Moduli of K3 Surfaces and Abelian Varieties

Moduli van K3 oppervlakken en abelse variëteiten

(met een samenvatting in het Nederlands)

Proefschrift

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Jordan Ivanov Rizov

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promotor: Prof. Dr. F. Oort
Faculteit Wiskunde en Informatica
Universiteit Utrecht

copromotor: Dr. B. J. J. Moonen
Korteweg-de Vries Instituut
Universiteit van Amsterdam

leescommissie: Prof. Dr. Y. André
Prof. Dr. S. J. Edixhoven
Prof. Dr. G. B. M. van der Geer
Prof. Dr. E. J. N. Looijenga

Faculteit Wiskunde en Informatica
Universiteit Utrecht

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To Lada, Ivan and Lazar

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Introduction

In order to study a class of algebraic varieties one constructs moduli spaces. A point of a moduli space corresponds to an isomorphism class of the varieties we want to study. In this way we parametrize all possible isomorphism classes of these varieties. Properties of the moduli space give information about the deformations of the varieties under consideration. Classically, moduli spaces over $\mathbb{C}$ are constructed via periods as quotient manifolds. In modern algebraic geometry one uses ideas of Grothendieck, Mumford, Artin and others to construct moduli spaces over an arbitrary base, such as $\mathbb{Z}$.

We will consider moduli spaces of K3 surfaces with a polarization. For a natural number d and an algebraically closed field k, a K3 surface with a polarization of degree $2d$ over k is a pair $(X, \mathcal{L})$ consisting of a K3 surface X over k and an ample line bundle $\mathcal{L}$ on X with self-intersection number $(\mathcal{L}, \mathcal{L}) = 2d$. The moduli space of polarized K3 surfaces with certain level structure over $\mathbb{C}$ is constructed as an open subspace of the Shimura variety associated with $\mathrm{SO}(2, 19)$. Over $\mathbb{Z}$ we use techniques developed by Artin to show the existence of such spaces.

When studying algebraic curves one constructs an abelian variety, called the Jacobian of the curve. The geometry of this abelian variety describes properties of the curve. Here we consider a similar construction for K3 surfaces. Namely, we assign to every polarized K3 surface an abelian variety with certain properties.

Kuga and Satake associate with every polarized complex K3 surface $(X, \mathcal{L})$ a complex abelian variety called the Kuga-Satake abelian variety of $(X, \mathcal{L})$. We use this construction to define morphisms between moduli spaces of polarized K3 surface with certain level structures and moduli spaces of polarized abelian varieties with level structure over $\mathbb{C}$. In this thesis we study these morphisms. We prove first that they are defined over finite extensions of $\mathbb{Q}$. Then we show that they extend in positive characteristic. In this was we give an indirect construction of Kuga-Satake abelian varieties over an arbitrary base.

Moduli spaces. In Chapter 1 we construct moduli spaces of polarized K3 surfaces using the language of algebraic stacks. We define the categories $\mathcal{F}_{2d}$ and $\mathcal{M}_{2d}$ of primitively polarized (respectively polarized) K3 surfaces of degree $2d$ over $\mathbb{Z}$ and show that they are Deligne-Mumford stacks over $\mathbb{Z}$.

For various technical reasons we will need to work with algebraic spaces rather than with Deligne-Mumford stacks. In the case of abelian varieties one introduces level n-structures using Tate modules and considers moduli functors of polarized abelian varieties with level n-structure for $n \in \mathbb{N}$, $n \geq 3$. These functors are representable by schemes. We adopt a similar strategy in order to define moduli functors which are representable by algebraic spaces. For a certain class of compact open subgroups $\mathbb{K}$ of $\mathrm{SO}(2,19)(\mathbb{A}_f)$ we introduce the notion of a level $\mathbb{K}$-structure on K3 surfaces using their second étale cohomology groups. Further, we introduce moduli spaces $\mathcal{F}_{2d,\mathbb{K}}$ of primitively polarized K3 surfaces with level $\mathbb{K}$-structure and show that these are smooth algebraic spaces over $\mathrm{Spec}(\mathbb{Z}[1/N_{\mathbb{K}}])$ where $N_{\mathbb{K}} \in \mathbb{N}$ depends on $\mathbb{K}$. These moduli spaces are finite unramified covers of $\mathcal{F}_{2d}$. Important examples of level structures are spin level n-structures. These are level structures defined by the images of some principal level n-subgroups of $\mathrm{CSpin}(2,19)(\mathbb{A}_f)$ under the adjoint representation homomorphism $\mathrm{CSpin}(2,19) \to \mathrm{SO}(2,19)$. We denote the corresponding moduli space by $\mathcal{F}_{2d,n^{\mathrm{sp}}}$.

Strata. In positive characteristic one can define interesting subvarieties of moduli spaces of abelian varieties and of curves. Such loci can be given by considering the collection of these objects having fixed certain discrete invariants, such as for instance filtrations on BT_1-groups or Newton polygons (see [Oor01a] and [Oor01b]). A similar approach can be taken when studying moduli spaces of K3 surfaces.

To every K3 surface over a perfect field k of characteristic $p > 0$ one associates a Newton polygon. By definition it is the Newton polygon of the F-crystal $H^2_{\mathrm{cris}}(X/W(k))$. Denote by α the smallest slope of the Newton polygon of X. We define the height of X to be infinite if $\alpha = 1$ and $1/(1-\alpha)$ otherwise. If finite, the height of a K3 surface takes integral values from 1 to 10. We look at the subspaces $\mathcal{M}^{(h)}_{2d,\mathbb{F}_p}$ of $\mathcal{M}_{2d} \otimes \mathbb{F}_p$ of K3 surfaces with height at least h. The collection of those 11 subspaces is called the height stratification of $\mathcal{M}_{2d} \otimes \mathbb{F}_p$. One further stratifies $\mathcal{M}^{(11)}_{2d,\mathbb{F}_p}$ by the Artin invariant (see for instance [Art74]). In this way we obtain a filtration of the moduli space $\mathcal{M}_{2d} \otimes \mathbb{F}_p$

$$\mathcal{M}_{2d} \otimes \mathbb{F}_p = \mathcal{M}^{(1)}_{2d,\mathbb{F}_p} \supset \mathcal{M}^{(2)}_{2d,\mathbb{F}_p} \supset \cdots \supset \mathcal{M}^{(11)}_{2d,\mathbb{F}_p} = \Sigma_1 \supset \cdots \supset \Sigma_{10}.$$

The following question rises naturally.

Question. *Are the all subspaces in the height strata of $\mathcal{M}_{2d} \otimes \mathbb{F}_p$ non-empty?*

This can be reformulated in the following way: For a given natural number d and a prime p determine all Newton polygons of polarized K3 surfaces of degree $2d$ over fields of characteristic p. This is an analogue of the Manin problem for Newton polygons of abelian varieties ([Man63, Conj. 2, p. 76]).

In Chapter 2 we answer partially the question posed above proving the following result.

Theorem. *For every d, large enough and prime to $p > 2$, the subspaces in the height strata of $\mathcal{M}_{2d} \otimes \mathbb{F}_p$ are non-empty.*

The idea of the proof is to start with a polarized abelian surface (A, λ) over k of certain degree and to use λ to construct an ample line bundle on the Kummer surface X associated to A. In this way we find $\bar{k}$-valued points of $\mathcal{M}_{2d} \otimes \mathbb{F}_p$. Making some appropriate choices of supersingular polarized abelian surfaces (A, λ) we are able to show that the height strata of $\mathcal{M}_{2d} \otimes \mathbb{F}_p$ are non-empty if d is large enough. The construction gives explicit bounds for d.

Complex multiplication for K3 surfaces. The construction of Kuga-Satake varieties given in [KS67] uses transcendental methods involving Betti cohomology groups. It gives a relation between the Betti cohomology groups of a K3 surface X and of its associated abelian variety A. In its simplest form this relation can be written as an inclusion of Hodge structures

$$H^2_B(X, \mathbb{Q}) \hookrightarrow H^1_B(A, \mathbb{Q}) \otimes H^1_B(A, \mathbb{Q}).$$

P. Deligne ([Del72]) shows, among other things, that a similar relation holds for étale cohomology groups. He uses it to prove the Weil conjecture for K3 surfaces over finite fields. In [And96a] Y. André studies the rationality properties of the Kuga-Satake construction, describing the motive of a K3 surface and computing the motivic Galois group.

The main goal of our work is to extend this construction to K3 surfaces in positive characteristic. We will do this by defining morphisms from moduli spaces of primitively polarized K3 surfaces with spin level n-structure to moduli spaces of polarized abelian varieties with level n-structure. An essential step in carrying out this program is to build a bridge between the two approaches to moduli of K3 surfaces, namely the one via algebraic stacks and the one using Shimura varieties.

For a certain class of compact open subgroups $\mathbb{K}$ of $\mathrm{SO}(2, 19)(\mathbb{A}_f)$ we define a period morphism

$$j_{d,\mathbb{K},\mathbb{C}} \colon \mathcal{F}_{2d,\mathbb{K}} \otimes \mathbb{C} \to Sh_{\mathbb{K}}(\mathrm{SO}(V_{2d}, \psi_{2d}), \Omega^{\pm})_{\mathbb{C}}.$$

This period morphism differs slightly from the ones considered for instance in [BBD85, Exposé XIII] and [Fri84, §1]. This is due to the fact that we work with moduli spaces of polarized K3 surfaces over $\mathbb{Q}$ and in general these have more than one connected component. In Chapter 3 we study the field of definition of this period morphism.

The main theorem for complex multiplication for abelian varieties describes the action of the elements of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ on the torsion points of an abelian variety A with complex multiplication. In [Del71] Deligne uses this description as a departure point for the definition of canonical models of Shimura varieties and proves that $\varprojlim \mathcal{A}_{g,1,n,\mathbb{Q}}$ is the canonical model of $Sh(\mathrm{CSp}_{2g}, \mathfrak{H}^{\pm})_{\mathbb{C}}$ (Théorème 4.21 in *loc. cit.*). The main result of Chapter 3 is the following.

Theorem. *The field of definition of the period morphism $j_{d,\mathbb{K},\mathbb{C}}$ is $\mathbb{Q}$. In other words $j_{d,\mathbb{K},\mathbb{C}}$ descends to a morphism*

$$j_{d,\mathbb{K}} \colon \mathcal{F}_{2d,\mathbb{K}} \otimes \mathbb{Q} \to Sh_{\mathbb{K}}(\mathrm{SO}(2,19), \Omega^{\pm})$$

where $Sh_{\mathbb{K}}(\mathrm{SO}(2,19), \Omega^{\pm})$ is the canonical model of $Sh_{\mathbb{K}}(\mathrm{SO}(2,19), \Omega^{\pm})_{\mathbb{C}}$.

