check{\mathbf{v}}_\sigma}^{\tilde{\sigma}}$ in $\underset{\sigma}{\times}_{\mathbb{O}_F} \bar{M}_{\mathbf{v}_\sigma}^{\tilde{\sigma}}$, where the superscript $\tilde{\sigma}$ denotes pull-back by $\tilde{\sigma}$, as before.*

(3) *If for each $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$ any two of the integers v_ψ with $\psi_0 = \sigma$ differ by at most 1, then $M_{\mathbf{v}}^{\mathrm{loc}} = M_{\mathbf{v}}$ provided either $v_\psi = 0$ or 1 for all ψ, or $e \leqslant 2$.*

Proof. By (2.2.5) we have $M_{\mathbf{v}}^{\mathrm{loc}} = \underset{\sigma}{\times}_{\mathbb{O}_F} M_{\mathbf{v}_\sigma}^{\mathrm{loc}, \tilde{\sigma}}$. Then the Cohen-Macaulay property in (1) follows immediately from (2.2.2)(1), and (2) follows from (2.2.2)(2) (cf. [GrD, IV 6.6.1]). For the normality in (1) it then suffices to check that $M_{\mathbf{v}}^{\mathrm{loc}}$ is regular in codimension 1, which follows from the fact that it has normal fibers over $\mathrm{Spec}\,\mathbb{O}_F$. Finally (3) follows from (2.2.2)(3). $\square$

(2.2.9) Fix $\mathbf{v}$ and F as in (2.2.4). We also fix an object $\mathfrak{M}_{\mathbb{F}}$ of $(\mathrm{Mod\,FI}/\mathfrak{S})_{\mathbb{F}}$ such that for $a \in \mathbb{O}_K$

$$(2.2.10) \qquad \underset{\mathbb{F}}{\det}(a | (1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}}))/u^e \mathfrak{M}_{\mathbb{F}}) = \prod_\psi \psi(a)^{v_\psi}$$

as polynomial functions on $\mathbb{O}_K$. We also fix an isomorphism of $\mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{F}$-modules

$$\iota : \mathfrak{M}_{\mathbb{F}}/u^e \mathfrak{M}_{\mathbb{F}} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}.$$

We now define three groupoids over $\mathfrak{AR}_{\mathbb{O}_F}$. Let A be in $\mathfrak{AR}_{\mathbb{O}_F}$. We define $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$ by declaring the objects of $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A)$ to be $\mathbb{O}_K \otimes_{\mathbb{Z}_p} A$-modules L equipped with an embedding $\varepsilon_A : L \hookrightarrow \Lambda \otimes_{\mathbb{Z}_p} A$ such that:

(1) L is an A-module direct summand in $\Lambda \otimes_{\mathbb{Z}_p} A$.

(2) For $a \in \mathbb{O}_K$ we have $\det_A(a|L) = \prod_\psi \psi(a)^{v_\psi}$, where, as usual, ψ runs over the $\mathbb{Q}_p$-algebra embeddings $K \hookrightarrow K_0^{\mathrm{sep}}$, and the formula is to be interpreted as an equality of polynomial functions on $\mathbb{O}_K$.

(3) The composite

$$L \otimes_A \mathbb{F} \overset{\varepsilon_A \otimes 1}{\hookrightarrow} \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F} \xrightarrow[\iota^{-1}]{\sim} \mathfrak{M}_{\mathbb{F}}/u^e \mathfrak{M}_{\mathbb{F}}$$

identifies $L \otimes_A \mathbb{F}$ with $(1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}}))/u^e \mathfrak{M}_{\mathbb{F}}$.

We define $D_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$ by declaring the objects of $D_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A)$ to consist of an object $\mathfrak{M}_A$ in $(\mathrm{Mod\,FI}/\mathfrak{S})_A$ equipped with an isomorphism $\psi_A : \mathfrak{M}_A \otimes_A \mathbb{F} \xrightarrow{\sim} \mathfrak{M}_{\mathbb{F}}$ in

(Mod FI/$\mathfrak{S}$)$_{\mathbb{F}}$, and such that for $a \in \mathbb{O}_K$

$$\det_A(a|(1 \otimes \phi)(\phi^*(\mathfrak{M}_A))/E(u)\mathfrak{M}_A) = \prod_\psi \psi(a)^{v_\psi}$$

as polynomial functions on $\mathbb{O}_K$ with values in A. Note that the left-hand side makes sense, since $(1 \otimes \phi)(\phi^*(\mathfrak{M}_A))/E(u)\mathfrak{M}_A$ is a finite free A-module by (1.2.2)(3).

Finally we define $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}$ by declaring the objects of $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}(A)$ to consist of an object $\mathfrak{M}_A$ in $D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}(A)$ equipped with an isomorphism of $\mathbb{O}_K \otimes_{\mathbb{Z}_p} A$-modules $\iota_A : \mathfrak{M}_A/E(u)\mathfrak{M}_A \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} A$ such that the diagram

$$
\begin{array}{ccc}
\mathfrak{M}_A/E(u)\mathfrak{M}_A \otimes_A \mathbb{F} & \xrightarrow{\iota_A \otimes 1} & \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F} \\
\downarrow{\scriptstyle \sim} & & \| \\
\mathfrak{M}_{\mathbb{F}}/u^e\mathfrak{M}_{\mathbb{F}} & \xrightarrow{\iota} & \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}
\end{array}
$$

commutes.

A morphism for the first (resp. second, resp. third) groupoid covering a morphism $A \to A'$ in $\mathfrak{AR}_{\mathbb{O}_F}$ consists of an isomorphism $L_A \otimes_A A' \xrightarrow{\sim} L_{A'}$ (resp. $\mathfrak{M}_A \otimes_A A' \xrightarrow{\sim} \mathfrak{M}_{A'}$) compatible with the embeddings ε_A and $\varepsilon_{A'}$ (resp. ψ_A and $\psi_{A'}$, resp. ψ_A, and $\psi_{A'}$, and ι_A and $\iota_{A'}$).

PROPOSITION (2.2.11). *We have morphisms of groupoids over $\mathfrak{AR}_{\mathbb{O}_F}$:*

(2.2.12) $$\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}} \to D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}; \quad \mathfrak{M}_A \mapsto \mathfrak{M}_A$$

and

(2.2.13) $$\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}} \to \bar{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}; \quad \mathfrak{M}_A \mapsto \iota_A((1 \otimes \phi)\phi^*(\mathfrak{M}_A)/E(u)\mathfrak{M}_A).$$

The first of these morphisms is relatively pro-representable, and both are formally smooth. That is, they give rise to formally smooth maps of functors on $\mathfrak{AR}_{\mathbb{O}_F}$.

Proof. Let A be in $\mathfrak{AR}_{\mathbb{O}_F}$, and $\xi = \mathfrak{M}_A$ be an object of $D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}(A)$. If A' is in $\mathfrak{AR}_{\mathbb{O}_F}$, an object of $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi}(A')$ consists of a morphism $A \to A'$, an object $\mathfrak{M}_{A'}$ of $D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}(A')$, an isomorphism $\mathfrak{M}_A \otimes_A A' \xrightarrow{\sim} \mathfrak{M}_{A'}$, and an isomorphism $\mathfrak{M}_{A'}/E(u)\mathfrak{M}_{A'} \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} A'$. Hence one sees that $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi}$ is represented by a complete local A-algebra R whose A'-points are given by the pre-image of ι under $\text{Hom}_{\mathbb{O}_K \otimes_{\mathbb{Z}_p} A'}((\mathfrak{M}_A/E(u)\mathfrak{M}_A) \otimes_A A', \Lambda \otimes_{\mathbb{Z}_p} A')$

$$\to \text{Hom}_{\mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{F}}(\mathfrak{M}_{\mathbb{F}}/E(u)\mathfrak{M}_{\mathbb{F}}, \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}).$$

Thus $\text{Spf}\,R$ is a torsor under the formal group G over A whose A' points are given by

$$G(A') = \ker\left(\text{Aut}_{\mathbb{O}_K \otimes_{\mathbb{Z}_p} A'}(\Lambda \otimes_{\mathbb{Z}_p} A') \to \text{Aut}_{\mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{F}}(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{F})\right).$$

In particular R is formally smooth over A, and this establishes the claims regarding (2.2.12).

Now suppose that $L_A \in \bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A)$, and that $I \subset A$ is a nilpotent ideal. Write $L_{A/I} = L_A \otimes_A A/I \in \bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A/I)$. Let $\mathfrak{M}_{A/I}$ in $\tilde{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A/I)$ be an object which maps to the isomorphism class of $L_{A/I}$. Let $\mathfrak{M}_A$ be a free $\mathfrak{S}_A$-module of rank d, and choose an isomorphism $\mathfrak{M}_A \otimes_A A/I \xrightarrow{\sim} \mathfrak{M}_{A/I}$, and an isomorphism $\iota_A : \mathfrak{M}_A / E(u)\mathfrak{M}_A \xrightarrow{\sim} \Lambda \otimes_{\mathbb{Z}_p} A$ making the diagram

$$
\begin{array}{ccc}
\mathfrak{M}_A / E(u)\mathfrak{M}_A & \xrightarrow[\iota_A]{\sim} & \Lambda \otimes_{\mathbb{Z}_p} A \\
\downarrow & & \\
\mathfrak{M}_A / E(u)\mathfrak{M}_A \otimes_A A/I & & \downarrow \\
\downarrow{\scriptstyle\sim} & & \\
\mathfrak{M}_{A/I} / E(u)\mathfrak{M}_{A/I} & \xrightarrow{\sim} & \Lambda \otimes_{\mathbb{Z}_p} A/I
\end{array}
$$

commute. Let L_A^+ denote the preimage of $\iota_A^{-1}(L_A)$ in $\mathfrak{M}_A$, and define $L_{A/I}^+ \subset \mathfrak{M}_{A/I}$ analogously. Then $1 \otimes \phi : \phi^*(\mathfrak{M}_{A/I}) \to \mathfrak{M}_{A/I}$ factors through $L_{A/I}^+$, since its image contains $E(u)\mathfrak{M}_{A/I}$. Since $\phi^*(\mathfrak{M}_A)$ is a free $\mathfrak{S}_A$-module, the composite

$$
\phi^*(\mathfrak{M}_A) \to \phi^*(\mathfrak{M}_{A/I}) \to \mathfrak{M}_{A/I}
$$

lifts to a map $\phi^*(\mathfrak{M}_A) \to L_A^+$. Since $L_A^+ \otimes_A A/I \xrightarrow{\sim} L_{A/I}^+$, this lift is a surjective map, and gives $\mathfrak{M}_A$ the structure of an object of $\tilde{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A)$, which maps to the isomorphism classes of $\mathfrak{M}_{A/I}$ and L_A in $\tilde{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A/I)$ and $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A)$ respectively. $\square$

LEMMA (2.2.14). *The groupoid $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$ over $\mathfrak{AR}_{\mathbb{O}_F}$ is pro-represented by a complete, local Noetherian $\mathbb{O}_F$-algebra.*

Proof. By definition, $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$ is represented by the completion of a local ring on $M_{\mathbf{v}}$. $\square$

(2.2.15) Let $\bar{R}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$ be the complete local $\mathbb{O}_F$-algebra pro-representing $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$. We denote by $\bar{R}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v},\mathrm{loc}}$ the quotient of $\bar{R}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$ by its ideal of p-power torsion elements, and by $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v},\mathrm{loc}}$ the corresponding full sub-groupoid of $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}$. That is, if A is in $\mathfrak{AR}_{\mathbb{O}_F}$, then an object of $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A)$ is in $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v},\mathrm{loc}}(A)$ if and only if the corresponding map $\bar{R}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}} \to A$ factors through $\bar{R}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v},\mathrm{loc}}$.

Note that it may happen that $\bar{R}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v},\mathrm{loc}}$ is the 0 ring, in which case $\bar{D}_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v},\mathrm{loc}}$ is empty.

(2.2.16) To end this subsection, we explain how one can use the techniques above to prove (for $p > 2$) a conjecture of Breuil which asserts that $(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})_{\mathbb{Z}_p}$ is equivalent to the category of p-divisible groups over $\mathbb{O}_K$ [Br5, 2.1.2].[2]

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[2]According to the introduction of [Br5] one of the main reasons for introducing the category $(\mathrm{Mod}\ \mathrm{FI}/\mathfrak{S})$ was to try to obtain a classification of p-divisible groups and finite flat group schemes

Let $\mathfrak{M}_{\mathbb{F}}$ be as before. For simplicity we assume from now on that all the v_ψ are equal to a common non-negative integer v, that $F = \mathbb{Q}_p$ (so that $\mathbb{F} = \mathbb{F}_p$), and that $(1 \otimes \phi)\phi^*\mathfrak{M}_{\mathbb{F}_p}/u^e\mathfrak{M}_{\mathbb{F}_p}$ is a free $k[\![u]\!]/u^e$-module of rank v.

Some of what we will do below goes through with weaker assumptions, and we leave it to the reader to formulate these results in the greatest generality.

Let $\mathcal{M}_{\mathbb{F}_p}$ in (Mod FI/S)$_{\mathbb{F}_p}$ be the image of $\mathfrak{M}_{\mathbb{F}}$ under the functor (1.2.4). We define a groupoid $D_{\mathcal{M}_{\mathbb{F}}}$ over $\mathfrak{AR}_{\mathbb{Z}_p}$, by declaring the objects of $D_{\mathcal{M}_{\mathbb{F}}}(A)$ to consist of an object $\mathcal{M}_A$ in (Mod FI/S)$_A$ equipped with an isomorphism $\psi : \mathcal{M}_A \otimes_A \mathbb{F}_p \xrightarrow{\sim} \mathcal{M}_{\mathbb{F}_p}$ in (Mod FI/S)$_{\mathbb{F}_p}$. Similarly, we define a groupoid $D_{\mathfrak{M}_{\mathbb{F}_p}}$ over $\mathfrak{AR}_{\mathbb{Z}_p}$, by declaring the objects of $D_{\mathfrak{M}_{\mathbb{F}_p}}(A)$ to consist of an object $\mathfrak{M}_A$ in (Mod FI/$\mathfrak{S}$)$_A$ equipped with an isomorphism $\psi : \mathfrak{M}_A \otimes_A \mathbb{F}_p \xrightarrow{\sim} \mathfrak{M}_{\mathbb{F}_p}$ in (Mod FI/$\mathfrak{S}$)$_{\mathbb{F}_p}$. As usual, in the first (resp. second) case a morphism $(\mathcal{M}_A, \psi) \to (\mathcal{M}_{A'}, \psi')$ (resp. $(\mathfrak{M}_A, \psi) \to (\mathfrak{M}_{A'}, \psi')$) covering a given morphism $A \to A'$ of $\mathfrak{AR}_{\mathbb{Z}_p}$, consists of an isomorphism $\mathcal{M}_A \otimes_A A' \xrightarrow{\sim} \mathcal{M}_{A'}$ in (Mod FI/S)$_{A'}$ (resp. $\mathfrak{M}_A \otimes_A A' \xrightarrow{\sim} \mathfrak{M}_{A'}$ in (Mod FI/$\mathfrak{S}$)$_{A'}$) compatible with ψ and ψ'.

LEMMA (2.2.17). *For A in $\mathfrak{AR}_{\mathbb{Z}_p}$ and*:

(1) *For $\mathfrak{M}_A$ in $D_{\mathfrak{M}_{\mathbb{F}_p}}(A)$, $(1 \otimes \phi)(\phi^*\mathfrak{M}_A)/E(u)\mathfrak{M}_A$ and $\mathfrak{M}_A/(1 \otimes \phi)(\phi^*\mathfrak{M}_A)$ are finite free $\mathbb{O}_K \otimes_{\mathbb{Z}_p} A$-modules.*

(2) *The natural morphism of groupoids*

$$D_{\mathfrak{M}_{\mathbb{F}_p}}^{\mathbf{v}} \to D_{\mathfrak{M}_{\mathbb{F}_p}}$$

is an equivalence.

Proof. Let $\mathfrak{m}_A$ denote the maximal ideal of A. By (1.2.2)

$$(1 \otimes \phi)(\phi^*\mathfrak{M}_A)/E(u)\mathfrak{M}_A$$

is a finite free A-module whose reduction modulo $\mathfrak{m}_A$ is

$$(1 \otimes \phi)(\phi^*\mathfrak{M}_{\mathbb{F}_p})/E(u)\mathfrak{M}_{\mathbb{F}_p}.$$

By assumption, the latter is finite free over $\mathbb{O}_K/p\mathbb{O}_K$ of rank v, and so by Nakayama's lemma $(1 \otimes \phi)(\phi^*\mathfrak{M}_A)/E(u)\mathfrak{M}_A$ is a quotient of $(\mathbb{O}_K \otimes_{\mathbb{Z}_p} A)^v$. However, since $(1 \otimes \phi)(\phi^*\mathfrak{M}_A)/E(u)\mathfrak{M}_A$ is A-flat, Nakayama's lemma applied to the kernel of the quotient map shows that this map is actually an isomorphism. This proves the claim in (1) regarding, $(1 \otimes \phi)(\phi^*\mathfrak{M}_A)/E(u)\mathfrak{M}_A$ and (2) follows immediately.

The same argument shows that $\mathfrak{M}_A/(1 \otimes \phi)(\phi^*\mathfrak{M}_A)$ is a finite free $\mathbb{O}_K \otimes_{\mathbb{Z}_p} A$-module, once we have checked that $\mathfrak{M}_{\mathbb{F}_p}/(1 \otimes \phi)(\phi^*\mathfrak{M}_{\mathbb{F}_p})$ is finite free over $k[\![u]\!]/u^e$. For this consider any exact sequence of finite $k[\![u]\!]/u^e$ modules

$$0 \to M' \to M \to M'' \to 0,$$

---

when $p = 2$. Unfortunately we do not yet have a complete understanding of the precise relationship between these objects when $p = 2$.

with M finite free. If M'' is finite free, then so is M'. Since $k[\![u]\!]/u^e$ is a Gorenstein-Artin ring, the functor $\mathrm{Hom}_{k[\![u]\!]/u^e}(\_, k[\![u]\!]/u^e)$ induces an exact involution of the category of finite $k[\![u]\!]/u^e$-modules. Hence we see that, conversely, if M' is free, then so is M''. $\qquad\square$

LEMMA (2.2.18). *The functors $|D_{\mathfrak{M}_{\mathbb{F}_p}}|$ and $|D_{\mathcal{M}_{\mathbb{F}_p}}|$ admit versal deformation rings $R^{\mathrm{ver}}_{\mathfrak{M}_{\mathbb{F}_p}}$ and $R^{\mathrm{ver}}_{\mathcal{M}_{\mathbb{F}_p}}$ respectively. Both these rings are Noetherian.*

Proof. This is a standard argument using Schlessinger's criterion [Ma, 1.2]. To see the finiteness of the reduced tangent spaces, write $\tilde{S}_1 = S/(p, u^{ie}/i!)_{i \geq p}$. Then [Br3, 2.1.2.2], implies that for any A in $\mathfrak{AR}_{\mathbb{F}_p}$, an element $\mathcal{M}_A$ in $D_{\mathcal{M}_{\mathbb{F}_p}}(A)$ is determined up to canonical isomorphism by the $\tilde{S}_1 \otimes_{\mathbb{F}_p} A$-module $\mathcal{M}_A \otimes_S \tilde{S}_1$ with its induced Frobenius and filtration. This shows that $|D_{\mathcal{M}_{\mathbb{F}_p}}|$ has a finite dimensional reduced tangent space, whence so does $|D_{\mathfrak{M}_{\mathbb{F}_p}}|$, by (1.2.4). $\qquad\square$

PROPOSITION (2.2.19). *There is an equivalence of groupoids over $\mathfrak{AR}_{\mathbb{Z}_p}$:*

$$(2.2.20) \qquad D_{\mathfrak{M}_{\mathbb{F}_p}} \to D_{\mathcal{M}_{\mathbb{F}_p}}; \quad \mathfrak{M}_A \mapsto S \otimes_{\phi, \mathfrak{S}} \mathfrak{M}_A.$$

Proof. The functor (2.2.20) is induced by that in (1.2.4), and its full faithfulness follows from that of (1.2.4). Hence it suffices to show that the corresponding map of functors

$$(2.2.21) \qquad |D_{\mathfrak{M}_{\mathbb{F}_p}}| \to |D_{\mathcal{M}_{\mathbb{F}_p}}|$$

is an isomorphism.

Denote by $|D_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$ and $|D_{\mathcal{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$ the functors on $\mathfrak{AR}_{\mathbb{Z}_p}$ represented by $R^{\mathrm{ver}}_{\mathfrak{M}_{\mathbb{F}_p}}$ and $R^{\mathrm{ver}}_{\mathcal{M}_{\mathbb{F}_p}}$ respectively. By (1.2.5), $|D_{\mathfrak{M}_{\mathbb{F}_p}}| \to |D_{\mathcal{M}_{\mathbb{F}_p}}|$ is an isomorphism on $\mathfrak{AR}_{\mathbb{F}_p}$, and so we may choose an isomorphism

$$|D_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}|_{\mathfrak{AR}_{\mathbb{F}_p}} \xrightarrow{\sim} |D_{\mathcal{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}|_{\mathfrak{AR}_{\mathbb{F}_p}}$$

which covers (2.2.21). Since $|D_{\mathcal{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$ is formally smooth over $|D_{\mathcal{M}_{\mathbb{F}_p}}|$ (this is one of the defining properties of the versal deformation), this isomorphism lifts to a map $|D_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}} \to |D_{\mathcal{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$ which covers (2.2.21).

Let $R^{\mathrm{ver}}_{\mathcal{M}_{\mathbb{F}_p}} \to R^{\mathrm{ver}}_{\mathfrak{M}_{\mathbb{F}_p}}$ be the corresponding map of rings. This map is an isomorphism modulo p. As in the proof of (2.2.17), it follows that it is an isomorphism, provided we can show that $R^{\mathrm{ver}}_{\mathfrak{M}_{\mathbb{F}_p}}$ is $\mathbb{Z}_p$-flat. Assuming this for a moment, we conclude that $|D_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}} \xrightarrow{\sim} |D_{\mathcal{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$. It follows that for any A in $\mathfrak{AR}_{\mathbb{Z}_p}$ the map (2.2.21) is surjective on A-valued points. However, the full faithfulness in (1.2.4) implies that it is injective on A-valued points, and this proves the proposition.

It remains to show that $R_{\mathfrak{M}_{\mathbb{F}_p}}^{\mathrm{ver}}$ is $\mathbb{Z}_p$-flat. We will show that $R_{\mathfrak{M}_{\mathbb{Z}_p}}^{\mathrm{ver}}$ is even formally smooth over $\mathbb{Z}_p$. For this we consider the Cartesian diagram of functors

$$
\begin{array}{ccc}
|\tilde{D}_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}} & \longrightarrow & |\tilde{D}_{\mathfrak{M}_{\mathbb{F}_p}}| \\
\downarrow & & \downarrow \\
|D_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}} & \longrightarrow & |D_{\mathfrak{M}_{\mathbb{F}_p}}|.
\end{array}
$$

Here we have written $\tilde{D}_{\mathfrak{M}_{\mathbb{F}_p}}$ for $\tilde{D}_{\mathfrak{M}_{\mathbb{F}_p}}^{\mathbf{v}}$, and the top left term is defined so that the diagram is Cartesian. Now the bottom map is formally smooth by the properties of the versal deformation, and hence so is the top map. On the other hand by (2.2.11) we have a formally smooth map

$$
\tilde{D}_{\mathfrak{M}_{\mathbb{F}_p}} \to \bar{D}_{\mathfrak{M}_{\mathbb{F}_p}} := \bar{D}_{\mathfrak{M}_{\mathbb{F}_p}}^{\mathbf{v}}.
$$

Moreover an argument as in the proof of (2.2.17) implies that, if A is in $\mathfrak{AR}_{\mathbb{Z}_p}$ and L_A is in $\bar{D}_{\mathfrak{M}_{\mathbb{F}_p}}$, then L_A and $\Lambda \otimes_{\mathbb{Z}_p} A/L_A$ are free $\mathbb{O}_K \otimes_{\mathbb{F}_p} A$-modules (this can also be deduced directly from (2.2.17)), and it follows easily that $|\bar{D}_{\mathfrak{M}_{\mathbb{F}_p}}|$, and hence $|\tilde{D}_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$, is formally smooth. On the other hand, the map on the right is formally smooth by (2.2.11) and (2.2.17), hence so is the map on the left. In particular this map is surjective on A-valued points for any A in $\mathfrak{AR}_{\mathbb{Z}_p}$; thus, the formal smoothness of $|\tilde{D}_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$ implies that of $|D_{\mathfrak{M}_{\mathbb{F}_p}}|^{\mathrm{ver}}$. $\square$

COROLLARY (2.2.22). *The category* $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_{\mathbb{Z}_p}$ *is equivalent to the category of p-divisible groups over* $\mathbb{O}_K$.

Proof. By [Br3, 4.2.2.9] the category $(\mathrm{Mod}\,\mathrm{FI}/S)_{\mathbb{Z}_p}$ is anti-equivalent to the category of p-divisible groups over $\mathbb{O}_K$, and we turn this into an equivalence by composing with Cartier duality. Now applying (1.2.4) with $A = \mathbb{Z}/p^n\mathbb{Z}$, and passing to the limit over n, we obtain a fully faithful functor $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_{\mathbb{Z}_p} \to (\mathrm{Mod}\,\mathrm{FI}/S)_{\mathbb{Z}_p}$.

To show this is essentially surjective, let $\mathcal{M}$ be in $(\mathrm{Mod}\,\mathrm{FI}/S)_{\mathbb{Z}_p}$, and write $\mathcal{M}_n = \mathcal{M}/p^n\mathcal{M}$. By (1.2.5) there exists $\mathfrak{M}_1$ in $(\mathrm{Mod}\,\mathrm{FI}/S)_{\mathbb{F}_p}$ which maps to $\mathcal{M}_1$ under the functor of (1.2.5). As above, we have a natural isomorphism

$$
(1 \otimes \phi)\phi^*(\mathfrak{M}_1)/u^e\mathfrak{M}_1 \xrightarrow{\sim} \mathcal{M}_1/\mathrm{Fil}^1\mathcal{M}_1 \xrightarrow{\sim} (\mathcal{M}/\mathrm{Fil}^1\mathcal{M}) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p\mathbb{Z},
$$

and the left-hand side is a free $k[\![u]\!]/u^e = \mathbb{O}_K/p\mathbb{O}_K$-module, because $\mathcal{M}/\mathrm{Fil}^1\mathcal{M}$ is $\mathbb{Z}_p$-free, and hence $\mathbb{O}_K$-free. It follows that we may apply (2.2.19) with $\mathfrak{M}_{\mathbb{F}_p} = \mathfrak{M}_1$. This tells us that (the class of) $\mathcal{M}$ in $|D_{\mathcal{M}_1}|(\mathbb{Z}_p)$ corresponds to an element $\mathfrak{M}$ in $|D_{\mathfrak{M}_1}|(\mathbb{Z}_p)$, and proves the required essential surjectivity. $\square$

(2.3) *Generic fibers of flat deformation rings.* In this section we will explain some techniques for analyzing the generic fibers of deformation rings, and in particular deformation rings of Galois representations.

(2.3.1) Let $E/\mathbb{Q}_p$ be a finite extension with residue field containing $\mathbb{F}$, and $\mathbb{O}_E \subset E$ the ring of integers. We denote by $\mathfrak{AR}_E$ the category of finite, local $W(\mathbb{F})[1/p]$-algebras B with residue field E. In particular B is then canonically an E-algebra.

We denote by Int_B the category of finite $\mathbb{O}_E$-subalgebras $A \subset B$, such that $A \otimes_{\mathbb{O}_E} E = B$. The morphisms in this category are given by the natural inclusions. Note that Int_B is ordered by inclusion, and that any two elements of Int_B are contained in a third.

Let $\mathbb{O}$ be a discrete valuation ring, finite over $W(\mathbb{F})$ and with $\mathbb{O} \subset \mathbb{O}_E$, and D a groupoid over $\mathfrak{AR}_\mathbb{O}$. As explained in (A.7), D extends to a groupoid over $\widehat{\mathfrak{AR}}_\mathbb{O}$. As in (A.8) we denote by $\widehat{\mathfrak{AR}}_{\mathbb{O},(\mathbb{O}_E)}$ the category consisting of an $\mathbb{O}$-algebra A in $\widehat{\mathfrak{AR}}_\mathbb{O}$, equipped with a map of $\mathbb{O}$-algebras $A \to \mathbb{O}_E$. In particular, we may regard Int_B as a subcategory of $\widehat{\mathfrak{AR}}_{\mathbb{O},(\mathbb{O}_E)}$.

Fix an object $\xi \in D(\mathbb{O}_E)$. The construction of (A.8) gives a groupoid $D_{(\xi)}$ over $\widehat{\mathfrak{AR}}_{\mathbb{O},(\mathbb{O}_E)}$, such that an object of the category $D_{(\xi)}(A)$ consists of an object η of $D(A)$ equipped with a morphism $\alpha : \eta \to \xi$ in D covering the given map $A \to \mathbb{O}_E$. We define a groupoid $D_{(\xi)}$ over $\mathfrak{AR}_E$ by setting

$$D_{(\xi)}(B) = \varinjlim_{A \in \text{Int}_B} D_{(\xi)}(A)$$

for B in $\mathfrak{AR}_E$. Here the limit is taken in the same sense as in the definition of $D_{V_\mathbb{F}}(A, I)$ in (2.1.1). More precisely, for $A \in \text{Int}_B$, denote by $\text{Int}_{B,\geq A}$ the subcategory of Int_B consisting of the subrings $A' \subset B$ which contain A. Then

$$D_{(\xi)}(B) = \varinjlim_{A \in \text{Int}_B} \text{Hom}_{\widehat{\mathfrak{AR}}_{\mathbb{O},(\mathbb{O}_E)}}(\text{Int}_{B,\geq A}, D_{(\xi)}).$$

The morphisms in this groupoid are also defined in an analogous fashion.

LEMMA (2.3.2). *Let $D \to D'$ be a formally smooth morphism of groupoids over $\mathfrak{AR}_\mathbb{O}$. Let ξ be in $D(\mathbb{O}_E)$, and ξ' in $D'(\mathbb{O}_E)$ the image of ξ. Then the morphism $D_{(\xi)} \to D'_{(\xi')}$ of groupoids over $\mathfrak{AR}_E$ is formally smooth.*

Proof. Let B be in $\mathfrak{AR}_E$, and $A \in \text{Int}B$. Choose an object $\eta = (\eta, \alpha)$ of $D'_{(\xi')}(A)$. The morphism $\alpha : \eta \to \xi'$ gives ξ the structure of an object of $D_\eta(\mathbb{O}_E)$. Unwinding definitions reveals that $(D_{(\xi)})_\eta = (D_\eta)_{(\xi)}$. Thus we may replace D' by the groupoid attached to η and D by D_η, and hence assume that D and D' are pro-representable. In this case, the lemma follows from (2.3.3) below. $\square$

LEMMA (2.3.3). *If D is pro-represented by a complete local $\mathcal{O}$-algebra R, then the groupoid $D_{(\xi)}$ on $\mathfrak{AR}_E$ is pro-represented by the complete local $\mathcal{O}[1/p]$-algebra $\hat{R}_\xi$ obtained by completing $R \otimes_{\mathcal{O}} E$ along the kernel I_ξ of the map $R \otimes_{\mathcal{O}} E \to E$ induced by ξ.*

Proof. Since D is representable as a groupoid over $\widehat{\mathfrak{AR}}_{\mathcal{O}}$, for any A in $\widehat{\mathfrak{AR}}_{\mathcal{O}}$, any two isomorphic objects of $D(A)$ are related by a unique isomorphism. Hence for any B in $\mathfrak{AR}_E$, any two isomorphic objects of $D_{(\xi)}$ are related by a unique isomorphism, and it suffices to check that $\hat{R}_\xi$ represents the functor $|D_{(\xi)}|$.

Let B be in $\mathfrak{AR}_E$. Any element of $|D_{(\xi)}|(B)$ is induced by a map $R \to A$, for some A in Int_B, such that the composite map $R \otimes_{\mathcal{O}} \mathcal{O}_E \to A \otimes_{\mathcal{O}} \mathcal{O}_E \to \mathcal{O}_E$ kills the preimage of I_ξ in $R \otimes_{\mathcal{O}} \mathcal{O}_E$. Hence we get a map $R \otimes_{\mathcal{O}} E \to B$ with kernel containing a power of I_ξ, since B is Artinian.

Conversely, any map $R \otimes_{\mathcal{O}} E \to B$ which kills a power of I_ξ sends $R \otimes_{\mathcal{O}} \mathcal{O}_E$ onto a compact subring A of B, so that $A \in \mathrm{Int}_B$, and the induced map $R \to A$ produces an element of $|D_{(\xi)}|(A) \subset |D_{(\xi)}|(B)$. $\square$

(2.3.4) Let $\mathbb{F}/\mathbb{F}_p$ be a finite extension. Consider a representation of G_K on a finite dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$ of dimension d. We will need a variant of the groupoid $D_{V_{\mathbb{F}}}$ introduced in (2.1.1). Namely we apply the discussion of (A.6) with $G = G_K$, and we obtain a groupoid $D_{V_{\mathbb{F}}}^{\square}$ over $\mathfrak{AR}_{W(\mathbb{F})}$ such that $D_{V_{\mathbb{F}}}^{\square}(A)$ consists of pairs (V_A, β_A) where V_A in $D_{V_{\mathbb{F}}}(A)$ is a deformation of the G_K-representation $V_{\mathbb{F}}$ to a finite free A-module, and β_A is an ordered A-basis of V_A lifting $\beta_{\mathbb{F}}$.

If $\rho_{\mathbb{F}} : G_K \to \mathrm{GL}_d(V_{\mathbb{F}})$ is the map corresponding to $\beta_{\mathbb{F}}$, then $|D_{V_{\mathbb{F}}}^{\square}|(A)$ is the set of liftings of $\rho_{\mathbb{F}}$ to a map $\rho_A : G_K \to \mathrm{GL}_d(A)$ (*not just up to equivalence*). We call ρ_A a *framed deformation* of $\rho_{\mathbb{F}}$ to A. By Schlessinger's criterion [Ma, 1.2], the functor $|D_{V_{\mathbb{F}}}^{\square}|$ is pro-representable by a complete local $W(\mathbb{F})$-algebra $R_{V_{\mathbb{F}}}^{\square}$, which we call the universal framed deformation ring of $V_{\mathbb{F}}$. Since the objects of $D_{V_{\mathbb{F}}}^{\square}$ have no non-trivial automorphisms, this groupoid is also pro-represented by $R_{V_{\mathbb{F}}}^{\square}$ (cf. (A.5)).

There is a natural map of groupoids $D_{V_{\mathbb{F}}}^{\square} \to D_{V_{\mathbb{F}}}$ which is easily seen to be formally smooth. We remark that the fibers of the map $|D_{V_{\mathbb{F}}}^{\square}|(\mathbb{F}[\epsilon]) \to |D_{V_{\mathbb{F}}}|(\mathbb{F}[\epsilon])$ are principal homogeneous spaces under $\mathrm{ad}V_{\mathbb{F}}/(\mathrm{ad}V_{\mathbb{F}})^{G_K}$, where $\mathrm{ad}V_{\mathbb{F}} = \mathrm{Hom}_{\mathbb{F}}(V_{\mathbb{F}}, V_{\mathbb{F}})$, and hence

$$\dim_{\mathbb{F}} |D_{V_{\mathbb{F}}}^{\square}|(\mathbb{F}[\epsilon]) = \dim_{\mathbb{F}} |D_{V_{\mathbb{F}}}|(\mathbb{F}[\epsilon]) + \dim_{\mathbb{F}} \mathrm{ad}V_{\mathbb{F}} - \dim_{\mathbb{F}} (\mathrm{ad}V_{\mathbb{F}})^{G_K}.$$

We will often have occasion to use this formula and its variants.

Suppose that $E/\mathbb{Q}_p$ is a finite extension whose residue field contains $\mathbb{F}$, and let $\tilde{\xi} = (V_{\mathcal{O}_E}, \beta_{\mathcal{O}_E}) \in D_{V_{\mathbb{F}}}^{\square}(\mathcal{O}_E)$. We denote by ξ the image of $\tilde{\xi}$ in $D_{V_{\mathbb{F}}}(\mathcal{O}_E)$. Write $V_\xi = V_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} E$. This is an E-vector space equipped with a continuous action of G_K, and an ordered basis β_E induced by $\beta_{\mathcal{O}_E}$.

Applying the construction of (A.3) with E in place of $\mathbb{F}$, and V_ξ in place of $V_\mathbb{F}$, we obtain a groupoid D_{V_ξ} over $\mathfrak{AR}_E$ such that the objects of $D_{V_\xi}(B)$ consist of a deformation of the G_K-representation V_ξ to a finite free B-module. Similarly, applying (A.6), we obtain a groupoid $D_{V_\xi}^\square$ over $\mathfrak{AR}_E$, such that for B in $\mathfrak{AR}_E$, $D_{V_\xi}^\square(B)$ is the category of deformations of the G_K-representation V_ξ to a finite free B-module V_B, together with an ordered B-basis β_B for V_B, which lifts β_E.