Just like in the case of abelian varieties it gives a modular interpretation of (an open part of) the canonical model of $Sh(\mathrm{SO}(2,19), \Omega^{\pm})_{\mathbb{C}}$ as a moduli space.

To prove that the field of definition of $j_{d,\mathbb{K},\mathbb{C}}$ is $\mathbb{Q}$ we need to find 'enough' points in $\mathcal{F}_{2d,\mathbb{K}} \otimes \mathbb{C}$ and $Sh_{\mathbb{K}}(\mathrm{SO}(2,19), \Omega^{\pm})_{\mathbb{C}}$ for which we can control the action of $\mathrm{Aut}(\mathbb{C})$. The definition of canonical models of Shimura varieties suggests a collection of such points, namely the set of special points. Here we will restrict this set a bit by working with points corresponding to exceptional K3 surfaces. By definition these are K3 surfaces X over $\mathbb{C}$ such that $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X) = 20$.

The proof of the preceding theorem splits up in two parts. We first prove a version of the main theorem for complex multiplication [Mil04, Thm. 11.2] for exceptional K3 surfaces. It gives a relation between the transcendental lattices of an exceptional K3 surface X and of its conjugate X^{σ} by an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$, fixing the reflex field of X. The main tool is a result of Shioda and Inose. Using their construction we reduce the proof of the main theorem of complex multiplication for exceptional K3 surfaces to a similar statement for abelian surfaces. The latter follows easily from the theorem of Shimura and Taniyama. Next we show that the set of points in $\mathcal{F}_{2d,\mathbb{K}} \otimes \mathbb{C}$ corresponding to exceptional K3 surfaces with a given reflex field is dense for the Zariski topology. The proof of the fact that the field of definition of the period morphism $j_{d,\mathbb{K},\mathbb{C}}$ is $\mathbb{Q}$ is a formal consequence of those two results.

We conclude Chapter 3 by showing that an analogue of the theorem of Shimura and Taniyama holds for all complex K3 surfaces with complex multiplication. Further, we prove that any such K3 surface can be defined over an abelian extension of its reflex field. In this way we complete a theory of complex multiplication for K3 surfaces.

We should mention that the proofs given in Chapter 3 are quite different from the ones given in [ST61] in the case of abelian varieties. Shimura and Taniyama work directly with a given abelian variety A using the geometric interpretation of $H^1_{\mathrm{et}}(A, \hat{\mathbb{Z}})$. We obtain our results from the general properties of canonical models of Shimura varieties using the period morphisms. We wonder if one can give proofs of the statements in Chapter 3 working directly with an exceptional K3 surface just like in the case of abelian varieties. More precisely, we wonder if argumentation of the type 'a Hodge cycle is an absolute Hodge cycle' on a K3 surface (see [And96b] and [Del82]) could lead to complete proofs of the results in question.

Kuga-Satake morphisms. Chapter 4 is devoted to Kuga-Satake abelian varieties.

Let us explain briefly the construction of these varieties over $\mathbb{C}$. Starting with a polarized complex K3 surface $(X, \mathcal{L})$ one considers the second primitive Betti cohomology group

$$P^2_B(X, \mathbb{Z}(1)) := c_1(\mathcal{L})^\perp \subset H^2_B(X, \mathbb{Z}(1)).$$

The orthogonal complement is taken with respect to the Poincaré pairing on $H^2_B(X, \mathbb{Z}(1))$. Using the polarized $\mathbb{Z}$-Hodge structure on $P^2_B(X, \mathbb{Z}(1))$ one defines a polarized $\mathbb{Z}$-Hodge structure of type $\{(1,0),(0,1)\}$ on the even Clifford algebra $C^+\big(P^2_B(X, \mathbb{Z}(1))\big)$. One might think of this construction as "taking a square root of a Hodge structure". Such a Hodge structure corresponds to a complex abelian variety A, called the Kuga-Satake abelian variety associated to $(X, \mathcal{L})$. Using Kuga-Satake varieties one can deduce some properties of K3 surfaces, mostly of motivic nature, from the corresponding properties of abelian varieties.

At this point one may ask whether one can use this construction to define Kuga-Satake abelian varieties over subfields of $\mathbb{C}$. Or whether one can construct Kuga-Satake abelian schemes starting with families of polarized K3 surfaces. We find some answers in [Del72] and [And96a]. One can go even further and ask whether one can define Kuga-Satake abelian varieties in positive characteristic. We combine all these questions in one, which was originally the motivation for our work.

Question. *Can one define Kuga-Satake abelian varieties using only methods of algebraic geometry without making use of complex analytic constructions?*

Up to isogeny a positive answer to this question can be found in [And96b, Thm 7.1] and [And96a, Thm. 1.5.1]. We refer also to Chapter 9 and 10.2.4 in [And04]. Starting with a polarized K3 surface $(X, \mathcal{L})$, Y. André constructs a "motive" which is isomorphic to the motive of the Kuga-Satke abelian variety of $(X, \mathcal{L})$. The construction is purely algebro-geometric.

In Chapter 4 we solve a modification of this problem. Namely, we will be interested in a Kuga-Satake construction over an arbitrary base without putting any restriction on the "methods". The reason is that we will use the existing transcendental construction as a starting point. We explain this in more detail.

P. Deligne gives an interpretation of the Kuga-Satake construction in terms of the adjoint representation homomorphism $\mathrm{CSpin}(2,19) \to \mathrm{SO}(2,19)$ and the spin representation homomorphism $\mathrm{CSpin}(2,19) \hookrightarrow \mathrm{CSp}_{2g}$ where $g = 2^{19}$ (see [Del72, §§3 & 4]). We consider the morphisms, between the Shimura varieties associated to the groups $\mathrm{CSpin}(2,19)$, $\mathrm{SO}(2,19)$ and CSp_{2g}, defined by the adjoint and the spin representations. Putting together these maps and the results of Chapter 3, for every $n \geq 3$, we define a Kuga-Satake morphism

$$f^{ks}_{d,a,n,\gamma,E_n} : \mathcal{F}_{2d,n^{\mathrm{sp}}} \otimes E_n \to \mathcal{A}_{g,d',n} \otimes E_n$$

where E_n is a finite abelian extension of $\mathbb{Q}$. For details we refer to Section 4.2.5. The morphism $f^{ks}_{d,a,n,\gamma,E_n}$ assigns to every primitively polarized complex K3 surface with spin level n-structure its associate Kuga-Satake abelian variety plus extra data (polarization and level n-structure). In this way the first step of our program is completed. Next we show that $f^{ks}_{d,a,n,\gamma,E_n}$ extends over an open part of $\mathrm{Spec}(\mathcal{O}_{E_n})$ where $\mathcal{O}_{E_n}$ is the ring of integers in E_n. In a sense this is the core of the matter. First we prove a general result on extension of morphisms.

Proposition. *Let U be a scheme smooth over a discrete valuation ring R of mixed characteristic $(0,p)$. Suppose given three natural numbers g,d,n such that $n \geq 3$ and p does not divide dn. Let $\{\eta, s\}$ be the generic and the special points of $\mathrm{Spec}(R)$ and let $f_\eta \colon U_\eta \to \mathcal{A}_{g,d,n,\eta}$ be a morphism. Assume that the total ramification index e of R satisfies $e < p-1$ and that all generic points x of the special fiber U_s of U satisfy the following:*

> *Let $\mathcal{O}_{U,x}$ be the local ring of x and denote by L the field of fractions of $\mathcal{O}_{U,x}$. Then the morphism $f_\eta \colon \mathrm{Spec}(L) \to \mathcal{A}_{g,d,n,\eta}$ extends to a morphism $\tilde{f} \colon \mathrm{Spec}(\mathcal{O}_{U,x}) \to \mathcal{A}_{g,d,n} \otimes R$.*

Then f_η extends uniquely to a morphism $f \colon U \to \mathcal{A}_{g,d,n} \otimes R$ over R.

The proof of this proposition is based on a result of G. Faltings on extension of abelian schemes (see for instance [Moo98, Lemma 3.6]). It is not difficult to see that one can apply this proposition to an atlas U of $\mathcal{F}_{2d,n^{\mathrm{sp}}}$ over an open part of $\mathrm{Spec}(\mathcal{O}_{E_n})$. Then using a descent argument one concludes that the Kuga-Satake morphism extends over that open part of $\mathrm{Spec}(\mathcal{O}_{E_n})$. More precisely we prove the following statement.

Theorem. *Let $d,n \in \mathbb{N}$ and suppose that $n \geq 3$. Then the Kuga-Satake morphism $f^{ks}_{d,n,a,\gamma,E_n} \colon \mathcal{F}_{2d,n^{\mathrm{sp}},E_n} \to \mathcal{A}_{g,d',n,E_n}$ extends uniquely to a morphism*

$$f^{ks}_{d,a,n,\gamma} \colon \mathcal{F}_{2d,n^{\mathrm{sp}}} \otimes \mathcal{O}_{E_n}[1/N] \to \mathcal{A}_{g,d',n} \otimes \mathcal{O}_{E_n}[1/N]$$

where $N = 2dd'nl$ and l is the product of the prime numbers p whose ramification index e_p in E_n is $\geq p-1$.

We conclude Chapter 4 with some applications. First we show that the étale cohomology relations from [Del72, (6.6.1)] hold for the Kuga-Satake abelian varieties we construct. Then we focus our attention on the ordinary locus of $\mathcal{F}_{2d,n^{\mathrm{sp}}} \otimes \mathbb{F}_p$, where p is a prime not dividing N. Suppose that k is a finite field of characteristic p. One can easily see that $f^{ks}_{d,a,n,\gamma}$ maps an ordinary point $x = (X, \mathcal{L}, \nu)$ in $\mathcal{F}_{2d,n^{\mathrm{sp}}} \otimes \mathbb{F}_p(k)$ to an ordinary point $y = (A, \mu, \epsilon)$ in $\mathcal{A}_{g,d',n} \otimes \mathbb{F}_p(k)$. For both ordinary K3 surfaces and ordinary abelian varieties we have the notion of canonical lifts. We denote by $x^{\mathrm{can}} = (X^{\mathrm{can}}, \mathcal{L}, \nu)$ the canonical lift of X over $W(k)$ and by $y^{\mathrm{can}} = (A^{\mathrm{can}}, \mu, \epsilon)$ the canonical lift of A. We prove that $f^{ks}_{d,a,n,\gamma}(x^{\mathrm{can}}) = y^{\mathrm{can}}$. A straightforward corollary of this is that the restriction of the Kuga-Satake morphism to the ordinary locus of $\mathcal{F}_{2d,n^{\mathrm{sp}}} \otimes \mathbb{F}_p$ is quasi-finite. One can also use this result to prove a special case of the André-Oort conjecture for the Shimura variety associated to $\mathrm{SO}(2,19)$. As we have not included the proof in this work we will omit any discussion in this direction.

Notations and Conventions

General

We write $\hat{\mathbb{Z}}$ for the profinite completion of $\mathbb{Z}$. We denote by $\mathbb{A}$ the ring of adèles of $\mathbb{Q}$ and by $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adèles of $\mathbb{Q}$. Similarly, for a number field E we denote by $\mathbb{A}_E$ and $\mathbb{A}_{E,f}$ the ring of adèles and the ring of finite adèles of E.

If A is a ring, $A \to B$ a ring homomorphism then for any A-module (A-algebra etc.) V we will denote by V_B the B-module (B-algebra etc.) $V \otimes_A B$.

For a ring A we denote by (Sch/A) the category of schemes over A. We will write Sch for the category of schemes over $\mathbb{Z}$.

By a variety over a field k we will mean a separated, geometrically integral scheme of finite type over k. For a variety X over $\mathbb{C}$ we will denote by X^{an} the associated analytic variety. For an algebraic stack $\mathcal{F}$ over a scheme S and a morphism of schemes $S' \to S$ we will denote by $\mathcal{F}_{S'}$ the product $\mathcal{F} \times_S S'$ and consider it as an algebraic stack over S'.

A Jacobi level n-structure on a polarized abelian scheme (A, λ) of relative dimension g over a base scheme S is an isomorphism of sheaves

$$\theta\colon A[n] \to (\mathbb{Z}/n\mathbb{Z})_S^{2g}$$

such that there exists an isomorphism of sheaves $\nu\colon (\mathbb{Z}/n\mathbb{Z})_S \to \mu_n$ making the following diagram

$$\begin{tikzcd}
A[n] \times A[n] \arrow[r, "\epsilon_\lambda"] \arrow[d, "\theta\times\theta"'] & \mu_{n,S} \\
(\mathbb{Z}/n\mathbb{Z})_S^{2g} \times (\mathbb{Z}/n\mathbb{Z})_S^{2g} \arrow[r, "\psi"] & (\mathbb{Z}/n\mathbb{Z})_S \arrow[u, "\nu"']
\end{tikzcd}$$

commutative. Here ϵ_λ is the Weil pairing and ψ denotes the standard alternating bilinear form on $(\mathbb{Z}/n\mathbb{Z})^{2g}$. We denote by $\mathcal{A}_{g,d,n}$ the moduli stack of g-dimensional abelian varieties with a polarization of degree d^2 and a Jacobi level n-structure. It is a Deligne-Mumford stack which is smooth over $\mathbb{Z}[1/dn]$. We will write $\mathcal{A}_{g,d}$ for $\mathcal{A}_{g,d,1}$.

Algebraic Groups

A superscript ^{0} usually indicates a connected component for the Zariski topology. For an algebraic group G will denote by G^0 the connected component of the identity. We will use the superscript $^+$ to denote connected components for other topologies.

For a reductive group G over $\mathbb{Q}$ we denote by G^{ad} the adjoint group of G, by G^{der} the derived group of G and by G^{ab} the maximal abelian quotient of G. We let $G(\mathbb{R})_+$ denote the group of elements of $G(\mathbb{R})$ whose image in $G^{\mathrm{ad}}(\mathbb{R})$ lies in its identity component $G^{\mathrm{ad}}(\mathbb{R})^+$, and we let $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$.

Let V be a vector space over $\mathbb{Q}$ and let $G \hookrightarrow \mathrm{GL}(V)$ be an algebraic group over $\mathbb{Q}$. Suppose given a full lattice L in V (i.e., $L \otimes \mathbb{Q} = V$). Then $G(\mathbb{Z})$ and $G(\hat{\mathbb{Z}})$ will denote the abstract groups consisting of the elements in $G(\mathbb{Q})$ and $G(\mathbb{A}_f)$ preserving the lattices L and $L_{\hat{\mathbb{Z}}}$ respectively.

Chapter 1

Moduli Stacks of Polarized K3 surfaces

In this chapter we set up the basic theory of moduli stacks of (primitively) polarized K3 surfaces. In various places in the literature one finds detailed accounts on coarse moduli schemes of primitively polarized complex K3 surfaces. We outline in Section 1.4.3 two approaches to the theory, one via geometric invariant theory ([Vie95]) and another via periods of complex K3 surfaces ([BBD85, Exposé XIII] and [Fri84, §1]). Here we take up a different point of view and work with moduli stacks rather than with coarse moduli schemes. In this way, our exposition is closer to [Ols04] where moduli stacks of primitively polarized K3 surfaces and their compactifications over $\mathbb{Q}$ are constructed. This approach turns out to be essential for our ultimate goal - the construction of Kuga-Satake morphisms in mixed characteristic.

Let us outline briefly the contents of this chapter. In the first few sections we review some basic properties of K3 surfaces. Then we continue with the study of the representability of Picard and automorphism functors arising from K3 surfaces. The core of the chapter is Section 1.4.3 in which we define various moduli functors of polarized K3 surfaces and prove that those define Deligne-Mumford stacks. In Section 1.5.1 we define level structures on K3 surfaces associated to compact open subgroups of $\mathrm{SO}(2, 19)(\mathbb{A}_f)$. In the last section we show that the moduli functors of primitively polarized K3 surfaces with level structure are representable by algebraic spaces.

1.1 Basic Results

1.1.1 Definitions and Examples

We will briefly recall some basic notions concerning families of K3 surfaces.

Definition 1.1.1. Let k be a field. A non-singular, proper surface X over k is called a

K3 surface if $\Omega^2_{X/k} \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Note that a K3 surface is automatically projective. Let us give some basic examples one can keep in mind:

Example 1.1.2. Let S be a non-singular sextic curve in $\mathbb{P}^2_k$ where k is a field and consider a double cover i.e., a finite generically étale morphism, $\pi\colon X \to \mathbb{P}^2_k$ which is ramified along S. Then X is a K3 surface.

Example 1.1.3. *Complete intersections*: Let X be a smooth surface which is a complete intersection of n hypersurfaces of degree $d_1, \ldots, d_n$ in $\mathbb{P}^{n+2}$ over a field k. The adjunction formula shows that $\Omega^2_{X/k} \cong \mathcal{O}_X(d_1 + \cdots + d_n - n - 3)$. So a necessary condition for X to be a K3 surface is $d_1 + \cdots + d_n = n + 3$. The first three possibilities are:

$$\begin{array}{lc} n = 1 & d_1 = 4 \\ n = 2 & d_1 = 2, d_2 = 3 \\ n = 3 & d_1 = d_2 = d_3 = 2. \end{array}$$

For a complete intersection M of dimension n one has that $H^i(M, \mathcal{O}_M(m)) = 0$ for all $m \in \mathbb{Z}$ and $1 \leq i \leq n - 1$. Hence in those three cases we have $H^1(X, \mathcal{O}_X) = 0$ and therefore X is a K3 surface.

Example 1.1.4. Let A be an abelian surface over a field k of characteristic different from 2. Let $A[2]$ be the kernel of the multiplication by-2-map, let $\pi\colon \tilde{A} \to A$ be the blow-up of $A[2]$ and let $\tilde{E}$ be the exceptional divisor. The automorphism $[-1]_A$ lifts to an involution $[-1]_{\tilde{A}}$ on $\tilde{A}$. Let X be the quotient variety of $\tilde{A}$ by the group of automorphisms $\{\mathrm{id}_{\tilde{A}}, [-1]_{\tilde{A}}\}$ and denote by $\iota : \tilde{A} \to X$ the quotient morphism. It is a finite map of degree 2. We have the following diagram

$$\begin{tikzcd} & \tilde{A} \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\ A & & X \end{tikzcd}$$

of morphisms over k. The variety X is a K3 surface and it is called the *Kummer surface* associated to A.

Definition 1.1.5. By a *K3 scheme* over a base scheme S we will mean a scheme X and a proper and smooth morphism $\pi\colon X \to S$ whose geometric fibers are K3 surfaces. A *K3 space over a scheme* S is an algebraic space X together with a proper and smooth morphism $\pi\colon X \to S$ such that there is an étale cover $S' \to S$ of S for which $\pi'\colon X' = X \times_S S' \to S'$ is a K3 scheme.

If $\pi\colon X \to S$ is a K3 space, then $\pi_* \mathcal{O}_X = \mathcal{O}_S$. Indeed, this is true since π is proper and its geometric fibers are reduced and connected.

1.1.2 Ample Line Bundles on K3 Surfaces

In order to construct the moduli stacks of polarized K3 spaces one needs a number of results on ample line bundles. We give them below.

Definition 1.1.6. Let X be a K3 surface over a field k. The self-intersection index $(\mathcal{L}, \mathcal{L})_X$ of a line bundle $\mathcal{L}$ on X will be called its *degree.* A line bundle $\mathcal{L}$ on X is called *primitive* if $\mathcal{L} \otimes \bar{k}$ is is not a positive power of a line bundle on $X_{\bar{k}}$.

Theorem 1.1.7. *Let X be a K3 surface over a field k.*

(a) If $\mathcal{L}$ is a line bundle on X, then $(\mathcal{L}, \mathcal{L})$ is even. If $\mathcal{L}$ is ample and $d := (\mathcal{L}, \mathcal{L})/2$, then the Hilbert polynomial of $\mathcal{L}$ is given by $h_{\mathcal{L}}(t) = dt^2 + 2$.

(b) Suppose $\mathcal{L}$ is an ample bundle. Then $\mathcal{L}$ is effective and $H^i(X, \mathcal{L}) = 0$ for $i > 0$. Further, $\mathcal{L}^n$ is generated by global sections if $n \geq 2$ and is very ample if $n \geq 3$.