PROPOSITION (2.3.5). *There are natural isomorphisms of groupoids over* $\mathfrak{AR}_E$,

$$D_{V_\mathbb{F},(\xi)} \overset{\sim}{\longrightarrow} D_{V_\xi} \text{ and } D_{V_\mathbb{F},(\tilde{\xi})}^\square \overset{\sim}{\longrightarrow} D_{V_\xi}^\square.$$

Proof. Let B be in $\mathfrak{AR}_E$. If $A \in \mathrm{Int}_B$, then an object of $D_{V_\mathbb{F},(\xi)}(A)$ corresponds to a continuous representation of G_K on a finite free A-module V_A, equipped with an isomorphism $V_A \otimes_A \mathbb{O}_E \overset{\sim}{\longrightarrow} V_{\mathbb{O}_E}$. Hence, we have natural maps

$$(2.3.6) \qquad D_{V_\mathbb{F},(\xi)}(B) = \varinjlim_{A \in \mathrm{Int}_B} D_{V_\mathbb{F},(\xi)}(A) \to D_{V_\xi}(B)$$

and this underlies a map of groupoids $D_{V_\mathbb{F},(\xi)} \to D_{V_\xi}$ over $\mathfrak{AR}_E$.

The argument of [Ki1, 9.5] shows that any element of $|D_{V_\xi}|(B)$ arises from an element of $|D_{V_\mathbb{F},(\xi)}|(A)$ for some $A \in \mathrm{Int}_B$. Hence the morphism of groupoids underlying (2.3.6) is essentially surjective, and it is fully faithful, because if $V_A, W_A \in D_{V_\mathbb{F},(\xi)}(A)$, then any isomorphism $V_A \otimes_A B \overset{\sim}{\longrightarrow} W_A \otimes_A B$ is induced by an isomorphism $V_A \otimes_A A' \overset{\sim}{\longrightarrow} W_A \otimes_A A'$ for some $A \subset A' \in \mathrm{Int}_B$.

The straightforward argument for the construction of the second isomorphism is similar, and left to the reader. $\square$

(2.3.7) Keeping the above notation, suppose now that $V_\mathbb{F}$ comes from a finite flat group scheme. As in (2.1.1), we denote by $D_{V_\mathbb{F}}^{\mathrm{fl}} \subset D_{V_\mathbb{F}}$ the full subcategory over $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}$ corresponding to deformations which are the generic fiber of a finite flat group scheme. It is again a groupoid. We define $D_{V_\mathbb{F}}^{\mathrm{fl},\square} \subset D_{V_\mathbb{F}}^{\mathrm{fl}}$ analogously, so that

$$D_{V_\mathbb{F}}^{\mathrm{fl},\square} \overset{\sim}{\longrightarrow} D_{V_\mathbb{F}}^{\mathrm{fl}} \times_{D_{V_\mathbb{F}}} D_{V_\mathbb{F}}^\square.$$

Since $D_{V_\mathbb{F}}^{\mathrm{fl}} \to D_{V_\mathbb{F}}$ is relatively representable, and $D_{V_\mathbb{F}}^\square$ is pro-representable, $D_{V_\mathbb{F}}^{\mathrm{fl},\square}$ is pro-representable by a complete local $\mathrm{W}(\mathbb{F})$-algebra $R_{V_\mathbb{F}}^{\mathrm{fl},\square}$.

Suppose that $\tilde{\xi} \in D_{V_\mathbb{F}}^{\mathrm{fl},\square}(\mathbb{O}_E)$ and ξ is its image in $D_{V_\mathbb{F}}^{\mathrm{fl}}(\mathbb{O}_E)$. Then we denote by $D_{V_\xi}^{\mathrm{cris}}$ the full subgroupoid of D_{V_ξ} corresponding to deformations which are crystalline, and we define a subgroupoid $D_{V_\xi}^{\mathrm{cris},\square}$ of $D_{V_\xi}^\square$ in a similar way.

PROPOSITION (2.3.8). *The isomorphisms of* (2.3.5) *induce isomorphisms*

$$D_{V_\mathbb{F},(\xi)}^{\mathrm{fl}} \overset{\sim}{\longrightarrow} D_{V_\xi}^{\mathrm{cris}} \text{ and } D_{V_\mathbb{F},(\tilde{\xi})}^{\mathrm{fl},\square} \overset{\sim}{\longrightarrow} D_{V_\xi}^{\mathrm{cris},\square}.$$

Proof. The second isomorphism follows easily from the first and (2.3.5), so we need only to establish the first isomorphism.

Let B be in $\mathfrak{AR}_E$ and $A \in \mathrm{Int}_B$. If V_A is in $D^{\mathrm{fl}}_{V_{\mathbb{F}}}(A)$, then $V_A/p^n V_A$ extends to a finite flat group scheme for every n, so that V_A is the Tate module of a p-divisible group [Ra, 2.3.1], and $V_B = V_A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline. This shows, in particular, that V_ξ is Barsotti-Tate, and that the first morphism of (2.3.6) induces a morphism $D^{\mathrm{fl}}_{V_{\mathbb{F}},(\xi)} \to D^{\mathrm{cris}}_{V_\xi}$ over $\mathfrak{AR}_E$.

Now suppose that $V_B \in D^{\mathrm{cris}}_{V_\xi}(B)$ is crystalline. As a $\mathbb{Q}_p$-representation it is a successive extension of copies of V_ξ, and hence its Hodge-Tate weights are all either 0 or 1. By (2.3.5), there exists a G_K-stable, free, A-submodule $V_A \subset V_B$ with $V_A \otimes_A B \xrightarrow{\sim} V_B$. On the other hand, by [Br3, 5.3.1], there exists a G_K-stable $\mathbb{Z}_p$-lattice $L \subset V_B$ such that L is the Tate module of a p-divisible group. We may assume that $V_A \subset L$. The quotient L/V_A is killed by p^r for some positive integer r. Hence for every integer $n > r$, $V_A/p^{n-r} V_A$ is a subquotient of $L/p^n L$. It follows that $V_A/p^{n-r} V_A$ extends to a finite flat group scheme [Ram, 2.1], which shows that $V_A \in D^{\mathrm{fl}}_{V_{\mathbb{F}},(\xi)}(A)$. It follows that $D^{\mathrm{fl}}_{V_{\mathbb{F}},(\xi)} \to D^{\mathrm{cris}}_{V_\xi}$ is essentially surjective, and it is fully faithful because the morphisms in (2.3.5) are. $\square$

LEMMA (2.3.9). *The functor $|D^{\mathrm{cris}}_{V_\xi}|$ is formally smooth.*

Proof. Let B be in $\mathfrak{AR}_E$ and $I \subset B$ an ideal with $I^2 = 0$. Suppose $[V_{B/I}]$ is in $|D^{\mathrm{cris}}_{V_\xi}|(B/I)$. Since $V_{B/I}$ is crystalline, it corresponds to a weakly admissible ϕ-module $\mathcal{M}_{B/I}$, which lies in $(\mathrm{Mod}/K_0)_{B/I\text{-fr}}$, by (1.3.4). Moreover, since $V_{B/I}$, considered as a $\mathbb{Q}_p$-representation, is a successive extension of copies of V_ξ, we have $\mathrm{Fil}^1 \mathcal{M}_{B/I,K} = 0$ and $\mathrm{Fil}^{-1} \mathcal{M}_{B/I,K} = \mathcal{M}_{B/I,K}$. Here the subscript K denotes tensoring by $\otimes_{K_0} K$.

By (1.3.2), $\mathcal{M}_{B/I}$ is free as a $B/I \otimes_{\mathbb{Q}_p} K_0$-module. Choose a free $B \otimes_{\mathbb{Q}_p} K_0$-module $\mathcal{M}_B$ and an isomorphism $\mathcal{M}_B \otimes_B B/I \xrightarrow{\sim} \mathcal{M}_{B/I}$. Since $\mathrm{gr}^\bullet \mathcal{M}_{B/I,K}$ is a projective $B/I \otimes_{\mathbb{Q}_p} K$-module, one sees easily that there exists a projective $B \otimes_{\mathbb{Q}_p} K$-submodule $\mathrm{Fil}^0 \mathcal{M}_{B,K} \subset \mathcal{M}_{B,K}$ such that $\mathcal{M}_{B,K}/\mathrm{Fil}^0 \mathcal{M}_{B,K}$ is also projective over $B \otimes_{\mathbb{Q}_p} K$, and $\mathrm{Fil}^0 \mathcal{M}_{B,K} \otimes_B B/I = \mathrm{Fil}^0 \mathcal{M}_{B/I,K} \subset \mathcal{M}_{B/I,K}$. Similarly, we can lift the morphism $\varphi^*(\mathcal{M}_{B/I}) \to \mathcal{M}_{B/I}$ to a morphism $\varphi^*(\mathcal{M}_B) \to \mathcal{M}_B$. Now we have an exact sequence

$$0 \to I \otimes_{B/I} \mathcal{M}_{B/I} \to \mathcal{M}_B \to \mathcal{M}_{B/I} \to 0$$

which is compatible with Frobenius maps, and remains exact after tensoring by $\otimes_{K_0} K$ and applying Fil^0. Now choosing a presentation of I by free B/I-modules, we see that the term on the left is admissible, since the category of weakly admissible modules, admits cokernels. Since this category is stable under extensions, this shows that $\mathcal{M}_B$ is weakly admissible.

Now setting $V_B = V_{\mathrm{cris},B}(\mathcal{M}_B)$ and using (1.3.5), we see that V_B lifts $V_{B/I}$.

$\square$

(2.3.10) Let $\mathbf{v}$ and F be as in (2.2.4), and E/F be a finite extension. If $\xi \in D^{\mathrm{fl}}_{V_{\mathbb{F}}}(\mathcal{O}_E)$, we say that ξ or V_ξ has p-adic Hodge type $\mathbf{v}$ if for all $a \in K$

$$\det_E(a \mid D_{\mathrm{cris}}(V_\xi)_K / \mathrm{Fil}^0 D_{\mathrm{cris}}(V_\xi)_K) = \prod_\psi \psi(a)^{v_\psi}.$$

Now, $\tilde{\xi}$ in $D^{\mathrm{fl},\square}_{V_{\mathbb{F}}}(\mathcal{O}_E)$ has p-adic Hodge type $\mathbf{v}$ if its image in $D^{\mathrm{fl}}_{V_{\mathbb{F}}}(\mathcal{O}_E)$ has p-adic Hodge type $\mathbf{v}$.

We denote by $\hat{R}^{\mathrm{fl},\square}_{\tilde{\xi}}$ the completion of $R^{\mathrm{fl},\square}_{V_{\mathbb{F}}} \otimes_{\mathrm{W}(\mathbb{F})} E$ along the kernel of the map $R^{\mathrm{fl},\square}_{V_{\mathbb{F}}} \otimes_{\mathrm{W}(\mathbb{F})} E \to E$ corresponding to $\tilde{\xi}$. Similarly, if $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, then $\hat{R}^{\mathrm{fl}}_{\xi}$ denotes the completion of $R^{\mathrm{fl}}_{V_{\mathbb{F}}} \otimes_{\mathrm{W}(\mathbb{F})} E$ along the kernel of the map $R^{\mathrm{fl}}_{V_{\mathbb{F}}} \otimes_{\mathrm{W}(\mathbb{F})} E \to E$ corresponding to ξ.

COROLLARY (2.3.11). *The E-algebra $\hat{R}^{\mathrm{fl},\square}_{\tilde{\xi}}$ is formally smooth over E. If ξ has p-adic Hodge type $\mathbf{v}$, and $d = \dim_{\mathbb{F}} V_{\mathbb{F}}$, then*

$$\dim_E \hat{R}^{\mathrm{fl},\square}_{\tilde{\xi}} = d^2 + \sum_\psi (d - v_\psi) v_\psi.$$

If $\mathrm{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, then $\hat{R}^{\mathrm{fl}}_{\xi}$ is formally smooth over E and

$$\dim_E \hat{R}^{\mathrm{fl}}_{\xi} = 1 + \sum_\psi (d - v_\psi) v_\psi.$$

Proof. The claims regarding formal smoothness follows from (2.3.8), (2.3.9) and (2.3.3).

Given this, to check that $\dim_E \hat{R}^{\mathrm{fl},\square}_{\tilde{\xi}}$ (resp. $\dim_E \hat{R}^{\mathrm{fl}}_{\xi}$) is given by the above formula, it suffices to check that this is the dimension of its tangent space. By what we have already shown this is equal to $\dim_E |D^{\mathrm{cris},\square}_{V_{\tilde{\xi}}}|(E[\epsilon])$ (resp. $\dim_E D^{\mathrm{cris}}_{V_\xi}(E[\epsilon])$) (where $\epsilon^2 = 0$). Now $|D^{\mathrm{cris}}_{V_\xi}|(E[\epsilon]) = H^1_f(G_K, \mathrm{ad}V_\xi)$ has E-dimension $\sum_\psi (d - v_\psi) v_\psi + \dim_E H^0(G_K, \mathrm{ad}V_\xi)$ by [Ne, 1.24], which, in particular, proves the second formula of the corollary. As in (2.3.4) we then have

$$\dim_E |D^{\mathrm{cris},\square}_{V_{\tilde{\xi}}}|(E[\epsilon]) = \dim_E |D^{\mathrm{cris}}_{V_\xi}|(E[\epsilon]) + d^2 - \dim_E H^0(G_K, \mathrm{ad}V_\xi)$$

$$= d^2 + \sum_\psi (d - v_\psi) v_\psi. \qquad \square$$

(2.4) *Connected components.* In this subsection we analyze the connected components of $\mathrm{Spec}\, R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}[1/p]$ and $\mathrm{Spec}\, R^{\mathrm{fl}}_{V_{\mathbb{F}}}[1/p]$ using $\mathscr{GR}_{V_{\mathbb{F}}}$ and the related schemes introduced in Section (2.1). We retain the notation of the previous section. As in Section (2.1), we denote by $M_{\mathbb{F}}$ the preimage of $V_{\mathbb{F}}(-1)$ under the functor $T_{\mathbb{F}}$.

(2.4.1) Let R be a complete, local, Noetherian $\mathbb{O}_F$-algebra with maximal ideal $\mathfrak{m}_R$, and residue field $\mathbb{F}$, and fix $\xi \in D_{V_\mathbb{F}}^\mathrm{fl}(R)$. Let E/F be a finite extension and $y : R \to \mathbb{O}_E$ a map of $\mathbb{O}_F$-algebras. We say that y has p-adic Hodge type $\mathbf{v}$ if the object of $D_{V_\mathbb{F}}^\mathrm{fl}(\mathbb{O}_E)$ obtained from ξ by applying $\otimes_R \mathbb{O}_E$ has p-adic Hodge type $\mathbf{v}$.

Let $\mathscr{X}^\mathrm{an}$ denote the p-adic analytic space attached to $\mathrm{Spf}\,R$ [deJ, §7]. The points of $\mathscr{X}^\mathrm{an}$ are in bijection with the maximal ideals of $R[1/p]$ [deJ, 7.1.9]. In particular, any y as above corresponds to a point of $\mathscr{X}^\mathrm{an}$ which we again denote by y. Denote by $(v_{\psi,y})_\psi$ the p-adic Hodge type of y. By the main result of [Se], for each $\mathbb{Q}_p$-algebra embedding $\psi : K \to K_0^\mathrm{sep}$, the coefficients of $X^{d-v_{\psi,y}}(X-1)^{v_{\psi,y}}$ are analytic functions of $y \in \mathscr{X}^\mathrm{an}$. Since they are in fact integers, they must be constant on the connected components of $\mathscr{X}^\mathrm{an}$, and the functions $y \mapsto v_{\psi,y}$ are likewise constant on these components. (Sen considers a more general situation involving families of representations which are not necessarily Hodge-Tate, and so his functions are in general non-constant.)

The connected components of $\mathscr{X}^\mathrm{an}$ coincide with those of $\mathrm{Spec}\,R[1/p]$. When R is normal this follows from [deJ, 7.4.1], which shows that any idempotent function on $\mathscr{X}^\mathrm{an}$ is contained in R. For the general case, let $\tilde{R}$ denote the normalization of R, and $\tilde{\mathscr{X}}^\mathrm{an}$ the associated analytic space. Then $\tilde{R}$ is a finite R-algebra, so that $\tilde{\mathscr{X}}^\mathrm{an} \to \mathscr{X}^\mathrm{an}$ is finite [deJ, 7.2.1]. Since the points of $\mathscr{X}^\mathrm{an}$ are in bijection with the closed points of $\mathrm{Spec}\,R[1/p]$, one sees that the images in $\mathrm{Spec}\,R[1/p]$ of two connected components of $\mathrm{Spec}\,\tilde{R}[1/p]$ meet if and only if the images in $\mathscr{X}^\mathrm{an}$ of the corresponding components of $\tilde{\mathscr{X}}^\mathrm{an}$ meet.

Hence there is a subset of the set of connected components of $\mathrm{Spec}\,R[1/p]$ such that y has p-adic Hodge type $\mathbf{v}$ if and only if (the image of) y lies on one of these connected components. We denote by $R^\mathbf{v}$ the quotient of R corresponding to the closure of these connected components in $\mathrm{Spec}\,R$.

(2.4.2) Restricting $D_{\mathfrak{S},M_\mathbb{F}}$ to $\mathfrak{Aug}_{\mathbb{O}_F}$ yields a groupoid $D_{\mathfrak{S},M_\mathbb{F}}|_{\mathfrak{Aug}_{\mathbb{O}_F}}$, and we denote by $D_{\mathfrak{S},M_\mathbb{F}}^\mathbf{v}$ the full subgroupoid whose value on (A, I) in $\mathfrak{Aug}_{\mathbb{O}_F}$ consists of those $\mathfrak{M}_A$ in $D_{\mathfrak{S},M_\mathbb{F}}(A, I)$ such that for $a \in \mathbb{O}_K$ we have

$$\det_A(a|(1 \otimes \phi)(\phi^*(\mathfrak{M}_A))/E(u)\mathfrak{M}_A) = \prod_\psi \psi(a)^{v_\psi}$$

as polynomial functions on $\mathbb{O}_K$. We set $D_{\mathfrak{S},M_\mathbb{F},\xi}^\mathbf{v} = \xi \times_{D_{V_\mathbb{F}}^\mathrm{fl}|_{\mathfrak{Aug}_{\mathbb{O}_F}}} D_{\mathfrak{S},M_\mathbb{F}}^\mathbf{v}$.

LEMMA (2.4.3). *The groupoid $D_{\mathfrak{S},M_\mathbb{F},\xi}^\mathbf{v}$ is representable in the sense of* (2.1.10) *by a closed subscheme*

$$\mathscr{GR}_{V_\mathbb{F},\xi}^\mathbf{v} \subset \mathscr{GR}_{V_\mathbb{F},\xi}.$$

Proof. Let T be any $\mathbb{O}_F$-scheme, and L be an $\mathbb{O}_K \otimes_{\mathbb{Z}_p} \mathbb{O}_T$-module which is locally $\mathbb{O}_T$-free. With the notation of the proof of (2.2.5), equating coefficients in

the equality

$$\det_{\mathbb{O}_T}\left(\sum_{i,\sigma}\epsilon_\sigma\pi^i X_{i,\sigma}|L\right) = \prod_\psi \sum_{i,\sigma}(\psi(\epsilon_\sigma\pi^i)X_{i,\sigma})^{v_\psi}$$

defines a closed subspace of T.

Now over $\mathcal{GR}_{V_{\mathbb{F}},\xi}$ we have a universal sheaf of $\mathfrak{S}\otimes\mathbb{O}_{\mathcal{GR}_{V_{\mathbb{F}},\xi}}$-modules $\mathfrak{M}_\xi^{\mathrm{univ}}$, equipped with a morphism $\phi^*(\mathfrak{M}_\xi^{\mathrm{univ}})\to\mathfrak{M}_\xi^{\mathrm{univ}}$. The closed subspace $\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}\subset\mathcal{GR}_{V_{\mathbb{F}},\xi}$ can be constructed by applying the remark of the previous paragraph with $T=\mathcal{GR}_{V_{\mathbb{F}},\xi}$, $L=\phi^*(\mathfrak{M}_\xi^{\mathrm{univ}})/E(u)\mathfrak{M}_\xi^{\mathrm{univ}}$. $\qquad\square$

(2.4.4) If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$ and $\mathbb{F}^{\mathrm{sep}}$ the residue field of K_0^{sep}, then following (2.2.5), for any $\sigma\in\mathrm{Gal}(K_0/\mathbb{Q}_p)$, we denote by $\epsilon_\sigma\in k\otimes_{\mathbb{F}_p}\mathbb{F}'$ the idempotent which is 1 modulo the kernel of the map $k\otimes_{\mathbb{F}_p}\mathbb{F}'\overset{\sigma\otimes 1}{\to}\mathbb{F}^{\mathrm{sep}}$ corresponding to σ, and 0 modulo the other maximal ideals of $k\otimes_{\mathbb{F}_p}\mathbb{F}'$.

From now on we make the following assumption:

(2.4.5) The morphism $\xi\to D_{V_{\mathbb{F}}}^{\mathrm{fl}}$ of groupoids over $\mathfrak{AR}_{\mathbb{O}_F}$ is formally smooth.

The two most important examples for applications are $R=R_{V_{\mathbb{F}}}^{\mathrm{fl}}\otimes_{\mathrm{W}(\mathbb{F})}\mathbb{O}_F$ if $\mathrm{End}_{\mathbb{F}[G_K]}V_{\mathbb{F}}=\mathbb{F}$, or more generally, $R=R_{V_{\mathbb{F}}}^{\mathrm{fl},\square}\otimes_{\mathrm{W}(\mathbb{F})}\mathbb{O}_F$.

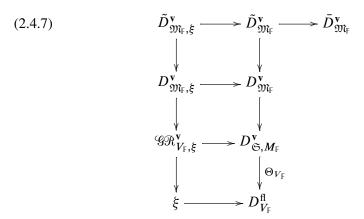PROPOSITION (2.4.6). *Let $\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ denote the closed subscheme of $\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}$ corresponding to the ideal sheaf of p-power torsion sections, and write $\overline{\mathcal{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}=\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}\otimes_{\mathbb{O}_F}\mathbb{F}$ and $\overline{\mathcal{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}=\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}\otimes_{\mathbb{O}_F}\mathbb{F}$. Then:*

(1) *$\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ is normal and Cohen-Macaulay.*

(2) *$\overline{\mathcal{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ is reduced and normal with rational singularities.*

(3) *If $\mathbb{F}'/\mathbb{F}$ is a finite extension, and $\mathfrak{M}_{\mathbb{F}'}$ in $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_{\mathbb{F}'}$ corresponds to a point of $y\in\overline{\mathcal{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}(\mathbb{F}')$, then $y\in\overline{\mathcal{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}(\mathbb{F}')$ if and only if for every $\sigma\in\mathrm{Gal}(K_0/\mathbb{Q}_p)$, the endomorphism $1\otimes\pi$ of $\epsilon_\sigma(1\otimes\phi)\phi^*(\mathfrak{M}_{\mathbb{F}'})/E(u)\mathfrak{M}_{\mathbb{F}'}$ has Jordan type $\mathbf{s}_\sigma$, satisfying $\mathbf{s}_\sigma\leqslant\check{\mathbf{v}}_\sigma$.*

(4) *If for any $\sigma\in\mathrm{Gal}(K_0/\mathbb{Q}_p)$ any two of the integers v_ψ with $\psi_0=\sigma$ differ by at most 1, then $\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}=\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}$ provided either $v_\psi=0$ or 1 for each ψ, or $e\leqslant 2$.*

Proof. Since each of the statements is local on $\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}$, it suffices to prove them in a neighborhood of a closed point y of $\mathcal{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}$ with residue field $\mathbb{F}'$. After replacing $\mathbb{F}$ by $\mathbb{F}'$, and R by $R\otimes_{\mathrm{W}(\mathbb{F})}\mathrm{W}(\mathbb{F}')$, we may assume that $\mathbb{F}'=\mathbb{F}$. We denote by $\mathfrak{M}_{\mathbb{F}}$ the object of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_{\mathbb{F}}$ corresponding to y.

If $(A,\mathfrak{m}_A)$ is in $\mathfrak{AR}_{\mathbb{O}_F}$, and $\mathfrak{M}_A$ is in $D_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}(A)$, then we can also regard $\mathfrak{M}_A$ as an object of $D_{\mathfrak{S},M_{\mathbb{F}}}^{\mathbf{v}}(A,\mathfrak{m}_A)$. Thus we get a morphism of groupoids $D_{\mathfrak{M}_{\mathbb{F}}}^{\mathbf{v}}\to D_{\mathfrak{S},M_{\mathbb{F}}}^{\mathbf{v}}$ on $\mathfrak{AR}_{\mathbb{O}_F}$. (Here we again denote by $D_{\mathfrak{S},M_{\mathbb{F}}}^{\mathbf{v}}$ the restriction of $D_{\mathfrak{S},M_{\mathbb{F}}}^{\mathbf{v}}$ to $\mathfrak{AR}_{\mathbb{O}_F}$,

and we will adopt the analogous convention for other groupoids below.) This fits into a commutative diagram of groupoids over $\mathfrak{AR}_{\mathbb{O}_F}$.

(2.4.7)
$$
\begin{array}{ccc}
\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi} \longrightarrow \tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}} \longrightarrow \bar{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}} \\
\downarrow \qquad\qquad \downarrow \\
D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi} \longrightarrow D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}} \\
\downarrow \qquad\qquad \downarrow \\
\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},\xi} \longrightarrow D^{\mathbf{v}}_{\mathfrak{S},M_{\mathbb{F}}} \\
\qquad\qquad \downarrow \Theta_{V_{\mathbb{F}}} \\
\xi \longrightarrow D^{\mathrm{fl}}_{V_{\mathbb{F}}}
\end{array}
$$

Here the lowest square is Cartesian by definition, and $D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi}$ and $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi}$ are defined so that the other two squares are Cartesian. The two maps with $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}$ as their source are given by (2.2.12) and (2.2.13).

From the definitions one sees that the completion of the local ring on $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ at the point y represents the groupoid $D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi}$. Let $R^{\mathbf{v}}_{V_{\mathbb{F}},y}$ denote this complete R-algebra. By (2.2.11) $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}},\xi}$ is also (pro)-represented by a complete local R-algebra $\tilde{R}^{\mathbf{v}}_{V_{\mathbb{F}},y}$. Let $R^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},y}$ and $\tilde{R}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},y}$ denote the quotient of $R^{\mathbf{v}}_{V_{\mathbb{F}},y}$ and $\tilde{R}^{\mathbf{v}}_{V_{\mathbb{F}},y}$ respectively by their ideals of p-power torsion elements. Since $\tilde{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}} \to D^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}$ is formally smooth by (2.2.11), $\tilde{R}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},y} = R^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},y} \otimes_{R^{\mathbf{v}}_{V_{\mathbb{F}},y}} \tilde{R}^{\mathbf{v}}_{V_{\mathbb{F}},y}$ is formally smooth over $R^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},y}$.

Now for any Noetherian local ring A, its completion $\hat{A}$ is flat over A, and this implies that $A[p^\infty] \cdot \hat{A} = \hat{A}[p^\infty]$. From this we deduce that $R^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},y}$ is the completion of the local ring of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}$ at y, if y lies in this closed subscheme, and is 0 otherwise.

On the other hand, $\bar{D}^{\mathbf{v},\mathrm{loc}}_{\mathfrak{M}_{\mathbb{F}}}$ and $\bar{D}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}$ are represented by complete local rings $\bar{R}^{\mathbf{v},\mathrm{loc}}_{\mathfrak{M}_{\mathbb{F}}}$ and $\bar{R}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}$ respectively. These are the complete local rings at points on the schemes $M^{\mathrm{loc}}_{\mathbf{v}}$ and $M_{\mathbf{v}}$ respectively, where $M^{\mathrm{loc}}_{\mathbf{v}}$ and $M_{\mathbf{v}}$ are the $\mathbb{O}_F$-schemes introduced in Section (2.2). By (2.2.11), and the formal smoothness of $\xi \to D^{\mathrm{fl}}_{V_{\mathbb{F}}}$, $\tilde{R}^{\mathbf{v}}_{V_{\mathbb{F}},y}$ is formally smooth over $\bar{R}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}$, and hence $\tilde{R}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},y} = \tilde{R}^{\mathbf{v}}_{V_{\mathbb{F}},y} \otimes_{\bar{R}^{\mathbf{v}}_{\mathfrak{M}_{\mathbb{F}}}} \bar{R}^{\mathbf{v},\mathrm{loc}}_{\mathfrak{M}_{\mathbb{F}}}$ is formally smooth over $\bar{R}^{\mathbf{v},\mathrm{loc}}_{\mathfrak{M}_{\mathbb{F}}}$.

The proposition now follows from (2.2.8). $\qquad\qquad\qquad\qquad\square$

PROPOSITION (2.4.8). *The map* $\Theta_{V_{\mathbb{F}}}$ *of* (2.1.4) *induces a projective map of* $\mathbb{O}_F$-*schemes*
$$
\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi} : \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi} \to \mathrm{Spec}\, R^{\mathbf{v}}.
$$
This map becomes an isomorphism after we apply $\otimes_{\mathbb{O}_F} F$.

Proof. To check that the composite

$$(2.4.9) \qquad \mathscr{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi} \hookrightarrow \mathscr{GR}_{V_{\mathbb{F}},\xi} \overset{\Theta_{V_{\mathbb{F}},\xi}}{\to} \operatorname{Spec} R$$

factors through $\operatorname{Spec} R^{\mathbf{v}}$, it suffices to check that the map obtained from (2.4.9) by inverting p factors through $\operatorname{Spec} R^{\mathbf{v}}[1/p]$. Since the latter scheme is a union of connected components of $\operatorname{Spec} R[1/p]$, it is enough to check that for any finite extension E/F, any E-valued point of $\mathscr{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi} \otimes_{\mathbb{O}_F} F$ maps to $\operatorname{Spec} R^{\mathbf{v}}[1/p](E)$.

Let y be such a point. Then y corresponds to a map $R \to \mathbb{O}_E$, and hence to a point of $\mathscr{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}$, by the valuative criterion for properness. These two $\mathbb{O}_E$-valued points give rise, respectively, to a continuous representation of G_K on a finite free $\mathbb{O}_E$-module $V_{\mathbb{O}_E}$, and an object $\mathfrak{M}_{\mathbb{O}_E}$ of $(\operatorname{Mod} \operatorname{FI}/\mathfrak{S})_{\mathbb{O}_E}$.

Let $\mathcal{M}$ denote the object of $(\operatorname{Mod} \operatorname{FI}/S)_{\mathbb{O}_E}$ corresponding to $\mathfrak{M}_{\mathbb{O}_E}$. Using (1.1.3), one can attach a p-divisible group $\operatorname{Gr}(\mathcal{M})$ to $\mathcal{M}$, whose Cartier dual we denote by $\operatorname{Gr}_D(\mathcal{M})$. The latter p-divisible group is the one attached to $\mathfrak{M}_{\mathbb{O}_E}$ by the functor of (2.2.22). $\operatorname{Gr}(\mathcal{M})$ and $\operatorname{Gr}_D(\mathcal{M})$ are equipped with an action of $\mathbb{O}_E$ by functoriality.

Now write $V_E = V_{\mathbb{O}_E} \otimes_{\mathbb{O}_E} E$, and for a p-divisible group $\mathcal{G}$, let $V_p(\mathcal{G}) = T_p(\mathcal{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $T_p(\mathcal{G})$ is the Tate module of $\mathcal{G}$. Given a continuous representation of G_K on a $\mathbb{Q}_p$-vector space V, we set

$$D_{\mathrm{cris}}(V)^* = \operatorname{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{\mathrm{cris}}).$$

We compute

$$D_{\mathrm{cris}}(V_E)_K/\operatorname{Fil}^0 D_{\mathrm{cris}}(V_E)_K$$

$$\overset{\sim}{\longrightarrow} D_{\mathrm{cris}}(V_p(\operatorname{Gr}_D(\mathcal{M}))(-1))_K/\operatorname{Fil}^0 D_{\mathrm{cris}}(V_p(\operatorname{Gr}_D(\mathcal{M}))(-1))_K$$

$$\overset{\sim}{\longrightarrow} D^*_{\mathrm{cris}}(V_p(\operatorname{Gr}(\mathcal{M})))_K/\operatorname{Fil}^1 D^*_{\mathrm{cris}}(V_p(\operatorname{Gr}(\mathcal{M})))_K$$

$$\overset{\sim}{\longrightarrow} (\mathcal{M}/\operatorname{Fil}^1 \mathcal{M}) \otimes_{\mathbb{O}_K} K \overset{\sim}{\longrightarrow} ((1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{O}_E}))/E(u)\mathfrak{M}_{\mathbb{O}_E}) \otimes_{\mathbb{O}_K} K.$$

The first isomorphism follows from (1.1.13), and the full faithfulness of [Br4, 3.4.3], the third from [Br3, 5.3.1], and the final isomorphism was noted in the proof of (1.2.4), and has been used several times above. It now follows from the definitions that y corresponds to an E-valued point of $R^{\mathbf{v}}$. This proves the existence of the map $\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$.

To check that $\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ becomes an isomorphism after $\otimes_{\mathbb{O}_F} F$, we begin by checking that it induces a bijection on $\mathbb{O}_E$-valued points for any finite extension E/F. Consider an $\mathbb{O}_E$-valued point y of $R^{\mathbf{v}}$, and let $V_{\mathbb{O}_E}$ be the corresponding G_K-representation. By definition $V_{\mathbb{O}_E}/p^n V_{\mathbb{O}_E}$ comes from a finite flat group scheme, so $V_{\mathbb{O}_E} = T_p(\mathcal{G})$ for a p-divisible group $\mathcal{G}$ [Ra, 2.3.1], which is unique by Tate's theorem [Ta, Thm. 4]. By (2.2.22) $\mathcal{G}$ arises from a unique module $\mathfrak{M}_{\mathbb{O}_E}$ in $(\operatorname{Mod} \operatorname{FI}/\mathfrak{S})_{\mathbb{Z}_p}$,

which carries an action of $\mathbb{O}_E$. Since $\mathfrak{M}_{\mathbb{O}_E}$ is free over $\mathfrak{S}$, $\mathfrak{M}_{\mathbb{O}_E}/u\mathfrak{M}_{\mathbb{O}_E}$ is free over $\mathfrak{S}/u\mathfrak{S}$, and hence it is finite flat over $\mathfrak{S}_{\mathbb{O}_E}/u\mathfrak{S}_{\mathbb{O}_E}$. This implies that $\mathfrak{M}_{\mathbb{O}_E}$ is finite flat over $\mathfrak{S}_{\mathbb{O}_E}$, so that $\mathfrak{M}_{\mathbb{O}_E}$ is in $(\mathrm{Mod\ FI}/\mathfrak{S})_{\mathbb{O}_E}$. Finally, if $\mathbb{F}'$ denotes the residue field of $\mathbb{O}_E$, the isomorphism $V_{\mathbb{O}_E} \otimes_{\mathbb{O}_E} \mathbb{F}' \xrightarrow{\sim} V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ induces an isomorphism $\mathfrak{M}_{\mathbb{O}_E} \otimes_{\mathbb{O}_E} \mathbb{F}'[1/u] \xrightarrow{\sim} M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$.

This, together with the calculation already done above, shows that $\mathfrak{M}_{\mathbb{O}_E}$ corresponds to an E-valued point $\tilde{y}$ of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ which maps to y. Any other such E-valued point $\tilde{y}'$ corresponds to an object $\mathfrak{M}'_{\mathbb{O}_E}$ of $(\mathrm{Mod\ FI}/\mathfrak{S})_{\mathbb{O}_E}$. Since $\mathfrak{M}_{\mathbb{O}_E}$ and $\mathfrak{M}'_{\mathbb{O}_E}$ both give rise to $V_{\mathbb{O}_E}$, the construction in (2.1.7) shows that we may identify $\mathfrak{M}_{\mathbb{O}_E}[1/u]$ with $\mathfrak{M}'_{\mathbb{O}_E}[1/u]$. On the other hand, $\mathfrak{M}'_{\mathbb{O}_E}$ gives rise to a p-divisible group $\mathcal{G}'$, and if we identify the generic fibers $\mathcal{G}_K$ and $\mathcal{G}'_K$ with the étale p-divisible group over K corresponding to $V_{\mathbb{O}_E}$, then this identification extends uniquely to an isomorphism $\mathcal{G}' \xrightarrow{\sim} \mathcal{G}$, by Tate's theorem. This implies that the isomorphism $\mathfrak{M}_{\mathbb{O}_E}[1/u] \xrightarrow{\sim} \mathfrak{M}'_{\mathbb{O}_E}[1/u]$ identifies $\mathfrak{M}_{\mathbb{O}_E}$ with $\mathfrak{M}'_{\mathbb{O}_E}$, so that $\tilde{y} = \tilde{y}'$.

It follows that $\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ induces a bijection on E-valued points. Let $\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi} \otimes_{\mathbb{O}_F} F$ denote the map obtained from $\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ by inverting p. It is a projective map, which induces a bijection on closed points. In particular, this implies that $\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi} \otimes_{\mathbb{O}_F} F$ is a finite map.