Proof. (a) First note that, by Serre duality, $h^2(\mathcal{O}_X) = h^0(\Omega^2_{X/k}) = h^0(\mathcal{O}_X) = 1$. Since $h^1(\mathcal{O}_X) = 1$ we find that $\chi(\mathcal{O}_X) = 2$. Hirzebruch-Riemann-Roch gives

$$\begin{aligned} \chi(\mathcal{L}) &= \chi(\mathcal{O}_X) + \frac{1}{2} \cdot \big((\mathcal{L}, \mathcal{L}) - (\mathcal{L}, \Omega^2_{X/k})\big) \\ &= 2 + \frac{1}{2} \cdot (\mathcal{L}, \mathcal{L}) \end{aligned}$$

as $\Omega^2_{X/k}$ is trivial. Hence $(\mathcal{L}, \mathcal{L}) = 2d$ is even. If $\mathcal{L}$ is ample then its Hilbert polynomial is $h_{\mathcal{L}}(t) = dt^2 + 2$

(b) By Serre duality and the fact that $\Omega^2_{X/k} \cong \mathcal{O}_X$ we have $h^i(\mathcal{L}) = h^{2-i}(\mathcal{L}^{-1})$. In particular $h^2(\mathcal{L}) = h^0(\mathcal{L}^{-1}) = 0$ as an anti-ample bundle is not effective. Since $d := (\mathcal{L}, \mathcal{L})/2 > 0$ it follows that $h^0(\mathcal{L}) = d + 2 + h^1(\mathcal{L}) > 0$, so $\mathcal{L}$ is effective. For the remaining assertions we refer to [SD74], Section 8. □

Example 1.1.8. Let $\pi \colon X \to \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ as in Example 1.1.2. The line bundle $\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is ample and one has that $(\mathcal{L}, \mathcal{L})_X = 2(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))_{\mathbb{P}^2} = 2$. Hence any K3 surface X which is a double cover of $\mathbb{P}^2$ ramified along a non-singular sextic curve has an ample line bundle $\mathcal{L}$ of degree 2.

Example 1.1.9. Let $X \subset \mathbb{P}^{n+2}$ be a K3 surface which is obtained as a complete intersection of multiple degree $(d_1, d_2, \ldots, d_n)$; see Example 1.1.3. Then $\mathcal{O}_X(1)$ degree $d_1 d_2 \cdots d_n$. Note that the equality $d_1 + d_2 + \cdots + d_n = n + 3$ implies that at least one of the d_i is even.

The following lemma shows that for a K3 scheme fiberwise ampleness is equivalent to relative ampleness. Note first that if $\pi \colon X \to S$ is a K3 scheme over a connected base S then for a line bundle $\mathcal{L}$ on X the intersection index $(\mathcal{L}_{\bar{s}}, \mathcal{L}_{\bar{s}})_{X_{\bar{s}}}$ is constant for any $\bar{s}$. This follows from the fact that π is flat and the relation $(\mathcal{L}_{\bar{s}}, \mathcal{L}_{\bar{s}})_{X_{\bar{s}}} = 2\chi(\mathcal{L}_{\bar{s}}) - 4$.

Lemma 1.1.10. *Let $\pi\colon X \to S$ be a K3 scheme and let $\mathcal{L}$ be a line bundle on X which is fiberwise ample on X i.e., $\mathcal{L}_{\bar{s}}$ is ample on $X_{\bar{s}}$ for every geometric point $\bar{s} \in S$. Let $2d = (\mathcal{L}_{\bar{s}}, \mathcal{L}_{\bar{s}})_{X_{\bar{s}}}$ for any point $\bar{s} \in S$. Then $\pi_*\mathcal{L}^n$ is a locally free sheaf of of rank $dn^2 + 2$ and $\mathcal{L}^n$ is relatively very ample over S if $n \geq 3$.*

Proof. By Theorem 1.1.7 (b) we have that for all $\bar{s} \in S$ the group $H^1(X_{\bar{s}}, \mathcal{L}^n_{\bar{s}})$ is trivial. It follows from [GD67, Ch. III, §7], that $\pi_*\mathcal{L}^n$ is a locally free sheaf and that $\pi_*\mathcal{L}_{\bar{s}} \cong H^0(\mathcal{X}_{\bar{s}}, \mathcal{L}^n_{\bar{s}})$. The rank statement follows from Theorem 1.1.7 (a). By part (a) of Theorem 1.1.7 one sees that for every geometric point $\bar{s} \in S$ and any $n \geq 3$ the line bundle $\mathcal{L}^n_{\bar{s}}$ gives a closed immersion $X_{\bar{s}} \hookrightarrow \mathbb{P}(\pi_*\mathcal{L}^n_{\bar{s}})$ over $\kappa(\bar{s})$. Hence the morphism $X \hookrightarrow \mathbb{P}(\pi_*\mathcal{L}^n)$ induced by $\mathcal{L}^n$ is a closed immersion. This finishes the proof. □

1.2 Cohomology Groups of K3 Surfaces

1.2.1 Quadratic Lattices Related to Cohomology Groups of K3 Surfaces

In this section we introduce some notations which will be used in the sequel. Let U be the hyperbolic plane and denote by E_8 the *positive* quadratic lattice associated to the Dynkin diagram of type E_8 (cf. [Ser73, Ch. V, 1.4 Examples]).

Notation 1.2.1. Denote by (L_0, ψ) the quadratic lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$. Further, let (V_0, ψ_0) be the quadratic space $(L_0, \psi) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We have that L_0 is a free $\mathbb{Z}$-module of rank 22. The form $\psi_{\mathbb{R}}$ has signature $(19+, 3-)$ on $L_0 \otimes \mathbb{R}$.

Let $\{e_1, f_1\}$ be a basis of the first copy of U in L_0 such that

$$\psi(e_1, e_1) = \psi(f_1, f_1) = 0 \quad\text{and}\quad \psi(e_1, f_1) = 1.$$

For a positive integer d we consider the vector $e_1 - df_1$ of L_0. It is a primitive vector i.e., the module $L_0/\langle e_1 - df_1\rangle$ is free and we have that $\psi(e_1 - df_1, e_1 - df_1) = -2d$. The orthogonal complement of $e_1 - df_1$ in L_0 with respect to ψ is $\langle e_1 + df_1\rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}$.

Notation 1.2.2. Denote the quadratic sublattice $\langle e_1 + df_1\rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}$ of L_0 by (L_{2d}, ψ_{2d}). Further, we denote by (V_{2d}, ψ_{2d}) the quadratic space $(L_{2d}, \psi_{2d}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The signature of the form $\psi_{2d,\mathbb{R}}$ is $(19+, 2-)$. We have that $\langle e_1 - df_1\rangle \oplus L_{2d}$ is a sublattice of L_0 of index $2d$. The inclusion of lattices $i\colon L_{2d} \hookrightarrow L_0$ defines injective homomorphisms of groups

$$i^{\mathrm{ad}}\colon \{g \in \mathrm{O}(V_0)(\mathbb{Z}) \mid g(e_1 - df_1) = e_1 - df_1\} \hookrightarrow \mathrm{O}(V_{2d})(\mathbb{Z}) \tag{1.1}$$

and

$$i^{\mathrm{ad}}\colon \{g \in \mathrm{SO}(V_0)(\mathbb{Z}) \mid g(e_1 - df_1) = e_1 - df_1\} \hookrightarrow \mathrm{SO}(V_{2d})(\mathbb{Z}). \tag{1.2}$$

Let L_{2d}^* denote the dual lattice $\mathrm{Hom}(L_{2d}, \mathbb{Z})$. Then the bilinear form ψ_{2d} defines an embedding $L_{2d} \hookrightarrow L_{2d}^*$ and we denote by A_{2d} the factor group L_{2d}^*/L_{2d}. It is an abelian group of order $2d$ ([LP81, §2, Lemma]). One can extend the bilinear form ψ_{2d} on L_{2d} to a $\mathbb{Q}$-valued form on L_{2d}^* and define

$$q_{2d}\colon A_{2d} \to \mathbb{Q}/2\mathbb{Z}$$

defined by

$$q_{2d}(x + L_{2d}) = \psi_{2d}(x, x) + 2\mathbb{Z}$$

for any $x \in L_{2d}^*$. Let $\mathrm{O}(q_{2d})$ denote the group of isomorphisms of A_{2d} preserving the form q_{2d}. Then one has a natural homomorphism $\tau\colon \mathrm{O}(V_{2d})(\mathbb{Z}) \to \mathrm{O}(q_{2d})$. It is shown in [Nik80] that

$$i^{\mathrm{ad}}\big(\{g \in \mathrm{O}(V_0)(\mathbb{Z}) \mid g(e_1 - df_1) = e_1 - df_1\}\big) = \ker(\tau).$$

1.2.2 De Rham Cohomology

Let X be a K3 surface over a field k. The following proposition will play an essential rôle when studying deformations of K3 surfaces (Section 1.4.1). We will use it also to show that the automorphism group $\mathrm{Aut}(X)$ of a K3 surface is reduced (see Theorem 1.3.13 below).

Proposition 1.2.3. *If X is a K3 surface over a field k, then*

(a) The Hodge-de Rham spectral sequence

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Longrightarrow H^{i+j}_{DR}(X, k)$$

degenerates at E_1. For the Hodge numbers $h^{i,j} = \dim_k H^j(X, \Omega^i_{X/k})$ of X we have

$$\begin{gathered} h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 0 \\ h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1 \\ h^{1,1} = 20. \end{gathered}$$

(b) Let $\Theta_{X/k} = \Omega^{1\vee}_{X/k}$ be the tangent bundle of X. Then $H^i(X, \Theta_{X/k}) = 0$ for $i = 0$ and 2 and $\dim_k H^1(X, \Theta_{X/k}) = 20$.

Proof. If k has characteristic zero, then one may assume that $k = \mathbb{C}$ and the proposition follows from [LP81, §1, Prop. 1.2]. The case $\mathrm{char}(k) = p > 0$ is treated in Proposition 1.1 in [Del81b]. □

Remark 1.2.4. Part (b) of the proposition is classical in the case $k = \mathbb{C}$. The proof in the general case is due to Rudakov and Shafarevich. It can be reformulated in following way: There exist no non-trivial regular vector fields on a K3 surface (see [RS76, §6, Thm. 7]).