By (2.4.6)(1) the source of this map is normal, and by (2.3.2), (2.3.8), (2.3.9) and the condition (2.4.5), its target is regular. Hence $\Theta^{\mathbf{v}}_{V_{\mathbb{F}},\xi} \otimes_{\mathbb{O}_F} F$ is an isomorphism. $\qquad\square$

COROLLARY (2.4.10). *For a topological space X we denote by $H_0(X)$ the set of connected components of X. Let $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}$ denote the fiber of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}$ over the closed point of $R^{\mathbf{v}}$. There is a bijection*

$$H_0(\mathrm{Spec}\ R^{\mathbf{v}}[1/p]) \xrightarrow{\sim} H_0(\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},0}).$$

Proof. Consider a connected component of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi} \otimes_{\mathbb{O}_F} F$, and let e be the idempotent which is 1 on this component, and vanishes on the others. If $\pi_F \in \mathbb{O}_F$ is a uniformiser, let n be the least non-negative integer n such that $\pi_F^n e$ extends to a section of $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}$. We have

$$(\pi_F^n e)^2 = \pi_F^n (\pi_F^n e).$$

If $n \geq 1$, this implies that $\pi_F^n e$ induces a non-zero, nilpotent function on $\overline{\mathcal{GR}}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi} = \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi} \otimes_{\mathbb{O}_F} \mathbb{F}$. However, this scheme is reduced by (2.4.6)(2). It follows that $n = 0$. This implies that

(2.4.11) $$H_0(\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi} \otimes_{\mathbb{O}_F} F) \xrightarrow{\sim} H_0(\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}).$$

On the other hand, if $\widehat{\mathscr{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ denotes the formal scheme obtained from $\mathscr{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ by completing along $\mathfrak{m}_R$, then we have

$$(2.4.12) \qquad H_0(\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc}}) \xrightarrow{\sim} H_0(\widehat{\mathscr{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}) \xrightarrow{\sim} H_0(\mathscr{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}})$$

where the first isomorphism follows from the fact that $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc}}$ and $\widehat{\mathscr{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ have the same underlying topological space, while the second isomorphism follows from [GrD, III, 5.5.1]; the key point is that the global idempotent functions on $\mathscr{GR}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ and $\widehat{\mathscr{GR}}_{V_{\mathbb{F}},\xi}^{\mathbf{v},\mathrm{loc}}$ are in bijection [GrD, III, 4.1.5]. Combining (2.4.11) and (2.4.12) with (2.4.8) proves the corollary. $\square$

(2.4.13) We will make a conjecture regarding the connected components of $\operatorname{Spec} R^{\mathbf{v}}[1/p]$ or, equivalently those of $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{loc}}$. To formulate it we need a strengthened version of (1.2.11).

PROPOSITION (2.4.14). *Let (A, I) be in $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$, and $\mathfrak{M}_A$ in $D_{\mathfrak{S},M_{\mathbb{F}}}(A, I)$. Then $\mathfrak{M}_A$ admits a maximal multiplicative subobject $\mathfrak{M}_A^m$ and a maximal étale quotient $\mathfrak{M}_A^{\text{ét}}$. Moreover*:

(1) *Both the quotient $\mathfrak{M}_A/\mathfrak{M}_A^m$ and the kernel of $\mathfrak{M}_A \to \mathfrak{M}_A^{\text{ét}}$ are objects of $(\operatorname{Mod} \mathrm{FI}/\mathfrak{S})_A$.*

(2) *If $(A, I) \to (B, J)$ is a morphism in $\mathfrak{Aug}_{\mathrm{W}(\mathbb{F})}$ then there are natural isomorphisms*

$$(\mathfrak{M}_A \otimes_A B)^m = \mathfrak{M}_A^m \otimes_A B \text{ and } (\mathfrak{M}_A \otimes_A B)^{\text{ét}} = \mathfrak{M}_A^{\text{ét}} \otimes_A B.$$

(3) *There are natural isomorphisms*

$$(\mathfrak{M}_A^*)^m \xrightarrow{\sim} (\mathfrak{M}_A^{\text{ét}})^* \text{ and } (\mathfrak{M}_A^*)^{\text{ét}} \xrightarrow{\sim} (\mathfrak{M}_A^m)^*.$$

Proof. Standard arguments show that it is enough to consider the case where A is finitely generated over $\mathrm{W}(\mathbb{F})$, and we assume this from now on.

Let x be a closed point of A/I, and $\kappa(x)$ be its residue field. By (1.2.11), the object $\mathfrak{M}_{A,x} = \mathfrak{M}_A \otimes_A \kappa(x)$ of $(\operatorname{Mod} \mathrm{FI}/\mathfrak{S})_{\kappa(x)}$ has a maximal multiplicative subobject $\mathfrak{M}_{A,x}^m$. By (1.2.2)(4), $\mathfrak{M}_{A,x}^m$ is finite free over $\mathfrak{S}_{\kappa(x)}$, and we denote its rank by $\mathbf{d}_m(x)$. Locally on A we may choose a basis for $\mathfrak{M}_A/u\mathfrak{M}_A$, and consider the characteristic polynomial $P(T)$ of the matrix for ϕ acting on $\mathfrak{M}_A/u\mathfrak{M}_A$ in this basis. Then $\mathbf{d}_m(x)$ is equal to the greatest integer i such that the coefficient of T^{d-i} in $P(T)$ does not vanish at x. Hence $\mathbf{d}_m$ is lower semi-continuous for the Zariski topology on the closed points of $\operatorname{Spec} A/I$. We will show that it is also upper semi-continuous, and hence constant.

We extend the definition of (1.2.6), and denote by $\Phi\mathbf{M}_{\mathcal{O}_{\mathcal{E}},A}$ the category of finite $\mathcal{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$-modules M_A equipped with an isomorphism $\phi^*(M_A) \xrightarrow{\sim} M_A$. Let W_A be a finitely generated A-module, equipped with an action of G_{K_∞} such that

the orbit of any element $y \in W_A$ under the action of G_{K_∞} generates a finite $W(\mathbb{F})$-module. Consider the functor from such $A[G_{K_\infty}]$-modules to $\Phi \mathbf{M}_{\mathbb{O}_{\mathcal{E}}, A}$ given by

$$D_A : W_A \mapsto (\mathbb{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} W_A)^{G_{K_\infty}}.$$

An argument as in (1.2.7) shows that D_A is exact, commutes with extensions of scalars for any morphism $A \to B$ in $\mathfrak{Aug}_{W(\mathbb{F})}$, and that W_A is locally free over A if and only if $D_A(W_A)$ is locally free over $\mathbb{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$.

Let V_A be the image of $\mathfrak{M}_A$ under the functor $\Theta_{V_{\mathbb{F}}}$ of (2.1.4). V_A corresponds to a Galois representation $V_{A'}$ in $D_{V_{\mathbb{F}}}^{\mathrm{fl}}(A')$ where A' is in $\mathfrak{AR}_{W(\mathbb{F})}^{A, I}$. We will again write V_A for the A-module $V_{A'} \otimes_{A'} A$ equipped with the induced A-linear action of G_K. Suppose that $L_A \subset V_A$ is a G_K-stable, projective A-submodule, such that V_A/L_A is also projective, and that the action of G_K on $L_A(-1)$ is unramified. Write $\bar{k}$ for the residue field of $\bar{K}$. The above remarks show that

$$D_A(L_A(-1)) = \mathbb{O}_{\mathcal{E}} \otimes_{W(k)} (W(\bar{k}) \otimes_{\mathbb{Z}_p} L_A(-1))^{G_{K_\infty}}$$
$$\subset D_A(V_A(-1)) \xrightarrow{\sim} \mathfrak{M}_A[1/u]$$

is a locally free $\mathbb{O}_{\mathcal{E}} \otimes_{\mathbb{Z}_p} A$-submodule, with locally free quotient, and that its formation commutes with extension of scalars in the same sense as above.

Now fix an integer $d_m \leqslant d$. We define a groupoid $D_{\mathfrak{S}, M_{\mathbb{F}}}^{d_m}$ over $\mathfrak{Aug}_{W(\mathbb{F})}$ as follows: For (A, I) in $\mathfrak{Aug}_{W(\mathbb{F})}$, an object of $D_{\mathfrak{S}, M_{\mathbb{F}}}^{d_m}(A, I)$ consists of an object $\mathfrak{M}_A$ of $D_{\mathfrak{S}, M_{\mathbb{F}}}$, together with a G_K-stable, projective A-submodule $L_A \subset V_A$ of rank d_m (here V_A is associated to $\mathfrak{M}_A$ as above), such that V_A/L_A is projective, the action of G_K on $L_A(-1)$ is unramified, and we have

$$U_A(L_A) := (W(\bar{k}) \otimes_{\mathbb{Z}_p} L_A(-1))^{G_{K_\infty}} \subset \mathfrak{M}_A \subset D_A(V_A(-1)).$$

Formation of $U_A(L_A)$ (as a functor in L_A) commutes with extension of scalars, because $D_A(L_A(-1))$ has this property, and $\mathbb{O}_{\mathcal{E}}$ is a faithfully flat $W(k)$-algebra. In particular $D_{\mathfrak{S}, M_{\mathbb{F}}}^{d_m}$ is a well defined groupoid.

The morphism $D_{\mathfrak{S}, M_{\mathbb{F}}}^{d_m} \to D_{\mathfrak{S}, M_{\mathbb{F}}}$ given by sending $(\mathfrak{M}_A, L_A)$ to $\mathfrak{M}_A$ is relatively representable and projective. Its fiber over an object $\mathfrak{M}_A$ in $D_{\mathfrak{S}, M_{\mathbb{F}}}$ is given by a closed subspace of the Grassmannian of d_m-dimensional subspaces of V_A. To see this it suffices to show that given a projective submodule $L_A \subset V_A$ of rank d_m, with V_A/L_A projective, there is an ideal $J \subset A$ such that for any map $A \to A'$ in $\mathfrak{Aug}_{W(\mathbb{F})}$, $L_A \otimes_A A'$ is in $D_{\mathfrak{S}, M_{\mathbb{F}}}^{d_m}(A')$ if and only if $JA' = 0$. The conditions that L_A is G_K-stable and that the action of G_K on $L_A(-1)$ is unramified obviously cut out a closed subscheme of $\operatorname{Spec} A$. Hence we may assume that L_A already satisfies these conditions. Noting that $\mathfrak{S}_A[1/u]/\mathfrak{S}_A$ is a free A-module, so that $\mathfrak{M}_A[1/u]/\mathfrak{M}_A$ is a projective A-module, we identify it with a direct summand in $\bigoplus_{i \in I} A$ for some

index set I. Let r denote the composite

$$r : U_A(L_A) \to D_A(V_A(-1)) \to D_A(V_A(-1))/\mathfrak{M}_A \overset{\sim}{\longrightarrow} \mathfrak{M}_A[1/u]/\mathfrak{M}_A \hookrightarrow \bigoplus_{i \in I} A.$$

Set J equal to the ideal generated by the co-ordinates of the elements $r(u)$ where u runs over $U_A(L_A)$. Using the fact that formation of $U_A(L_A)$ commutes with extension of scalars, one sees immediately that J has the required property.

Let A be a finitely generated $W(\mathbb{F})$-algebra, $\xi = \mathfrak{M}_A$ in $D_{\mathfrak{S},M_{\mathbb{F}}}(A)$, and $D_{\mathfrak{S},M_{\mathbb{F}},\xi}^{d_m}$ the fiber of $D_{\mathfrak{S},M_{\mathbb{F}}}^{d_m}$ over ξ.

Suppose first that A is a local $\mathbb{Z}_p$-algebra with $|A| < \infty$. Then we have the maximal multiplicative submodule $\mathfrak{M}_A^m = \bigcap_{r=1}^\infty (\phi^*)^r (\mathfrak{M}_A)$ defined in (1.2.11). Since $\mathfrak{M}_A^m$ and $\mathfrak{M}_A/\mathfrak{M}_A^m$ are finite free $\mathfrak{S}_A$-modules by (1.2.2)(4),

$$L_A^m = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathbb{Z}_p} \mathfrak{M}_A^m[1/u])^{\phi=1}(1) = (W(\bar{k}) \otimes_{\mathbb{Z}_p} \mathfrak{M}_A^m)^{\phi=1}(1)$$

and V_A/L_A^m are finite free A-modules by (1.2.7)(4), the former module having rank $\mathrm{rk}_{\mathfrak{S}_A} \mathfrak{M}_A^m$. Hence for any finite local A-algebra A', $D_{\mathfrak{S},M_{\mathbb{F}},\xi}^{d_m}(A')$ consists of the finite projective submodules $L_{A'} \subset L_A^m \otimes_A A'$ of rank d_m such that $(L_A^m \otimes_A A')/L_{A'}$ is projective. Here we have used that the formation of $\mathfrak{M}_A^m$ and L_A^m commutes with extension of scalars $A \to A'$ by (1.2.11)(2) and (1.2.9)(2).

In particular, we see that for any finitely generated $W(\mathbb{F})$-algebra A, and $\xi = \mathfrak{M}_A$ in $D_{\mathfrak{S},M_{\mathbb{F}}}(A)$, a closed point x of $\mathrm{Spec}\, A$ (or equivalently of A/I) is in the image of $D_{\mathfrak{S},M_{\mathbb{F}},\xi}^{d_m}$ if and only if $\mathbf{d}_m(x) \geq d_m$. This shows that the function $\mathbf{d}_m$ is upper semi-continuous, and hence constant on the closed points of $\mathrm{Spec}\, A$. Moreover, if $\mathbf{d}_m(x) = d_m$, then the morphism $D_{\mathfrak{S},M_{\mathbb{F}},\xi}^{d_m} \to \mathrm{Spec}\, A$ is an isomorphism in a neighborhood of any closed point in its image. It follows that this map is an isomorphism onto those components of $\mathrm{Spec}\, A$ where the function $\mathbf{d}_m$ is equal to d_m. In particular, using the universal A-submodule $L_A \subset V_A$, we obtain a multiplicative submodule

$$\mathfrak{M}_A^m = \mathfrak{S} \otimes_{\mathbb{Z}_p} (W(\bar{k}) \otimes_{\mathbb{Z}_p} L_A(-1))^{G_{K_\infty}} \subset \mathfrak{M}_A.$$

When $|A| < \infty$ this agrees with the definition of $\mathfrak{M}_A^m$ in (1.2.11). To see this, temporarily denote by $\mathfrak{M}_A^{m'} = \bigcap_{r=1}^\infty (\phi^*)^r (\mathfrak{M}_A)$ the submodule defined in (1.2.11). Then $\mathfrak{M}_A^m \subset \mathfrak{M}_A^{m'}$, and

$$(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathbb{Z}_p} \mathfrak{M}_A^m[1/u])^{\phi=1}(1) = L_A^m = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathbb{Z}_p} \mathfrak{M}_A^{m'}[1/u])^{\phi=1}(1).$$

Hence $\mathfrak{M}_A^m[1/u] = \mathfrak{M}_A^{m'}[1/u]$. Since the map $\phi^*(\mathfrak{M}_A^m) \to \mathfrak{M}_A^m$ is an isomorphism, this implies that $\mathfrak{M}_A^m = \mathfrak{M}_A^{m'}$ (cf. the proof of (2.1.7)).

For any A, the formation of $\mathfrak{M}_A^m$ commutes with extension of scalars. In particular, if A' is any Artinian quotient of A, with $|A'| < \infty$, then

$$\mathfrak{M}_A^m \otimes_A A' = (\mathfrak{M}_A \otimes_A A')^m = \bigcap_{r=1}^{\infty} (\phi^*)^r (\mathfrak{M}_A \otimes_A A').$$

Hence $\mathfrak{M}_A^m \otimes_A A'$ contains the image of $\bigcap_{r=1}^{\infty} (\phi^*)^r (\mathfrak{M}_A)$ in $\mathfrak{M}_A \otimes_A A'$. It follows that for any maximal ideal $\mathfrak{m}$ of A, and any positive integer i, we have

$$\left(\bigcap_{r=1}^{\infty} (\phi^*)^r (\mathfrak{M}_A) \right) / \mathfrak{M}_A^m \subset \mathfrak{m}^i (\mathfrak{M}_A / \mathfrak{M}_A^m),$$

and so $\mathfrak{M}_A^m = \bigcap_{r=1}^{\infty} (\phi^*)^r (\mathfrak{M}_A)$. This proves that $\mathfrak{M}_A^m$ is the maximal multiplicative submodule of $\mathfrak{M}_A$. Its other properties follow by construction. The existence and properties of $\mathfrak{M}_A^{\text{ét}}$ now follow by duality as in (1.2.11). $\qquad\square$

(2.4.15) Let $\mathfrak{M}_0^{\mathbf{v},\text{loc}}$ denote the universal sheaf of $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}}$ -modules on $\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$. We may apply (2.4.14) to obtain a submodule $\mathfrak{M}_0^{\mathbf{v},\text{loc},m} \subset \mathfrak{M}_0^{\mathbf{v},\text{loc}}$ and a quotient module $\mathfrak{M}_0^{\mathbf{v},\text{loc},\text{ét}}$. On any connected component of $\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ the ranks of these modules are well defined, and constant.

Let $\mathbf{d} = \{d_{\text{ét}}, d_m\}$ be a pair of non-negative integers. We denote by $\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},\mathbf{d}} \subset \mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ the union of the connected components on which $\mathfrak{M}_0^{\mathbf{v},\text{loc},m}$ has rank d_m and $\mathfrak{M}_0^{\mathbf{v},\text{loc},\text{ét}}$ has rank $d_{\text{ét}}$.

Suppose that E/F is a finite extension, and that $x \in \operatorname{Spec} R_{V_{\mathbb{F}}}^{\mathbf{v}}(E)$. Let V_E denote the corresponding representation of G_K. It is not hard to check that (the image of) x lies on a connected component of $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\mathbf{v}}[1/p]$ which corresponds via the isomorphism of (2.4.10) to a connected component of $\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},\mathbf{d}}$, if and only if the maximal unramified subrepresentation of $V_E(-1)$ has E-dimension d_m, and the maximal unramified quotient of V_E has dimension $d_{\text{ét}}$.

CONJECTURE (2.4.16). *Suppose that* $\operatorname{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$. *Then* $\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},\mathbf{d}}$ *is connected.*

(2.4.17) We want to give a conjectural description of the normalization of $R^{\mathbf{v}}$. Given (A, I) in $\mathfrak{Aug}_{\mathcal{O}_F}$, and a point $y \in D_{\mathfrak{S}, M_{\mathbb{F}}, \xi}^{\mathbf{v}}(A, I)$ we obtain an object $\mathfrak{M}_A$ of $(\operatorname{Mod} FI/\mathfrak{S})_A$. Let $\mathfrak{M}_{A,0} = \mathfrak{M}_A / u\mathfrak{M}_A$. This is a finite free $W \otimes_{\mathbb{Z}_p} A$-module equipped with a Frobenius semi-linear map $\varphi : \mathfrak{M}_{A,0} \to \mathfrak{M}_{A,0}$. Hence $\varphi^{[k:\mathbb{F}_p]}$ induces a linear endomorphism of $\mathfrak{M}_{A,0}$. We denote by $P_y^{\mathbf{v}}(T)$ its characteristic polynomial.

Since the construction of $P_y^{\mathbf{v}}(T)$ is functorial in y, there is a polynomial

$$P_{V_{\mathbb{F}}}^{\mathbf{v}}(T) \in \Gamma(\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}, \mathcal{O}_{\mathscr{G}\mathscr{R}_{V_{\mathbb{F}},\xi}^{\mathbf{v}}})[T]$$

which pulls back to $P_y^{\mathbf{v}}(T)$ for every point y as above.

We note for later reference that if $E/\mathbb{Q}_p$ is a finite extension, $y \in (\operatorname{Spec} R^{\mathbf{v}})(E)$ and V_y denotes the G_K-representation corresponding to y, then y may be viewed as an E-valued point of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ by (2.4.8), and the pull-back of $P^{\mathbf{v}}_{V_{\mathbb{F}}}(T)$ by y is the characteristic polynomial of $\varphi^{[k:\mathbb{F}_p]}$ on $D_{\mathrm{cris}}(V_y^*)$. This follows from the construction and [Br3, 5.3.1].

Let $\tilde{R}^{\mathbf{v}}$ denote the subring of $\Gamma(\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}, \mathcal{O}_{\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}})$ generated by $R^{\mathbf{v}}$ (via the map of (2.4.8)) and the images of the coefficients of $P^{\mathbf{v}}_{V_{\mathbb{F}}}(T)$.

CONJECTURE (2.4.18). *The natural inclusion* $\tilde{R}^{\mathbf{v}} \subset \Gamma(\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}, \mathcal{O}_{\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}})$ *is an isomorphism. In particular*, $\tilde{R}^{\mathbf{v}}$ *is the normalization of* $R^{\mathbf{v}}$.

(2.4.19) Note that the two statements in the conjecture are equivalent, because $\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}$ is normal by (2.4.6), so that $\Gamma(\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}, \mathcal{O}_{\mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}})$ is normal and finite over $R^{\mathbf{v}}$. Because of the formal smoothness condition (2.4.5), the truth of (2.4.18) depends only on $V_{\mathbb{F}}$ and not on the choice of R.

(2.5) *Rank 2 calculations.* In this subsection we assume that $d = 2$, that the integers $v_{\psi} = 1$ for all ψ and, unless explicitly stated otherwise, that $K_0 = \mathbb{Q}_p$. In this situation we will verify the conjecture (2.4.16).

Since $v_{\psi} = 1$ for all ψ, we have $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},\xi} = \mathcal{GR}^{\mathbf{v},\mathrm{loc}}_{V_{\mathbb{F}},\xi}$ by (2.4.6)(4). Following the notation of the previous section, we regard $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},\xi}$ as an $R^{\mathbf{v}}$-scheme via the map $\Theta_{V_{\mathbb{F}}}$ of (2.1.10), and we denote by $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ its fiber over the closed point of $R^{\mathbf{v}}$. This is of course equal to $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},z}$ where z is the object $V_{\mathbb{F}}$ of $D^{\mathrm{fl}}_{V_{\mathbb{F}}}(\mathbb{F})$.

We will assume that $\operatorname{Spec} R^{\mathbf{v}}$ is non-empty.

LEMMA (2.5.1). *If* $\mathbb{F}'$ *is a finite extension of* $\mathbb{F}$, *then the elements of* $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ *naturally correspond to finite free* $\mathfrak{S} \otimes_{\mathbb{F}_p} \mathbb{F}' = \mathbb{F}'[\![u]\!]$ *submodules* $\mathfrak{M}_{\mathbb{F}'} \subset M_{\mathbb{F}'} := M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ *of rank 2 such that*:

(1) $\mathfrak{M}_{\mathbb{F}'}$ *is stable under the map* $\phi : M_{\mathbb{F}'} \to M_{\mathbb{F}'}$.

(2) *For some* (*and hence any*) *choice of an* $\mathbb{F}'[\![u]\!]$-*basis of* $\mathfrak{M}_{\mathbb{F}'}$, *the induced map* $\phi : \mathfrak{M}_{\mathbb{F}'} \to \mathfrak{M}_{\mathbb{F}'}$ *has determinant* αu^e *where* $\alpha \in \mathbb{F}'[\![u]\!]^{\times}$.

Proof. By definition, $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ corresponds to finite free $\mathbb{F}'[\![u]\!]$ submodules $\mathfrak{M}_{\mathbb{F}'} \subset M_{\mathbb{F}'}$ of rank 2 such that $u^e \mathfrak{M}_{\mathbb{F}'} \subset (1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}'})) \subset \mathfrak{M}_{\mathbb{F}'}$, and satisfying the condition of (2.4.2). Since $K_0 = \mathbb{Q}_p$, this condition becomes (cf. (2.2.1))

$$\det_{\mathbb{F}'}(T - \pi | (1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}'}))/u^e \mathfrak{M}_{\mathbb{F}'}) = T^e,$$

which is equivalent to the condition $\dim_{\mathbb{F}'}((1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}'}))/u^e \mathfrak{M}_{\mathbb{F}'}) = e$, because π induces a nilpotent endomorphism of $(1 \otimes \phi)\phi^*(\mathfrak{M}_{\mathbb{F}'})/u^e \mathfrak{M}_{\mathbb{F}'}$.

Now any ϕ-stable, finite free $\mathbb{F}'[\![u]\!]$ submodule $\mathfrak{M}_{\mathbb{F}'} \subset M_{\mathbb{F}'}$ of rank 2 admits a basis $\{e_1, e_2\}$ such that $(1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}'})) = \langle u^i e_1, u^j e_2 \rangle$ for some non-negative

integers i, j. (Note that $\phi^* M_{\mathbb{F}} \to M_{\mathbb{F}}$ is an isomorphism). If v_u denotes the u-adic valuation on $\mathbb{F}'[\![u]\!]$ then we have $v_u(\det(\phi|_{\mathfrak{M}_{\mathbb{F}'}})) = i + j$, where the determinant is computed with respect to any choice of basis of $\mathfrak{M}_{\mathbb{F}'}$. Thus, $\mathfrak{M}_{\mathbb{F}'}$ satisfies (2) above if and only if $u^e \mathfrak{M}_{\mathbb{F}'} \subset (1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}'}))$, and

$$\dim_{\mathbb{F}'}((1 \otimes \phi)(\phi^*(\mathfrak{M}_{\mathbb{F}'}))/u^e \mathfrak{M}_{\mathbb{F}'}) = 2e - i - j = e. \qquad \square$$

(2.5.2) Let $I_K \subset G_K$ denote the inertia subgroup, and $\bar{k}$ the residue field of $\bar{K}$. Recall that the fundamental character of level n, $\omega_n = \omega_{n,K}$ is given by

$$\omega_n : I_K \to \bar{k}^\times; \qquad g \mapsto g(\sqrt[p^n-1]{\pi})/\sqrt[p^n-1]{\pi}.$$

The formation of fundamental characters is not, in general, compatible with change of field. Namely, if K'/K is a finite extension with ramification degree e, then π is a product of e uniformisers of K', so that $\omega_{n,K}|_{I_{K'}} = \omega_{n,K'}^e$.

If K'/K is a finite unramified extension, which contains the (p^n-1)-th roots of unity, then ω_n extends to a character of $G_{K'}$ given by the same formula as above, which we again denote ω_n. Note however, that this extension depends on the choice of uniformiser π.

LEMMA (2.5.3). *If $V_{\mathbb{F}}$ is reducible its semi-simplification $V_{\mathbb{F}}^{ss}$ satisfies*

$$V_{\mathbb{F}}^{ss}|_{I_K} \sim \omega_1^i \oplus \omega_1^j$$

with $i, j \in [0, e]$, and $p - 1 \mid e - i - j$.

If $V_{\mathbb{F}}$ is irreducible and $\mathbb{F}_{p^2} \subset \mathbb{F}$, then

$$V_{\mathbb{F}}|_{I_K} \sim \omega_2^i \oplus \omega_2^{pi}$$

for an integer i such that $i = i_0 + pi_1$, for integers $i_0, i_1 \in [0, e]$ with $p + 1 \nmid i$ and $p - 1 \mid e - i$. Here we regard ω_1 and ω_2 as $\mathbb{F}$-valued via any embedding $\bar{\mathbb{F}} \hookrightarrow \bar{k}$.

Proof. The image of G_K in $\mathrm{Aut}_{\mathbb{F}} V_{\mathbb{F}}$ is either contained in a Borel subgroup, or is dihedral [Ra, 3.2]. This corresponds to the two cases in the lemma. In the first case, it follows from local class field theory and the assumption $K_0 = \mathbb{Q}_p$ that $V_{\mathbb{F}}^{ss}|_{I_K} \sim \omega_1^i \oplus \omega_1^j$ for some integers i, j. Since $V_{\mathbb{F}}$ extends to a finite flat group scheme, i, j can be chosen in $[0, e]$, by [Ra, 3.4.3]. In the second case local class field theory and $K_0 = \mathbb{Q}_p$ imply that $V_{\mathbb{F}}|_{I_K} \sim \omega_2^i \oplus \omega_2^{pi}$ for some integer i. Since $V_{\mathbb{F}}$ extends to a finite flat group scheme, i can be chosen to be of the form $i_0 + pi_1$ for some integers $i_0, i_1 \in [0, e]$. If $p + 1 \mid i$, then $\omega_2^i = \omega_2^{pi}$, and $V_{\mathbb{F}}$ cannot be irreducible, hence $p + 1 \nmid i$.

It remains to check the claims regarding divisibility by $p - 1$. These are equivalent to asking that I_K acts on $\det_{\mathbb{F}} V_{\mathbb{F}}$ via ω_1^e. On the other hand, if $\mathrm{Spec}\, R_{V_{\mathbb{F}}}^{\mathrm{fl},\mathbf{v}}$ is non-empty then $V_{\mathbb{F}}$ lifts to a characteristic 0 representation whose restriction to I_K has cyclotomic determinant. (This follows easily from the definition of $R_{V_{\mathbb{F}}}^{\mathrm{fl},\mathbf{v}}$ in

(2.4.1), and the fact that $v_\psi = 1$ for all ψ.) Thus it remains to show that ω_1^e is the reduction mod p of the cyclotomic character. When $K = \mathbb{Q}_p$ this is a standard fact, and the general case follows from the remarks of (2.5.2). □

(2.5.4) If $\mathfrak{M}_\mathbb{F}$ is an object of $(\text{Mod FI}/\mathfrak{S})_\mathbb{F}$ of rank 2 over $\mathbb{F}[[u]]$, then we call $\mathfrak{M}_\mathbb{F}$ *ordinary* if it is an extension of an étale object by a multiplicative object, each of rank 1. By (1.1.15), if $\mathcal{M}_\mathbb{F}$ is the image of $\mathfrak{M}_\mathbb{F}$ under the functor of (1.2.4), then $\mathfrak{M}_\mathbb{F}$ is ordinary, if and only if $\text{Gr}_D(\mathcal{M}_\mathbb{F})$ is ordinary.

For any sublattice $\mathfrak{M}_\mathbb{F} \subset M_\mathbb{F}$ in $(\text{Mod FI}/\mathfrak{S})_\mathbb{F}$ and any $A \in M_2(\mathbb{F}((u)))$ we will write $\mathfrak{M}_\mathbb{F} \sim A$ if $\mathfrak{M}_\mathbb{F}$ has an $\mathbb{F}[[u]]$-basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ such that the corresponding $\mathbb{F}((u))$-basis of $M_\mathbb{F}$, satisfies $\phi\left(\begin{smallmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{smallmatrix}\right) = A\left(\begin{smallmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{smallmatrix}\right)$. Similarly, we will write $M_\mathbb{F} \sim A$ if $M_\mathbb{F}$ has a basis in which ϕ is given by A, in the sense just explained.

Finally, we will make the convention that if $\mathfrak{M}_{\mathbb{F},1} \subset M_\mathbb{F}$ is an $\mathbb{F}[[u]]$-sub-module with a chosen basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, and $A \in M_2(F((u)))$, then we denote by $\mathfrak{M}_{\mathbb{F},2} = A \cdot \mathfrak{M}_{\mathbb{F},1}$ the $\mathbb{F}[[u]]$-span of the entries of $\langle A\left(\begin{smallmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{smallmatrix}\right)\rangle$, and we will consider $\mathfrak{M}_{\mathbb{F},2}$ with the basis given by these entries.

We will denote by v_u the u-adic valuation on $\mathbb{F}((u))$.

LEMMA (2.5.5). *Suppose that $V_\mathbb{F}$ is irreducible, and let $M_\mathbb{F}$ be the module in $\Phi\mathbf{M}_{\mathcal{O}_\mathcal{E},\mathbb{F}}$ assigned to $V_\mathbb{F}(-1)$ by the equivalence of (1.2.7). If $\mathbb{F}'$ denotes the quadratic extension of $\mathbb{F}$, then*

$$\mathbb{F}' \otimes_\mathbb{F} M_\mathbb{F} \sim \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix}$$

for some $a, d \in \mathbb{F}'[[u]]$ such that $v_u(ad) = e$, and $p + 1 \nmid v_u(d) - v_u(a)$.

Proof. Fix an embedding $\mathbb{F}' \hookrightarrow \bar{k}$. Let K' be the unramified extension of K of degree 2. By (2.5.3), I_K acts on $\det_\mathbb{F} V_\mathbb{F}$ via the cyclotomic character. Hence $V_\mathbb{F}(-1)|_{I_K}$ is isomorphic to the dual of $V_\mathbb{F}|_{I_K}$. Since $V_\mathbb{F}(-1)$ is irreducible, it is induced from a character of $G_{K'}$, and by (2.5.3) we have

$$\mathbb{F}' \otimes_\mathbb{F} V_\mathbb{F}(-1)|_{G_{K'}} \sim \lambda\omega_2^{-i} \oplus \lambda\omega_2^{-pi}$$

for some integer i such that $i = i_0 + pi_1$ for integers $i_0, i_1 \in [0, e]$, $p - 1 \mid e - i$, and $p + 1 \nmid i$, and some unramified character $\lambda : G_{K'} \to \mathbb{F}'^\times$. After replacing $V_\mathbb{F}$ by $\mathbb{F}' \otimes_\mathbb{F} V_\mathbb{F}$, and $\mathbb{F}$ by $\mathbb{F}'$, we may assume that λ is $\mathbb{F}^\times$-valued, and that $\mathbb{F}_{p^2} \subset \mathbb{F}$.

Note that replacing (i_0, i_1) by (i_1, i_0), interchanges ω_2^i and ω_2^{pi}, so we may assume that either $i_1 \in [1, e - 1]$ or $i_1 = 0$ and $i_0 = e$. Now choose an integer $j \in [0, e]$ such that $j = \frac{i-e}{p-1}$ modulo $p + 1$. If $e \geq p + 1$ this is obviously possible. If $e \leq p$, then our choice of i_0 and i_1 guarantees that $\frac{i-e}{p-1} \in [0, e]$.

Let $k' \xrightarrow{\sim} \mathbb{F}_{p^2}$ denote the residue field of K', and denote by $\sigma \in \text{Gal}(\bar{k}/k')$ the geometric Frobenius. Let $N_\mathbb{F}$ be the object of $\Phi\mathbf{M}_{\mathcal{O}_\mathcal{E},\mathbb{F}}$ with $\mathbb{F}((u))$-basis $\{\mathbf{e}_1, \mathbf{e}_2\}$,

and ϕ given by

$$\phi(\mathbf{e}_1) = \lambda(\sigma)u^j\mathbf{e}_2 \quad \text{and} \quad \phi(\mathbf{e}_2) = u^{e-j}\mathbf{e}_1.$$

Since $N_{\mathbb{F}} \sim \left(\begin{smallmatrix} 0 & u^{e-j} \\ \lambda(\sigma)u^j & 0 \end{smallmatrix}\right)$, it will suffice to check that $T_{\mathbb{F}}(N_{\mathbb{F}}) \xrightarrow{\sim} V_{\mathbb{F}}(-1)|_{G_{K_\infty}}$. Note that $v_u(u^{e-j}) - v_u(\lambda(\sigma)u^j) = \frac{(p+1)e-2i}{p-1} \mod (p+1)$, and that $p + 1 \nmid \frac{(p+1)e-2i}{p-1}$. To see the second condition, observe that if it fails, then $i = (\frac{p+1}{2})k$ for some integer k, and $p - 1 | e - \frac{2i}{p+1} = e - k$. Since $p - 1 | e - i$ this implies $p - 1 | i - k = (\frac{p-1}{2})k$, so that k is even, and $p + 1 | i$, a contradiction. Hence the condition $p + 1 \nmid v_u(d) - v_u(a)$ is satisfied.

Using the notation of (1.1.12), let $u_2 \in R$ be an element with $u_2^{p^2-1} = u$. Then $k((u))[u_2] \subset \mathrm{Fr} R$ is a separable extension of $k((u))$, so that $u_2 \in \mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}/p\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}$. Note that $g(u_2) = \omega_2(g)u_2$ for $g \in G_{K'}$. We also choose an element $w_\lambda \in (\bar{k} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$, such that $g(w_\lambda) = 1 \otimes \lambda(g) \cdot w_\lambda$ for $g \in G_{K'}$. The element w_λ is uniquely determined up to multiplication by an element of $(\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} \mathbb{F})^\times$. Note that $\phi^2(w_\lambda) = \sigma^{-1}(w_\lambda) = 1 \otimes \lambda(\sigma)^{-1}w_\lambda$.

Let $\epsilon_1 \in k' \otimes_{\mathbb{F}_p} \mathbb{F}$ be the idempotent which is 1 modulo the kernel of the map $k' \otimes_{\mathbb{F}_p} \mathbb{F} \to \bar{k}$ induced by the chosen inclusion $\mathbb{F} \subset \bar{k}$, and is contained in the other maximal ideal of $k' \otimes_{\mathbb{F}_p} \mathbb{F}$. Set $\epsilon_2 = \phi(\epsilon_1)$. Then for $a \in k'$ we have $(a \otimes 1)\epsilon_1 = (1 \otimes a)\epsilon_1$, and $(a^p \otimes 1)\epsilon_2 = (1 \otimes a)\epsilon_2$.

Let $i' = pj + e - j$, and define elements $v_1, v_2 \in \mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}/p\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}} \otimes_{k((u))} N_{\mathbb{F}}$ by

$$v_1 = w_\lambda \epsilon_1(u_2^{-i'} \otimes \mathbf{e}_1) + \phi(w_\lambda)\lambda(\sigma)\epsilon_2(u_2^{-pi'}u^j \otimes \mathbf{e}_2)$$

and

$$v_2 = w_\lambda \epsilon_2(u_2^{-i'} \otimes \mathbf{e}_1) + \phi(w_\lambda)\lambda(\sigma)\epsilon_1(u_2^{-pi'}u^j \otimes \mathbf{e}_2).$$

Here we view $\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}/p\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}} \otimes_{k((u))} N_{\mathbb{F}}$ as a $\bar{k} \otimes_{\mathbb{F}_p} \mathbb{F}$-module via the inclusion of $\bar{k}$ in $\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}/p\mathbb{O}_{\mathscr{E}^{\mathrm{ur}}}$ and the natural action of $\mathbb{F}$ on $N_{\mathbb{F}}$. A straightforward calculation shows that v_1 and v_2 are invariant under ϕ. Since $T_{\mathbb{F}}(N_{\mathbb{F}})$ is 2-dimensional over $\mathbb{F}$, it must be spanned by v_1 and v_2. Write $K'_\infty = K' \cdot K_\infty$. By the choice of j, we have that $i = i'$ modulo $p^2 - 1$, so that for $g \in G_{K'_\infty}$,

$$g(v_1) = v_1 \cdot \lambda(g)\omega_2^{-i}(g) \quad \text{and} \quad g(v_2) = v_2 \cdot \lambda(g)\omega_2^{-pi}(g).$$

This implies that $T(N_{\mathbb{F}}) \xrightarrow{\sim} V_{\mathbb{F}}(-1)$ as $G_{K'_\infty}$-representations, and hence as G_{K_∞}-representations, since $V_{\mathbb{F}}(-1)$ is irreducible, and obtained by inducing $\lambda\omega_2^{-i}$. $\square$

PROPOSITION (2.5.6). *Let $\mathbb{F}'/\mathbb{F}$ be a finite extension. Suppose $x_1, x_2 \in \mathscr{GR}^{\mathsf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ and that the corresponding objects of $(\mathrm{Mod}/\mathfrak{S})_{\mathbb{F}'}$, $\mathfrak{M}_{\mathbb{F}',1}$ and $\mathfrak{M}_{\mathbb{F}',2}$ are both non-ordinary. Then (the images of) x_1 and x_2 lie on the same component of $\mathscr{GR}^{\mathsf{v}}_{V_{\mathbb{F}},0}$.*

Proof. After replacing $V_{\mathbb{F}}$ by $\mathbb{F}' \otimes_{\mathbb{F}} V_{\mathbb{F}}$, we may assume that $\mathbb{F}' = \mathbb{F}$. Suppose then that $\mathfrak{M}_{\mathbb{F},1} \sim A$ for some A. If we fix the corresponding choice of basis for $\mathfrak{M}_{\mathbb{F},1}$, then $\mathfrak{M}_{\mathbb{F},2} = B \cdot \mathfrak{M}_{\mathbb{F},1}$ for some $B \in \mathrm{GL}_2(\mathbb{F}((u)))$, and $\mathfrak{M}_{\mathbb{F},2} \sim \phi(B)AB^{-1}$. Note that B is uniquely determined up to multiplication on the left by elements of $\mathrm{GL}_2(\mathbb{F}[[u]])$. Thus, by the Iwasawa decomposition, we may assume that B is upper triangular. By (2.5.1)(2), $\det \phi(B) \det B^{-1} = \gamma$ for some γ in $\mathbb{F}[[u]]^{\times}$. Thus, the diagonal entries of B are of the form $\mu_1 u^i$ and $\mu_2 u^{-i}$ for $\mu_1, \mu_2 \in \mathbb{F}[[u]]^{\times}$, and some $i \in \mathbb{Z}$. After replacing B by $\mathrm{diag}(\mu_1, \mu_2)^{-1}B$, we may assume B has diagonal entries u^{-i} and u^i. Note that although B depends on the choice of basis for $\mathfrak{M}_{\mathbb{F},1}$, we may choose B with diagonal entries u^{-i} and u^i for any such choice of basis. Thus, we will often choose a basis, and then assume B is of the above form.

We will show that x_1 and x_2 are connected by a chain of rational curves. For this we need the following result:

LEMMA (2.5.7). *Keeping the above notation, suppose that N is a nilpotent element of $M_2(\mathbb{F}((u)))$ such that $\mathfrak{M}_{\mathbb{F},2} = (1 + N) \cdot \mathfrak{M}_{\mathbb{F},1}$. Let $N^{\mathrm{ad}} = -N$ denote the adjoint matrix of N. If $\phi(N)AN^{\mathrm{ad}} \in M_2(\mathbb{F}[[u]])$ then there exists a map $\mathbb{P}^1 \to \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ sending 0 to x_1 and 1 to x_2.*

Proof. Let $M_{\mathbb{F}[T]} = \mathbb{F}[T] \otimes_{\mathbb{F}} M_{\mathbb{F}}$, and $\mathfrak{M}_{\mathbb{F}[T],1} = \mathbb{F}[T] \otimes_{\mathbb{F}} \mathfrak{M}_{\mathbb{F},1}$. We consider $\mathfrak{M}_{\mathbb{F}[T],1}$ with the basis induced by the chosen basis of $\mathfrak{M}_{\mathbb{F},1}$. Define an $\mathbb{F}[[u]][T]$-submodule $\mathfrak{M}_{\mathbb{F}[T]} \subset M_{\mathbb{F}[T]}$ by $\mathfrak{M}_{\mathbb{F}[T]} = (1 + NT) \cdot \mathfrak{M}_{\mathbb{F}[T],1}$. This submodule induces a map from $\mathbb{A}^1$ into the affine Grassmannian parametrizing $\mathbb{F}[[u]]$-sublattices of $M_{\mathbb{F}}$, and takes 0 to x_1 and 1 to x_2. It remains to check that this map actually factors through $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. For this, note that multiplying the chosen basis of $\mathfrak{M}_{\mathbb{F},1}$ by $1 + NT$ gives an $\mathbb{F}[[u]][T]$-basis of $\mathfrak{M}_{\mathbb{F}[T]}$, and hence a $\mathbb{F}((u))[T]$-basis of $M_{\mathbb{F}[T]}$. In this basis ϕ is given by

(2.5.8) $\phi(1 + TN)A(1 + TN)^{-1}$
$$= A + T\phi(N)A + TAN^{\mathrm{ad}} + T^2\phi(N)AN^{\mathrm{ad}}$$
$$= A + T(\phi(N)A + AN^{\mathrm{ad}} + \phi(N)AN^{\mathrm{ad}}) + (T^2 - T)(\phi(N)AN^{\mathrm{ad}}).$$

Now specializing the right-hand side of (2.5.8) to $T = 1$, yields an element of $M_2(\mathbb{F}[[u]])$ because $\mathfrak{M}_{\mathbb{F},2}$ is ϕ-stable. Thus, $\phi(N)A + AN^{\mathrm{ad}} + \phi(N)AN^{\mathrm{ad}} \in M_2(\mathbb{F}[[u]])$, and $\phi(N)AN^{\mathrm{ad}} \in M_2(\mathbb{F}[[u]])$ by assumption. It follows that the right-hand side of (2.5.8) is an element of $M_2(\mathbb{F}[[u]][T])$, so that $\mathfrak{M}_{\mathbb{F}[T]}$ is ϕ-stable. On the other hand, $\det \phi(1 + TN)A(1 + TN)^{-1} = \det A$, since N is nilpotent, and so (2.5.1) shows that for any finite extension $\mathbb{F}'$ of $\mathbb{F}$, specializing T by an $\mathbb{F}'$-valued point of $\mathbb{A}^1$, yields a point of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. If follows that the map, constructed above, from $\mathbb{A}^1$ into the affine Grassmannian does indeed factor through $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$, and any such map extends to $\mathbb{P}^1$, as $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ is projective. $\square$

(2.5.9) We return to the proof of (2.5.6). Suppose first that $V_\mathbb{F}$ is reducible. In this case $M_\mathbb{F}$ contains a ϕ-stable $\mathbb{F}((u))$-line $L_\mathbb{F}$. If we choose an ordered basis for $\mathfrak{M}_{\mathbb{F},1}$ such that the first basis vector lies on $L_\mathbb{F}$, then the matrix A is upper triangular. Write $A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} u^{-i} & v \\ 0 & u^i \end{smallmatrix}\right)$. Interchanging $\mathfrak{M}_{\mathbb{F},1}$ and $\mathfrak{M}_{\mathbb{F},2}$, if necessary (and hence replacing B by B^{-1}), we may assume that $i \geq 0$. Now,

$$
\mathfrak{M}_{\mathbb{F},2} \sim \phi(B)AB^{-1} = \begin{pmatrix} u^{-pi} & \phi(v) \\ 0 & u^{pi} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} u^i & -v \\ 0 & u^{-i} \end{pmatrix}
$$
$$
= \begin{pmatrix} u^{i-pi}a & -avu^{-ip} + u^{-i-ip}b + \phi(v)u^{-i}d \\ 0 & u^{pi-i}d \end{pmatrix}.
$$

Thus, $v_u(d) \geq 0$, $v_u(a) \geq pi - i$, and

(2.5.10) $v_u(-avu^i + b + \phi(v)u^{ip}d) \geq i + ip.$

In fact, the first two inequalities can be improved to $v_u(d) \geq 1$, and $v_u(a) \geq pi - i + 1$, since if $v_u(d) = 0$, then $v_u(a) = e$ by (2.5.1), and $\mathfrak{M}_{\mathbb{F},1}$ would be an extension of an étale module by a multiplicative module, which contradicts the assumption that $\mathfrak{M}_{\mathbb{F},1}$ is non-ordinary. Similarly, if $v_u(a) = pi - i$, then $v_u(u^{i-pi}a) = 0$, so that $\mathfrak{M}_{\mathbb{F},2}$ is an extension of a multiplicative module by an étale module. It is easy to see that any such extension splits. (One can of course also deduce this from the analogous fact for finite flat group schemes.) This implies that $\mathfrak{M}_{\mathbb{F},2}$ is ordinary, which is again a contradiction.

Now set $\mathfrak{M}_{\mathbb{F},3} = \left(\begin{smallmatrix} 1 & vu^i \\ 0 & 1 \end{smallmatrix}\right) \cdot \mathfrak{M}_{\mathbb{F},1}$. Then
(2.5.11)
$$
\mathfrak{M}_{\mathbb{F},3} \sim \begin{pmatrix} 1 & \phi(v)u^{pi} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & -vu^i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -avu^i + b + \phi(v)u^{ip}d \\ 0 & d \end{pmatrix}
$$

and $\mathfrak{M}_{\mathbb{F},2} = \left(\begin{smallmatrix} u^{-i} & 0 \\ 0 & u^i \end{smallmatrix}\right) \cdot \mathfrak{M}_{\mathbb{F},3}$. Note that the right-hand side of (2.5.11) is in $M_2(\mathbb{F}[\![u]\!])$, by (2.5.10), and because $i \geq 0$. By (2.5.1) $\mathfrak{M}_{\mathbb{F},3}$ corresponds to a point $x_3 \in \mathscr{GR}^{\mathsf{v}}_{V_\mathbb{F},0}(\mathbb{F})$, and x_3 lies on the same connected component as x_1, by (2.5.7), since if $N = \left(\begin{smallmatrix} 0 & vu^i \\ 0 & 0 \end{smallmatrix}\right)$, then $\phi(N)AN^{\mathrm{ad}} = 0$. Moreover $\mathfrak{M}_{\mathbb{F},3}$ is non-ordinary (for example because the corresponding group scheme is an extension of two connected group schemes).

It follows that we may replace $\mathfrak{M}_{\mathbb{F},1}$ by $\mathfrak{M}_{\mathbb{F},3}$ and B by $\left(\begin{smallmatrix} u^{-i} & 0 \\ 0 & u^i \end{smallmatrix}\right)$. By (2.5.10) this implies that we may assume $v_u(b) \geq i + ip$. For $j = 0, 1, 2, \ldots, i$ we have

$$
\mathfrak{N}_{\mathbb{F},j} := \begin{pmatrix} u^{-j} & 0 \\ 0 & u^j \end{pmatrix} \cdot \mathfrak{M}_{\mathbb{F},1} \sim \begin{pmatrix} u^{-pj} & 0 \\ 0 & u^{pj} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} u^j & 0 \\ 0 & u^{-j} \end{pmatrix}
$$
$$
= \begin{pmatrix} au^{j-pj} & bu^{-j-pj} \\ 0 & du^{pj-j} \end{pmatrix} \in M_2(\mathbb{F}[\![u]\!])
$$

since $v_u(b) \geq i + pi \geq j + pj$. Hence $\mathfrak{N}_{\mathbb{F},j}$ corresponds to a point of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$, and is non-ordinary. Replacing $(\mathfrak{M}_{\mathbb{F},1}, \mathfrak{M}_{\mathbb{F},2})$ by $(\mathfrak{N}_{\mathbb{F},j-1}, \mathfrak{N}_{\mathbb{F},j})$ for successive values of j, it suffices to consider the case $B = \left(\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix} \right)$, when $v_u(a) \geq p$, $v_u(d) \geq 1$ and $v_u(b) \geq p + 1$.

In this case, since $\left(\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 2u \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & -u \\ u^{-1} & 0 \end{smallmatrix} \right)$, we have $\mathfrak{M}_{\mathbb{F},2} = \left(\begin{smallmatrix} 2 & -u \\ u^{-1} & 0 \end{smallmatrix} \right) \cdot \mathfrak{M}_{\mathbb{F},1}$. Thus, by (2.5.7), to prove that x_1 and x_2 lie on the same connected component of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$, it suffices to check that if $N = \left(\begin{smallmatrix} 1 & -u \\ u^{-1} & -1 \end{smallmatrix} \right)$, then $\phi(N)AN^{\mathrm{ad}} \in M_2(\mathbb{F}[[u]])$. However, for any $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in M_2(\mathbb{F}[[u]])$, a direct calculation shows that
(2.5.12)
$$\phi(N) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} N^{\mathrm{ad}} = \begin{pmatrix} -\beta u^{-1} & 0 \\ -\alpha u^{-p} - \beta u^{-p-1} + \delta u^{-1} & \alpha u^{1-p} + \beta u^{-p} \end{pmatrix} + W$$

for some $W \in M_2(\mathbb{F}[[u]])$. It follows that $\phi(N)AN^{\mathrm{ad}} \in M_2(\mathbb{F}[[u]])$, and this completes the proof of the proposition if $V_{\mathbb{F}}$ is reducible.

(2.5.13) Suppose that $V_{\mathbb{F}}$ is irreducible. By (2.5.5) it suffices to consider the case where $A = \left(\begin{smallmatrix} 0 & d \\ a & 0 \end{smallmatrix} \right)$, with $p + 1 \nmid v_u(d) - v_u(a)$. As before, we may assume $B = \left(\begin{smallmatrix} u^{-i} & v \\ 0 & u^i \end{smallmatrix} \right)$. Then
(2.5.14)
$$\mathfrak{M}_{\mathbb{F},2} \sim \begin{pmatrix} u^{-pi} & \phi(v) \\ 0 & u^{pi} \end{pmatrix} \begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix} \begin{pmatrix} u^i & -v \\ 0 & u^{-i} \end{pmatrix} = \begin{pmatrix} au^i \phi(v) & du^{-pi-i} - a\phi(v)v \\ au^{pi+i} & -au^{pi}v \end{pmatrix}$$

and, in particular, the right-hand side of (2.5.14) is in $M_2(\mathbb{F}[[u]])$.

Suppose first that $v = 0$, so that $B = \left(\begin{smallmatrix} u^{-i} & 0 \\ 0 & u^i \end{smallmatrix} \right)$. In this case we may interchange $\mathfrak{M}_{\mathbb{F},1}$ and $\mathfrak{M}_{\mathbb{F},2}$, while retaining our assumption about the form of A. In particular, we may assume that $i \geq 0$. For $j = 0, 1, 2, \ldots i$ set $\mathfrak{N}_{\mathbb{F},j} = \left(\begin{smallmatrix} u^{-j} & 0 \\ 0 & u^j \end{smallmatrix} \right) \cdot \mathfrak{M}_{\mathbb{F},1}$. Then in the basis of $\mathfrak{N}_{\mathbb{F},j}$ obtained by multiplying the chosen basis of $\mathfrak{M}_{\mathbb{F},1}$ by $\left(\begin{smallmatrix} u^{-j} & 0 \\ 0 & u^j \end{smallmatrix} \right)$, ϕ is given by $\left(\begin{smallmatrix} 0 & du^{-pj-j} \\ au^{pj+j} & 0 \end{smallmatrix} \right)$.

Comparing this to the right-hand side of (2.5.14), we see that $\mathfrak{N}_{\mathbb{F},j}$ is ϕ-stable, and gives rise to a point $y_j \in \mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$.

Combining (2.5.7) and (2.5.12) with the fact that the right-hand side of (2.5.14) has entries in $\mathbb{F}[[u]]$, one sees that for $j = 0, 1, 2, \ldots, i$, there is map $\mathbb{P}^1 \to \mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ sending 0 to y_{j-1} and 1 to y_j. Hence $x_1 = y_0$ and $y_i = x_2$ lie on the same component of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$.

Now we return to the case of arbitrary v. We distinguish two cases. Suppose first that either $i \leqslant 0$ or that $v_u(v) \geq 0$. In this case $v_u(u^{-pi-i}d - a\phi(v)v) \geq 0$, implies that $v_u(u^{-ip-i}d) \geq 0$, and $v_u(a\phi(v)v) \geq 0$.

Hence, if we set $\mathfrak{M}_{\mathbb{F},3} = \left(\begin{smallmatrix} u^{-i} & 0 \\ 0 & u^i \end{smallmatrix} \right) \cdot \mathfrak{M}_{\mathbb{F},1}$, and consider the basis of $\mathfrak{M}_{\mathbb{F},3}$ induced by the chosen basis for $\mathfrak{M}_{\mathbb{F},1}$ then (2.5.14) shows that $\mathfrak{M}_{\mathbb{F},3}$ is stable by ϕ and corresponds to a point $x_3 \in \mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$. By the case $v = 0$, considered above, x_3 lies on the same component of $\mathscr{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ as x_1.

Replacing, x_1 by x_3, we may assume that $A = \begin{pmatrix} 0 & u^{-pi-i}d \\ u^{pi+i}a & 0 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & u^{-i}v \\ 0 & 1 \end{pmatrix}$. If $N = \begin{pmatrix} 0 & u^{-i}v \\ 0 & 0 \end{pmatrix}$, then we compute that $\phi(N)AN^{\mathrm{ad}} = \begin{pmatrix} 0 & -a\phi(v)v \\ 0 & 0 \end{pmatrix}$, and as we saw above, $v_u(a\phi(v)v) \geq 0$. Hence x_1 and x_2 lie on the same component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ by (2.5.7).

It remains to treat the case when $i > 0$, and $v_u(v) < 0$. In this case, we have $v_u(d - au^{ip+i}\phi(v)v) \geq ip + i > 0$. In particular this implies that $v_u(au^{ip+i}\phi(v)v)$ is non-negative.

Set $\mathfrak{M}_{\mathbb{F},3} = \begin{pmatrix} 1 & u^i v \\ 0 & 1 \end{pmatrix} \mathfrak{M}_{\mathbb{F},1}$. As usual, the chosen basis of $\mathfrak{M}_{\mathbb{F},1}$ induces a basis of $\mathfrak{M}_{\mathbb{F},3}$, in which the Frobenius is given by $\begin{pmatrix} au^{ip}\phi(v) & d - au^{ip+i}\phi(v)v \\ a & -au^i v \end{pmatrix}$ (this can be seen from the computation (2.5.14), by replacing u^i by 1 and v by $u^i v$). This matrix is easily seen to have entries in $M_2(\mathbb{F}[\![u]\!])$ since the right-hand side of (2.5.14) does, and because we have $i > 0$ and $v_u(v) < 0$. Hence $\mathfrak{M}_{\mathbb{F},3}$ corresponds to a point x_3 of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. Moreover, if $N = \begin{pmatrix} 0 & u^i v \\ 0 & 0 \end{pmatrix}$, then $\phi(N)AN^{\mathrm{ad}} = \begin{pmatrix} 0 & -au^{ip+i}\phi(v)v \\ 0 & 0 \end{pmatrix}$ lies in $M_2(\mathbb{F}[\![u]\!])$. Hence x_3 and x_1 lie on the same component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ by (2.5.7).

Now for $j = 0, 1, 2 \ldots, i$, set $\mathfrak{N}_{\mathbb{F},j} = \begin{pmatrix} u^{-j} & 0 \\ 0 & u^j \end{pmatrix} \cdot \mathfrak{M}_{\mathbb{F},3}$, with the basis induced by the chosen basis of $\mathfrak{M}_{\mathbb{F},3}$. Then ϕ is given by

$$\begin{pmatrix} au^{ip-pj+j}\phi(v) & u^{-pj-j}(d - au^{ip+i}\phi(v)v) \\ au^{pj+j} & -au^{i+pj-j}v \end{pmatrix}$$

in this basis. Since $\mathfrak{N}_{\mathbb{F},i} = \mathfrak{M}_{\mathbb{F},2}$ and $\mathfrak{N}_{\mathbb{F},0} = \mathfrak{M}_{\mathbb{F},3}$, when $j = 0$ or i, this matrix has integral entries, and is true for all j. Hence $\mathfrak{N}_{\mathbb{F},j}$ corresponds to a point $y_j \in \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. To check that y_j and y_{j+1} lie on the same component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ for $j \in [0, i-1]$, it suffices by (2.5.7) and (2.5.12) to show that

$$v_u(au^{i+p-1}\phi(v)) \geq p, \; v_u(u^{-pi-i}(d - au^{ip+i}\phi(v)v)) \geq 0 \text{ and } v_u(au^i v) \geq 1.$$

We have already seen the second of these inequalities, while the third follows from the fact that $v_u(au^i\phi(v)) \geq 0$ using (2.5.14) and $v_u(v) < 0$. If the first inequality does not hold, then we must have $v_u(au^i\phi(v)) = 0$. In particular $v_u(au^{ip+i}\phi(v)v) = v_u(u^{ip}v) < ip < ip + i$. This implies that

$$v_u(d) = v_u(au^{ip+i}\phi(v)v),$$

whence $p + 1 \mid v_u(d) - v_u(a)$, which contradicts our assumptions. This completes the proof of (2.5.6). □

PROPOSITION (2.5.15). *We no longer assume that $K_0 = \mathbb{Q}_p$. Let $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0} \subset \mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ denote the union of components corresponding to the ordinary points of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$. (This was denoted by $\mathscr{GR}^{\mathbf{v},\mathrm{loc},\mathbf{d}}_{V_{\mathbb{F}},0}$ with $\mathbf{d} = \{1, 1\}$ in the notation of (2.4.15).) If $\mathscr{GR}^{\mathbf{v},\mathrm{ord}}_{V_{\mathbb{F}},0}$ is non-empty then it consists of a single point, unless $V_{\mathbb{F}} \sim \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ where χ_1 and χ_2 are unramified characters of G_K. In the latter case:*

(1) If $\chi_1 \neq \chi_2$, then $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{ord}}$ consists of two points corresponding to its finite flat models $D(\mathscr{G}_{\chi_1^{-1}\omega}) \oplus \mathscr{G}_{\chi_2}$ and $D(\mathscr{G}_{\chi_2^{-1}\omega}) \oplus \mathscr{G}_{\chi_1}$, where ω denotes the mod p cyclotomic character, and for an unramified mod p character χ, $\mathscr{G}_\chi$ denotes the unique extension of χ to a finite étale group scheme.

(2) If $\chi_1 = \chi_2$ then $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{ord}} \xrightarrow{\sim} \mathbb{P}^1$ and all the finite flat models of $V_{\mathbb{F}}$ are isomorphic to $D(\mathscr{G}_{\chi_1^{-1}\omega}) \oplus \mathscr{G}_{\chi_1}$.

Proof. Let $\mathbb{F}'/\mathbb{F}$ be a finite extension, and A a local Artin ring with residue field $\mathbb{F}'$, and such that $p \cdot A = \{0\}$. Keeping our previous notation, let $V_A = V_{\mathbb{F}} \otimes_{\mathbb{F}} A$, and $M_A = M_{\mathbb{F}} \otimes_{\mathbb{F}} A$. Consider a point of $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{ord}}(A)$, and let $\mathfrak{M}_A \subset M_A$ be the corresponding element of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_A$. Thus $\mathfrak{M}_A$ is free of rank 2 over $\mathfrak{S}_A$, and spans the $\mathfrak{S}_A[1/u]$-module M_A. By the discussion in (2.4.13) $\mathfrak{M}_A$ is an extension

$$0 \to \mathfrak{M}_A^m \to \mathfrak{M}_A \to \mathfrak{M}_A^{\mathrm{\acute{e}t}} \to 0$$

of an étale object of $(\mathrm{Mod}\,\mathrm{FI}/\mathfrak{S})_A$ of $\mathfrak{S}_A$-rank 1, by a multiplicative one. Composing the functors (1.1.3) and (1.1.11), we see that this corresponds to an extension of finite flat group schemes equipped with an action of A

$$0 \to \mathscr{G}_A^m \to \mathscr{G}_A \to \mathscr{G}_A^{\mathrm{\acute{e}t}} \to 0$$

with $\mathscr{G}_A^m$ multiplicative and $\mathscr{G}_A^{\mathrm{\acute{e}t}}$ étale by (1.1.15). Finally $L_A = \mathscr{G}_A^m(\bar{K}) \subset V_A$ gives rise to a G_K-stable A-line in V_A, such that the action of G_K on V_A/L_A is unramified. (Here by an A-line we mean a free A-submodule of rank 1, with A-free quotient).

Conversely, given such an A-line, $L_A \subset V_A$, consider an ordinary finite flat group scheme $\mathscr{G}_A$ equipped with an isomorphism $\mathscr{G}_A(\bar{K}) \xrightarrow{\sim} V_A$, under which $\mathscr{G}_A^m(\bar{K})$ is mapped onto L_A. Since $\mathscr{G}_A^m$ is multiplicative while $\mathscr{G}_A^{\mathrm{\acute{e}t}}$ is étale, they are uniquely determined by L_A and V_A/L_A respectively, and an argument as in [BCDT, 4.1.2] shows that $\mathscr{G}_A$ is uniquely determined by L_A. There is more than one possibility for L_A only if $V_{\mathbb{F}}$ is decomposable, with unramified diagonal characters χ_1 and χ_2. The possibilities for L_A are then the two A-lines in V_A corresponding to χ_1 and χ_2 in (1) and to any A-line in (2).

This establishes the proposition except possibly in (2), when we have only shown a functorial isomorphism $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{ord}}(A) \xrightarrow{\sim} \mathbb{P}(V_{\mathbb{F}})(A)$. However this implies that $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{ord}}$ is smooth, and that its zeta function is $(1-T)^{-1}(1-|\mathbb{F}|T)^{-1}$. This implies that $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{ord}}$ is connected, 1-dimensional of genus 0, and has a rational point, and so $\mathscr{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\mathrm{ord}} \xrightarrow{\sim} \mathbb{P}^1$. $\qquad \square$

COROLLARY (2.5.16). *Let $R = R_{V_{\mathbb{F}}}^{\mathrm{fl},\square} \otimes_{W(\mathbb{F})} \mathcal{O}_F$. As above assume that $d = 2$, and that all the v_ψ are equal to 1, but drop the assumption that $K_0 = \mathbb{Q}_p$.*

(1) *The complete local ring $R^{\mathbf{v}}$ is flat over $\mathbb{Z}_p$ of pure relative dimension $4 + [K : \mathbb{Q}_p]$, and $R^{\mathbf{v}}[1/p]$ is formally smooth over F.*

(2) *Let E/F be a finite extension and $x_1, x_2 \in (\operatorname{Spec} R^{\mathbf{v}})(E)$. If (the images of) x_1 and x_2 lie on the same irreducible component of $\operatorname{Spec} R^{\mathbf{v}}$ then the corresponding E-representations of G_K, V_{x_1} and V_{x_2} are both ordinary or both non-ordinary. The converse holds in the following two cases:*

   (i) *V_{x_1} and V_{x_2} are both non-ordinary and $K_0 = \mathbb{Q}_p$.*
   (ii) *V_{x_1} and V_{x_2} are both ordinary. If $L_1 \subset V_{x_1}$ and $L_2 \subset V_{x_2}$ denote the (unique) E-lines on which I_K acts via the cyclotomic character, then G_K acts on L_1 and L_2 via $\mathbb{O}_E^{\times}$-valued characters with the same reduction modulo π_E.*

If $\operatorname{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$ then the same results hold for $R = R^{\mathrm{fl}}_{V_{\mathbb{F}}} \otimes_{\mathrm{W}(\mathbb{F})} \mathbb{O}_F$, except that the dimension in (1) is $1 + [K : \mathbb{Q}_p]$.

Proof. $R^{\mathbf{v}}$ is flat over $\mathbb{Z}_p$ by construction, and the claims concerning dimensions and formal smoothness follow from (2.3.11).

The second part of the corollary follows from (2.5.6), (2.5.15) and (2.4.10). Note that since $\operatorname{Spec} R^{\mathbf{v}}[1/p]$ is formally smooth over F, its connected components coincide with its irreducible components, and since $R^{\mathbf{v}}$ is flat over $\mathbb{Z}_p$, these are in bijection with the irreducible components of $\operatorname{Spec} R^{\mathbf{v}}$.

Finally the last claim is easily deduced using the formal smoothness of the map $R^{\mathrm{fl}}_{V_{\mathbb{F}}} \to R^{\mathrm{fl},\square}_{V_{\mathbb{F}}}$. $\qquad\square$

(2.6) *The case of residue characteristic $l \neq p$.* In this section we will apply the techniques developed above to study deformation rings of Galois representations of local fields of residue characteristic $l \neq p$. Since this case is significantly simpler than that considered above, we will develop the theory only as far as it is needed in this paper.

(2.6.1) Suppose that $l \neq p$ is a prime, and that L is a finite extension of $\mathbb{Q}_l$. We fix an algebraic closure $\bar{L}$ of L, and write $G_L = \operatorname{Gal}(\bar{L}/L)$. We write $I_L \subset G_L$ for the inertia subgroup. As above, we denote by $\mathbb{F}$ a finite extension of $\mathbb{F}_p$, and we consider a two dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$ equipped with a continuous action of G_L, and a fixed choice of basis. We make two assumptions:

(1) $\det V_{\mathbb{F}}$ is equal to the cyclotomic character $\chi : G_L \to \mathbb{Z}_p^{\times}$ (modulo p).

(2) $V_{\mathbb{F}}(-1)^{G_L} \neq \{0\}$.

We define a groupoid $D^{\chi}_{V_{\mathbb{F}}}$ on $\mathfrak{AR}_{\mathrm{W}(\mathbb{F})}$ by declaring the objects of $D^{\chi}_{V_{\mathbb{F}}}(A)$ to be finite free A-modules V_A equipped with a continuous action of G_L such that $\det_A V_A = \chi$, and an isomorphism of $\mathbb{F}[G_L]$-modules $V_A \otimes_A A/\mathfrak{m}_A \xrightarrow{\sim} V_{\mathbb{F}}$. Similarly we define $D^{\chi,\square}_{V_{\mathbb{F}}}$ by declaring an object of $D^{\chi,\square}_{V_{\mathbb{F}}}(A)$ to consist of an object V_A of $D^{\chi}_{V_{\mathbb{F}}}(A)$ together with a choice of A-basis lifting the chosen basis on $V_{\mathbb{F}}$. We denote by $D_{V_{\mathbb{F}}}$ and $D^{\square}_{V_{\mathbb{F}}}$ the groupoids defined in the same way, except that

we drop the condition on the determinant. As in (2.1.1), we extend each of these functors to $\mathfrak{Aug}_{W(\mathbb{F})}$.

Finally we define a groupoid $L_{V_{\mathbb{F}}}^{\chi}$ (resp. $L_{V_{\mathbb{F}}}^{\chi,\square}$) on $\mathfrak{Aug}_{W(\mathbb{F})}$ by declaring $L_{V_{\mathbb{F}}}^{\chi}(A, I)$ (resp. $L_{V_{\mathbb{F}}}^{\chi,\square}(A, I)$) to consist of pairs (V_A, L_A) where V_A is an object of $D_{V_{\mathbb{F}}}^{\chi}(A, I)$ (resp. $D_{V_{\mathbb{F}}}^{\chi,\square}(A, I)$), and $L_A \subset V_A$ is an A-line on which G_L acts via χ (that is a projective A-submodule of rank 1, such that V_A/L_A is projective over A, with a trivial action of G_L).

LEMMA (2.6.2). (1) *The functors $|D_{V_{\mathbb{F}}}^{\square}|$ and $|D_{V_{\mathbb{F}}}^{\chi,\square}|$ are pro-representable by complete local $W(\mathbb{F})$-algebras $R_{V_{\mathbb{F}}}^{\square}$ and $R_{V_{\mathbb{F}}}^{\chi,\square}$ respectively.*

(2) *The morphism $|L_{V_{\mathbb{F}}}^{\chi,\square}| \to |D_{V_{\mathbb{F}}}^{\chi,\square}|$ given by sending (V_A, L_A) to V_A is represented by a projective morphism $\Theta_{V_{\mathbb{F}}} : \mathscr{L}_{V_{\mathbb{F}}}^{\chi,\square} \to \operatorname{Spec} R_{V_{\mathbb{F}}}^{\chi,\square}$.*

Proof. (1) is clear. For (2), we note that $|L_{V_{\mathbb{F}}}^{\chi,\square}|$ is pro-representable by a closed subspace of $\widehat{\mathbb{P}} \to \operatorname{Spec} R_{V_{\mathbb{F}}}^{\chi,\square}$, where $\mathbb{P}$ denotes the projectivization of the universal rank 2 $R_{V_{\mathbb{F}}}^{\chi,\square}$-module, and $\widehat{\mathbb{P}}$ denotes its completion along the maximal ideal of $R_{V_{\mathbb{F}}}^{\chi,\square}$. By formal GAGA this subspace corresponds to a unique projective $R_{V_{\mathbb{F}}}^{\chi,\square}$-scheme. $\square$

LEMMA (2.6.3). *$\mathscr{L}_{V_{\mathbb{F}}}^{\chi,\square}$ is formally smooth over $W(\mathbb{F})$. The $W(\mathbb{F})[1/p]$-scheme $\mathscr{L}_{V_{\mathbb{F}}}^{\chi,\square} \otimes_{W(\mathbb{F})} W(\mathbb{F})[1/p]$ is connected.*

Proof. To begin with we observe that for any finite group M of p-power order the natural map

$$H^1(G_L, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} M \to H^1(G_L, \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} M)$$

is an isomorphism. To see this is suffices to consider the case $M = \mathbb{Z}/p^n\mathbb{Z}$, in which case the above map is injective with cokernel equal to $H^2(G_L, \mathbb{Z}_p(1))[p^n]$. Since $H^2(G_L, \mathbb{Z}_p(1))$ is Pontryagin dual to $\mathbb{Q}_p/\mathbb{Z}_p$, it is a free $\mathbb{Z}_p$-module of rank 1, and has no p-torsion. In particular we see that for any finite, Artinian $\mathbb{Z}_p$-algebra A, the composite

$$\operatorname{Ext}^1_{\mathbb{Z}_p[G_L]}(\mathbb{Z}_p, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} A \xrightarrow{\sim} H^1(G_L, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} A$$
$$\to H^1(G_L, A(1)) \xrightarrow{\sim} \operatorname{Ext}^1_{A[G_L]}(A, A(1))$$

is an isomorphism.

Now let $A \to A'$ be a surjection of local Artin rings with residue field $\mathbb{F}'$, a finite extension of $\mathbb{F}$. For the first claim it suffices to show that the map $|L_{V_{\mathbb{F}}}^{\chi}|(A) \to |L_{V_{\mathbb{F}}}^{\chi}|(A')$ is a surjection. Suppose that $(V_{A'}, L_{A'})$ is an object of $L_{V_{\mathbb{F}}}^{\chi}(A')$. Then the discussion of the previous paragraph shows that $V_{A'}$ corresponds

to a class in $\mathrm{Ext}^1_{\mathbb{Z}_p[G_L]}(1,\chi) \otimes_{\mathbb{Z}_p} A'$. Lifting this class to $\mathrm{Ext}^1_{\mathbb{Z}_p[G_L]}(1,\chi) \otimes_{\mathbb{Z}_p} A$ produces an object $V_A = (V_A, L_A)$ of $L^\chi_{V_{\mathbb{F}}}(A)$ such that $V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$.