1.2.3 Betti Cohomology

Let X be a complex K3 surface. Then the Betti cohomology groups $H^i_B(X, \mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$ respectively. One has a non-degenerate bilinear form (given by the Poincaré duality pairing):

$$\psi\colon H^2_B(X, \mathbb{Z})(1) \times H^2_B(X, \mathbb{Z})(1) \to \mathbb{Z}$$

given by

$$\psi(x, y) = -\mathrm{tr}(x \cup y)$$

where $x \cup y$ is the cup product of x and y and $\mathrm{tr}\colon H^4_B(X, \mathbb{Z}(2)) \to \mathbb{Z}$ is the trace map. It has signature $(19+, 3-)$ over $\mathbb{R}$. The quadratic lattice $\big(H^2_B(X, \mathbb{Z})(1), \psi\big)$ is isometric to (L_0, ψ) (cf. Section 1.2.1). For proofs of those results we refer to [LP81, §1, Prop. 1.2].

The group $H^2_B(X, \mathbb{Z})$ carries a natural $\mathbb{Z}$-Hodge structure (which we will abbreviate as $\mathbb{Z}$-HS) of type $\{(2, 0), (1, 1), (0, 2)\}$ with $h^{2,0} = h^{0,2} = 1$ and $h^{1,1} = 20$ as we see from Proposition 1.2.3.

For a complex K3 surface $H^1(X, \mathcal{O}_X)$ is trivial so the first Chern class map

$$c_1\colon \mathrm{Pic}(X) \to H^2_B(X, \mathbb{Z})(1)$$

is injective. Exactly in the same way we see that for a K3 space $\pi\colon X \to S$, where S is a scheme over $\mathbb{C}$, one has a short exact sequence of sheaves

$$0 \to R^1\pi^{\mathrm{an}}_* \mathcal{O}^*_X \to R^2\pi^{\mathrm{an}}_* \mathbb{Z}(1)$$

as $R^1\pi^{\mathrm{an}}_* \mathcal{O}_X$ is trivial.

Notation 1.2.5. Let $\mathcal{L}$ be an ample line bundle on X. We denote by $P^2_B(X, \mathbb{Z})(1)$ the orthogonal complement of $c_1(\mathcal{L})$ with respect to ψ. It is a free $\mathbb{Z}$-module of rank 21 called the *primitive part* (or the *primitive cohomology group*) of $H^2_B(X, \mathbb{Z})(1)$ with respect to $c_1(\mathcal{L})$. The restriction of ψ defines a non-degenerate bilinear form:

$$\psi_{\mathcal{L}}\colon P^2_B(X, \mathbb{Z})(1) \times P^2_B(X.\mathbb{Z})(1) \to \mathbb{Z}.$$

The group $P^2_B(X, \mathbb{Z}(1))$ carries a natural $\mathbb{Z}$-HS induced by the one on $H^2_B(X, \mathbb{Z}(1))$ of type $\{(-1, 1), (0, 0), (1, -1)\}$ with $h^{-1,1} = h^{1,-1} = 1$ for which $\psi_{\mathcal{L}}$ is a polarization.

Remark 1.2.6. Let $\mathcal{L}$ be an ample line bundle for which $(\mathcal{L}, \mathcal{L})_X = 2d$ and assume that it is primitive. Let $\{e_1, f_1\}$ be a basis of the first copy of U in L_0 as in Section 1.2.1. By [BBD85, Exp. IX, §1, Prop. 1] one can find an isometry

$$a\colon \big(H^2_B(X, \mathbb{Z}(1)), \psi\big) \to L_0$$

such that $a(c_1(\mathcal{L})) = e_1 - df_1$. Therefore a induces an isometry

$$a\colon \big(P^2_B(X, \mathbb{Z}(1)), \psi_{\mathcal{L}}\big) \to (L_{2d}, \psi_{2d}).$$

1.2.4 Étale Cohomology

Let k be a field of characteristic $p \geq 0$ and fix a prime l which is different from p. Suppose given a K3 surface X over k. Then the étale cohomology group $H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)$ is a free $\mathbb{Z}_l$-module of rank $1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$. One sees this in the following way: If k has characteristic zero, then the claim follows from the corresponding result for Betti cohomology and the comparison theorem between Betti and étale cohomology ([Mil80, Ch. III, §3, Thm. 3.12]). Assume that $p > 0$. By [Del81b, §1, Cor. 1.8] there exists a discrete valuation ring R with residue field $\bar{k}$ and a smooth lift $\mathcal{X}$ over R of X. If η is the generic point of $\operatorname{Spec}(R)$, then by the smooth base change theorem for étale cohomology ([Mil80, Ch. VI, §4, Cor. 4.2]) one has that

$$H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Z}/l^n\mathbb{Z}) \cong H^i_{\text{et}}(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/l^n\mathbb{Z}) \tag{1.3}$$

for every $i = 0, \ldots, 4$ and every n. Hence $H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l) \cong H^i_{\text{et}}(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_l)$ and we deduce the claim from the characteristic zero result.

Further, one has a non-degenerate bilinear form

$$\psi_{\mathbb{Z}_l}\colon H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(1) \times H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(1) \to \mathbb{Z}_l$$

given by

$$\psi_{\mathbb{Z}_l}(x, y) = -\operatorname{tr}_{\mathbb{Z}_l}(x \cup y)$$

where $\operatorname{tr}_{\mathbb{Z}_l}\colon H^4_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(2) \to \mathbb{Z}_l$ is the trace isomorphism. This is simply Poincaré duality for étale cohomology ([Mil80, Ch. VI, §11, Cor. 11.2]).

The Kummer short exact sequence of étale sheaves on X

$$1 \to \boldsymbol{\mu}_{l^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

gives an exact sequence of cohomology groups

$$H^1_{\text{et}}(X_{\bar{k}}, \boldsymbol{\mu}_{l^n}) \to H^1_{\text{et}}(X_{\bar{k}}, \mathbb{G}_m) \to H^1_{\text{et}}(X_{\bar{k}}, \mathbb{G}_m) \to H^2_{\text{et}}(X_{\bar{k}}, \boldsymbol{\mu}_{l^n}).$$

By (1.3) the group $H^1_{\text{et}}(X_{\bar{k}}, \boldsymbol{\mu}_{l^n})$ is trivial we have an injection

$$0 \to \operatorname{Pic}(X)/l^n \operatorname{Pic}(X) \to H^2_{\text{et}}(X_{\bar{k}}, \boldsymbol{\mu}_{l^n}).$$

Taking the projective limit over n one sees that the first Chern class map

$$c_1\colon \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_l \hookrightarrow H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(1)$$

is injective. In particular, since $H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1))$ is free, $\operatorname{Pic}(X)$ has no l-torsion for any l different from p.

Similarly, if $\pi\colon X \to S$ is a K3 space then one can consider the long exact sequence of higher direct images, coming from the Kummer sequence

$$R^1_{\text{et}}\pi_*\boldsymbol{\mu}_{l^n} \to R^1_{\text{et}}\pi_*\mathbb{G}_m \to R^1_{\text{et}}\pi_*\mathbb{G}_m \to R^2_{\text{et}}\pi_*\boldsymbol{\mu}_{l^n}.$$

Further, since the stalk of $R^1_{\text{et}}\pi_*\boldsymbol{\mu}_{l^n}$ at any geometric point of S is zero (one uses here the proper base change theorem), the sheaf itself is zero ([Mil80, Ch. II, §2, Prop. 2.10]). Hence passing again to the projective limit over n we obtain the exact sequence of $\mathbb{Z}_l$-sheaves

$$0 \to R^1_{\text{et}}\pi_*\mathbb{G}_m \otimes \mathbb{Z}_l \to R^2_{\text{et}}\pi_*\mathbb{Z}_l(1).$$

Notation 1.2.7. Let $\mathcal{L}$ be a primitive ample line bundle on X with $(\mathcal{L},\mathcal{L})_X = 2d$. Denote by $P^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1))$ the *primitive part* of $H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(1)$ with respect to $c_1(\mathcal{L})$ i.e., the orthogonal complement of $c_1(\mathcal{L})$ in $H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(1)$ with respect to $\psi_{\mathbb{Z}_l}$. Denote the restriction of $\psi_{\mathbb{Z}_l}$ to $P^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1))$ by $\psi_{\mathcal{L},\mathbb{Z}_l}$.

If k has characteristic 0, then by the comparison theorem between Betti and étale cohomology one has that $\big(H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1)), \psi_{\mathbb{Z}_l}\big)$ is isometric to $\big(H^2_B(X_{\mathbb{C}}, \mathbb{Z}(1)), \psi\big) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ which is isometric to $(L_0, \psi) \otimes_{\mathbb{Z}} \mathbb{Z}_l$. Moreover since the comparison isomorphism respects algebraic cycles, the same holds for the primitive parts with respect to $\mathcal{L}$ i.e., we have that $\big(P^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1)), \psi_{\mathcal{L},\mathbb{Z}_l}\big) \cong (L_{2d}, \psi_{2d}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$.

Assume that $\operatorname{char}(k) = p > 0$. Then the pair $(X, \mathcal{L}) \otimes \bar{k}$ has a lift $(\mathcal{X}, \mathcal{L})$ over a discrete valuation ring R with $\operatorname{char}(R) = 0$ and with residue field $\bar{k}$ (see [Del81b, §1, Cor. 1.8]). Using the same argument as above one concludes that

$$H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(m) \cong H^i_{\text{et}}(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_l)(m)$$

and that $\big(H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l)(m), \psi_{\mathbb{Z}_l}\big)$ is isometric to $\big(H^2_{\text{et}}(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_l)(m), \psi_{\mathbb{Z}_l}\big)$, where η is the generic point of $\operatorname{Spec}(R)$. Consequently the two quadratic lattices $\big(P^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1)), \psi_{\mathcal{L},\mathbb{Z}_l}\big)$ and $\big(P^2_{\text{et}}(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_l(1)), \psi_{\mathcal{L}_{\bar{\eta}}}\big) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ are also isometric. Thus, if $\mathcal{L}$ is primitive, then there is an isometry

$$a\colon \big(H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1)), \psi_l\big) \to L_0 \otimes \mathbb{Z}_l$$

such that $a(c_1(\mathcal{L})) = e_1 - df_1$. It induces an isometry

$$a\colon \big(P^2_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_l(1)), \psi_{\mathcal{L},\mathbb{Z}_l}\big) \to (L_{2d}, \psi_{2d}) \otimes \mathbb{Z}_l.$$

Remark 1.2.8. Let k be a field of characteristic p. We make the following notations

$$\hat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l \quad \text{and} \quad \mathbb{A}_f^{(p)} = \hat{\mathbb{Z}}^{(p)} \otimes \mathbb{Q}.$$

In the sequel we will be considering étale cohomology with $\hat{\mathbb{Z}}^{(p)}$ or $\mathbb{A}_f^{(p)}$ coefficients. Then we have that for a K3 surface over a field k one has isometries

$$\big(H^2_{\text{et}}(X_{\bar{k}}, \hat{\mathbb{Z}}^{(p)}(1)), \psi_f\big) \cong (L_0, \psi) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{(p)}$$

and for a primitive ample line bundle $\mathcal{L}$ of degree $2d$ on X one has

$$\big(P^2_{\text{et}}(X_{\bar{k}}, \hat{\mathbb{Z}}^{(p)}(1)), \psi_{\mathcal{L},f}\big) \cong (L_{2d}, \psi_{2d}) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{(p)}.$$

Here ψ_f and $\psi_{\mathcal{L},f}$ are the corresponding bilinear forms coming from the Poincaré duality on $H^2_{\text{et}}(X_{\bar{k}}, \hat{\mathbb{Z}}^{(p)}(1))$.