For the second claim, the smoothness implies, in particular, that $\mathscr{L}^{\chi,\square}_{V_{\mathbb{F}}} \otimes_{W(\mathbb{F})} \mathbb{F}$ is reduced, so as in (2.4.10), it suffices to show that $\mathscr{L}^{\chi,\square}_{V_{\mathbb{F}}} \otimes_{R^{\chi,\square}_{V_{\mathbb{F}}}} \mathbb{F}$ is connected. However, exactly as in (2.5.15) one sees that this scheme is isomorphic to $\mathbb{P}^1$ if $\mathbb{F} \xrightarrow{\sim} \mathbb{F}(1)$ and $V_{\mathbb{F}}$ is split (i.e. if the action of G_L on $V_{\mathbb{F}}$ is trivial), and a reduced point otherwise. $\square$

LEMMA (2.6.4). *Let $E/\mathbb{Q}_p$ be a finite extension, and $\xi \in L^{\chi,\square}_{V_{\mathbb{F}}}(\mathbb{O}_E)$. We again denote by ξ the image of ξ in $D^{\chi,\square}_{V_{\mathbb{F}}}(\mathbb{O}_E)$.*

The morphism of groupoids on $\mathfrak{AR}_E$, $L^{\chi,\square}_{V_{\mathbb{F}},(\xi)} \to D^{\chi,\square}_{V_{\mathbb{F}},(\xi)}$ is fully faithful. If the E-representation V_ξ of G_L corresponding to ξ is indecomposable, then this morphism is an equivalence.

Proof. Suppose that B is in $\mathfrak{AR}_E$, and let V_B be an object of $D^{\chi,\square}_{V_{\mathbb{F}},(\xi)}$. As in (2.3.5) V_B corresponds to a deformation of V_ξ to B together with a choice of basis. To prove the first assertion we have to show that if V_B admits a B-line $L_B \subset V_B$ such that $L_B(-1)$ is G_L-invariant, then L_B is the unique such line. Since $\det_B V_B = \chi$, we have $\mathrm{Hom}_{B[G_L]}(B(1), V_B/L_B) = \{0\}$, so that

$$\mathrm{Hom}_{B[G_L]}(B(1), V_B) = \mathrm{Hom}_{B[G_L]}(B(1), L_B),$$

and the uniqueness of L_B follows.

Suppose that V_ξ is indecomposable. We have to show that V_B admits a unique B-line $L_B \subset V_B$ on which G_L acts via χ. It suffices to show that V_B is isomorphic to the trivial deformation $V_\xi \otimes_E B$. This follows by induction on the length of B, and the fact that, by local Tate duality, we have

$$\dim_E H^1(G_L, \mathrm{ad}^0 V_\xi) = \dim_E H^0(G_L, \mathrm{ad}^0 V_\xi) + \dim_E H^0(G_L, \mathrm{ad}^0 V_\xi(1)) = 0.$$
                                                                                                 $\square$

(2.6.5) For ξ as above, we define a groupoid D_{V_ξ} (resp. $D^{\chi,\square}_{V_\xi}$) on $\mathfrak{AR}_E$ by declaring $D^\chi_{V_\xi}(B)$ (resp. $D^{\chi,\square}_{V_\xi}(B)$) to consist of deformations of V_ξ to B with determinant χ (resp. determinant χ and a lift of the given basis on V_ξ).

PROPOSITION (2.6.6). *Let $\mathrm{Spec}\, R^{\chi,1,\square}_{V_{\mathbb{F}}}$ denote the scheme theoretic image of the morphism $\Theta_{V_{\mathbb{F}}}$ of (2.6.2). Then*

(1) $R^{\chi,1,\square}_{V_{\mathbb{F}}}$ *is a domain of dimension 4, and $R^{\chi,1,\square}_{V_{\mathbb{F}}}[1/p]$ is formally smooth over $W(\mathbb{F})$.*

(2) *If $E/\mathbb{Q}_p$ is a finite extension, then a morphism $\xi: R^{\chi,\square}_{V_{\mathbb{F}}} \to E$ factors through $R^{\chi,1,\square}_{V_{\mathbb{F}}}$ if and only if the corresponding two dimensional E-representation V_ξ is an extension of E by $E(1)$.*

Proof. That $R_{V_{\mathbb{F}}}^{\chi,1,\square}$ is a domain follows from (2.6.3). Let ξ be as in (2.6.4). An argument as in (2.3.3) together with (2.6.4) shows that the map between the complete local rings at the image of ξ on $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\chi,\square}$ and $\mathscr{L}_{V_{\mathbb{F}}}^{\chi,\square}$ is a surjection, and is an isomorphism if V_ξ is indecomposable. Hence we see that $\Theta_{V_{\mathbb{F}}}$ becomes a closed immersion after inverting p. In particular (2.6.3) implies that $R_{V_{\mathbb{F}}}^{\chi,1,\square}[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$. To compute its dimension, suppose that V_ξ is indecomposable. Then the dimension of $R_{V_{\mathbb{F}}}^{\chi,1,\square}[1/p]$ is equal to

$$\dim_E |D_{V_\xi}^{\chi,\square}|(E[\epsilon]) = \dim_E |D_{V_\xi}^{\chi}|(E[\epsilon]) + 4 - \dim_E(\operatorname{ad}V_\xi)^{G_L}$$
$$= \dim_E H^1(G_L, \operatorname{ad}^0 V_\xi) + 3 = 3.$$

Finally for (2), by what we have seen, ξ factors through $R_{V_{\mathbb{F}}}^{\chi,1,\square}$ if and only if it lifts to a (necessarily unique) point of $\mathscr{L}_{V_{\mathbb{F}}}^{\chi,\square}$, that is, if and only if V_ξ admits an E-line $L_E \subset V_\xi$ such that $L_E(-1)$ is G_L-invariant. $\qquad\square$

COROLLARY (2.6.7). *Let $\mathbb{O}$ be the ring of integers in a finite extension of* $W(\mathbb{F})[1/p]$*, and $\gamma : G_L \to \mathbb{O}^\times$ a continuous unramified character. Write $R_{V_{\mathbb{F}},\mathbb{O}}^{\square} = R_{V_{\mathbb{F}}}^{\square} \otimes_{W(\mathbb{F})} \mathbb{O}$. Then there exists a quotient $R_{V_{\mathbb{F}},\mathbb{O}}^{\chi\gamma,\gamma,\square}$ of $R_{V_{\mathbb{F}},\mathbb{O}}^{\square}$ with the following properties.*

(1) *$R_{V_{\mathbb{F}},\mathbb{O}}^{\chi\gamma,\gamma,\square}$ is a domain of dimension 4, and $R_{V_{\mathbb{F}},\mathbb{O}}^{\chi\gamma,\gamma,\square}[1/p]$ is formally smooth over $\mathbb{O}$.*

(2) *If $E/\mathbb{O}[1/p]$ is a finite extension, a map $\xi : R_{V_{\mathbb{F}},\mathbb{O}}^{\square} \to E$ factors through $R_{V_{\mathbb{F}},\mathbb{O}}^{\chi\gamma,\gamma,\square}$ if and only if V_ξ is an extension of γ by $\gamma(1)$.*

Proof. We may replace $\mathbb{F}$ by the residue field of $\mathbb{O}$. Then twisting by γ^{-1} induces an isomorphism $R_{V_{\mathbb{F}},\mathbb{O}}^{\square} \xrightarrow{\sim} R_{V_{\mathbb{F}}\otimes\gamma^{-1},\mathbb{O}}^{\square}$, and the quotient $R_{V_{\mathbb{F}},\mathbb{O}}^{\chi\gamma,\gamma,\square}$ corresponds to $R_{V_{\mathbb{F}}\otimes\gamma^{-1}}^{\chi,1,\square} \otimes_{W(\mathbb{F})} \mathbb{O}$ under this isomorphism. The corollary follows from (2.6.6). $\quad\square$

3. Modularity

(3.1) *Quaternionic forms.* We continue to assume $p > 2$. We need some results on automorphic forms on definite quaternion algebras, following [Tay1]. In that paper there is a standing assumption that p is unramified in the totally real number field over which one works, and that $p > 3$. We will use certain results from [Tay1] without assuming this; however the reader will easily check that when we do this the proofs of *loc. cit.* go over unchanged, though they do sometimes rely on the hypothesis (3.1.2) made below (in the situation of [Tay1] this hypothesis is always satisfied). Another difference with [Tay1] is that Taylor considers only totally definite quaternion algebras which are unramified at all finite primes, whereas we allow ramification at finite primes. This makes no difference to the proofs, and so we make no further mention of it.

There are three types of result we will establish in this section: raising the level at primes dividing p, raising the level at primes not dividing p, and a result on the freeness of the space of quaternionic forms over the ring of diamond operators. We work in greater generality than needed for the applications of this paper, because this entails relatively little extra work, and may prove useful in future papers.

(3.1.1) We recall the definitions of [Tay1, §1], or rather a slight variant of them.

Let F be a totally real field, and D a quaternion algebra with center F which is ramified at all the infinite places of F and at a set of finite places Σ, which does not contain any primes dividing p. We fix a maximal order $\mathbb{O}_D$ of D, and for each finite place $v \notin \Sigma$, an isomorphism of F_v-algebras $(\mathbb{O}_D)_v \xrightarrow{\sim} M_2(\mathbb{O}_{F_v})$. We will denote by $\mathbf{N}(v)$ the order of the residue field at v.

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup contained in $\prod_v (\mathbb{O}_D)_v^\times$. We assume that if $v \in \Sigma$, then $U_v = (\mathbb{O}_D)_v^\times$, and that $U_v = \mathrm{GL}_2(\mathbb{O}_{F_v})$ for $v | p$.

Let A be a topological $\mathbb{Z}_p$-algebra. For each $v | p$, we fix a continuous representation $\tau_v : U_v \to \mathrm{Aut}(W_{\tau_v})$ on a finite A-module. (In applications this will usually be a free A-module). We write $W_\tau = \otimes_{v|p,A} W_{\tau_v}$ and we denote by $\tau : \prod_{v|p} U_v \to \mathrm{Aut}(W_\tau)$ the corresponding representation. We regard τ as being a representation of U by letting U_v act trivially if $v \nmid p$.

Finally, we fix a continuous character $\psi : (\mathbb{A}_F^f)^\times / F^\times \to A^\times$ such that for any place v of F, τ on $U_v \cap \mathbb{O}_{F_v}^\times$ is given by multiplication by ψ^{-1}. We think of $(\mathbb{A}_F^f)^\times$ as acting on W_τ via ψ^{-1}, so that W_τ becomes a $U(\mathbb{A}_F^f)^\times$-module. Note that, given τ and U, such a ψ need not exist. A necessary condition for the existence of ψ is that τ be trivial on $U \cap \mathbb{O}_F^\times$.

Let $S_{\tau,\psi}(U, A)$ denote the set of functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \to W_\tau$$

such that for $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have $f(gu) = \tau(u)^{-1} f(g)$ for $u \in U$, and $f(gz) = \psi(z) f(g)$ for $z \in (\mathbb{A}_F^f)^\times$. If we write $(D \otimes_F \mathbb{A}_F^f)^\times = \coprod_{i \in I} D^\times t_i U(\mathbb{A}_F^f)^\times$ for some $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$ and some finite index set I, then

$$S_{\tau,\psi}(U, A) \underset{f \mapsto \{f(t_i)\}}{\xrightarrow{\sim}} \bigoplus_{i \in I} W_\tau^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i)/F^\times}.$$

We make the following assumption:

(3.1.2) For all $t \in (D \otimes_F \mathbb{A}_F^f)^\times$ the group $(U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t)/F^\times$

has prime to p-order.

Note that the calculations of [Tay1, 1.1] show that $(U(\mathbb{A}_F^f)^\times \cap t^{-1}D^\times t)/F^\times$ is automatically finite, and that if U is sufficiently small then it is a 2-group. Thus (3.1.2) holds for U sufficiently small.

(3.1.3) Let S be a set of primes containing Σ, the primes dividing p, and the primes v of F such that $U_v \subset D_v^\times$ is not maximal compact. Let $\mathbb{T}_{S,A}^{\mathrm{univ}} = A[T_v, S_v]_{v \notin S}$, be a commutative polynomial ring in the indicated formal variables. For each finite prime v of F we fix a uniformiser π_v of F_v. We consider the left action of $(D \otimes_F \mathbb{A}_F^f)^\times$ on W_τ-valued functions on $(D \otimes_F \mathbb{A}_F^f)^\times$ given by the formula $(gf)(z) = f(zg)$. As explained in [Tay1, §1], $S_{\tau,\psi}(U, A)$ has a natural action of $\mathbb{T}_{S,A}^{\mathrm{univ}}$, with S_v acting via the double coset $U \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & \pi_v \end{smallmatrix} \right) U$ and T_v via $U \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix} \right) U$. These operators do not depend on the choice of π_v.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{S,A}^{\mathrm{univ}}$ with residue field a finite field of characteristic p. Also, $\mathfrak{m}$ is in the support of (τ, ψ) if $S_{\tau,\psi}(U, A)_\mathfrak{m}$ is non-zero. Also, $\mathfrak{m}$ is *Eisenstein* if $T_v - 2 \in \mathfrak{m}$ for all but finitely many primes which split in some fixed abelian extension of F.

If A is a local ring with maximal ideal $\mathfrak{m}_A$, and $f \in S_{\tau,\psi}(U, A)$ is an eigenfunction (that is, a function such that for each $T \in \mathbb{T}_{S,A}^{\mathrm{univ}}$, $Tf = a_T f$ for some $a_T \in A$) whose image $\bar{f}$ in $S_{\tau,\psi}(U, A) \otimes_A A/\mathfrak{m}_A$ is non-zero, then the kernel of $\mathbb{T}_{S,A}^{\mathrm{univ}}$ acting on $\bar{f}$ is a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{S,A}^{\mathrm{univ}}$. We call $\mathfrak{m}$ the maximal ideal associated to f.

We will need a variant of the results of [Kh] for totally real fields, and these are relatively easy to deduce using definite quaternion algebras.

LEMMA (3.1.4). *Let $I \subset A$ be an ideal, and write $\bar{\psi}$ for the composite of ψ with the projection $A \to A/I$. For $v|p$ fix a representation τ_v' of U_v on a finite free A/I-module $W_{\tau_v'}$. Denote by τ' the $\prod_{v|p} U_v$-representation $W_{\tau'} = \otimes_{v|p, A/I} W_{\tau_v'}$, and assume that, on $U_v \cap \mathbb{O}_{F_v}^\times$, τ' is given by $\bar{\psi}^{-1}$.*

Suppose that $W_{\tau'}$ occurs as a $\prod_{v|p} U_v$-module subquotient of $W_{\bar{\tau}} := W_\tau \otimes_A A/I$. If $\mathfrak{m}$ is in the support of $(\tau', \bar{\psi})$, then $\mathfrak{m}$ is in the support of (τ, ψ).

Proof. The hypothesis (3.1.2) implies that $S_{\tau,\psi}(U, A) \xrightarrow{\sim} \bigoplus_{j \in J} W_\tau^{G_j}$ for some index set J and a group G_j which is of prime to p order, and does not depend on τ. Hence the functor $W_\tau \mapsto S_{\tau,\psi}(U, A)$ is exact. Applying this with A/I in place of A, we find that $S_{\tau', \bar{\psi}}(U, A/I)_\mathfrak{m} \neq \{0\}$ implies $S_{\bar{\tau}, \bar{\psi}}(U, A/I)_\mathfrak{m} \neq \{0\}$. Similarly, we see that the surjection $W_{\tau,\psi} \to W_{\bar{\tau}, \bar{\psi}}$ induces a surjection $S_{\tau,\psi}(U, A)_\mathfrak{m} \to S_{\bar{\tau}, \bar{\psi}}(U, A/I)_\mathfrak{m}$, so that $S_{\tau,\psi}(U, A)_\mathfrak{m} \neq \{0\}$. $\qquad \square$

LEMMA (3.1.5). *Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathbb{O}_K$ and residue field $\mathbb{F}_q$, and E a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathbb{O}$, and residue field $\mathbb{F}$. Denote by $B \subset \mathrm{GL}_2(\mathbb{F}_q)$ the subgroup of upper triangular matrices. Let $\mathbf{1}$ denote the trivial representation of B on an $\mathbb{F}$-vector space of dimension*

1, *write $W_0 \subset \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)} \mathbf{1}$ for the subspace of constant functions, and set $W_1 = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)} \mathbf{1}/W_0$.*

If E is sufficiently large, then for $i = 0, 1$ there exists a representation of $\mathrm{GL}_2(\mathbb{O}_K)$ on a finite free $\mathbb{O}$-module W with the following properties.

(1) *$W \otimes_{\mathbb{O}} \mathbb{F}$ has a subquotient isomorphic to W_i.*

(2) *W is a smooth representation (i.e. it has locally constant orbit maps) and is a cuspidal K-type. That is, there exists a cuspidal representation α of $\mathrm{GL}_2(K)$ such that $\alpha|_{\mathrm{GL}_2(\mathbb{O}_K)}$ contains $W \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as a subrepresentation. If α' is any irreducible admissible representation with the same property then α' is obtained from α by twisting by an unramified character.*

(3) *$W|_{\mathbb{O}_{K^\times}}$ is trivial.*

Proof. Let H be the quadratic unramified extension of K, and write $\mathbb{O}_H$ for its ring of integers, and $\mathfrak{p}_H$ for its maximal ideal. We fix an embedding $\mathbb{O}_H \subset M_2(\mathbb{O}_K)$, and we denote by $\sigma \in M_2(\mathbb{O}_K)$ an element of order 2 which induces the non-trivial Galois automorphism of $\mathbb{O}_H$ over $\mathbb{O}_K$.

Let $\theta : \mathbb{O}_H^\times \to \mathbb{O}^\times$ be a character which has even conductor $2m > 0$, and such that for any character $\chi : \mathbb{O}_K^\times \to \mathbb{O}^\times$, $\theta \cdot (\chi^{-1} \circ N_{H/K})$ has conductor $\geq 2m$. Here $N_{H/K} : H^\times \to K^\times$ denotes the norm. Such a θ exists if E is sufficiently large. (One could for example take $m = 1$, and θ of order dp, where d is any divisor of $q + 1$.) By [GK, 3.4] $\mathbb{O}_H^\times(1 + \sigma(\mathfrak{p}_H^m)) \subset \mathrm{GL}_2(\mathbb{O}_K)$ is a subgroup, and we may extend θ to a character of this group (again denoted θ) by requiring the extension to be trivial on $1 + \sigma(\mathfrak{p}_H^m)$.

Let $\tilde{\theta}$ be any extension of θ to a character $\tilde{\theta} : K_\theta = H^\times(1 + \sigma(\mathfrak{p}_H^m)) \to E^\times$. By *loc. cit.* $\alpha = \mathrm{Ind}_{K_\theta}^{\mathrm{GL}_2(K)} \tilde{\theta}$ is an irreducible, admissible cuspidal representation of $\mathrm{GL}_2(K)$. Hence

$$W' = \mathrm{Ind}_{K_\theta}^{K^\times \mathrm{GL}_2(\mathbb{O}_K)} \tilde{\theta}$$

is also irreducible. Moreover by using [Cas, 2.1] one sees that the $K^\times \mathrm{GL}_2(\mathbb{O}_K)$-representation W' is not contained in any other admissible irreducible representation of $\mathrm{GL}_2(K)$ (cf. [Kh, 3.2]). Hence $W = \mathrm{Ind}_{\mathbb{O}_H^\times(1+\sigma(\mathfrak{p}_H^m))}^{\mathrm{GL}_2(\mathbb{O}_K)} \theta$ is an $\mathbb{O}$-lattice in W', and satisfies (2) of the lemma.

We have to check that θ can be chosen so that W satisfies (1). Let

$$\theta_0 : \mathbb{F}_{q^2}^\times \to \mathbb{F}^\times$$

denote the character induced by θ. If $i = 0$, then we choose θ so that θ_0 is trivial. In this case

$$W \otimes_{\mathbb{O}} \mathbb{F} = \mathrm{Ind}_{\mathbb{O}_H^\times(1+\sigma(\mathfrak{p}_H^m))}^{\mathrm{GL}_2(\mathbb{O}_K)} \mathbf{1},$$

and in particular it contains a copy of the trivial representation.

If $i = 1$, we choose θ so that θ_0 is a non-trivial character whose kernel contains $\mathbb{F}_q^\times$. Then $W \otimes_\mathbb{O} \mathbb{F}$ admits $\mathrm{Ind}_{\mathbb{F}_{q^2}^\times}^{\mathrm{GL}_2(\mathbb{F}_q)} \theta_0$ as a subquotient, where the inclusion

$$\mathbb{F}_{q^2}^\times \subset \mathrm{GL}_2(\mathbb{F}_q)$$

is induced by $\mathbb{O}_H^\times \subset \mathrm{GL}_2(\mathbb{O}_K)$. The restriction of W_1 to $\mathbb{F}_{q^2}^\times$ is a direct sum over the non-trivial characters of $\mathbb{F}_{q^2}^\times / \mathbb{F}_q^\times$. Hence there exists a non-zero map

$$W_1 \to \mathrm{Ind}_{\mathbb{F}_{q^2}^\times}^{\mathrm{GL}_2(\mathbb{F}_q)} \theta_0$$

which is an injection as W_1 is irreducible [CDT, 3.1.1].

Finally, since $p > 2$, we may choose θ so that $\theta|_{\mathbb{O}_K^\times}$ is trivial (as well as satisfying the condition involving conductors imposed above). With such a choice of θ, W satisfies (3). $\qquad\square$

COROLLARY (3.1.6). *Suppose that $E/\mathbb{Q}_p$ is as above, and that $A = \mathbb{O}$. For each $v | p$ let $W_{\tau_{0,v}^{\mathrm{sm}}}$ be a representation of $\mathrm{GL}_2(\mathbb{O}_{F_v})$ on a finite free $\mathbb{O}$-module which is either trivial of rank 1 or isomorphic to $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbb{O}_{F_v})} \mathbb{O}$ modulo the constant functions, where $I \subset \mathrm{GL}_2(\mathbb{O}_{F_v})$ denotes the subgroup of matrices which are upper triangular modulo π_v.*

Let τ_0^{alg} be an algebraic representation of $\prod_{v | p} \mathrm{GL}_2(\mathbb{O}_{F_v})$ on a finite free $\mathbb{O}$-module $W_{\tau_0^{\mathrm{alg}}}$, and let $W_{\tau_0} = W_{\tau_0^{\mathrm{alg}}} \otimes_\mathbb{O} W_{\tau_0^{\mathrm{sm}}}$ where $W_{\tau_0^{\mathrm{sm}}} = \otimes_{v | p} W_{\tau_{0,v}^{\mathrm{sm}}}$. We suppose that for $v | p$, the restriction of τ_0 to $U_v \cap \mathbb{O}_{F_v}^\times$ is given by ψ^{-1}. Suppose that $f \in S_{\tau_0, \psi}(U, \mathbb{O})$ is an eigenfunction for the action of $\mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}$, whose image in $S_{\tau_0, \psi}(U, \mathbb{O}) \otimes_\mathbb{O} \mathbb{F}$ is non-zero, and let $\mathfrak{m} \subset \mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}$ be the associated maximal ideal.

Then after increasing E, there exists a τ as in (3.1.1), such that $\tau = \tau_0^{\mathrm{alg}} \otimes \tau^{\mathrm{sm}}$, with $W_{\tau^{\mathrm{sm}}} = \otimes_{v|p} W_{\tau_v^{\mathrm{sm}}}$ and each $W_{\tau_v^{\mathrm{sm}}}$ a cuspidal K-type, and a non-zero $\mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}$-eigenfunction $g \in S_{\tau, \psi}(U, \mathbb{O})_\mathfrak{m}$.

Proof. For $v | p$ let $W_{\tau_v^{\mathrm{sm}}} = W$ be the representation constructed in (3.1.5) corresponding to the case $i = 0$ if $W_{\tau_{0,v}^{\mathrm{sm}}}$ is trivial and $i = 1$ otherwise. Set

$$W_{\tau^{\mathrm{sm}}} = \otimes_{v | p} W_{\tau_v^{\mathrm{sm}}} \quad \text{and} \quad W_\tau = W_{\tau_0^{\mathrm{alg}}} \otimes_\mathbb{O} W_{\tau^{\mathrm{sm}}}.$$

Note that, since $\mathbb{O}_{F_v}^\times$ acts trivially on $W_{\tau_{0,v}^{\mathrm{sm}}}$ for $v | p$, the condition (3.1.5)(3) implies that $\mathbb{O}_{F_v}^\times$ acts via ψ^{-1} on W_τ.

Since $\mathfrak{m}$ is in the support of $S_{\tau_0, \psi}(U, \mathbb{O})$ it is in the support of $S_{\bar{\tau}_0, \bar{\psi}}(U, \mathbb{F})$, where $\bar{\tau}_0$ and $\bar{\psi}$ denote the reductions of τ_0 and ψ modulo the radical of $\mathbb{O}$ (cf. the proof of (3.1.4)). Hence by (3.1.4) and (3.1.5)(1), $\mathfrak{m}$ is in the support of $S_{\tau, \psi}(U, \mathbb{O})$. Since $S_{\tau, \psi}(U, \mathbb{O})_\mathfrak{m}$ is a non-zero finitely generated $\mathbb{O}$-module, after enlarging E if necessary, we may assume it contains a non-zero $\mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}$-eigenfunction g. By (3.1.5)(2) the automorphic representation generated by g is cuspidal at all places $v | p$. More precisely, for each v the corresponding local factor π_v is a twist of the representation α of (3.1.5) by an unramified character. $\qquad\square$

(3.1.7) Let $\lambda \nmid p$ be a finite place of F, such that $U_\lambda = \mathrm{GL}_2(\mathbb{O}_{F_\lambda})$. Define a compact open subgroup U' of $\prod_v (\mathbb{O}_D)_v^\times$ by $U'_v = U_v$ if $v \neq \lambda$, and

$$U'_\lambda = \{g \in \mathrm{GL}_2(\mathbb{O}_{F_\lambda}) : g = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)(\pi_\lambda)\}.$$

As in [DT, §2], we define a map

$$i_\lambda : S_{\tau,\psi}(U, A)^2 \to S_{\tau,\psi}(U', A); \quad (f_1, f_2) \mapsto f_1 + \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\lambda \end{smallmatrix}\right) f_2.$$

We will assume from now on that $\lambda \in S$; however, note that since $U_\lambda = \mathrm{GL}_2(\mathbb{O}_{F_\lambda})$, there is a well defined operator T_λ on $S_{\tau,\psi}(U, A)$.

LEMMA (3.1.8). *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}$, and suppose that $A = \mathbb{F}$. Then the map i_λ above is injective after localizing at $\mathfrak{m}$.*

Proof. By dévissage it suffices to consider the case when W_τ is irreducible. The irreducible case follows from the argument given in the second part of the proof of [Tay1, 3.1]. □

(3.1.9) Let $\bar{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$, and $E \subset \bar{\mathbb{Q}}_p$ a finite extension of $\mathbb{Q}_p$ with integer ring $\mathbb{O}$. We will assume that E contains the image of every embedding $\sigma : F \hookrightarrow \bar{\mathbb{Q}}_p$. Let $k \geq 2$ be an integer. The same argument as in [CDT, 3.1.1] shows that, after possibly replacing E by a larger field, there exists an $\mathbb{O}$-lattice $L_k \subset \mathrm{Sym}^{k-2} E^2$ which is stable under the natural action of $M_2(\mathbb{O})$ on $\mathrm{Sym}^{k-2} E^2$, and is equipped with a perfect pairing $\langle \cdot, \cdot \rangle$ such that for any $g \in M_2(\mathbb{O})$ and $x, y \in L_k$

$$\langle gx, gy \rangle = (\det g)^{k-2} \langle x, y \rangle.$$

Now let $(k_\sigma, w_\sigma)_\sigma$ be a collection of pairs of integers, where σ runs over embeddings $F \hookrightarrow E$, $k_\sigma \geq 2$, and $w = k_\sigma + 2w_\sigma - 1$ is independent of σ.

Suppose that τ has the form

$$W_\tau = \bigotimes_{\sigma : F \hookrightarrow E} L_{k_\sigma} \otimes \det{}^{w_\sigma}.$$

For each $v | p$, W_{τ_v} corresponds to the product of the factors for which σ induces a continuous embedding $F_v \to E$, and $M_2(\mathbb{O}_{F_v})$ acts on the factor corresponding to σ via the induced map $M_2(\mathbb{O}_{F_v}) \to M_2(\mathbb{O})$. The perfect pairings on each of the L_{k_σ} induce a perfect pairing $\langle \cdot, \cdot \rangle$ on W_τ.

Now fix a character ψ as in (3.1.1). Such a character exists provided that $N_{F/\mathbb{Q}}(U \cap \mathbb{O}_F^\times)^{w+1} = 1$, which we assume from now on, this condition being automatic if w is odd. Then for $g \in U(\mathbb{A}_F^f)^\times$,

$$\langle gf_1, gf_2 \rangle = \psi^{-1}(\det g)\langle f_1, f_2 \rangle.$$

In particular, for any $t \in (D \otimes_F \mathbb{A}_F^f)^\times$, the pairing $\langle \cdot, \cdot \rangle$ is invariant under the action of $U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t / F^\times$, since the determinant of an element in this group lies in

$F^\times$. Hence $\langle \cdot, \cdot \rangle$ restricts to a perfect pairing on $W_\tau^{U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t / F^\times}$. We define a perfect pairing on $S_{\tau,\psi}(U, \mathbb{O})$ by

$$\langle h, g \rangle_U = \sum_{[t] \in D^\times \backslash (D \otimes \mathbb{A}_F^f)^\times / U(\mathbb{A}_F^f)^\times} \langle h(t), g(t) \rangle \psi^{-1}(\det t) \big| U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t / F^\times \big|^{-1}.$$

Similarly, if λ is as in (3.1.7), we can define a perfect pairing $\langle \cdot, \cdot \rangle_{U'}$ on $S_{\tau,\psi}(U', \mathbb{O})$.

For each $v | p$, there is a natural action of the semigroup $\mathrm{GL}_2(F_v) \cap M_2(\mathbb{O}_{F_v})$ on W_τ-valued functions on $(D \otimes_F \mathbb{A}_F^f)^\times$, which is *smooth* on functions in $S_{\tau,\psi}(U', \mathbb{O})$ for any compact open subgroup U' of $(D \otimes_F \mathbb{A}_F^f)^\times$. It is given by $u^{\mathrm{sm}}(f)(g) = \tilde\tau(u) f(gu)$, where $\tilde\tau$ denotes the natural representation of $\prod_{v|p} \mathrm{GL}_2(F_v)$ on $W_\tau \otimes_{\mathbb{O}} E$. Under this action $S_{\tau,\psi}(U, \mathbb{O})$ is invariant under U_v, and hence for $g \in \mathrm{GL}_2(F_v) \cap M_2(\mathbb{O}_{F_v})$ there is a well defined Hecke operator $U_v g U_v$ on $S_{\tau,\psi}(U, \mathbb{O})$. We extend the action of $\mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}$ on $S_{\tau,\psi}(U, A)$ to $\mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}} = \mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}[T_v, S_v]_{v|p}$, by requiring that T_v and S_v act by the Hecke operators corresponding to $\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix}$ respectively. As before T_v and S_v are formal variables.

We call a maximal ideal of $\mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}}$ Eisenstein if it induces an Eisenstein maximal ideal of $\mathbb{T}_{S,\mathbb{O}}^{\mathrm{univ}}$. As in (3.1.3), if $f \in S_{\tau,\psi}(U, \mathbb{O})$ is a $\mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}}$-eigenfunction, whose image $\bar{f}$ in $S_{\tau,\psi}(U, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{F}$ is non-zero, we call the kernel of the action of $\mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}}$ on $\bar{f}$ the maximal ideal associated to f.

LEMMA (3.1.10). *Let $i_\lambda^\dagger$ denote the adjoint of the map i_λ of (3.1.7) with respect to the pairings $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_{U'}$. The maps i_λ and $i_\lambda^\dagger$ are compatible with the action of $\mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}}$, and then*

$$i_\lambda^\dagger \circ i_\lambda = \begin{pmatrix} \mathbf{N}(\lambda) + 1 & T_\lambda \\ T_\lambda & \psi(\pi_\lambda)(\mathbf{N}(\lambda) + 1) \end{pmatrix} : \quad S_{\tau,\psi}(U, \mathbb{O})^2 \to S_{\tau,\psi}(U, \mathbb{O})^2.$$

Proof. The compatibility of i_λ with the action of $\mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}}$ is clear. A calculation shows T_v is self-adjoint for the above pairing, and S_v is obviously self-adjoint, since it acts via the scalar $\psi(\pi_v)$. It follows that $i_\lambda^\dagger$ is also compatible with the action of $\mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}}$.

Finally the formula for $i_\lambda^\dagger \circ i_\lambda$ is proved just as in [Tay2, Lemma 2]. Note that, by definition, S_λ acts on $S_{\tau,\psi}(U, \mathbb{O})$ by $\psi(\pi_\lambda)$. $\square$

COROLLARY (3.1.11). *Let $\mathfrak{m} \subset \mathbb{T}_{S^p,\mathbb{O}}^{\mathrm{univ}}$ be a non-Eisenstein maximal ideal. Suppose that $S_{\tau,\psi}(U, \mathbb{O})_{\mathfrak{m}} \neq \{0\}$ and that*

$$(T_\lambda^2 - (\mathbf{N}(\lambda) + 1)^2 \psi(\pi_\lambda))(S_{\tau,\psi}(U, \mathbb{O})) \subset \mathfrak{m} S_{\tau,\psi}(U, \mathbb{O}).$$

Then the quotient $S_{\tau,\psi}(U', \mathbb{O})_{\mathfrak{m}} / i_\lambda(S_{\tau,\psi}(U, \mathbb{O})_{\mathfrak{m}}^2)$ is a non-zero free $\mathbb{O}$-module.

Proof. That the quotient is torsion-free follows from (3.1.8). If it were zero then the localization of i_λ at $\mathfrak{m}$ would be an isomorphism, and hence so would those of $i_\lambda^\dagger$ and $i_\lambda^\dagger \circ i_\lambda$. This contradicts the fact that the determinant of the matrix in (3.1.10), giving $i_\lambda^\dagger \circ i_\lambda$, annihilates $S_{\tau,\psi}(U, \mathbb{O})/\mathfrak{m}S_{\tau,\psi}(U, \mathbb{O})$

To see this, denote this matrix by A, and let $\mathbb{O}[T_\lambda]$ denote the $\mathbb{O}$-algebra of endomorphisms of $S_{\tau,\psi}(U, \mathbb{O})_\mathfrak{m}$ generated by T_λ. Note that the localization of $i_\lambda^\dagger \circ i_\lambda$ is obtained by applying $\otimes_{\mathbb{O}[T_\lambda]} S_{\tau,\psi}(\mathbb{O}, U)_\mathfrak{m}$ to the endomorphism of $\mathbb{O}[T_\lambda]^2$ given by A. Since $S_{\tau,\psi}(U, \mathbb{O})_\mathfrak{m}$ is a finite faithful $\mathbb{O}[T_\lambda]$-module, if the localization of $i_\lambda^\dagger \circ i_\lambda$ is invertible then Nakayama's lemma implies that $A \in M_2(\mathbb{O}[T_\lambda])$ is invertible. But then $\det A$ induces an automorphism of $S_{\tau,\psi}(U, \mathbb{O})_\mathfrak{m}$, which contradicts (3.1.10). $\square$

(3.1.12) Keeping the above assumptions (so in particular W_τ is as above), let Q be a finite set of finite primes of F, such that for $v \in Q$, D is unramified at v and $v \nmid p$. Suppose that for each $v \in Q$,

$$U_v = \{g \in \mathrm{GL}_2(\mathbb{O}_{F_v}) : g = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)(\pi_v)\}.$$

For each $v \in Q$ fix a quotient Δ_v of $(\mathbb{O}_{F_v}/\pi_v\mathbb{O}_{F_v})^\times$ of p-power order, and write $\Delta = \prod_{v \in Q} \Delta_v$. Define a compact open subgroup $U_\Delta = \prod_v (U_\Delta)_v \subset U$ by setting $(U_\Delta)_v = U_v$ if $v \notin Q$, and $(U_\Delta)_v$ the set of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U_v$ such that ad^{-1} maps to 1 in Δ_v. The group Δ acts on $S_{\tau,\psi}(U_\Delta, \mathbb{O})$. For $h \in \Delta$ we denote by $\langle h \rangle$ the corresponding endomorphism of $S_{\tau,\psi}(U_\Delta, \mathbb{O})$.

LEMMA (3.1.13). (1) *The operator* $\sum_{h \in \Delta} \langle h \rangle$ *on* $S_{\tau,\psi}(U_\Delta, \mathbb{O})$ *induces an isomorphism*

$$\sum_{h \in \Delta} \langle h \rangle : S_{\tau,\psi}(U_\Delta, \mathbb{O})_\Delta \xrightarrow{\sim} S_{\tau,\psi}(U, \mathbb{O}).$$

(2) $S_{\tau,\psi}(U_\Delta, \mathbb{O})$ *is a free* $\mathbb{O}[\Delta]$-*module.*

Proof. This is proved as in [Tay1, 2.3]. The proof uses (3.1.2). $\square$

(3.1.14) We end this section by explaining the relationship between the spaces $S_{\tau,\psi}(U, \mathbb{O})$ and classical automorphic forms on $D^\times$.

Suppose $A = E$ in (3.1.1) and that W_τ has the form $W_{\tau^{\mathrm{alg}}} \otimes_E W_{\tau^{\mathrm{sm}}}$ where $W_{\tau^{\mathrm{sm}}}$ is a smooth irreducible E-representation of $\prod_{v|p} (\mathbb{O}_D)_v^\times$,

$$W_{\tau^{\mathrm{alg}}} = \bigotimes_{\sigma:F \hookrightarrow E} \mathrm{Sym}^{k_\sigma - 2} E^2 \otimes \det{}^{w_\sigma}$$

is an irreducible algebraic representation of $D_p^\times = (D \otimes_\mathbb{Q} \mathbb{Q}_p)^\times$, and (we assume) E contains the image of all embeddings of F into $\bar{\mathbb{Q}}_p$. The existence of ψ satisfying the conditions of (3.1.1) implies that $w = k_\sigma + 2w_\sigma - 1$ is independent of σ.

Fix an isomorphism $\bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. In particular, this induces an embedding $E \hookrightarrow \mathbb{C}$, and we may view $W_{\tau_\mathbb{C}^{\mathrm{alg}}} := W_{\tau^{\mathrm{alg}}} \otimes_E \mathbb{C}$ as a representation of $D_\infty^\times = (D \otimes_\mathbb{Q} \mathbb{R})^\times$

(and even of $(D \otimes_{\mathbb{Q}} \mathbb{C})^{\times} \xrightarrow{\sim} (D \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p)^{\times})$. Similarly we set $W_{\tau_{\mathbb{C}}^{sm}} = W_{\tau^{sm}} \otimes_E \mathbb{C}$ and regard this as a representation of $\prod_{v|p}(\mathcal{O}_D)_v^{\times}$, so that $W_{\tau_{\mathbb{C}}} = W_{\tau} \otimes_E \mathbb{C}$ is a representation of $\prod_{v|p,\infty}(\mathcal{O}_D)_v^{\times}$.

Choose $U' = \prod_v U'_v$ so that $U'_v = U_v$ if $v \nmid p$, and U'_v acts trivially on $W_{\tau^{sm}}$ if $v|p$, and denote by $C^{\infty}(D^{\times} \backslash D(\mathbb{A}_F)^{\times}/U')$ the space of smooth $\mathbb{C}$-valued functions on $D^{\times} \backslash D(\mathbb{A}_F)^{\times}$ which are invariant by U'. Then the map

$$\mathfrak{A}: S_{\tau,\psi}(U, E) \to \mathrm{Hom}_{D_{\infty}^{\times}}(W_{\tau_{\mathbb{C}}}^*, C^{\infty}(D^{\times} \backslash D(\mathbb{A}_F)^{\times}/U'))$$

which sends $f \in S_{\tau,\psi}(U, E)$ to $w \mapsto (g \mapsto w(\tau_{\mathbb{C}}^{alg}(g_{\infty})^{-1} \tau^{alg}(g_p) f(g^{\infty})))$ identifies $S_{\tau,\psi}(U, E) \otimes_E \mathbb{C}$ with a space of automorphic forms on $D^{\times}$ with central character $\psi_{\mathbb{C}}$ given by $\psi_{\mathbb{C}}(g) = N_{F/\mathbb{Q}}(g_{\infty})^{1-w} N_{F/\mathbb{Q}}(g_p)^{w-1} \psi(g^{\infty})$. Here $W_{\tau_{\mathbb{C}}}^*$ denotes the $\mathbb{C}$-linear dual of $W_{\tau_{\mathbb{C}}}$.

If $\pi = \otimes_v \pi_v$ is an irreducible automorphic representation of $D^{\times}$, then π is generated by a function in $\mathfrak{A}(f)(W_{\tau_{\mathbb{C}}}^*)$ for some $f \in S_{\tau,\psi}(U, E)$ (with U sufficiently small and E sufficiently large) if and only if π_{∞} is isomorphic to $W_{\tau^{alg},\mathbb{C}}^*$ and $\pi_p = \otimes_{v|p} \pi_v$ contains $W_{\tau^{sm},\mathbb{C}}^*$ as a $\prod_{v|p}(\mathcal{O}_D)_v^{\times}$-representation.

(3.2) *Galois cohomology calculations.* In this section we explain how to control the number of generators necessary to present certain deformation rings using Galois cohomology.

(3.2.1) Fix a finite extension $E/\mathbb{Q}_p$, with ring of integers $\mathcal{O}$, uniformiser π_E, and residue field $\mathbb{F}$. Let S be a finite set of primes of F containing the primes dividing p, and the infinite primes. We denote by $G_{F,S}$ the Galois group of the maximal extension of F unramified outside S, and for each $v \in S$, we fix a choice of decomposition group $G_{F_v} \subset G_{F,S}$. We also fix a subset $\Sigma \subset S$ consisting of finite primes not dividing p, and we write $\Sigma_p = \Sigma \cup \{\mathfrak{p}\}_{\mathfrak{p}|p}$. Although in most of this section the primes of Σ and those dividing p will play a rather similar role, in applications this will not be the case, and this is the reason we have distinguished them in the notation.

Let $\psi : (\mathbb{A}_F^f)^{\times}/F^{\times} \to \mathcal{O}^{\times}$ be a character which is unramified outside S, and which we regard as a character of $G_{F,S}$ via the class field theory map

$$G_{F,S}^{ab} \xrightarrow{\sim} \mathbb{A}_F^{\times}/\overline{F^{\times}(F \otimes_{\mathbb{Q}} \mathbb{R})^{+,\times} \prod_{v \notin S} \mathcal{O}_{F_v}^{\times}} \to (\mathbb{A}_F^f)^{\times}/\overline{F^{\times} \prod_{v \notin S} \mathcal{O}_{F_v}^{\times}} \xrightarrow{\psi} \mathcal{O}^{\times}.$$

Here $(F \otimes_{\mathbb{Q}} \mathbb{R})^{+,\times}$ denotes the totally positive elements of $F \otimes_{\mathbb{Q}} \mathbb{R}$ and the first map sends an *arithmetic* Frobenius at a prime $v \notin S$ to a uniformiser π_v at v.

Let $\bar{\rho}: G_{F,S} \to \mathrm{GL}_d(\mathbb{F})$ be a continuous representation on an $\mathbb{F}$-vector space V, equipped with a choice of basis. For each prime $\mathfrak{p}$ of F lying over p, we consider, as in (2.3.4), the functor which to A in $\mathfrak{AR}_{\mathcal{O}}$ assigns the set of framed deformations of $\bar{\rho}|_{G_{F_{\mathfrak{p}}}}$ to A, and we denote by $R_{\mathfrak{p}}^{\square}$ the $\mathcal{O}$-algebra which represents this functor.

Denote by $R_{\mathfrak{p}}^{\square,\psi}$ the quotient of $R_{\mathfrak{p}}^{\square}$ corresponding to deformations with determinant $\psi\chi$, where χ denotes the p-adic cyclotomic character, and set $R_p^{\square} = \widehat{\otimes}_{\mathfrak{p}|p} R_{\mathfrak{p}}^{\square}$ and $R_p^{\square,\psi} = \widehat{\otimes}_{\mathfrak{p}|p} R_{\mathfrak{p}}^{\square,\psi}$.

Similarly, if $v \in \Sigma$ we have the universal framed deformation $\mathbb{O}$-algebras $R_v^{\square}$ and $R_v^{\square,\psi}$ of $\bar{\rho}|_{G_{F_v}}$. We set $R_{\Sigma}^{\square} = \widehat{\otimes}_{v\in\Sigma} R_v^{\square}$ (resp. $R_{\Sigma}^{\square,\psi} = \widehat{\otimes}_{v\in\Sigma} R_v^{\square,\psi}$), and $R_{\Sigma,p}^{\square} = R_{\Sigma}^{\square}\widehat{\otimes}_{\mathbb{O}} R_p^{\square}$ (resp. $R_{\Sigma,p}^{\square,\psi} = R_{\Sigma}^{\square,\psi}\widehat{\otimes}_{\mathbb{O}} R_p^{\square,\psi}$).

We denote by $R_{F,S}^{\square}$ the $\mathbb{O}$-algebra representing the functor which to A in $\mathfrak{AR}_{\mathbb{O}}$ assigns the set of (isomorphism classes of) deformations of V to a finite free A-module V_A together with a collection $\{\beta_v\}_{v\in\Sigma_p}$ where each β_v is an ordered A-basis of V_A lifting the chosen basis of V. If $\mathrm{End}_{\mathbb{F}[G_{F,S}]}V = \mathbb{F}$, we denote by $R_{F,S}$ the universal deformation $\mathbb{O}$-algebra of V. Finally we denote by $R_{F,S}^{\square,\psi}$ (resp. $R_{F,S}^{\psi}$) the quotient of $R_{F,S}^{\square}$ (resp. $R_{F,S}$) corresponding to deformations with determinant $\psi\chi$.

LEMMA (3.2.2). *Suppose that $p \nmid d$, denote by $h_{\Sigma_p'}^1(G_{F,S}, \mathrm{ad}^0 V)$ the $\mathbb{F}$-dimension of*

$$H_{\Sigma_p'}^1(G_{F,S}, \mathrm{ad}^0 V) := \ker(\theta^1 : H^1(G_{F,S}, \mathrm{ad}^0 V) \to \prod_{v\in\Sigma_p} H^1(G_{F_v}, \mathrm{ad}^0 V)),$$

and for $v \in \Sigma_p$ let $\delta_v = \dim_{\mathbb{F}} H^0(G_{F_v}, \mathrm{ad}V)$ and $\delta_F = \dim_{\mathbb{F}} H^0(G_{F,S}, \mathrm{ad}V)$.

Then $R_{F,S}^{\square,\psi}$ is a quotient of a power series ring over $R_{\Sigma,p}^{\square,\psi}$ in

$$g = h_{\Sigma_p'}^1(G_{F,S}, \mathrm{ad}^0 V) + \sum_{v\in\Sigma_p} \delta_v - \delta_F \text{ variables.}$$

Proof. Denote the maximal ideal of $R_{\Sigma,p}^{\square,\psi}$ by $\mathfrak{m}_{\Sigma,p}$. An element in the dual tangent space of $R_{F,S}^{\square,\psi}/\mathfrak{m}_{\Sigma,p}$ corresponds to a deformation of V to a finite free $\mathbb{F}[\epsilon]$-module $V_{\mathbb{F}[\epsilon]}$ ($\epsilon^2 = 0$), together with a collection of bases $\{\beta_v\}_{v\in\Sigma_p}$ lifting the chosen basis of V, such that for each $v \in \Sigma_p$, the pair $(V_{\mathbb{F}[\epsilon]}|_{G_{F_v}}, \beta_v)$ is isomorphic to $V \otimes_{\mathbb{F}} \mathbb{F}[\epsilon]$ equipped with the basis induced by the chosen basis of V. The space of such deformations $V_{\mathbb{F}[\epsilon]}$ is given by

$$H_{\Sigma_p'}^1(G_{F,S}, \mathrm{ad}^0 V).$$

(Here we use the fact that $p \nmid d$, which implies that $\mathrm{ad}^0 V$ is a direct summand of $\mathrm{ad}V$ as a $G_{F,S}$-representation.) Given such a deformation $V_{\mathbb{F}[\epsilon]}$ the space of possible choices for the basis β_v is given by $H^0(G_{F_v}, \mathrm{ad}V)$. Finally, two such sets of choices $\{\beta_v\}_{v\in\Sigma_p}$ and $\{\beta_v'\}_{v\in\Sigma_p}$ are equivalent if there is an automorphism of $V_{\mathbb{F}[\epsilon]}$ respecting the action of $G_{F,S}$, reducing to the identity on V, and taking β_v to β_v'. The lemma follows. $\square$

(3.2.3) We now specialize to the case where $\dim_{\mathbb{F}} V = 2$, and we will make the following assumptions.

(1) $\bar{\rho}$ is unramified outside the primes of F dividing p, and has odd determinant.

(2) The restriction of $\bar{\rho}$ to $G_{F(\zeta_p)}$ is absolutely irreducible.

(3) If $p = 5$, and $\bar{\rho}$ has projective image isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, then the kernel of proj $\bar{\rho}$ does not fix $F(\zeta_5)$.[3] This condition holds if $[F(\zeta_5) : F]=4$.

(4) If $v \in S \setminus \Sigma_p$, then

$$(3.2.4) \qquad (1 - \mathbf{N}(v))((1 + \mathbf{N}(v))^2 \det \bar{\rho}(\mathrm{Frob}_v) - (\mathbf{N}(v))(\mathrm{tr}\bar{\rho}(\mathrm{Frob}_v))^2) \in \mathbb{F}^{\times}.$$

Here, Frob_v denotes an arithmetic Frobenius at v.

Note that (2) is equivalent to asking that $\bar{\rho}$ remain absolutely irreducible when restricted to the unique quadratic extension of F contained in $F(\zeta_p)$. When F is linearly disjoint from $\mathbb{Q}(\zeta_p)$ - for example if p is unramified in F - then this extension is equal to $F(\sqrt{(-1)^{(p-1)/2}p})$ and (2) agrees with the familiar condition found in [Wi, Thm 3.1] and [TW].

PROPOSITION (3.2.5). *Set* $g = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar{\rho}(1)) - [F : \mathbb{Q}] + |\Sigma_p| - 1$. *For each positive integer* n, *there exists a finite set of primes* Q_n *of* F, *which is disjoint from* S, *and such that*

(1) *If* $v \in Q_n$, *then* $\mathbf{N}(v) = 1(p^n)$ *and* $\bar{\rho}(\mathrm{Frob}_v)$ *has distinct eigenvalues.*

(2) $|Q_n| = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar{\rho}(1))$. *If* $S_{Q_n} = S \cup Q_n$, *then as an* $R_{\Sigma,p}^{\square,\psi}$-*algebra* $R_{F,S_{Q_n}}^{\square,\psi}$ *is topologically generated by* g *elements.*

Proof. Given a set of primes Q_n of F, let $S_{Q_n} = S \cup Q_n$. For $v \in S_{Q_n}$, given a subspace $\mathbb{L}_v \subset H^1(G_{F_v}, \mathrm{ad}^0\bar{\rho})$, denote by $\mathbb{L}_v^* \subset H^1(G_{F_v}, \mathrm{ad}^0\bar{\rho}(1))$ the annihilator of $\mathbb{L}_v$ under Tate local duality. Fixing an $\mathbb{L}_v$ for each $v \in S_{Q_n}$, we write $\mathbb{L}_{Q_n}$ (resp. $\mathbb{L}_{Q_n}^*$) for the subspace of $H^1(G_{F,S_{Q_n}}, \mathrm{ad}^0\bar{\rho})$ (resp. $H^1(G_{F,S_{Q_n}}, \mathrm{ad}^0\bar{\rho}(1))$) consisting of classes which map to $\mathbb{L}_v$ (resp. $\mathbb{L}_v^*$) for each $v \in S_{Q_n}$. We will apply this with $\mathbb{L}_v = \{0\}$ if $v \in \Sigma_p$, and $\mathbb{L}_v = H^1(G_{F_v}, \mathrm{ad}^0\bar{\rho})$ otherwise.

Now an argument as in [DDT, 2.49] shows that we may choose a set of finite primes Q_n of F satisfying (1) and the first condition of (2), and such that $\mathbb{L}_{Q_n}^* = \{0\}$. This uses the conditions (2) and (3) introduced in (3.2.3) above, the latter condition being needed only to apply [DDT, 2.47].

It remains to show that the final condition of (2) is satisfied. By [DDT, 2.19] we have

$$(3.2.6) \qquad |\mathbb{L}_{Q_n}| = \frac{|H^0(G_{F,S}, \mathrm{ad}^0\bar{\rho})|}{|H^0(G_{F,S}, \mathrm{ad}^0\bar{\rho}(1))|} \prod_{v \in S_{Q_n}} \frac{|\mathbb{L}_v|}{|H^0(G_{F_v}, \mathrm{ad}^0\bar{\rho})|}.$$

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[3] We thank the referee for pointing out that an assumption on $F(\zeta_p)$ is needed only in this special case.

Now (3.2.3)(2) implies that the global terms on the right-hand side of (3.2.6) are equal to 1, while (3.2.4) implies that the terms corresponding to the finite places of $S \setminus \Sigma_p$ are 1. Our assumption that $\mathbb{L}_v = \{0\}$ if $v \in \Sigma_p$ implies the terms corresponding to such v are $|\mathbb{F}|^{1-\delta_v}$ where we use the notation of (3.2.2).

Finally we compute the contribution of the terms coming from infinite primes and primes in Q_n. Keeping in mind that $\bar{\rho}$ has odd determinant at every infinite prime, and that the primes in Q_n satisfy (1) of the proposition, we find $|\mathbb{L}_{Q_n}| = |\mathbb{F}|^{|Q_n|-[F:\mathbb{Q}]+|\Sigma_p|-\sum_{v \in \Sigma_p} \delta_v}$, so that

$$(3.2.7) \quad h^1_{\Sigma'_p}(G_{F,S_{Q_n}}, \mathrm{ad}^0\bar{\rho})$$
$$= \dim_{\mathbb{F}} \mathbb{L}_{Q_n} = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar{\rho}(1)) - [F:\mathbb{Q}] + |\Sigma_p| - \sum_{v \in \Sigma_p} \delta_v.$$

The proposition now follows from (3.2.2). □

(3.2.8) For each positive integer n fix a set of primes Q_n satisfying the conditions of (3.2.5). Let $\mathfrak{n}_{Q_n} \subset \mathbb{O}_F$ denote the product of the primes in Q_n, write Δ_{Q_n} for the maximal p-quotient of $(\mathbb{O}_F/\mathfrak{n}_{Q_n}\mathbb{O}_F)^\times$, and $\mathfrak{a}_{Q_n}$ for the augmentation ideal of $\mathbb{O}[\Delta_{Q_n}]$. After replacing $\mathbb{F}$ by a quadratic extension, we may and do assume that elements in the image of $\bar{\rho}$ have their eigenvalues contained in $\mathbb{F}$.

As in [Tay1, 2.1], we make $R^\psi_{F,S_{Q_n}}$ (and hence also $R^{\square,\psi}_{F,S_{Q_n}}$) into a $\mathbb{O}[\Delta_{Q_n}]$-algebra as follows: If $v \in Q_n$, let $\bar{\alpha}_v, \bar{\beta}_v \in \mathbb{F}$ denote the distinct eigenvalues of $\bar{\rho}(\mathrm{Frob}_v)$. The restriction of the universal representation

$$\rho_{R^\psi_{G_{F,S_{Q_n}}}} : G_{F,S_{Q_n}} \to \mathrm{GL}_2(R^\psi_{F,S_{Q_n}})$$

to a decomposition group G_{F_v} at v is (equivalent to) a direct sum of two characters

$$\chi_{\bar{\alpha}_v} \oplus \chi_{\bar{\beta}_v} : G_{F_v} \to R^{\psi\times}_{F,S_{Q_n}} \oplus R^{\psi\times}_{F,S_{Q_n}},$$

where $\chi_{\bar{\alpha}_v}$ composed with the projection $R^{\psi\times}_{F,S_{Q_n}} \to \mathbb{F}^\times$ is an unramified character which takes Frob_v to $\bar{\alpha}_v$. By local class field theory (normalized to take uniformisers to the arithmetic Frobenii) $\chi_{\bar{\alpha}_v}$ induces a map

$$(3.2.9) \qquad \prod_{v \in Q_n} \chi_{\bar{\alpha}_v} : \prod_{v \in Q_n} \mathbb{O}^\times_{F_v} \to R^{\psi\times}_{F,S_{Q_n}}.$$

This map factors though Δ_{Q_n}, and induces an $\mathbb{O}[\Delta_{Q_n}]$-algebra structure on $R^\psi_{F,S_{Q_n}}$.

For $v \in Q_n$, denote by $\xi_v \in \Delta_{Q_n}$ the image of some fixed generator of $(\mathbb{O}_{F_v}/\pi_v)^\times$. Let $h^1 = |Q_n| = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar{\rho}(1))$, and order the elements of Q_n as $v_1, \ldots v_{h^1}$. Then we can write $\mathbb{O}[\Delta_{Q_n}]$ as a quotient of $\mathbb{O}[\![y_1, \ldots, y_{h^1}]\!]$ by sending y_i to $\xi_{v_i} - 1$.

(3.3) *The patching criterion.* In this section we explain a criterion for a map of $\mathbb{O}$-algebras to be an isomorphism up to p-torsion. This will be applied to establish that certain Galois deformation rings and Hecke rings are isomorphic up to p-torsion. The argument is analogous to that of Taylor-Wiles [TW], as modified by Diamond [Di2] and Fujiwara [Fu], however one of the differences with the approach of *loc. cit.* is that the same criterion will be used to treat the minimal and non-minimal cases.

PROPOSITION (3.3.1). *Let B be a complete local, flat $\mathbb{O}$-algebra, which is a domain of dimension $d + 1$, and such that $B[1/p]$ is formally smooth over E. Suppose that R is a B-algebra and M is a non-zero R-module, and that there are non-negative integers h and j such that for each non-negative integer n, there are maps of $\mathbb{O}$-algebras*

$$(3.3.2) \qquad \mathbb{O}[\![y_1, \ldots, y_{h+j}]\!] \to R_n \to R$$

and a map of R_n-modules $M_n \to M$ satisfying the following conditions:

(1) *The maps $R_n \to R$ and $M_n \to M$ are surjective, and the first is a map of B-algebras.*

(2) $(y_1, \ldots, y_h)R_n = \ker(R_n \to R)$ *and* $(y_1, \ldots, y_h)M_n = \ker(M_n \to M)$.

(3) *If $\mathfrak{b}_n \subset \mathbb{O}[\![y_1, \ldots, y_{h+j}]\!]$, is the annihilator of M_n, then*

$$\mathfrak{b}_n \subset ((1 + y_1)^{p^n} - 1, \ldots, (1 + y_h)^{p^n} - 1),$$

and M_n is finite free over $\mathbb{O}[\![y_1, \ldots, y_{h+j}]\!]/\mathfrak{b}_n$. (So, in particular, M is finite free over $\mathbb{O}[\![y_{h+1}, \ldots, y_{h+j}]\!]$.)

(4) *R_n is a quotient of $B[\![x_1, \ldots, x_{h+j-d}]\!]$.*

Then R is a finite $\mathbb{O}[\![y_{h+1}, \ldots, y_{h+j}]\!]$-algebra, and $M \otimes_{\mathbb{O}} E$ is a finite projective and faithful $R[1/p]$-module.

Proof. For a complete local ring A, we will denote by $\mathfrak{m}_A$ its maximal ideal. For a non-negative integer n, we denote by $\mathfrak{m}_A^{(n)} \subset \mathfrak{m}_A$ the ideal generated by the elements of $\mathfrak{m}_A$ which are n-th powers.

Let s denote the $\mathbb{O}[\![y_{h+1}, \ldots, y_{h+j}]\!]$-rank of M. This is equal to the rank of M_n as an $\mathbb{O}[\![y_1, \ldots, y_{h+j}]\!]/\mathfrak{b}_n$-module. For a non-negative integer m, write $r_m = smp^m(h + j)$, and

$$\mathfrak{c}_m = (\pi_E^m, (y_1 + 1)^{p^m} - 1, \ldots, (y_h + 1)^{p^m} - 1, y_{h+1}^{p^m}, \ldots, y_{h+j}^{p^m}) \subset \mathbb{O}[\![y_1, \ldots, y_{h+j}]\!].$$

For $m \geq 1$ a patching datum (D, L) of level m consists of

(1) A sequence of maps of complete local $\mathbb{O}$-algebras

$$(3.3.3) \qquad \mathbb{O}[\![y_1, \ldots, y_{h+j}]\!]/\mathfrak{c}_m \to D \to R/(\mathfrak{c}_m R + \mathfrak{m}_R^{(r_m)})$$

where the second map is a surjective map of B-algebras and $\mathfrak{m}_D^{(r_m)} = 0$.

(2) A surjection of B-algebras $B[\![x_1, \ldots, x_{h+j-d}]\!] \to D$.

(3) A D-module L which is finite free over $\mathbb{O}[\![y_1, \ldots, y_{h+j}]\!]/\mathfrak{c}_m$ of rank s, and a surjection of $B[\![x_1, \ldots, x_{h+j-d}]\!]$-modules $L \to M/\mathfrak{c}_m M$.

A morphism of patching data $(D_1, L_1) \to (D_2, L_2)$ is a pair of morphisms $D_1 \to D_2$ and $L_1 \to L_2$, where $D_1 \to D_2$ is a map of $B[\![x_1, \ldots, x_{h+j-d}]\!]$-algebras which is compatible with the morphisms in (3.3.3), and $L_1 \to L_2$ is a surjection of D_1-modules which is compatible with the surjections of L_1 and L_2 onto $M/\mathfrak{c}_m M$. Since the number of elements of D is bounded by

$$|B[\![x_1, \ldots, x_{h+j-d}]\!]/\mathfrak{m}_{B[\![x_1, \ldots, x_{h+j-d}]\!]}^{(r_m)}|,$$

there are only finitely many isomorphism classes of patching data of level m.

Given positive integers, $n \geq m$, we define a patching datum $(D_{m,n}, L_{m,n})$ of level m, by taking $D_{m,n} = R_n/\mathfrak{c}_m R_n + \mathfrak{m}_{R_n}^{(r_m)}$ and $L_{m,n} = M_n/\mathfrak{c}_m M_n$. To check that $L_{m,n}$ is a $D_{m,n}$ module we have to show that $\mathfrak{m}_{R_n}^{(r_m)} M_n \subset \mathfrak{c}_m M_n$. To see this, let $a \in \mathfrak{m}_{R_n}$. Then a induces a nilpotent $\mathbb{F}$-linear endomorphism of $M/(\pi_E, y_{h+1}, \ldots, y_{h+j})M$, so that a^s induces the zero endomorphism. Hence $a^s M_n \in (\pi_E, y_1, \ldots, y_{h+j})M_n$, and

$$a^{sp^m(h+j)} M_n \subset (\pi_E, y_1^{p^m}, \ldots, y_{h+j}^{p^m})M_n$$
$$= (\pi_E, (y_1+1)^{p^m} - 1, \ldots, (y_h+1)^{p^m} - 1, y_{h+1}^{p^m}, \ldots, y_{h+j}^{p^m})M_n.$$

Finally $a^{r_m} M_n = a^{sp^m(h+j)m} M_n \subset \mathfrak{c}_m M_n$.

Since there are only finitely many isomorphism classes of patching data of level m, after replacing the sequence (R_n, M_n) by a subsequence, we may assume that for each $m \geq 1$, and $n \geq m$, the datum $(D_{m,n}, L_{m,n})$ is equal to $(D_{m,m}, L_{m,m})$. Denoting this common value by (D_m, L_m), we have in particular an isomorphism of patching data

$$(D_{m+1}/(\mathfrak{c}_m D_{m+1} + \mathfrak{m}_{D_{m+1}}^{(r_m)}), L_{m+1}/\mathfrak{c}_m L_{m+1}) \xrightarrow{\sim} (D_m, L_m).$$

Now set $R_\infty = \varprojlim D_{m,m}$ and $M_\infty = \varprojlim L_{m,m}$. By construction we have a surjection

$$B[\![x_1, \ldots, x_{h+j-d}]\!] \to R_\infty,$$

and maps of complete local $\mathbb{O}$-algebras

$$\mathbb{O}[\![y_1, \ldots, y_{h+j}]\!] \to R_\infty \to R$$

where the second map is a map of B-algebras, and identifies

$$R_\infty/(y_1, \ldots, y_h)R_\infty$$

with R. We also have that M_∞ is a finite free $\mathbb{O}[[y_1, \dots, y_{h+j}]]$-module of rank $s > 0$, and that the R-module $M_\infty/(y_1, \dots, y_h)M_\infty$ is isomorphic to M. Note that

$$\dim \mathbb{O}[[y_1, \dots, y_{h+j}]] = h + j + 1 = d + 1 + h + j - d = \dim B[[x_1, \dots, x_{h+j-d}]].$$

By (3.3.4) below, $B[[x_1, \dots, x_{h+j-d}]]$ is a finite $\mathbb{O}[[y_1, \dots, y_{h+j}]]$-module, and $M_\infty \otimes_\mathbb{O} E$ is a finite projective, faithful $B[[x_1, \dots, x_{h+j-d}]][1/p]$-module. In particular we see that $B[[x_1, \dots, x_{h+j-d}]] \xrightarrow{\sim} R_\infty$, so that

$$M \otimes_\mathbb{O} E = (M_\infty \otimes_\mathbb{O} E)/(y_1, \dots, y_h)(M_\infty \otimes_\mathbb{O} E)$$

is a finite projective, faithful $R[1/p] = R_\infty[1/p]/(y_1, \dots, y_h)R_\infty[1/p]$-module, and R is finite over $\mathbb{O}[[y_{h+1}, \dots, y_{h+j}]]$ □

LEMMA (3.3.4). *Let $A \xrightarrow{\varphi} D$ be a map of Noetherian domains of the same finite dimension d, and L a non-zero D-module which is finite and projective over A. Then φ is a finite map. If A and D are regular then L is a finite projective, faithful D-module.*

Proof. Let D' be the image of D in $\text{End}_A L$. Then D' is finite over A, since L is finite over A. Since L is a faithful A-module, so is D'. It follows that $\dim D' \geq d$, so that $D = D'$.

To show the second statement, we first remark that since A is a domain, L has the same rank $s > 0$ at all points of A. Similarly if L is finite projective over D it is a faithful D-module. Now let $\mathfrak{p}$ be a prime of A, and $\mathfrak{q}$ a prime of D lying over $\mathfrak{p}$, and write $\hat{A}_\mathfrak{p}$ and $\hat{D}_\mathfrak{q}$ for the completions of A and D at $\mathfrak{p}$ and $\mathfrak{q}$ respectively. It suffices to show that $L \otimes_D \hat{D}_\mathfrak{q}$ is a finite free $\hat{D}_\mathfrak{q}$-module. Thus we may replace A, D and L by $\hat{A}_\mathfrak{p}$, $\hat{D}_\mathfrak{q}$ and $L \otimes_D \hat{D}_\mathfrak{q}$ respectively (note that $\hat{D}_\mathfrak{q}$ is finite over $\hat{A}_\mathfrak{p}$, and $L \otimes_D \hat{D}_\mathfrak{q}$ is an $\hat{A}_\mathfrak{p}$-direct summand in $L \otimes_A \hat{A}_\mathfrak{p}$), and we may assume that A and D are complete local regular rings.

Now the A-depth of L is d, since L is A-free, hence the D-depth of L is $\geq d$, and therefore equal to d. The Auslander-Buchsbaum theorem now implies that L is D-free (cf. [Di2, Thm. 2.1]).   □

(3.4) *Deformation rings and Hecke algebras.* We keep the notation of Section (3.1). In particular, D is a totally definite quaternion algebra with center F ramified at a set of finite primes Σ, none of which lie over p, $\mathbb{O}_D$ is a maximal order of D, and $U = \prod_v U_v$ is a compact open subgroup of $\prod_v (\mathbb{O}_D)_v^\times$. We assume that U is maximal at primes in Σ and at primes dividing p, and satisfies the condition (3.1.2).

We will assume from now on that $W_\tau = \mathbb{O}$ is the trivial representation. Although this hypothesis is not needed immediately, it is necessary for most of what we do in this section. We fix a character ψ satisfying the conditions explained in Section (3.1) with respect to this choice of τ. Thus, in our situation, ψ has finite order and

is unramified outside $S \backslash \Sigma$, and at primes dividing p. We will write $S_{2,\psi}(U, \mathbb{O})$ for $S_{\tau,\psi}(U, \mathbb{O})$.

(3.4.1) As in Section (3.2), we write $\Sigma_p = \Sigma \cup \{\mathfrak{p}\}_{\mathfrak{p}|p}$. We denote by $\mathbb{T}'_{\psi,\mathbb{O}}(U)$ (resp. $\mathbb{T}_{\psi,\mathbb{O}}(U)$) the image of $\mathbb{T}^{\mathrm{univ}}_{S,\mathbb{O}}$ (resp. $\mathbb{T}^{\mathrm{univ}}_{S^p,\mathbb{O}}$) in the endomorphisms of $S_{2,\psi}(U, \mathbb{O})$.

Let $\mathfrak{m}'$ be a non-Eisenstein maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S,\mathbb{O}}$ which is induced by a maximal ideal of $\mathbb{T}'_{\psi,\mathbb{O}}(U)$. Then $S_{2,\psi}(U, \mathbb{O})_{\mathfrak{m}'}$ contains no non-zero functions which factor through the reduced norm on $(D \otimes_F \mathbb{A}^f_F)^\times$. Write $\mathbb{F}$ for the residue field of $\mathbb{T}^{\mathrm{univ}}_{S,\mathbb{O}}$ at $\mathfrak{m}'$. By [Tay2], and the Jacquet-Langlands correspondence (see [Tay1, 1.3]), there exists a continuous representation

$$\rho_{\mathfrak{m}'} : G_{F,S} \to \mathrm{GL}_2(\mathbb{T}'_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'})$$

such that for $v \notin S$, the characteristic polynomial of $\rho_{\mathfrak{m}'}(\mathrm{Frob}_v)$ is $X^2 - T_v X + \mathbf{N}(v) S_v$. Here Frob_v denotes an *arithmetic* Frobenius at v. We denote by $\bar{\rho}_{\mathfrak{m}'} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$ the representation obtained by reducing $\rho_{\mathfrak{m}'}$ modulo $\mathfrak{m}'$. Since $\mathfrak{m}'$ is non-Eisenstein, $\bar{\rho}_{\mathfrak{m}'}$ is an absolutely irreducible representation.

As in Section (3.2), for each finite prime v of F, we fix a decomposition group $G_{F_v} \subset G_{F,S}$ at v. We denote by $\bar{V}_{\mathfrak{m}'}$ the underlying $\mathbb{F}$-vector space of $\bar{\rho}_{\mathfrak{m}'}$, and by $L_{\mathfrak{m}',\mathbb{O}}$ the underlying $\mathbb{O}$-module of $\rho_{\mathfrak{m}'}$. We set $V_{\mathfrak{m}',\mathbb{O}} = L_{\mathfrak{m}',\mathbb{O}}[1/p]$, and write $V^*_{\mathfrak{m}',\mathbb{O}}$ for the E-dual of $V_{\mathfrak{m}',\mathbb{O}}$.

LEMMA (3.4.2). *If $\mathfrak{p}$ is a prime of F lying over p, then*

(1) $V_{\mathfrak{m}',\mathbb{O}}|_{G_{F_\mathfrak{p}}}$ *is a Barsotti-Tate representation. If $I_\mathfrak{p} \subset G_{F_\mathfrak{p}}$ denotes the inertia subgroup, then the determinant of $V_{\mathfrak{m}',\mathbb{O}}|_{I_\mathfrak{p}}$ is equal to the cyclotomic character.*

(2) *The Hecke operator $T_\mathfrak{p} \in (\mathbb{T}_{\psi,\mathbb{O}}(U))_{\mathfrak{m}'}$ lies in $\mathbb{T}'_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'}[1/p]$, and*

$$(3.4.3) \qquad T_\mathfrak{p} = \mathrm{tr}_{\mathbb{T}'_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'}[1/p] \otimes_{\mathbb{Z}_p} \mathbb{W}(\kappa(\mathfrak{p}))} (\varphi^{[\kappa(\mathfrak{p}):\mathbb{F}_p]} | D_{\mathrm{cris}}(V^*_{\mathfrak{m}',\mathbb{O}})),$$

where $\kappa(\mathfrak{p})$ denotes the residue field of F at $\mathfrak{p}$, and tr denotes the indicated trace.

Proof. The first statement in (2) follows from the strong multiplicity one theorem, and the fact that $\mathbb{T}'_{\psi,\mathbb{O}}(U)$ is semi-simple, being generated by self-adjoint operators for the pairing in (3.1.9).

Let $R^{\mathrm{fl},\square}_\mathfrak{p}$ denote the quotient of $R^\square_\mathfrak{p}$ corresponding to flat deformations, and write $R^\mathbf{v} = (R^{\mathrm{fl},\square}_\mathfrak{p})^\mathbf{v}$ for the quotient of $R^{\mathrm{fl},\square}_\mathfrak{p}$ defined in (2.4.1), where $\mathbf{v}$ is given by setting all $v_\psi = 1$. Let $t_\mathfrak{p} \in R^\mathbf{v}[1/p]$ denote -1 times the coefficient of T in the polynomial $P^\mathbf{v}_{V_{\mathfrak{m}'}}(T)$ introduced in (2.4.17). Fix a positive integer $r \geq 1$ such that $p^r t_\mathfrak{p} \in R^\mathbf{v}$.

Consider the composite $\theta : R^\square_\mathfrak{p} \to \mathbb{T}'_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'} \to \mathbb{T}_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'}$, where the first map is induced by $\rho_{\mathfrak{m}'}$. It suffices to show that if $\mathbb{O}'$ is the ring of integers in a finite

extension of E, and $\kappa : \mathbb{T}_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'} \to \mathbb{O}'$ is a map of $\mathbb{O}$-algebras, then $\kappa \circ \theta$ factors through $R^{\mathbf{v}}$, and the induced map $R^{\mathbf{v}} \to \mathbb{O}'$ sends $p^r t_{\mathfrak{p}}$ to $\kappa(p^r T_{\mathfrak{p}})$.

If F has odd degree over $\mathbb{Q}$, or the Hilbert modular form corresponding to κ is cuspidal or special at some place v of F, then the Galois representation corresponding to κ is as constructed by Carayol in the Tate module of a Jacobian of a Shimura curve with good reduction at $\mathfrak{p}$ [Ca, §2] (here we use that $k = 2$). The statements in the previous paragraph follow easily from the Eichler-Shimura relation.