1.2.5 Crystalline Cohomology

Let k be a perfect field of characteristic $p > 0$ and let $W = W(k)$ be the ring of Witt vectors with coefficients in k. Consider a K3 surface X over k. Then by [Del81b, Prop. 1.1] the crystalline cohomology group $H^i_{\text{cris}}(X/W)$ is a free W-module of rank 1,0,22,0,2 for $i = 0, 1, 2, 3, 4$ respectively. We consider next the crystalline Chern class map

$$c_1\colon\ \operatorname{Pic}(X) \to H^2_{\text{cris}}(X/W).$$

As pointed out in [Del81b, Appendice, Rem. 3.5] the Chern class map defines an injection

$$c_1\colon\ \operatorname{NS}(X_{\bar{k}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow H^2_{\text{cris}}(X/W(\bar{k}))$$

where $\operatorname{NS}(X_{\bar{k}}) = \operatorname{Pic}(X_{\bar{k}})/\operatorname{Pic}^0(X_{\bar{k}})$ is the Néron-Severi group of $X_{\bar{k}}$. In particular this means that the Néron-Severi group of $X_{\bar{k}}$ has no p-torsion.

If K is the fraction field of W then we shall denote by $H^i_{\text{cris}}(X/K)$ the K-vector space $H^i_{\text{cris}}(X/W) \otimes_W K$.

1.3 Picard Schemes and Automorphisms of K3 Surfaces

1.3.1 Picard and Néron-Severi Groups of K3 Surfaces.

In this section we will study Picard functors of K3 spaces. Those functors will play an important rôle in two aspects in the construction of moduli spaces of (primitively) polarized K3 surfaces. First, we will define (quasi-) polarizations on K3 surfaces using Picard spaces (cf. Definition 1.3.10 below). Later, in Section 1.4.2, we will use Picard spaces in the construction of the Hilbert scheme parameterizing K3 subschemes of $\mathbb{P}^N$.

For a separated algebraic space X over a scheme S we denote by $\mathrm{Pic}(X)$ the group of isomorphism classes of invertible sheaves on X. Let $\pi\colon X \to S$ be a K3 space and consider the *relative Picard functor*

$$\mathrm{Pic}_{X/S}\colon (\mathrm{Sch}/S)^0 \to \mathrm{Groups}.$$

By definition it is the fppf-sheafification of the functor

$$P_{X/S}\colon (\mathrm{Sch}/S)^0 \to \mathrm{Groups} \quad \text{given by} \quad T \mapsto \mathrm{Pic}(X \times_S T).$$

For every $g\colon T \to S$ we have that $\mathrm{Pic}_{X/S}(T) = H^0(T, R^1\pi'_*\mathbb{G}_m)$ where $\pi'\colon X \times_S T \to T$ is the product morphism and all derived functors are taken with respect to the fppf-topology.

Theorem 1.3.1. *For a K3 space $\pi\colon X \to S$ the relative Picard functor $\mathrm{Pic}_{X/S}$ is represented by a separated algebraic space locally of finite presentation over S.*

Proof. The representability follows form [Art69, §7, Thm. 7.3]. The proof of the separatedness property goes exactly in the same way as the proof of Theorem 3 in [BLR90, Ch. 8, §8.4]. □

Let $S = \mathrm{Spec}(k)$ be a spectrum of a field. Then $\mathrm{Pic}_{X/k}$ is represented by a group scheme (cf. [Oor62] or Lemma 1.3.2 below) and shall denote by $\mathrm{Pic}^0_{X/k}$ its identity component. We set further

$$\mathrm{Pic}^\tau_{X/k} = \bigcup_{n>0} n^{-1}\big(\mathrm{Pic}^0_{X/k}\big)$$

where $n\colon \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X/k}$ is the multiplication by n.

Lemma 1.3.2. *Let X be a K3 surface over a field k. Then $\mathrm{Pic}_{X/k}$ is represented by a separated, smooth, zero dimensional scheme over k. In particular $\mathrm{Pic}^0_{X/k}$ is trivial. Further, we have also that $\mathrm{Pic}^\tau_{X/k}$ is trivial.*

Proof. Combining Theorem 3 and Theorem 1, with $S = \mathrm{Spec}(k)$, of [BLR90, Ch. 8, §8.2] one concludes that $\mathrm{Pic}_{X/k}$ is representable by a separated scheme, locally of finite type over k.

By Theorem 1 of [BLR90, Ch. 8, §8.4] one has that

$$\dim_k \mathrm{Pic}_{X/k} \leq \dim_k H^1(X, \mathcal{O}_X) = 0$$

and hence $\mathrm{Pic}_{X/k}$ is smooth over k. This shows the validity of all assertions except for the claim about $\mathrm{Pic}^{\tau}_{X/k}$.

The scheme $\mathrm{Pic}^{\tau}_{X/k}$ is proper and of finite type over k. Since its dimension is zero it is a finite commutative group scheme over k. The injectivity of the étale Chern class map shows that $\mathrm{Pic}(X)$ has no l-torsion for $l \neq p$. By the first part of the lemma we have that $\mathrm{NS}(X) = \mathrm{Pic}(X)$. Then the injectivity of the crystalline Chern class map shows that $\mathrm{Pic}(X)$ has no p-torsion either. Thus $\mathrm{Pic}(X)$ is torsion free and therefore $\mathrm{Pic}^{\tau}_{X/k}(\bar{k})$ is trivial. Since in this case $\mathrm{Pic}^{\tau}_{X/k}$ is reduced we conclude it is trivial. □

If X is a K3 surface over a field k, then $\mathrm{NS}(X) = \mathrm{Pic}(X)$, which follows from the fact that in this case $\mathrm{Pic}^0(X)$ is trivial. Hence $\mathrm{Pic}(X)$ is a free abelian group of rank at most 22 (use [Mil80, Ch. V, §3, Cor 3.28]). If the characteristic of the ground field is zero, then $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X) \leq 20$.

Let $\pi\colon X \to S$ be a K3 scheme. Define $\mathrm{Pic}^0_{X/S}$ and $\mathrm{Pic}^{\tau}_{X/S}$ as the subfunctors of $\mathrm{Pic}_{X/S}$ consisting of all elements whose restrictions to all fibers X_s belong to $\mathrm{Pic}^0_{X_s/\kappa(s)}$ and $\mathrm{Pic}^{\tau}_{X_s/\kappa(s)}$ respectively.

Proposition 1.3.3. *For a K3 scheme $\pi\colon X \to S$ over a quasi-compact base S one has that $\mathrm{Pic}_{X/S}$ is an algebraic space which is unramified over S. Further, we have that $\mathrm{Pic}^0_{X/S}$ and $\mathrm{Pic}^{\tau}_{X/S}$ are trivial.*

Proof. The first part of the proposition follows from the preceding lemma as it is enough to check the $\mathrm{Pic}_{X/S}$ is unramified in the case S is a spectrum of a field. To prove the second part we notice that according to [BLR90, Ch. 8, §8.3, Thm. 4] we have open immersions $\mathrm{Pic}^0_{X/S} \hookrightarrow \mathrm{Pic}_{X/S}$ and $\mathrm{Pic}^{\tau}_{X/S} \hookrightarrow \mathrm{Pic}_{X/S}$. By Lemma 1.3.2 above for every geometric point $\bar{s} \in S$ the subspaces $\mathrm{Pic}^0_{X_{\bar{s}}/\kappa(\bar{s})}$ and $\mathrm{Pic}^{\tau}_{X_{\bar{s}}/\kappa(\bar{s})}$ are trivial hence $\mathrm{Pic}^0_{X/S}$ and $\mathrm{Pic}^{\tau}_{X/S}$ are trivial. □

Remark 1.3.4. Let $\pi\colon X \to S$ be a K3 scheme and let $\mathcal{L}$ and $\mathcal{M}$ be two line bundles on X. If $\mathcal{L}^n = \mathcal{M}^n$ for some $n \in \mathbb{N}$, then $\mathcal{L}$ is isomorphic to $\mathcal{M} \otimes \pi^*\mathcal{N}$ where $\mathcal{N}$ is a line bundle on S. Indeed, we have that $cl(\mathcal{L})^n = cl(\mathcal{M})^n$ in $\mathrm{Pic}_{X/S}$. Since $\mathrm{Pic}^{\tau}_{X/S}$ is trivial we have that the multiplication by n-morphism $[n]\colon \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S}$ is an injective homomorphism of group schemes. Since $cl(\mathcal{L})^n = cl(\mathcal{M})^n$ we conclude that $cl(\mathcal{L} \otimes \mathcal{M}^{-1})$ is trivial, so $\mathcal{M}$ and $\mathcal{L}$ differ by an invertible sheaf coming from the base S ([BLR90, Ch. 8, §8.1, Prop. 4]).

Remark 1.3.5. It is easy to see that the statement of Proposition 1.3.3 remains true for K3 spaces.

Recall that a morphism of schemes $\pi\colon X \to S$ is called *strongly projective* (respectively *strongly quasi-projective*) if there exists a locally free sheaf $\mathcal{E}$ on S of constant finite rank such that X is S-isomorphic to a closed subscheme (respectively a subscheme) of $\mathbb{P}(\mathcal{E})$.

Lemma 1.3.6. *Let S be a noetherian scheme and suppose given a K3 scheme $\pi\colon X \to S$. If π is a strongly projective morphism, then we have that*

(i) for any $n \in \mathbb{N}$ the multiplication by n-morphism

$$[n]\colon \operatorname{Pic}_{X/S} \to \operatorname{Pic}_{X/S}$$

is a closed immersion of group schemes over S.