Suppose that neither of these conditions holds. If $\lambda \notin \Sigma_p$, is a prime where U is maximal, we define $U_\lambda = \prod_v (U_\lambda)_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ by setting $(U_\lambda)_\lambda$ equal to the subgroup of matrices in $\mathrm{GL}_2(\mathbb{O}_{F_\lambda})$ whose reduction mod λ is upper triangular, and $(U_\lambda)_v = U_v$ if $v \neq \lambda$. We denote by $\mathbb{T}_{\psi,\mathbb{O}}(U_\lambda)$ the $\mathbb{O}$-algebra of endomorphisms of $S_{2,\psi}(U_\lambda, \mathbb{O})$ generated by the image of $\mathbb{T}^{\mathrm{univ}}_{S\cup\{\lambda\},\mathbb{O}}$ and $T_{\mathfrak{p}}$ for $\mathfrak{p}|p$. We set $\mathfrak{m}'_\lambda = \mathfrak{m}' \cap \mathbb{T}^{\mathrm{univ}}_{S\cup\{\lambda\},\mathbb{O}}$.

Taylor [Tay2] constructs the Galois representation corresponding to κ by showing that for each $s \geq 1$ there exists λ, as above, such that the composite

$$\mathbb{T}_{\psi,\mathbb{O}}(U_\lambda)_{\mathfrak{m}'_\lambda} \to \mathbb{T}_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'} \xrightarrow{\kappa} \mathbb{O}' \to \mathbb{O}'/p^s\mathbb{O}'$$

factors through the quotient $\mathbb{T}_{\psi,\mathbb{O}}(U_\lambda)^{\lambda\text{-new}}_{\mathfrak{m}'_\lambda}$ corresponding to forms in $S_{2,\psi}(U_\lambda, \mathbb{O})$ which are new at λ (see Theorem 1, and the final part of the proof of Theorem 2 in [Tay2]).[4] The cases of the lemma covered by Carayol's construction imply that the map $R^{\square}_{\mathfrak{p}} \to \mathbb{T}_{\psi,\mathbb{O}}(U_\lambda)^{\lambda\text{-new}}_{\mathfrak{m}'_\lambda}$ factors through $R^{\mathbf{v}}$ and that the induced map $R^{\mathbf{v}} \to \mathbb{T}_{\psi,\mathbb{O}}(U_\lambda)^{\lambda\text{-new}}_{\mathfrak{m}'_\lambda}$ maps $p^r t_{\mathfrak{p}}$ to $p^r T_{\mathfrak{p}}$. In particular, we see that the composite

$$R^{\square}_{\mathfrak{p}} \to \mathbb{T}_{\psi,\mathbb{O}}(U_\lambda)_{\mathfrak{m}'_\lambda} \to \mathbb{T}_{\psi,\mathbb{O}}(U_\lambda)^{\lambda\text{-new}}_{\mathfrak{m}'_\lambda} \to \mathbb{O}'/p^s\mathbb{O}'$$

factors through $R^{\mathbf{v}}$ and the induced map sends $p^r t_{\mathfrak{p}} \in R^{\mathbf{v}}$ to the reduction of $\kappa \circ \theta(p^r T_{\mathfrak{p}})$ modulo p^s. $\square$

(3.4.4) Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S^p,\mathbb{O}}$ which lies over $\mathfrak{m}'$, and arises from a maximal ideal of $\mathbb{T}_{\psi,\mathbb{O}}(U)$. If σ' is a subset of the set of primes $\mathfrak{p}$ of F dividing p, then we call $\mathfrak{m}$ σ'-ordinary, if $T_{\mathfrak{p}} \in \mathfrak{m}$ for a prime $\mathfrak{p}$ of F dividing p, if and only if $\mathfrak{p} \notin \sigma'$.

For each $\mathfrak{p} \in \sigma'$ let $\chi_{\mathfrak{p}}$ denote an unramified character of $G_{F_{\mathfrak{p}}}$, which gives the action of $G_{F_{\mathfrak{p}}}$ on a one dimensional quotient of $\bar{V}_{\mathfrak{m}'}|_{G_{F_{\mathfrak{p}}}}$. By (3.4.2) $\chi_{\mathfrak{p}}$ exists provided that there is a σ'-ordinary ideal $\mathfrak{m}$ lying over $\mathfrak{m}'$. The choice of $\chi_{\mathfrak{p}}$ is unique if $\bar{\rho}_{\mathfrak{m}'}|_{G_{F_{\mathfrak{p}}}}$ is indecomposable. We set $\sigma = (\sigma', \{\chi_{\mathfrak{p}}\}_{\mathfrak{p}\in\sigma'})$. We call $\mathfrak{m}$ σ-ordinary

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[4]In fact the level structure used in [Tay2] is more specific than the one used here. This makes no difference to the proofs. Moreover the statement of the lemma for more general level structure follows from the case considered by Taylor.

if it is σ'-ordinary, and for each $\mathfrak{p} \in \sigma'$, the image of $T_\mathfrak{p}$ in $\mathbb{T}_{\psi,\mathbb{O}}(U)_\mathfrak{m}$ is equal to $\chi_\mathfrak{p}(\mathrm{Frob}_\mathfrak{p})$ modulo $\mathfrak{m}$. By (3.4.2), if $\bar{\rho}_{\mathfrak{m}'}|_{G_{F_\mathfrak{p}}}$ is indecomposable, then the second condition is automatic.

In general, given a maximal ideal $\mathfrak{m}$ there is always a choice of σ such that $\mathfrak{m}$ is σ-ordinary. The corresponding set σ' is always unique, and σ is unique unless for some $\mathfrak{p} \in \sigma'$ we have $\bar{V}_{\mathfrak{m}'}|_{G_{F_\mathfrak{p}}} \sim \left(\begin{smallmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{smallmatrix}\right)$ with χ_1, χ_2 distinct unramified characters of $G_{F_\mathfrak{p}}$. In the latter case the two choices for the character $\chi_\mathfrak{p}$ are χ_1 and χ_2.

(3.4.5) Suppose now that $\bar{\rho}_{\mathfrak{m}'}$ satisfies the conditions (1)–(3) of (3.2.3), and fix for each positive integer n, a set of primes Q_n satisfying the conclusion of (3.2.5), with $\bar{\rho} = \bar{\rho}_{\mathfrak{m}'}$. We will use the notation of Section (3.2). Define compact open subgroups U_{Q_n} and $U_{Q_n}^-$ of $\prod_v (\mathbb{O}_D)_v^\times$, by setting $(U_{Q_n})_v = (U_{Q_n}^-)_v = U_v$ if $v \notin Q_n$, and defining

$$(U_{Q_n}^-)_v = \{g \in \mathrm{GL}_2(\mathbb{O}_{F_v}) : g = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)(\pi_v)\}$$

and

$$(U_{Q_n})_v = \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in (U_{Q_n}^-)_v : ad^{-1} \mapsto 1 \in \Delta_{Q_n}\}.$$

Finally, after replacing $\mathbb{F}$ by a quadratic extension, we may and do assume that eigenvalues of elements in the image of $\bar{\rho}_{\mathfrak{m}'}$ are contained in $\mathbb{F}$.

As before, we may define Hecke algebras $\mathbb{T}'_{\psi,\mathbb{O}}(U_{Q_n})$ and $\mathbb{T}_{\psi,\mathbb{O}}(U_{Q_n})$ and similarly for $U_{Q_n}^-$. These are quotients of $\mathbb{T}^{\mathrm{univ}}_{S_{Q_n},\mathbb{O}}$ and $\mathbb{T}^{\mathrm{univ}}_{S^p_{Q_n},\mathbb{O}}$ respectively, where

$$S_{Q_n} = S \cup Q_n.$$

We denote by $\tilde{\mathbb{T}}'_{\psi,\mathbb{O}}(U_{Q_n})$ and $\tilde{\mathbb{T}}_{\psi,\mathbb{O}}(U_{Q_n})$, respectively, the rings of endomorphisms of $S_{2,\psi}(U_{Q_n}, \mathbb{O})$ generated by $\mathbb{T}'_{\psi,\mathbb{O}}(U_{Q_n})$ and $\mathbb{T}_{\psi,\mathbb{O}}(U_{Q_n})$, and the endomorphisms induced by the Hecke operators $U_{\pi_v} = U_{Q_n} \left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) U_{Q_n}$ for $v \in Q_n$. We define analogously subrings $\tilde{\mathbb{T}}'_{\psi,\mathbb{O}}(U_{Q_n}^-)$ and $\tilde{\mathbb{T}}_{\psi,\mathbb{O}}(U_{Q_n}^-)$ of the $\mathbb{O}$-endomorphisms of $S_{2,\psi}(U_{Q_n}^-, \mathbb{O})$. We will again denote by $\mathfrak{m}$ the ideal $\mathfrak{m} \cap \mathbb{T}^{\mathrm{univ}}_{S^p_{Q_n},\mathbb{O}}$ of $\mathbb{T}^{\mathrm{univ}}_{S^p_{Q_n},\mathbb{O}}$.

LEMMA (3.4.6). *The localization* $\tilde{\mathbb{T}}_{\psi,\mathbb{O}}(U_{Q_n}^-)_\mathfrak{m}$ *has* $2^{|Q_n|}$ *distinct maximal ideals corresponding to the* $2^{|Q_n|}$ *ways of choosing one of the two eigenvalues* $\bar{\alpha}_v$ *and* $\bar{\beta}_v$ *of* $\bar{\rho}_{\mathfrak{m}'}(\mathrm{Frob}_v)$ *for* $v \in Q_n$. *If* $\mathfrak{m}_{Q_n}^-$ *denotes the maximal ideal corresponding to* $\{\bar{\alpha}_v : v \in Q_n\}$ *and* $\mathfrak{m}_{Q_n}$ *denotes the induced maximal ideal of* $\tilde{\mathbb{T}}_{\psi,\mathbb{O}}(U_{Q_n})$, *then for* $v \in Q_n$, U_{π_v} *reduces to* $\bar{\alpha}_v$ *modulo* $\mathfrak{m}_{Q_n}$, *and moreover:*

(1) *There is a natural isomorphism* $S_{2,\psi}(U, \mathbb{O})_\mathfrak{m} \xrightarrow{\sim} S_{2,\psi}(U_{Q_n}^-, \mathbb{O})_{\mathfrak{m}_{Q_n}^-}$.

(2) $S_{2,\psi}(U_{Q_n}, \mathbb{O})$ *is a free* $\mathbb{O}[\Delta_{Q_n}]$-*module, and there is a natural isomorphism*

$$S_{2,\psi}(U_{Q_n}, \mathbb{O})_{\mathfrak{m}_{Q_n}} / \mathfrak{a}_{Q_n} \xrightarrow{\sim} S_{2,\psi}(U_{Q_n}^-, \mathbb{O})_{\mathfrak{m}_{Q_n}^-},$$

   where $\mathfrak{a}_{Q_n}$ *denotes the augmentation ideal of* $\mathbb{O}[\Delta_{Q_n}]$.

Proof. The description of the maximal ideals of $\tilde{\mathbb{T}}_{\psi,\mathbb{O}}(U_{Q_n})_{\mathfrak{m}}$ follows from [Tay1, 1.6, 1.8]. Part (1) follows from [Tay1, 2.2] and (2) from (3.1.13). $\qquad\square$

(3.4.7) Set $\mathfrak{m}'_{Q_n} = \mathfrak{m}_{Q_n} \cap \tilde{\mathbb{T}}'_{\psi,\mathbb{O}}(U_{Q_n})$. Write $\mathbb{T}$, $\mathbb{T}'$, $\mathbb{T}_{Q_n}$ and $\mathbb{T}'_{Q_n}$ for the localizations $\mathbb{T}_{\psi,\mathbb{O}}(U)_{\mathfrak{m}}$, $\mathbb{T}'_{\psi,\mathbb{O}}(U)_{\mathfrak{m}'}$, $\tilde{\mathbb{T}}_{\psi,\mathbb{O}}(U_{Q_n})_{\mathfrak{m}_{Q_n}}$, and $\tilde{\mathbb{T}}'_{\psi,\mathbb{O}}(U_{Q_n})_{\mathfrak{m}'_{Q_n}}$ respectively. As in (3.4.1), we have a representation $G_{F,S_{Q_n}} \to \mathrm{GL}_2(\mathbb{T}'_{Q_n})$, and hence a map $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}'_{Q_n}$, which is a surjective map of $\mathbb{O}[\Delta_{Q_n}]$-algebras by [Tay1, 1.8]. Here we regard ψ as a character of $G_{F,S}$ as explained in Section (3.2). In particular, $\mathbb{T}'_{Q_n}$ is reduced, being a localization of $\mathbb{T}'_{\psi,\mathbb{O}}(U_{Q_n})$.

For each $\mathfrak{p}|p$ let $R^{\mathrm{fl},\square}_{\mathfrak{p}}$ denote the quotient of $R^{\square}_{\mathfrak{p}}$, which parametrizes flat deformations of $\bar{\rho}_{\mathfrak{m}'}|_{G_{F_{\mathfrak{p}}}}$ to finite, local, Artinian $\mathbb{O}$-algebras. In the notation of (2.4.1), we have the quotient $R^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}} = (R^{\mathrm{fl},\square}_{\mathfrak{p}})^{\mathbf{v}}$ of $R^{\mathrm{fl},\square}_{\mathfrak{p}}$, where $\mathbf{v} = (1)_{\varphi}$ and φ runs through embeddings $F_{\mathfrak{p}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $\sigma = (\sigma', \{\mathfrak{p}\}_{\mathfrak{p}\in\sigma'})$ be as in (3.4.4), and suppose that $\mathfrak{m}$ is σ-ordinary. By (2.4.10), and the discussion of (2.4.13), if $\mathfrak{p} \notin \sigma'$ (resp. $\mathfrak{p} \in \sigma'$) there exists a collection of connected components of $\mathrm{Spec}\, R^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}}$ such that for any finite extension E' of E, an E'-valued point x of $\mathrm{Spec}\, R^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}}$ lies on one of these components if and only if the corresponding E'-representation $V_{x'}$ of $G_{F_{\mathfrak{p}}}$ is non-ordinary (resp. ordinary). We denote by $R^{\sigma'}_{\mathfrak{p}}$ the quotient of $\mathrm{Spec}\, R^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}}$ corresponding to the union of these components.

We now define a quotient $R^{\sigma}_{\mathfrak{p}}$ of $R^{\sigma'}_{\mathfrak{p}}$. If $\mathfrak{p} \notin \sigma'$, we set $R^{\sigma}_{\mathfrak{p}} = R^{\sigma'}_{\mathfrak{p}}$. Suppose that $\mathfrak{p} \in \sigma'$. By (2.5.16) $\mathrm{Spec}\, R^{\sigma'}_{\mathfrak{p}}[1/p]$ is connected unless $\bar{V}_{\mathfrak{m}'}|_{G_{F_{\mathfrak{p}}}} \sim \left(\begin{smallmatrix} \chi_{\mathfrak{p}} & 0 \\ 0 & \chi'_{\mathfrak{p}} \end{smallmatrix}\right)$ with $\chi_{\mathfrak{p}}$ and $\chi'_{\mathfrak{p}}$ distinct unramified characters of $G_{F_{\mathfrak{p}}}$. In the former case, we set $R^{\sigma}_{\mathfrak{p}} = R^{\sigma'}_{\mathfrak{p}}$. In the latter case $\mathrm{Spec}\, R^{\sigma'}_{\mathfrak{p}}[1/p]$ has two components $C_{\chi_{\mathfrak{p}}}$ and $C_{\chi'_{\mathfrak{p}}}$, where $C_{\chi_{\mathfrak{p}}}$ has the property that for any finite extension E' of E, (the image of) a point $x \in \mathrm{Spec}\, R^{\sigma'}_{\mathfrak{p}}[1/p](E')$ lies on $C_{\chi_{\mathfrak{p}}}$ if and only if G_K acts on the (unique) unramified, one dimensional quotient of V_x via an $\mathbb{O}^{\times}_{E'}$-valued character which reduces to $\chi_{\mathfrak{p}}$. In this case, we denote by $R^{\sigma}_{\mathfrak{p}}$ the quotient of $R^{\sigma'}_{\mathfrak{p}}$ corresponding to $C_{\chi_{\mathfrak{p}}}$.

We denote by $R^{\sigma,\psi}_{\mathfrak{p}}$ the quotient of $R^{\sigma}_{\mathfrak{p}}$ corresponding to deformations with determinant equal to ψ times the cyclotomic character. (Here we have dropped the $\square$ from the notation, but the reader should remember that we are working with framed deformations.)

Applying (2.4.17) with $R = R^{\mathrm{fl},\square}_{\mathfrak{p}}$, we also obtain a finite $R^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}}$-algebra $\tilde{R}^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}} \subset R^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}}[1/p]$. We denote by $\tilde{R}^{\sigma,\psi}_{\mathfrak{p}}$ the image of $\tilde{R}^{\mathrm{fl},\square,\mathbf{v}}_{\mathfrak{p}}$ in $R^{\sigma,\psi}_{\mathfrak{p}}[1/p]$. Finally we set $R^{\sigma,\psi}_{p} = \widehat{\otimes}_{\mathfrak{p}} R^{\sigma,\psi}_{\mathfrak{p}}$ and $\tilde{R}^{\sigma,\psi}_{p} = \widehat{\otimes}_{\mathfrak{p}} \tilde{R}^{\sigma,\psi}_{\mathfrak{p}}$, where the completed tensor products are taken over $\mathbb{O}$.

(3.4.8) Let $v \in \Sigma$. There is a unique unramified character $\gamma : G_{F_v} \to \mathbb{O}^{\times}$, such that $\gamma^2 = \psi|_{G_{F_v}}$, and such that $\bar{\rho}_{\mathfrak{m}'}|_{G_{F_v}}$ is an extension of γ by $\gamma(1)$. This

follows from the compatibility between the global and local Langlands and Jacquet-Langlands correspondences (cf. [Tay1, 1.7]).

By (2.6.7) there is a $\mathbb{O}$-flat quotient R_v^ψ of $R_v^\square$, such that for any finite extension E'/E, a map $\xi : R_v^\square \to E'$ factors through R_v^ψ if and only if the induced G_{F_v} representation V_ξ is an extension of γ by $\gamma(1)$.

Write $R_\Sigma^\psi = \widehat{\otimes}_{v \in \Sigma} R_v^\psi$, and set $R_{\Sigma,p}^{\sigma,\psi} = R_p^{\sigma,\psi} \widehat{\otimes}_{\mathbb{O}} R_\Sigma^\psi$ and $\tilde{R}_{\Sigma,p}^{\sigma,\psi} = \tilde{R}_p^{\sigma,\psi} \widehat{\otimes}_{\mathbb{O}} R_\Sigma^\psi$.

LEMMA (3.4.9). (1) *The composite map*

$$R_p^\square \widehat{\otimes}_{\mathbb{O}} R_\Sigma^\square \to R_{F,S_{Q_n}}^\square \to R_{F,S_{Q_n}}^\square \otimes_{R_{F,S_{Q_n}}} \mathbb{T}_{Q_n}$$

factors through $R_{\Sigma,p}^{\sigma,\psi}$.

(2) *The image of the induced map*

$$\tilde{R}_{\Sigma,p}^{\sigma,\psi} \otimes_{R_{\Sigma,p}^\square} R_{F,S_{Q_n}}^\square \to R_{F,S_{Q_n}}^\square \otimes_{R_{F,S_{Q_n}}} \mathbb{T}_{Q_n}[1/p]$$

is equal to $\mathbb{T}_{Q_n}^\square := R_{F,S_{Q_n}}^\square \otimes_{R_{F,S_{Q_n}}} \mathbb{T}_{Q_n}$. *The analogous statements with S in place of S_{Q_n} also hold.*

Proof. Since $\mathfrak{m}$ is σ-ordinary, so is $\mathfrak{m}_{Q_n}$. Hence (3.4.2) and the fact that $\det \rho_{\mathfrak{m}'} = \psi\chi$ imply that the map $R_p^\square \to R_{F,S_{Q_n}}^\square \otimes_{R_{F,S_{Q_n}}} \mathbb{T}_{Q_n}$ factors through $R_p^{\sigma,\psi}$. To see that the map $R_\Sigma^\square \to R_{F,S_{Q_n}}^\square \otimes_{R_{F,S_{Q_n}}} \mathbb{T}_{Q_n}$ factors through R_Σ^ψ, note that, since $\mathbb{T}_{Q_n}$ is reduced, it suffices to check that for any finite extension E'/E, and any map $\xi : \mathbb{T}_{Q_n} \to E'$, the composite of the above map with $1 \otimes \xi$ factors through R_Σ^ψ. This follows from the compatibility of the local and global Langlands correspondence. This proves (1).

From the construction in (2.4.17), $\tilde{R}_{\mathfrak{p}}^{\mathrm{fl},\square,\mathbf{v}}$ is generated as an $R_{\mathfrak{p}}^{\mathrm{fl},\square,\mathbf{v}}$-algebra by the coefficients of a quadratic polynomial $P_{\bar{V}_{\mathfrak{m}'}}^{\mathbf{v}}(X) = X^2 - t_{\mathfrak{p}}X + d_{\mathfrak{p}}$. To show (2), it suffices to prove that under the map

$$\tilde{R}_{\mathfrak{p}}^{\mathrm{fl},\square,\mathbf{v}} \to \mathbb{T}_{Q_n}^\square[1/p]$$

the element $t_{\mathfrak{p}}$ is mapped to $T_{\mathfrak{p}}$ while $d_{\mathfrak{p}}$ is mapped to an element of $\mathbb{O}$. Since $\mathbb{T}_{Q_n}[1/p]$ is reduced, it suffices to check that this is true after composing with a map $\mathbb{T}_{Q_n}[1/p] \to E'$ where E'/E is a finite extension. But this follows from (3.4.2) and the description in (2.4.17) of the pull-back of $P_{\bar{V}_{\mathfrak{m}'}}^{\mathbf{v}}(T)$ by an E'-valued point of $R_{\mathfrak{p}}^{\mathrm{fl},\square,\mathbf{v}}$. $\qquad\square$

(3.4.10) We set $\tilde{R}_{F,S_{Q_n}}^{\sigma,\psi} = \tilde{R}_{\Sigma,p}^{\sigma,\psi} \otimes_{R_{\Sigma,p}^{\square,\psi}} R_{F,S_{Q_n}}^{\square,\psi}$, and similarly with S in place of S_{Q_n}. Again, the reader should note that we are suppressing $\square$ from the notation, but that $\tilde{R}_{F,S_{Q_n}}^{\sigma,\psi}$ is a deformation ring for framed representations (in fact the functor defined in (3.2.1) involves a choice of frame for each $v \in \Sigma_p$). As

in (3.4.9) we set $\mathbb{T}^\square := R_{F,S}^\square \otimes_{R_{F,S}} \mathbb{T}$. We can now state the main result of this section.

THEOREM (3.4.11). *With the above notation and assumptions, suppose also that if $\mathfrak{p}$ is a prime of F dividing p, and $\mathfrak{p} \notin \sigma'$ then $\mathfrak{p}$ has residue field $\mathbb{F}_p$. Then the map $\tilde{R}_{F,S}^{\sigma,\psi} \to \mathbb{T}^\square$ induced by the map of (3.4.9)(2) has p-power torsion kernel.*

Proof. We apply (3.3.1) with $B = \tilde{R}_{\Sigma,p}^{\sigma,\psi}$, $R = \tilde{R}_{F,S}^{\sigma,\psi}$, $R_n = \tilde{R}_{F,S_{Q_n}}^{\sigma,\psi}$, and with

$$M = S_{2,\psi}(U,\mathbb{O})_{\mathfrak{m}} \otimes_{\mathbb{T}} \mathbb{T}^\square \text{ and } M_n = S_{2,\psi}(U_{Q_n},\mathbb{O})_{\mathfrak{m}_{Q_n}} \otimes_{\mathbb{T}_{Q_n}} \mathbb{T}_{Q_n}^\square.$$

We write $h = |Q_n|$, and fix a surjection $\mathbb{O}[\![y_1,\ldots,y_h]\!] \to \mathbb{O}[\Delta_{Q_n}]$, as in (3.2.8). The morphism $R_{F,S} \to R_{F,S}^\square$ is formally smooth of relative dimension $j = 4|\Sigma_p| - 1$. Thus we may identify $R_{F,S}^\square$ with a power series ring $R_{F,S}[\![w_1,\ldots,w_j]\!]$, so that $R_{F,S}^\square$ becomes an $\mathbb{O}[\![y_1,\ldots,y_{h+j}]\!]$-algebra via

$$\mathbb{O}[\![y_1,\ldots,y_{h+j}]\!] \to \mathbb{O}[\![y_{h+1},\ldots,y_{h+j}]\!] \overset{y_{h+i} \mapsto w_i}{\longrightarrow} R_{F,S}^\square.$$

Similarly, for any $n \geq 1$, $R_{F,S_{Q_n}}^\square$ is an $\mathbb{O}[\Delta_{Q_n}]$-algebra, and hence an $\mathbb{O}[\![y_1,\ldots,y_h]\!]$-algebra. We may extend this to an $\mathbb{O}[\![y_1,\ldots,y_{h+j}]\!]$-algebra structure, such that the projection $R_{F,S_{Q_n}}^\square \to R_{F,S}^\square$ becomes a map of $\mathbb{O}[\![y_1,\ldots,y_{h+j}]\!]$-algebras.

The maps in (3.3.2) then correspond to the induced maps

$$\mathbb{O}[\![y_1,\ldots,y_{h+j}]\!] \to \tilde{R}_{F,S_{Q_n}}^{\sigma,\psi} \to \tilde{R}_{F,S}^{\sigma,\psi}.$$

We now check the conditions in (3.3.1). The conditions on $R_n \to R$ in (1) and (2) follow from the description of the $\mathbb{O}[\Delta_{Q_n}]$-algebra structure on $R_{F,S_{Q_n}}$ given in (3.2.8). The conditions on M_n in (1), (2) and (3) follow from (3.4.6).

To check the conditions on B, we remark that if $\mathfrak{p}|p$, then $R_\mathfrak{p}^\sigma \overset{\sim}{\longrightarrow} R_\mathfrak{p}^{\sigma,\psi}[\![X]\!]$, where the isomorphism depends only on a choice of topological generator for the maximal unramified quotient $G_{F_\mathfrak{p}}$. Hence (2.5.16) applied with $R = R_{\tilde{V}_{\mathfrak{m}'}}^{\mathrm{fl},\square}$ implies that $R_\mathfrak{p}^{\sigma,\psi}$ is a domain which is flat over $\mathbb{O}$ of relative dimension $3 + [F_\mathfrak{p} : \mathbb{Q}_p]$, and that $R_\mathfrak{p}^{\sigma,\psi}[1/p]$ is formally smooth over E. More precisely, since E is an arbitrary sufficiently large finite extension of $\mathbb{Q}_p$, $R_\mathfrak{p}^{\sigma,\psi}[1/p]$ is *geometrically integral* in the sense that it remains a domain after any finite extension of scalars $E \to E'$. The same statements hold with $\tilde{R}_\mathfrak{p}^{\sigma,\psi}$ in place of $R_\mathfrak{p}^{\sigma,\psi}$, because $R_\mathfrak{p}^{\sigma,\psi} \to \tilde{R}_\mathfrak{p}^{\sigma,\psi}$ is a finite map of flat $\mathbb{O}$-algebras, which becomes an isomorphism after inverting p.

By (2.6.7), if $v \in \Sigma$, then R_v^ψ is an $\mathbb{O}$-flat domain of relative dimension 3 over $\mathbb{O}$, and $R_v^\psi[1/p]$ is geometrically integral and formally smooth over $\mathbb{O}$. It follows from (3.4.12) below that $R_{\Sigma,p}^{\sigma,\psi}$ is a domain which is flat over $\mathbb{O}$ of relative dimension

$$d = \sum_{\mathfrak{p}|p}([F_\mathfrak{p} : \mathbb{Q}_p] + 3) + 3|\Sigma| = [F : \mathbb{Q}] + 3|\Sigma_p|,$$

and that $R_{\Sigma,p}^{\sigma,\psi}[1/p] = \tilde{R}_{\Sigma,p}^{\sigma,\psi}[1/p]$ is formally smooth over E. In particular, $\tilde{R}_{\Sigma,p}^{\sigma,\psi}$ is a domain of dimension $d + 1$, and

$$h + j - d = |Q_n| + 4|\Sigma_p| - 1 - [F : \mathbb{Q}] - 3|\Sigma_p| = |Q_n| - [F : \mathbb{Q}] + |\Sigma_p| - 1,$$

so that condition (3.3.1)(4) follows from (3.2.5).

We conclude that $M \otimes_{\mathbb{O}} E$ is a faithful $\tilde{R}_{F,S}^{\sigma,\psi} \otimes_{\mathbb{O}} E$-module. Since the action of $\tilde{R}_{F,S}^{\sigma,\psi}$ on M factors through $\mathbb{T}^{\square}$, the theorem follows. $\square$

LEMMA (3.4.12). *Let R_1, R_2 be complete, local, flat $\mathbb{O}$-algebras with residue field $\mathbb{F}$, and suppose that $R_i[1/p]$ is geometrically integral for $i = 1, 2$. Then $R = R_1 \widehat{\otimes}_{\mathbb{O}} R_2$ is $\mathbb{O}$-flat and $R[1/p]$ is geometrically integral. If $R_1[1/p]$ and $R_2[1/p]$ are formally smooth over E, then so is $R[1/p]$.*

Proof. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ denote the radicals of R_1 and R_2 respectively. For $n \geq 1$, $R/\mathfrak{m}_1^n R = R_1/\mathfrak{m}_1^n \otimes_{\mathbb{O}} R_2$ is a flat $R_1/\mathfrak{m}_1^n R_1$-algebra. Hence R is a flat R_1-algebra by [GrD, 0_{III} 10.2.1], and hence a flat $\mathbb{O}$-algebra.

To show the two other statements, we remark that if S is any quotient of a power series ring $\mathbb{O}[\![x_1, \ldots, x_r]\!]$ for some $r \geq 0$, then $S[1/p]$ is a Jacobson ring [GrD, IV 10.5.7] and the residue field at any maximal ideal of $S[1/p]$ is a finite extension of E [deJ, p. 78–79].

Let X_1, X_2 and X denote the spectra of $R_1[1/p]$, $R_2[1/p]$ and $R[1/p]$ respectively. To show that X is irreducible we have to show that if $U_1, U_2 \subset X$ are disjoint open subsets, then one of U_1 and U_2 is empty. Suppose that both are non-empty. After replacing $\mathbb{O}$ by the ring of integers $\mathbb{O}'$ in a finite extension of E, and R_1 and R_2 by $R_1 \otimes_{\mathbb{O}} \mathbb{O}'$ and $R_2 \otimes_{\mathbb{O}} \mathbb{O}'$ respectively, we may assume that both U_1 and U_2 admit $\mathbb{O}[1/p]$-valued points and, in particular, that for $i = 1, 2$ there is a section $s_i : X_1 \to X$, such that $s_i(X_1)$ meets U_i. Then $s_i^{-1}(U_i) \subset X_1$ are non-empty open subsets of X_1 and hence they intersect as X_1 is irreducible. Let $y \in X_1$ be a closed point in the intersection. Then both U_1 and U_2 meet X_y, the fiber over y of the projection $X \to X_1$. This is a contradiction because X_2 is geometrically integral, so X_y is irreducible.

Next we remark that if $x_1 \in X_1$ and $x_2 \in X_2$ are closed points at which X_1 and X_2 are formally smooth over E, then the formal smoothness of X at (x_1, x_2) is a simple exercise using (2.3.3). This proves the final statement of the lemma, and shows that, in general, X is generically reduced.

To show that X is integral, it remains to show that it is reduced. If X_1 and X_2 are both 0-dimensional (so that R_1 and R_2 are finite extensions of $\mathbb{O}$) then this is clear. Thus we may assume that $\dim X_1 \geq 1$. Now X is reduced if and only if it is generically reduced and satisfies Serre's condition S_1 [GrD, IV 5.8.5]. Since X_1 is reduced, it satisfies S_1, and therefore has depth ≥ 1 at every point. Since X is a

flat X_1-scheme, it follows that X has depth ≥ 1 at every point [GrD, IV 6.3.5], and hence satisfies S_1.

This shows that $R[1/p]$ is integral, and since the extension $\mathcal{O}'$ was the ring of integers in an arbitrary extension, that $R[1/p]$ is geometrically integral. $\qquad\square$

(3.5) *Applications to modularity.* In this section we use (3.4.11) to prove modularity of certain two-dimensional Galois representations.

(3.5.1) We keep the notation of previous sections. It will be convenient to fix algebraic closures $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively, and embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and $\bar{\mathbb{Q}} \overset{\lambda}{\hookrightarrow} \bar{\mathbb{Q}}_p$. We denote by $\bar{\mathbb{Z}}_p$ the ring of integers of $\bar{\mathbb{Q}}_p$, and by $\bar{\mathbb{F}}_p$ its residue field.

Given a cuspidal Hilbert modular eigenform f over F (thought of as a complex valued function) we denote by $E_{f,\lambda}$ the λ-adic completion of its coefficient field, by $\mathcal{O}_{f,\lambda}$ the ring of integers of $E_{f,\lambda}$, and by $\pi_{f,\lambda}$ a uniformiser of $\mathcal{O}_{f,\lambda}$. Attached to f is a continuous representation of $G_{F,S}$ on a two-dimensional $E_{f,\lambda}$-vector space $V_{f,\lambda}$, where S is a set of primes containing the infinite primes, the primes dividing p, and the primes at which f is ramified. Choosing a $G_{F,S}$-stable $\mathcal{O}_{f,\lambda}$-lattice in $V_{f,\lambda}$, we obtain a representation $\rho_{f,\lambda} : G_{F,S} \to \mathrm{GL}_2(\mathcal{O}_{f,\lambda})$ (defined up to conjugation). We denote by $\bar{\rho}_{f,\lambda} \to \mathrm{GL}_2(\mathcal{O}_{f,\lambda}/\pi_{f,\lambda})$ the corresponding residual representation. The semi-simplification of $\bar{\rho}_{f,\lambda}$ does not depend on the choice of lattice.

As in (3.1.3), we can associate a maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S,\bar{\mathbb{Z}}_p}$ to f. If f is unramified at primes dividing p, we can associate a maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S^p,\bar{\mathbb{Z}}_p}$ to f, as in (3.1.9). Given two such eigenforms f and g, which are both unramified at primes dividing p, we will write $f \sim g$, if f and g have the same associated maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S^p,\bar{\mathbb{Z}}_p}$. In particular, this implies that $\bar{\rho}_{f,\lambda} \sim \bar{\rho}_{g,\lambda}$. (That is, the semi-simplifications of the two representations are isomorphic after an extension of scalars, and the set S is taken so as to contain the primes of ramification of f and g).

LEMMA (3.5.2). *Let f be a cuspidal Hilbert modular eigenform over F of parallel weight 2, which is unramified at primes dividing p. Let T be any finite set of finite primes at which f is unramified. There exists a finite, solvable, totally real extension F' of F, and a cuspidal Hilbert modular eigenform g over F' of parallel weight 2, such that*:

(1) *Each prime $v \in T$ is totally split in F'.*

(2) *If f is unramified at a place v of F, then g is unramified at every place v' of F' lying over v.*

(3) *If f' denotes the base change of f to F', then $f' \sim g$.*

(4) *If $v' \nmid p$ is a place of F', and $\bar{\rho}_{f',\lambda} \sim \bar{\rho}_{g,\lambda}$ is unramified at v', then g is unramified at v'.*

Proof. The main theorem of [SW1] produces an extension F'/F and a form g over F' which satisfies (1), (2) and (4), and such that $\rho_{f',\lambda} \sim \rho_{g,\lambda}$. To ensure the condition (3) we have to arrange that for $\mathfrak{p}|p$ a prime of F', the eigenvalues of $T_\mathfrak{p}$ acting on g and f' have the same image in $\bar{\mathbb{F}}_p$. This can be achieved by a very slight modification of the argument in [SW1]. Namely, on p. 21 of that paper (and with the notation used there), one should define the algebra $\mathbf{T}$ as in the *ordinary case* (that is, $\mathbf{T}$ includes the operators $t(\mathfrak{p})$ for $\mathfrak{p}|p$), and the maximal ideal $\mathfrak{m}$ of $\mathbf{T} \otimes \mathbb{O}$ should be defined as the ideal generated by

$$\{\tilde{\lambda}; \, t(\mathfrak{q}) - c(\mathfrak{q}, f), \, s(\mathfrak{q}) - \chi_f(\mathfrak{q}) \text{ for } \mathfrak{q} \nmid \mathfrak{n}_f \mathfrak{r} \ell p; \, t(\mathfrak{p}) - c(\mathfrak{p}, f) \text{ for } \mathfrak{p}|p\}.$$

The rest of the proof then proceeds unchanged.   □

LEMMA (3.5.3). *Suppose that $[F : \mathbb{Q}]$ is even. Let f be a cuspidal Hilbert modular eigenform over F of parallel weight 2 such that f is unramified at primes dividing p. Let Σ be a finite set of finite primes of F not containing any primes dividing p, and $F = F_0 \subset F_1 \subset \ldots F_r$ a tower of totally real fields satisfying the following conditions*:

(i) *$\Sigma = \{v_1, \ldots, v_r\}$, and for $i = 1, 2, \ldots r$, F_i/F_{i-1} is a quadratic extension such that for any $j \in \{1, 2, \ldots r\}$ any prime w of F_{i-1} over v_j is inert in F_i if $i \neq j$, and splits in F_i if $i = j$.*

(ii) *For each $v \in \Sigma$, f is unramified at v, and for $v \in \Sigma$, $\bar{\rho}_{f,\lambda}|_{G_{F_v}}$ is an extension of $\bar{\gamma}$ by $\bar{\gamma}(1)$ for some unramified character $\bar{\gamma}$.*

Then for any prime $\ell \notin \Sigma \cup \{v|p\}$, there exists a cuspidal Hilbert modular eigenform g over F_r of parallel weight 2 such that

(1) *If $v \neq \ell$ is a finite prime of F at which f is unramified and v' a prime of F_r over v, then (the automorphic representation generated by) g is special of conductor 1 at v' if $v \in \Sigma$, and unramified otherwise.*

(2) *If f' denotes the base change of f to F_r, then $f' \sim g$.*

Proof. We prove the lemma by induction on r. Let $\Sigma_{r-1} = \{v_1, \ldots, v_{r-1}\}$, and let g_{r-1} be a Hilbert modular eigenform on F_{r-1} which satisfies the conclusion of the lemma with Σ_{r-1} in place of Σ. Let D be the quaternion algebra with center F_{r-1} which is ramified at the infinite places of F_{r-1}, and at the finite primes lying over a prime of Σ_{r-1}. Note that the latter set has cardinality $2(r-1)$.