(ii) for any $\lambda \in \operatorname{Pic}_{X/S}(S)$ the set of points

$$S^o = \{s \in S \mid \lambda_s \text{ is primitive on } X_s\}$$

is open in S.

Proof. (i): By definition we have a closed immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ on S. Let $\mathcal{O}_X(1)$ denote the pull-back of the canonical bundle $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ via this inclusion. For a polynomial $\Phi \in \mathbb{Q}[t]$ let $\operatorname{Pic}^{\Phi}_{X/S}$ be the subfunctor of $\operatorname{Pic}_{X/S}$ which is induced by the line bundles $\mathcal{L}$ on X with a given Hilbert polynomial Φ (with respect to $\mathcal{O}_X(1)$) on the fibers of X over S. Then $\operatorname{Pic}^{\Phi}_{X/S}$ is representable by a strongly quasi-projective scheme over S and $\operatorname{Pic}_{X/S}$ is the disjoint union of the open and closed subschemes $\operatorname{Pic}^{\Phi}_{X/S}$ for all $\Phi \in \mathbb{Q}[t]$. For a proof of this result we refer to [BLR90, Ch. 8, §8.2, Thm. 5].

Since all schemes $\operatorname{Pic}^{\Phi}_{X/S}$ are quasi-compact we have that for a given Φ the image $[n](\operatorname{Pic}^{\Phi}_{X/S})$ is contained in a finite union $\bigcup_{i \in \mathcal{C}^n_{\Phi}} \operatorname{Pic}^{\Phi_i}_{X/S}$. We will show first that for a given $\Phi \in \mathbb{Q}[t]$ the morphism

$$[n]\colon \operatorname{Pic}^{\Phi}_{X/S} \to \bigcup_{i \in \mathcal{C}^n_{\Phi}} \operatorname{Pic}^{\Phi_i}_{X/S}$$

is proper. As all schemes involved are noetherian we can apply the valuative criterion for properness. We may assume that S is a spectrum of a discrete valuation ring R and that X admits a section over S and let η and s be the generic and the special point of S. Under those assumptions any element of $\operatorname{Pic}_{X/S}$ comes from a class of a line bundle ([BLR90, Ch. 8, §8.1, Prop. 4]). To show that the restriction of $[n]$ to $\operatorname{Pic}^{\Phi}_{X/S}$ is proper we have to show that if $\mathcal{L}$ is a line bundle over the generic fiber X_η of X, then $\mathcal{L}^n$ extends uniquely to a line bundle on X which is a n-th power of a line bundle. This follows from [BLR90, Ch. 8, §8.4, Thm. 3] as both $\mathcal{L}$ and $\mathcal{L}^n$ extend uniquely over X.

Further, the morphism $[n]\colon \operatorname{Pic}_{X/S} \to \operatorname{Pic}_{X/S}$ is an immersion of the corresponding topological spaces and as it is proper on every open and closed $\operatorname{Pic}^{\Phi}_{X/S}$, the image $[n](\operatorname{Pic}_{X/S})$ is closed in $\operatorname{Pic}_{X/S}$. We are left to show that the natural homomorphism of sheaves $\mathcal{O}_{\operatorname{Pic}_{X/S}} \to [n]_*\mathcal{O}_{\operatorname{Pic}_{X/S}}$ is surjective. As this can be checked on stalks we see further that it is enough to show the surjectivity assuming that S is a spectrum of a field. But under this condition the claim follows from Lemma 1.3.2. Indeed, $\mathrm{Pic}_{X/k}$ is a reduced, zero dimensional scheme. Hence all subschemes $\mathrm{Pic}^{\Phi}_{X/k}$ being reduced, quasi-projective and zero dimensional, are finite unions of points. Then the restrictions $[n]\colon \mathrm{Pic}^{\Phi}_{X/k} \to \bigcup_{i\in\mathcal{C}^n_\Phi} \mathrm{Pic}^{\Phi_i}_{X/k}$ are closed immersions and hence $[n]\colon \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X/k}$ is also a closed immersion. Therefore $\mathcal{O}_{\mathrm{Pic}_{X/k}} \to [n]_*\mathcal{O}_{\mathrm{Pic}_{X/k}}$ is surjective.

(ii): We may assume that S is connected. Then the intersection index $(\lambda_{\bar{s}}, \lambda_{\bar{s}})$ is constant on S, say $(\lambda_{\bar{s}}, \lambda_{\bar{s}}) = 2d$. For any natural number n consider the closed subscheme S_n of S defined by the following Cartesian diagram

$$\begin{tikzcd} S_n \arrow[r] \arrow[d] & S \arrow[d, "\lambda"] \\ \mathrm{Pic}_{X/S} \arrow[r, "{[n]}"] & \mathrm{Pic}_{X/S}\,. \end{tikzcd}$$

Then the subset S^o of S can be identified with $S \setminus \bigcup_n S_n$ where the union is taken over all $n \in \mathbb{N}$ such that n^2 divides d. So it has a structure of an open subscheme of S. $\square$

Remark 1.3.7. Note that if $\pi\colon X \to S$ is a K3 scheme, then the Picard functor $\mathrm{Pic}_{X/S}$ can be constructed using the étale topology on S instead of the fppf-topology. In other words $\mathrm{Pic}_{X/S}$ is also the étale sheafification of $P_{X/S}$. This follows from the fact that π is a proper morphism, using the Leray spectral sequence for π and the sheaf $\mathbb{G}_m$. For a proof we refer to the comments on p. 203 in [BLR90, Ch. 8, §8.1].

Example 1.3.8. Let A be an abelian surface over an algebraically closed field k of characteristic different from 2 and let X be the associated Kummer surface. Then one has that

$$\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}} \cong \mathrm{NS}(A)^{[-1]_A}_{\mathbb{Q}} \oplus \mathbb{Q}^{\oplus 16}$$

where $\mathrm{NS}(A)^{[-1]_A}$ denotes the elements of $\mathrm{NS}(A)$ invariant under the action of $[-1]_A$. We refer to [Shi79, §3, Prop. 3.1] for a proof.

1.3.2 Polarizations of K3 Surfaces

Here we will define the notion of a polarization on a K3 space.

Definition 1.3.9. Let k be a field. A *polarization* on a K3 surface X/k is a global section $\lambda \in \mathrm{Pic}_{X/k}(k)$ which over $\bar{k}$ is the class of an ample line bundle $\mathcal{L}_{\bar{k}}$. The degree of $\mathcal{L}_{\bar{k}}$ is called the *polarization degree* of λ. A *quasi-polarization* on X is a global section $\lambda \in \mathrm{Pic}_{X/k}(k)$ which over $\bar{k}$ comes from a line bundle $\mathcal{L}_{\bar{k}}$ with the following property:

(i) $\mathcal{L}_{\bar{k}}$ is nef i.e., $\big(\mathcal{L}_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(C)\big) \geq 0$ for all irreducible curves in $X_{\bar{k}}$,

(ii) if $\big(\mathcal{L}_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(C)\big) = 0$ for a curve C in $X_{\bar{k}}$ then $(C, C)_{X_{\bar{k}}} = (-2)$.

If (X, λ) is a polarized K3 surface over k, then one can find a finite separable extension k' of k such that λ comes from a line bundle $\mathcal{L}_{k'}$ over k'. Indeed, this follows either from Remark 1.3.7 or from Proposition 4 in [BLR90, Ch. 8, §8.1] taking $T = \mathrm{Spec}(k^{\mathrm{sp}})$ and the fact that $\mathrm{Br}(k^{\mathrm{sp}})$ is trivial.

Definition 1.3.10. Let S be scheme. A *polarization* on a K3 space $\pi\colon X \to S$ is a global section $\lambda \in \mathrm{Pic}_{X/S}(S)$ such that for every geometric point $\bar{s}$ of S the section $\lambda_{\bar{s}} \in \mathrm{Pic}_{X_{\bar{s}}/\kappa(\bar{s})}(\kappa(\bar{s}))$ is a polarization of $X_{\bar{s}}$. A *quasi-polarization* on X/S is a global section $\lambda \in \mathrm{Pic}_{X/S}(S)$ such that for every geometric point $\bar{s}$ of S the section $\lambda_{\bar{s}} \in \mathrm{Pic}_{X_{\bar{s}}/\kappa(\bar{s})}(\kappa(\bar{s}))$ is a quasi-polarization of $X_{\bar{s}}$.

Definition 1.3.11. A polarization (respectively quasi-polarization) λ on a K3 space $\pi\colon X \to S$ is called *primitive* if for every geometric point $\bar{s}$ of S the polarization (respectively the quasi-polarization) $\lambda_{\bar{s}} \in \mathrm{Pic}_{X_{\bar{s}}/\kappa(\bar{s})}(\kappa(\bar{s}))$ is primitive i.e., it is not a positive power of any element in $\mathrm{Pic}_{X_{\bar{s}}/\kappa(\bar{s})}(\kappa(\bar{s}))$.

Lemma 1.3.12. *Let $(\pi\colon X \to S, \lambda)$ be a K3 space over S with a polarization λ. Then one can find an étale covering $S' \to S$ such that $\pi_{S'}\colon X_{S'} \to S'$ is a K3 scheme and $\lambda_{S'}$ is the class of a relatively ample line bundle $\mathcal{L}_{S'}$ on $X_{S'}$.*

Proof. By definition one can find an étale covering $S_1 \to S$ such that $\pi_1\colon X_{S_1} \to S_1$ is a K3 scheme. The pull-back λ_{S_1} of λ is a polarization on X_{S_1}. By Remark 1.3.7 the Picard functor $\mathrm{Pic}_{X_{S_1}/S_1}$ can be computed using the étale topology on S_1. Hence one can find an étale covering $S' \to S$ such that $\lambda_{S'}$ is equal to the class of a line bundle $\mathcal{L}_{S'}$ on $X_{S'}$. By definition $\mathcal{L}_{S'}$ is pointwise ample hence using Lemma 1.1.10 we conclude that it is relatively ample. This finishes the proof. □

The self-intersection $(\mathcal{L}_{\bar{s}'}, \mathcal{L}_{\bar{s}'})$ for a geometric point $\bar{s}'$ on S' is constant on every connected component of S'. We say that λ is a *polarization of degree* $2d$ if $(\mathcal{L}_{\bar{s}'}, \mathcal{L}_{\bar{s}'}) = 2d$ for every geometric point $\bar{s}'$ of S'.