Fix a maximal order $\mathbb{O}_D$ of D, and let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup such that $U_v = (\mathbb{O}_D)_v^\times$ if v lies over a prime of Σ_{r-1}, or if g_{r-1} is unramified at v and $v \nmid \ell$. Note that the set of such primes contains the primes dividing p, and the primes of Σ. For the other finite primes v we choose

U_v small enough that g_{r-1} corresponds to a Hecke eigenform $h_{r-1} \in S_{2,\psi}(U,\mathbb{O})$ via the Jacquet-Langlands correspondence and U satisfies (3.1.2). This is possible, since we may choose U_v arbitrarily small if $v|\ell$. Here the character ψ is determined by the central character of g_{r-1}, and $\mathbb{O} \subset \bar{\mathbb{Z}}_p$ is the ring of integers of a sufficiently large, finite extension of $\mathbb{Q}_p$, $E \subset \bar{\mathbb{Q}}_p$.

Define $U' = \prod_v U'_v$ by $U'_v = U_v$ unless $v|v_r$ in which case we set

$$U'_v = \{g \in \mathrm{GL}_2(\mathbb{O}_{(F_{r-1})_v}) : g = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)(\pi_v)\}$$

where we have identified $(\mathbb{O}_D)_v$ with $M_2(\mathbb{O}_{(F_{r-1})_v})$. Note that there is exactly one prime $\tilde{v}_r$ of F_{r-1} dividing v_r.

Now let S be the union of the infinite primes of F_{r-1}, the primes dividing p or a prime in Σ, and the primes v such that U_v is not maximal compact. Let $\mathfrak{m} \subset \mathbb{T}^{\mathrm{univ}}_{S^p,\mathbb{O}}$ denote the maximal ideal associated to g_{r-1}. The compatibility between the local and global Langlands correspondence and the condition (ii) of the lemma implies that

$$(T^2_{\tilde{v}_r} - (\mathbf{N}(\tilde{v}_r)+1)^2 \psi(\pi_{\tilde{v}_r}))(S_{2,\psi}(U,\mathbb{O})) \subset \mathfrak{m}S_{2,\psi}(U,\mathbb{O}).$$

Hence, by (3.1.11), there exists a non-zero eigenform $h_r \in S_{2,\psi}(U',\mathbb{O})_\mathfrak{m}$ which is not in the image of $i_{\tilde{v}_r}$. If g_r is the Hilbert modular form over F_{r-1} associated to h_r via the Jacquet-Langlands correspondence, then we may take g to be the base change of g_r to F_r. $\qquad\square$

(3.5.4) Let $E \subset \bar{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathbb{O}$, and residue field $\mathbb{F}$. Let $\rho : G_{F,S} \to \mathrm{GL}_2(E)$ be a continuous representation. After conjugation by an element of $\mathrm{GL}_2(E)$, we may assume that ρ factors through $\mathrm{GL}_2(\mathbb{O})$, and we denote by $\bar{\rho}$ the composite

$$\bar{\rho} : G_{F,S} \xrightarrow{\rho} \mathrm{GL}_2(\mathbb{O}) \to \mathrm{GL}_2(\mathbb{F}).$$

Up to semi-simplification $\bar{\rho}$ is independent of the choice of element we used to conjugate the image of ρ into $\mathrm{GL}_2(\mathbb{O})$. In particular, if $\bar{\rho}$ is irreducible, then it is independent of this choice. We assume from now on that this is the case.

As usual, we call ρ residually modular if there exists a Hilbert modular eigenform f over F, such that $\bar{\rho}_{f,\lambda} \sim \bar{\rho}$.

We call ρ *strongly residually modular* if ρ is potentially Barsotti-Tate at all primes $\mathfrak{p}$ of F over p, with determinant equal to the cyclotomic character times a finite order character, and f can be chosen so that

(1) $\bar{\rho}_{f,\lambda} \sim \bar{\rho}$, and f has parallel weight 2.

(2) The automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ generated by f is not special at any place dividing p.

(3) For any $\mathfrak{p}|p$, $\rho_{f,\lambda}|_{G_{F_\mathfrak{p}}}$ is potentially ordinary (i.e. ordinary when restricted to an open subgroup of $G_{F_\mathfrak{p}}$) if and only if $\rho|_{G_{F_\mathfrak{p}}}$ is.

THEOREM (3.5.5). *Let* $\rho : G_{F,S} \to \mathrm{GL}_2(E)$ *be a continuous representation, which satisfies the following conditions:*[5]

(1) ρ *is strongly residually modular.*

(2) *If* $\mathfrak{p}|p$, *and* $\rho|_{G_{F_\mathfrak{p}}}$ *is not potentially ordinary, then the residue field of* $\mathfrak{p}$ *is* $\mathbb{F}_p$.

(3) *The restriction of* $\bar\rho$ *to* $G_{F(\zeta_p)}$ *is absolutely irreducible. If* $p = 5$, *and* $\bar\rho$ *has projective image isomorphic to* $\mathrm{PGL}_2(\mathbb{F}_5)$, *then the kernel of* $\mathrm{proj}\,\bar\rho$ *does not fix* $F(\zeta_5)$.

Then ρ *is modular. That is,* $\rho \sim \rho_{g,\lambda}$ *for some Hilbert modular form g.*

Proof. Let f be a Hilbert modular eigenform over F satisfying the conditions (1), (2) and (3) of (3.5.4), and denote by π_f the corresponding automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let F'/F be a totally real extension. Suppose that there exists a tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n$ with F_{i+1}/F_i a finite abelian extension, and such that $F' \subset F_n$. By Langlands base change it suffices to show that $\rho|_{G_{F'}}$ is modular. By class field theory we may choose F' with the following properties:

(i) The base change of π_f to F' is unramified or special at every finite place of F'.

(ii) $\rho|_{G_{F'_\mathfrak{p}}}$ is Barsotti-Tate for every prime $\mathfrak{p}$ of F' dividing p.

(iii) If $\mathfrak{p}|p$ and $\rho|_{G_{F'_\mathfrak{p}}}$ is ordinary then $\bar\rho|_{G_{F'_\mathfrak{p}}}$ is either indecomposable or has trivial image. If $\rho|_{G_{F'_\mathfrak{p}}}$ is non-ordinary then the residue field at $\mathfrak{p}$ is $\mathbb{F}_p$.

(iv) $\bar\rho|_{G_{F'}}$ is unramified outside the primes of F dividing p.

(v) If $v \nmid p$ is a prime of F where ρ is ramified then the image of the inertia subgroup $I_v \subset G_{F_v}$ under ρ is unipotent.

(vi) $[F' : \mathbb{Q}]$ is even and the images of $G_{F(\zeta_p)}$ and $G_{F'(\zeta_p)}$ under $\bar\rho$ are equal.

That F' can be chosen to satisfy (i)–(v) and the two conditions in (vi) follows from [Tay3, 2.2] and its proof (cf. also [JM, §2]), since these properties can be forced by imposing local conditions on the extension F'/F at a finite set of primes of F. To see that F' can be chosen to satisfy the last condition in (vi) note that in constructing the intermediate field F_{i+1} from F_i we can always choose a finite set of primes T of F_i, which are disjoint from any given finite set of primes, and such that $v \in T$ splits in $G_{F_i(\zeta_p)}$, and the elements $\bar\rho(\mathrm{Frob}_v)$ with $v \in T$ generate

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[5]After the writing of this paper was completed Toby Gee was able to prove (2.5.6) for $\bar\rho$ with trivial image, without the restriction that $K_0 = \mathbb{Q}_p$ [Ge]. This allows one to remove the condition (2) below.

$\bar{\rho}(G_{F_i(\zeta_p)})$. Requiring that $v \in T$ split in F_{i+1} then ensures the third condition in (vi).

After replacing ρ by $\rho|_{G_{F'}}$ and π_f by its base change to F', we may assume that (i)–(vi) hold with F in place of F'. (We remark that in (iii) we could make the more brutal assumption that $\bar{\rho}|_{G_{F'_\mathfrak{p}}}$ is trivial for all $\mathfrak{p}|p$ where ρ is ordinary, and this would suffice for the proof which follows.)

By (3.5.2), after making a further base change, we may assume that π_f is unramified at all finite places of F. Choose a prime $\ell \nmid p$ of F such that

$$(3.5.6) \qquad \mathrm{tr}\bar{\rho}(\mathrm{Frob}_\ell)^2 / \det \bar{\rho}(\mathrm{Frob}_\ell) \neq (1 + \mathbf{N}(\ell))^2 / \mathbf{N}(\ell).$$

This is possible by [DDT, 4.11], applied with χ equal to the cyclotomic character, because $[F(\zeta_p) : F]$ is even and $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. Now (v) implies that at any place v where ρ is ramified, $\rho|_{G_{F_v}}$ is an extension of γ by $\gamma(1)$ for some unramified character γ of G_{F_v}. Hence by (3.5.3), we may assume that (instead of the previous condition), for each finite prime v of F, π_f is unramified at v if $v|p$, is special with conductor 1 if ρ is ramified at $v \nmid p$, and is not special at any $v \nmid p$ where ρ is unramified. This last condition is guaranteed by (3.5.6) provided we choose the extension F_r in (3.5.3) so that ℓ splits in F_r. Hence, after a further base change, we may assume that π_f is also unramified at all $v \nmid p$ where ρ is unramified. Moreover we may suppose that the set Σ, consisting of primes $v \nmid p$ where ρ is ramified, has even order. We may also assume that (ii)–(vi) continue to hold. This is automatic except for (iii) and (vi), and these two conditions can be preserved, as above, by choosing the extensions F'/F and F_r/F in (3.5.2) and (3.5.3) so that a suitable set of primes splits completely in F' and F_r.

Finally, after twisting f by a character, we may assume that $\rho_{f,\lambda}$ and ρ have the same determinant.

Let D be the quaternion algebra with center F which is ramified at the infinite primes and at the primes in Σ. Fix a maximal order $\mathcal{O}_D$ of D, and define a compact open subgroup $U = \prod_v U_v$ of $(D \otimes_F \mathbb{A}_F^f)^\times$ by setting $U_v = (\mathcal{O}_D)_v^\times$. By the Jacquet-Langlands correspondence (enlarging $\mathcal{O}$ if necessary), f corresponds to a Hecke eigenform $f^D \in S_{2,\psi}(U, \mathcal{O})$ where ψ is determined by the central character of f (see (3.1.14)).

Now let $\ell \notin \Sigma$ be a finite prime of F with $\ell \nmid p$, $\mathbf{N}(\ell) \not\equiv 1(p)$ and satisfying (3.5.6). We define $U' = \prod_v U'_v$ by $U'_v = U_v$ if $v \neq \ell$, and

$$U'_\ell = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_\ell}); \ g = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}(\pi_\ell)\}.$$

We may assume that ℓ is sufficiently large that U' satisfies the condition (3.1.2), and we regard f^D as an element of $S_{2,\psi}(U', \mathcal{O})$.

Let S be the set of primes of F consisting of the infinite primes, the primes dividing $p\ell$, and the primes contained in Σ. Let $\mathfrak{m} \subset T^{\mathrm{univ}}_{S^p,\mathcal{O}}$ be the maximal ideal

associated to f^D, so that $f^D \in S_{2,\psi}(U', \mathbb{O})_\mathfrak{m}$. We can now apply (3.4.11) to $\mathfrak{m}$. To check that the hypotheses of that theorem are satisfied, note that the conditions (1)-(3) of (3.2.3) hold for $\bar\rho_{\mathfrak{m}'} = \bar\rho$ by (iv) and (vi) above, and the fact that $\bar\rho \sim \bar\rho_{f,\lambda}$ has odd determinant, while (3.2.3)(4) holds by the choice of ℓ.

Now let σ' be the set of primes of F dividing p at which f^D (or equivalently f) is ordinary. The assumption (iii) and the discussion of (3.4.7) then imply that there is a unique choice of $\sigma = \{\sigma', \{\chi_\mathfrak{p}\}\}$, and that $R_\mathfrak{p}^\sigma = R_\mathfrak{p}^{\sigma'}$. Now (ii), (v) and the fact that $\det \rho = \det \rho_{f,\lambda}$, imply that ρ arises from a homomorphism $\kappa_\rho : R_{F,S}^{\sigma,\psi} \to \mathbb{O}$, and this extends uniquely to a map $\tilde R_{F,S}^{\sigma,\psi} \to \mathbb{O}$. The required Hilbert modular form g then corresponds to the kernel of the composite map

$$\mathbb{T}_{\psi,\mathbb{O}}(U')_\mathfrak{m} = \mathbb{T} \to \mathbb{T}^\square \underset{(3.4.11)}{\xrightarrow{\sim}} \tilde R_{F,S}^{\sigma,\psi}/\tilde R_{F,S}^{\sigma,\psi}[p^\infty] \xrightarrow[\kappa_\rho]{} \mathbb{O}. \qquad \square$$

THEOREM (3.5.7). *Let $\rho : G_{F,S} \to \mathrm{GL}_2(E)$ be a continuous representation, and suppose that*:

(1) $\det \rho$ *is equal to the cyclotomic character times a character of finite order.*

(2) *For all primes $\mathfrak{p}$ of F dividing p, $\rho|_{G_{F_\mathfrak{p}}}$ is potentially Barsotti-Tate and not potentially ordinary, and the residue field at $\mathfrak{p}$ is equal to $\mathbb{F}_p$.[6]*

(3) $\bar\rho \sim \bar\rho_{f,\lambda}$ *for a Hilbert modular form f over F of parallel weight 2.*

(4) $\bar\rho|_{G_{F(\zeta_p)}}$ *is absolutely irreducible. If $p = 5$ and $\bar\rho$ has projective image isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, then the kernel of $\mathrm{proj}\,\bar\rho$ does not fix $F(\zeta_p)$.*

Then $\rho \sim \rho_{g,\lambda}$ for some Hilbert modular form g.

Proof. Let f be a Hilbert modular form such that $\bar\rho \sim \bar\rho_{f,\lambda}$. Applying a base change argument as in the proof of (3.5.5), we may assume that $[F : \mathbb{Q}]$ is even and that at each place $\mathfrak{p}|p$ of F, f is either unramified or special of conductor 1.

Applying the Jacquet-Langlands correspondence and the discussion of (3.1.14), as in the proof of (3.5.5), and using (3.1.6), we find that there exists a Hilbert modular eigenform f' of parallel weight 2 whose corresponding automorphic representation $\pi_{f'}$ is cuspidal at all primes $\mathfrak{p}|p$, and with $\bar\rho_{f,\lambda} \sim \bar\rho_{f',\lambda}$. Hence the Weil group representation corresponding to the local factor of $\pi_{f'}$ at $\mathfrak{p}$ is irreducible. The main result of [Sa] (or [CDT, B.4.2] together with the main result of [Ca]) imply that this Weil group representation can be obtained from the potentially crystalline representation $\rho_{f'}|_{G_{F_\mathfrak{p}}}$ by Fontaine's construction. This is only possible if $\rho_{f'}|_{G_{F_\mathfrak{p}}}$ is not potentially ordinary. This implies that ρ is strongly residually modular, and so the theorem follows from (3.5.5). $\qquad \square$

COROLLARY (3.5.8). *Let $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(E)$ be a continuous representation. Suppose that*

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[6] Again, the results of Gee make the last assumption unnecessary.

(1) ρ *is potentially Barsotti-Tate at* p, *and* $\det \rho$ *is the cyclotomic character times a finite order character.*

(2) ρ *is residually modular.*

(3) $\bar{\rho}|_{\mathbb{Q}(\sqrt{(-1)^{p-1/2}p})}$ *is absolutely irreducible.*

Then ρ *is modular.*

Proof. If $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible, then this follows from (3.5.7). So suppose that ρ is potentially ordinary at p. By [Di1, 6.4] there exists a modular eigenform f, such that $\rho_{f,\lambda}$ is potentially Barsotti-Tate and potentially ordinary at p, and such that $\bar{\rho}_{f,\lambda} \sim \bar{\rho}$. It follows that ρ is strongly residually modular in this case also. $\square$

Appendix on groupoids

(A.1) The purpose of the appendix is to recall some definitions involving categories cofibered in groupoids which are needed for the arguments in Section 2. Most of what we need is contained in [SGA1, §5] (which uses fibered categories) and [Ar]. The basic point is that the deformation theory of Galois representations is usually studied by considering functors whose values are the sets of *isomorphism classes* of liftings. In the presence of non-trivial automorphisms this definition can lead to pathologies, and it is better to consider functors with values in groupoids, that is, to consider the *category* of liftings and the isomorphisms between them, rather than just the set of isomorphism classes. For technical reasons, having to do with the fact that categories form a 2-*category*, it is more convenient to deal with categories cofibered in groupoids rather than functors with values in groupoids. None of this will come as any surprise to readers familiar with stacks, although we do not use that language here, since we will make no use of Grothendieck topologies.

(A.2) Let $\mathcal{E}$ be a category. We denote by $\mathrm{Ob}(\mathcal{E})$ the objects of $\mathcal{E}$, and $\mathrm{Mor}(\mathcal{E})$ its morphisms. A category over $\mathcal{E}$ is a category $\mathcal{F}$, together with a functor $\Theta : \mathcal{F} \to \mathcal{E}$. For $T \in \mathrm{Ob}(\mathcal{E})$ we denote by $\mathcal{F}(T)$ the subcategory of $\mathcal{F}$ consisting of objects ξ with $\Theta(\xi) = T$, and morphisms α with $\Theta(\alpha) = \mathrm{id}_T$.

A category $\mathcal{F}$ over $\mathcal{E}$ can be defined by specifying the objects of each of the categories $\mathcal{F}(T)$, and the morphisms $\eta \to \xi$ which cover (i.e. are taken by Θ to) a given morphism of $\mathcal{E}$ (as well of course as the composition laws for morphisms). It will often be convenient to define categories $\mathcal{F}$ over $\mathcal{E}$ in this way. The definition of the composition law will usually be obvious and left to the reader.

DEFINITION (A.2.1) (cf. [SGA1, 5.1]). Let $\alpha : \eta \to \xi$ be a morphism in $\mathcal{F}$, and set $T = \Theta(\eta)$, $S = \Theta(\xi)$, and $f = \Theta(\alpha)$. Then α is called *co-Cartesian* if for any $\xi' \in \mathrm{Ob}(\mathcal{F}(S))$, and any morphism $u : \eta \to \xi'$, with $\Theta(u) = f$, there exists a unique $\bar{u} : \xi \to \xi'$ in $\mathrm{Mor}(\mathcal{F}(S))$ such that $u = \bar{u} \circ \alpha$.

DEFINITION (A.2.2). $\mathscr{F}$ is said to be a *groupoid over* $\mathscr{E}$ (or less succinctly, a *category cofibered in groupoids over* $\mathscr{E}$) if

(1) Every morphism in $\mathscr{F}$ is co-Cartesian

(2) For any morphism $f : T \to S$ in $\mathrm{Mor}(\mathscr{E})$, and each $\eta \in \mathrm{Ob}(\mathscr{F}(T))$, there exists a morphism $\alpha : \eta \to \xi$ in $\mathrm{Mor}(\mathscr{F})$ with $\Theta(\alpha) = f$.

As a consequence of the definition one sees that if $\mathscr{F}$ is a groupoid over $\mathscr{E}$, then for any T in $\mathrm{Ob}(\mathscr{E})$, the category $\mathscr{F}(T)$ is a groupoid.

(A.3) The following example is the prototype for situations in which we will make use of the above definition.

Let $\mathbb{F}$ be a field with the discrete topology, and $\mathcal{O}$ a complete local ring with residue field $\mathbb{F}$. We denote by $\mathfrak{AR}_{\mathcal{O}}$ the category of Artinian local $\mathcal{O}$-algebras, A with maximal ideal $\mathfrak{m}_A$, such that $\mathbb{F} \xrightarrow{\sim} A/\mathfrak{m}_A$. Let G be a topological group, and $V_{\mathbb{F}}$ a finite dimensional $\mathbb{F}$-vector space equipped with a continuous G-action. We define a groupoid $D_{V_{\mathbb{F}}}$ over $\mathfrak{AR}_{\mathcal{O}}$ as follows

(1) For A in $\mathfrak{AR}_{\mathcal{O}}$, the objects of $D_{V_{\mathbb{F}}}(A)$ are pairs (V_A, ψ), where V_A is a finite free A-module equipped with a continuous action of G, and $\psi : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$ is an $\mathbb{F}$-linear isomorphism respecting the action of G.

(2) A morphism $(V_A, \psi) \to (V_{A'}, \psi')$ covering a given morphism $A \to A'$ of $\mathfrak{AR}_{\mathcal{O}}$, consists of an equivalence class $[\alpha]$, where $\alpha : V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ is an A'-linear isomorphism, compatible with the morphisms ψ and ψ', and with the action of G, and two morphisms are equivalent if they differ by an element of $A'^{\times}$.

(A.4) Let $\Theta : \mathscr{F} \to \mathscr{E}$ and $\Theta' : \mathscr{F}' \to \mathscr{E}$ be two categories over $\mathscr{E}$. A morphism of categories over $\mathscr{E}$, $\Psi : \mathscr{F} \to \mathscr{F}'$, is a functor such that $\Theta' \circ \Psi = \Theta$.

Two such functors Ψ, Ψ' are said to be equivalent if there is an isomorphism of functors $u : \Psi \xrightarrow{\sim} \Psi'$ such that for ξ in $\mathrm{Ob}(\mathscr{F})$, $\Theta'(u(\xi)) = \mathrm{id}_{\Theta(\xi)}$.

Let $\Theta : \mathscr{F} \to \mathscr{E}$, $\Theta' : \mathscr{F}' \to \mathscr{E}$ and $\Theta'' : \mathscr{F}'' \to \mathscr{E}$ be categories over $\mathscr{E}$, and suppose we are given morphisms $\Phi' : \mathscr{F}' \to \mathscr{F}$ and $\Phi'' : \mathscr{F}'' \to \mathscr{F}$. The 2-fiber product $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}''$ is a category defined as follows:

(1) An object of $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}''$ is a triple (ξ', ξ'', α) where $\xi' \in \mathrm{Ob}(\mathscr{F}')$, $\xi'' \in \mathrm{Ob}(\mathscr{F}'')$, and $\alpha : \Phi'(\xi') \xrightarrow{\sim} \Phi''(\xi'')$ is an isomorphism in $\mathscr{F}$ such that $\Theta(\alpha)$ is an identity morphism in $\mathscr{E}$.

(2) A morphism $(\xi', \xi'', \alpha) \to (\eta', \eta'', \beta)$ is a pair of morphisms $\gamma' : \xi' \to \eta'$ and $\gamma'' : \xi'' \to \eta''$, such that $\beta \circ \Phi'(\gamma') = \Phi''(\gamma'') \circ \alpha$.

If $\mathscr{F}, \mathscr{F}'$, and $\mathscr{F}''$ are groupoids over $\mathscr{E}$, then so is $\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}''$.

(A.5) Suppose that $\Theta : \mathscr{F} \to \mathscr{E}$ is a groupoid over $\mathscr{E}$. Let $\eta \in \mathrm{Ob}(\mathscr{F})$. We define a category $\tilde{\eta}$ associated to η as follows: An object of $\tilde{\eta}$ is a morphism $\alpha : \eta \to \xi$ in $\mathscr{F}$. A morphism $(\eta \overset{\alpha}{\to} \xi) \to (\eta \overset{\alpha'}{\to} \xi')$ is a morphism $\beta : \xi \to \xi'$ in $\mathscr{F}$ such that $\alpha' = \beta \circ \alpha$. The morphism $\tilde{\eta} \to \mathscr{E}$ sending $\eta \to \xi$ to $\Theta(\xi)$ gives $\tilde{\eta}$ the structure of a groupoid over $\mathscr{E}$. Note that $\tilde{\eta}$ is equipped with a functor $\tilde{\eta} \to \mathscr{F}$ given by sending $\eta \to \xi$ to ξ. This a morphisms of groupoids over $\mathscr{E}$. We will sometimes identify η with its associated category, and write η rather than $\tilde{\eta}$.

DEFINITION (A.5.1). A groupoid $\Theta : \mathscr{F} \to \mathscr{E}$ is called *representable* if there is an $\eta \in \mathrm{Ob}(\mathscr{F})$ such that the canonical functor $\tilde{\eta} \to \mathscr{F}$ is an equivalence of categories.

We remark that for $T \in \mathrm{Ob}(\mathscr{E})$, two isomorphic objects of $\tilde{\eta}(T)$ are related by a unique isomorphism. Hence $\tilde{\eta}(T)$ is equivalent to the category whose objects are morphisms $\Theta(\eta) \to T$, and whose only morphisms are the identity. More precisely, Θ induces an equivalence of categories between $\tilde{\eta}$ and $\widetilde{\Theta(\eta)}$. From this one sees easily that if $\mathscr{F}$ is representable, then the representing object η is defined up to canonical isomorphism. In particular, $\Theta(\eta)$ is well defined up to canonical isomorphism. We say that $\Theta(\eta)$ represents $\mathscr{F}$.

If **P** is a property of objects of $\mathscr{E}$ which is stable under isomorphism, we say that $\mathscr{F}$ has property **P** if it is representable, and the corresponding object T of $\mathscr{E}$ has property **P**.

Suppose that for each T in $\mathrm{Ob}(\mathscr{E})$ the isomorphism classes of $\mathscr{F}(T)$ form a set. We define a set-valued functor $|\mathscr{F}|$ on $\mathscr{E}$ by sending T to the set of isomorphism classes of $\mathscr{F}(T)$.

Suppose that $\mathscr{F}$ is representable, and let η be a corresponding object of $\mathrm{Ob}(\mathscr{F})$. Write $T = \Theta(\eta)$. Then there is an isomorphism of functors $\mathrm{Hom}_{\mathscr{E}}(T, \_) \overset{\sim}{\longrightarrow} |\mathscr{F}|$, so that T represents $|\mathscr{F}|$ in the usual set theoretic sense. Conversely, if $|\mathscr{F}|$ is representable, and for any T in $\mathrm{Ob}(\mathscr{E})$ two isomorphic objects of $\mathscr{F}(T)$ are related by a unique isomorphism, then $\mathscr{F}$ is representable.

DEFINITION (A.5.2). Suppose that $\Phi : \mathscr{F}' \to \mathscr{F}$ is a morphism of groupoids over $\mathscr{E}$.

(1) Φ is called relatively representable if for each η in $\mathrm{Ob}(\mathscr{F})$, the 2-fiber product

$$\mathscr{F}_\eta := \tilde{\eta} \times_{\mathscr{F}} \mathscr{F}'$$

is representable.

(2) Φ is called formally smooth if the associated map of functors $|\mathscr{F}'| \to |\mathscr{F}|$ is formally smooth. That is, for any morphism $T \to S$ in $\mathscr{E}$, the map

$$|\mathscr{F}'|(T) \to |\mathscr{F}'|(S) \times_{|\mathscr{F}|(S)} |\mathscr{F}|(T)$$

is surjective. This is equivalent to asking that for every η in $\mathrm{Ob}(\mathscr{F})$, the morphism $|\mathscr{F}_\eta| \to |\eta|$ is formally smooth.

As above, write $T = \Theta(\eta)$. If $\mathscr{F}_\eta$ is represented by $\xi \in \mathrm{Ob}(\mathscr{F}_\eta)$, write S for the image of ξ in $\mathscr{E}$. By definition of $\mathscr{F}_\eta$, there is a canonical morphism $T \to S$. Suppose now that **P** is a property of *morphisms* of $\mathscr{E}$, which is stable under composition with isomorphisms. We say that a morphism Φ as above has property **P** if it is relatively representable, and for each η in $\mathrm{Ob}(\mathscr{F})$, the corresponding morphism $T \to S$ has property **P**.

In fact we will not make much use of this last notion here. On the other hand, we will sometimes work in a situation where there is a notion of formal smoothness for morphisms in $\mathscr{E}$ (e.g. the category of schemes). In such situations we emphasize that saying a morphism of groupoids $\Phi : \mathscr{F}' \to \mathscr{F}$ is formally smooth does *not* indicate relative representability.

(A.6) The main reason we are forced to introduce the language of groupoids in this article, is that formation of fiber products is not compatible with the passage from a groupoid $\mathscr{F}$ over $\mathscr{E}$, to its associated functor $|\mathscr{F}|$. This is a serious technical issue since the definition of relative representability depends on formation of fiber products. The practical effect is that to construct certain geometric objects it is much more natural to work with groupoids. We illustrate this with a simple example.

Using the notation of (A.3), we fix an ordered basis of $V_\mathbb{F}$, and we define a groupoid $D_{V_\mathbb{F}}^\square$ over $\mathfrak{AR}_\mathbb{O}$ as follows: An object of $D_{V_\mathbb{F}}^\square(A)$ is a pair (V_A, β) where $V_A = (V_A, \psi)$ is an object of $D_{V_\mathbb{F}}(A)$, and β is an ordered A-basis for V_A lifting the chosen basis of $V_\mathbb{F}$. A morphism $(V_A, \psi, \beta) \to (V_{A'}, \psi', \beta')$ is an A'-linear isomorphism $V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ compatible with ψ, ψ', and the action of G, and taking β to β'. There is an obvious morphism $\Phi : D_{V_\mathbb{F}}^\square \to D_{V_\mathbb{F}}$.

Now consider the situation when the group G is trivial. If A is in $\mathfrak{AR}_\mathbb{O}$, and $\eta = (V_A, \psi)$ is in $D_{V_\mathbb{F}}(A)$, then $(D_{V_\mathbb{F}}^\square)_\eta$ is a principal homogeneous space for the formal group obtained by completing $\mathrm{PGL}_{d/\mathbb{O}}$ along its identity section, where $d = \dim_\mathbb{F} V_\mathbb{F}$. In other words, for any $A \to A'$ in $\mathfrak{AR}_\mathbb{O}$, $|(D_{V_\mathbb{F}}^\square)_\eta|(A')$ is a principal homogeneous space for $\ker(\mathrm{PGL}_d(A') \to \mathrm{PGL}_d(\mathbb{F}))$. On the other hand, $|D_{V_\mathbb{F}}^\square|(A')$ consists of a single element, and hence so does its fiber over the isomorphism class of η.

(A.7) We will need a slight extension of the above notions in a special case. Suppose that $\mathscr{F}$ is a groupoid over $\mathscr{E}$, and that $\mathscr{E}$ is a full subcategory of a category $\mathscr{E}'$. An *extension* of $\mathscr{F}$ to $\mathscr{E}'$ is a fully faithful embedding $\mathscr{F} \to \mathscr{F}'$ into a groupoid $\mathscr{F}'$ over $\mathscr{E}'$, such that for $T \in \mathscr{E}$, the induced morphism $\mathscr{F}(T) \to \mathscr{F}'(T)$ is an equivalence.

Now, using the notation of (A.3), suppose that $\mathscr{F}$ is a groupoid over $\mathfrak{AR}_\mathbb{O}$. We denote by $\widehat{\mathfrak{AR}_\mathbb{O}}$ the category of complete local $\mathbb{O}$-algebras, with residue field $\mathbb{F}$. We extend $\mathscr{F}$ to a groupoid $\widehat{\mathscr{F}}$ over $\widehat{\mathfrak{AR}_\mathbb{O}}$ as follows. If R in $\widehat{\mathfrak{AR}_\mathbb{O}}$ has radical $\mathfrak{m}_R$, we set $\widehat{\mathscr{F}}(R) = \varprojlim \mathscr{F}(R/\mathfrak{m}_R^i)$. More formally, if $\mathfrak{AR}_\mathbb{O}^R \subset \mathfrak{AR}_\mathbb{O}$ denotes the subcategory,

whose objects are the quotients $R/\mathfrak{m}_R^i$, for $i \geq 1$, and whose morphisms are the natural projection maps, then

$$\widehat{\mathscr{F}}(R) = \operatorname{Hom}_{\mathfrak{AR}_{\mathbb{O}}}(\mathfrak{AR}_{\mathbb{O}}^R, \mathscr{F}).$$

Given a morphism $h : (R, \mathfrak{m}_R) \to (R', \mathfrak{m}_{R'})$ in $\widehat{\mathfrak{AR}}_{\mathbb{O}}$, we write $\mathfrak{AR}_{\mathbb{O}}^h : \mathfrak{AR}_{\mathbb{O}}^R \to \mathfrak{AR}_{\mathbb{O}}^{R'}$, for the corresponding functor which sends $R/\mathfrak{m}_R^i$ to $R'/\mathfrak{m}_{R'}^i$. Then a morphism

$$(\mathfrak{AR}_{\mathbb{O}}^R \xrightarrow{\eta} \mathscr{F}) \to (\mathfrak{AR}_{\mathbb{O}}^{R'} \xrightarrow{\eta'} \mathscr{F})$$

in $\widehat{\mathscr{F}}$ covering h, is a natural transformation $\eta \to \eta' \circ \mathfrak{AR}_{\mathbb{O}}^h$ covering the natural morphism $\mathrm{id} \to \mathfrak{AR}_{\mathbb{O}}^h$ of functors on $\mathfrak{AR}_{\mathbb{O}}^R$, induced by h. (Here we regard id and $\mathfrak{AR}_{\mathbb{O}}^h$ as taking values in $\mathfrak{AR}_{\mathbb{O}}$.)

DEFINITION (A.7.1). (1) If $\mathscr{F}$ is a groupoid over $\mathfrak{AR}_{\mathbb{O}}$ we say that $\mathscr{F}$ is pro-representable if $\widehat{\mathscr{F}}$ is representable.

(2) Suppose that $\Phi : \mathscr{F}' \to \mathscr{F}$ is a morphism of groupoids over $\mathscr{E}$. We say that Φ is relatively pro-representable if the induced morphism $\widehat{\Phi} : \widehat{\mathscr{F}'} \to \widehat{\mathscr{F}}$ is relatively representable.

(A.8) Suppose that $\Theta : \mathscr{F} \to \mathscr{E}$ is a category over $\mathscr{E}$. Fix a $\xi \in \mathrm{Ob}(\mathscr{F})$. We denote by $\mathscr{F}_{(\xi)}$ the category whose objects consist of pairs (η, α) where η is in $\mathrm{Ob}(\mathscr{F})$, and $\alpha : \eta \to \xi$ is in $\mathrm{Mor}(\mathscr{F})$. A morphism $(\eta, \alpha) \to (\eta', \alpha')$ is a morphism $\beta : \eta \to \eta'$ in $\mathscr{F}$ such that $\alpha = \alpha' \circ \beta$.

Write $T = \Theta(\xi)$. Then Θ induces a functor $\Theta_{(\xi)} : \mathscr{F}_{(\xi)} \to \mathscr{E}_{(T)}$ given by sending (η, α) to $(\Theta(\eta), \Theta(\alpha))$. If $\mathscr{F}$ is a groupoid over $\mathscr{E}$, then $\mathscr{F}_{(\xi)}$ is a groupoid over $\mathscr{E}_{(T)}$.

In the situation of (A.3), if A is in $\mathfrak{AR}_{\mathbb{O}}$, and $\xi = V_A$ is an object of $D_{V_{\mathbb{F}}}(A)$, then $\mathfrak{AR}_{\mathbb{O},(A)}$ is the category of $\mathbb{O}$-algebras A' in $\mathfrak{AR}_{\mathbb{O}}$ equipped with a map of $\mathbb{O}$-algebras $A' \to A$, while $D_{V_{\mathbb{F}},(\xi)}(A' \to A)$ consists of objects $V_{A'}$ in $D_{V_{\mathbb{F}}}(A')$ equipped with an isomorphism $V_{A'} \otimes_A A' \xrightarrow{\sim} V_A$.

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