1.3.3 Automorphism Groups

Let S be a scheme and $\pi\colon X \to S$ be an algebraic space over S. Define the automorphism functor in the following way:

$$\mathrm{Aut}_S(X)\colon (\mathrm{Sch}/S)^0 \to \mathrm{Groups}$$
$$\mathrm{Aut}_S(X)(T) = \mathrm{Aut}_T(X_T)$$

for every S-scheme T.

Theorem 1.3.13. *If $\pi\colon X \to S$ is a K3 space over S, then $\mathrm{Aut}_S(X)$ is representable by a separated group scheme which is unramified and locally of finite type over S.*

Proof. Let $S' \to S$ be an étale cover such that $\pi'\colon X' = X \times_S S' \to S'$ is a K3 scheme over S'. Denote by π_i the projection morphisms $\pi_i\colon X' \times_X X' \to X' \to X \to S$ for $i = 1, 2$. By definition $X' \times_X X'$ is representable by a quasi-compact subscheme of $X' \times_S X'$.

According to Proposition 1.4 in [Knu71, Ch. II] we have an exact sequence of groups

$$0 \longrightarrow \operatorname{Aut}_S(X)(T) \longrightarrow \operatorname{Aut}_S(X')(T) \rightrightarrows \operatorname{Aut}_S(X' \times_X X')(T). \tag{1.4}$$

It follows from [Gro62, Exp. 221, §4.c] that the functors $\operatorname{Aut}_S(X')$ and $\operatorname{Aut}_S(X' \times_X X')$ are representable by group schemes locally of finite type over S. For simplicity we denote them by $\mathcal{Y}$ and $\mathcal{W}$ respectively. Then from the exact sequence (1.4) we see that $\operatorname{Aut}_S(X)$ is representable by the fiber product

$$\begin{tikzcd} \operatorname{Aut}_S(X) \arrow[r] \arrow[d] & \mathcal{W} \arrow[d, "\Delta"] \\ \mathcal{Y} \arrow[r, "(pr_1^*, pr_2^*)"] & \mathcal{W} \times_S \mathcal{W} \end{tikzcd}$$

where $\Delta\colon \mathcal{W} \to \mathcal{W} \times_S \mathcal{W}$ is the diagonal morphism.

The fact that the $\operatorname{Aut}_S(X)$ is separated follows directly from the valuative criterion for separatedness.

To check that $\operatorname{Aut}_S(X)$ is unramified we may take S to be the spectrum of an algebraically closed field k. A point in $\operatorname{Aut}_k(X)(k[\epsilon]/(\epsilon^2))$, which under the natural homomorphism maps to the the identity in $\operatorname{Aut}_k(X)(k)$, may be identified with a vector field on X. By Proposition 1.2.3 (1) a K3 surface has no non-trivial vector fields hence we conclude that $\operatorname{Aut}_k(X)$ is reduced. □

Remark 1.3.14. The proof of the theorem shows that $\operatorname{Aut}_S(X)$ is 0-dimensional over S. Its fibers are constant group schemes.

Let $\pi\colon X \to S$ be a K3 space and let λ be a polarization of X. Define the subfunctor $\operatorname{Aut}_S(X, \lambda)$ of $\operatorname{Aut}_S(X)$ in the following way

$$\operatorname{Aut}_S(X, \lambda)\colon (\mathrm{Sch}\,/S)^0 \to \mathrm{Groups}$$
$$\operatorname{Aut}_S(X, \lambda)(T) = \{\alpha \in \operatorname{Aut}_S(X)(T) \mid \alpha^*\lambda = \lambda \in \operatorname{Pic}_{X/S}(T)\}$$

for every S-scheme T.

Proposition 1.3.15. *The functor* $\operatorname{Aut}_S(X, \lambda)$ *is a closed subfunctor of* $\operatorname{Aut}_S(X)$. *It is represented by a separated group scheme which is unramified and of finite type over S. Its relative dimension over S is zero.*

Proof. The functor $\operatorname{Aut}_S(X, \lambda)$ is a closed subfunctor of $\operatorname{Aut}_S(X)$. It is representable by the subgroup scheme of $G = \operatorname{Aut}_S(X)$ (locally of finite type over S) given by the following (Cartesian) diagram:

$$\begin{tikzcd}
\mathrm{Aut}_S(X,\lambda) \arrow[rr] \arrow[d] & & S \arrow[d, "(\lambda,\mathrm{id})"] \\
G = G \times_S S \arrow[rr, "\psi"] & & \mathrm{Pic}_{X/S} \times_S S = \mathrm{Pic}_{X/S} .
\end{tikzcd}$$

Here we have that $\lambda\colon S \to \mathrm{Pic}_{X/S}$ is the section given by λ and ψ is the composition $\sigma \circ (\mathrm{id}, \lambda)$ where

$$\sigma\colon G \times \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S}$$

is the action of G on $\mathrm{Pic}_{X/S}$.

Just as in the proof of the preceding theorem we may take S to be the spectrum of an algebraically closed field k in order to check that $\mathrm{Aut}_S(X,\lambda)$ is unramified. If $\alpha \in \mathrm{Aut}_k(X,\lambda)(k[\epsilon]/\epsilon^2)$ which is the identity in $\mathrm{Aut}_k(X,\lambda)(k)$, then by Theorem 1.3.13 above we see that α is the identity element of the group $\mathrm{Aut}_k(X)(k[\epsilon]/\epsilon^2)$. Since by definition we have an inclusion

$$\mathrm{Aut}_k(X,\lambda)(k[\epsilon]/\epsilon^2) \subset \mathrm{Aut}_k(X)(k[\epsilon]/\epsilon^2)$$

we conclude that $\mathrm{Aut}_S(X,\lambda)$ is unramified over S.

Let $\bar{s}\colon \mathrm{Spec}(\Omega) \to S$ be a geometric point. Then by [Mat58] (see also Corollary 2 in [MM64]) the set $\mathrm{Aut}_S(X,\lambda)(\Omega)$ is finite. Hence $\mathrm{Aut}_S(X,\lambda)$ is of finite type over S. □

Note that in general, for a K3 surface X over a field k, the group $\mathrm{Aut}_k(X)(k)$ might be infinite.

Example 1.3.16. For any complex K3 surface X with $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X) = 20$ one has that $\mathrm{Aut}_{\mathbb{C}}(X)(\mathbb{C})$ is infinite. For a proof see [SI77, §5, Thm. 5].

There are also examples of K3 surfaces X having a finite group of automorphisms. An example of a complex K3 surface with $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X) = 18$ and finite automorphism group is given in the remark on page 132 in [SI77].

1.3.4 Automorphisms of Finite Order

In this section k will be an algebraically closed field. If it is a field of characteristic p, then we will denote by W the ring of Witt vectors with coefficients in k and K will be the field of fractions of W.

Let X be a K3 surface over k. If $k = \mathbb{C}$, then it is a well-known theorem that $\mathrm{Aut}_{\mathbb{C}}(X)(\mathbb{C})$ acts faithfully on $H^2_B(X,\mathbb{Z})$. Here we prove a similar result for the automorphisms of finite order of X acting trivially on $H^2_{\mathrm{et}}(X,\mathbb{Z}_l)$ where l is a prime number different from $\mathrm{char}(k)$. The only restriction we impose is that $\mathrm{char}(k) \neq 2$. Later on in Section 1.5.1 we will introduce level structures on K3 surfaces and we will use this result to show that the corresponding moduli stacks are algebraic spaces.

Lemma 1.3.17. *Let X be a K3 surface over k and assume that* $\mathrm{char}(k) = 0$. *Then* $\mathrm{Aut}_k(X)(k)$ *acts faithfully on* $H^2_{\mathrm{et}}(X, \mathbb{Z}_l)$ *for every prime l.*

Proof. Without loss of generality we may assume that the field k can be embedded into $\mathbb{C}$. Fix an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$. By the comparison theorem between Betti and étale cohomology we have an isomorphism $H^2_{\mathrm{et}}(X, \mathbb{Z}_l) \cong H^2_B(X \otimes_\sigma \mathbb{C}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$. Let $\alpha \in \mathrm{Aut}_k(X)(k)$ be an automorphism acting trivially on $H^2_{\mathrm{et}}(X, \mathbb{Z}_l)$. Then $\alpha_{\mathbb{C}}$ acts trivially on $H^2_B(X \otimes_\sigma \mathbb{C}, \mathbb{Z}) \otimes \mathbb{Z}_l$. Since $H^2_B(X \otimes_\sigma \mathbb{C}, \mathbb{Z})$ is a free $\mathbb{Z}$-module we conclude from [LP81, Prop. 7.5] that $\alpha = \mathrm{id}_X$. □

Proposition 1.3.18. *Let (X, λ) be a polarized K3 surface over k and assume that* $\mathrm{char}(k) = p$ *is different from* 2. *Then the finite group* $\mathrm{Aut}_k(X, \lambda)(k)$ *acts faithfully on* $H^2_{\mathrm{et}}(X, \mathbb{Z}_l)$ *for any* $l \neq p$.

Remark 1.3.19. This result can be viewed as an analogue of Theorem 3 in [Mum74, Ch. IV] for (polarized) K3 surfaces.

We will reduce the proof of Proposition 1.3.18 to the preceding lemma. To do so we will use crystalline cohomology and compare the action of an element in $\mathrm{Aut}_k(X, \lambda)(k)$ on $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ and $H^2_{\mathrm{cris}}(X/K)$.

Let X be a K3 surface over a field k. We denote by $H^n(X)$ and $H^n(X \times X)$ either $H^n_{\mathrm{et}}(X, \mathbb{Q}_l)$ and $H^n_{\mathrm{et}}(X \times X, \mathbb{Q}_l)$ for any l prime to $\mathrm{char}(k)$ or $H^n_{\mathrm{cris}}(X/K)$ and $H^n_{\mathrm{cris}}(X \times X/K)$. Note that we will be working with classes of certain algebraic cycles on X and $X \times X$ so we should consider some Tate twists of these cohomology groups. But since k is algebraically closed and the Galois action does not play any rôle in our consideration (we shall only consider some characteristic polynomials of automorphisms of X) we will omit these twists.

For an isomorphism $\alpha\colon X \to Y$ we will denote by α^*_l and α^*_{cris} the isomorphisms induced on $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ and $H^2_{\mathrm{cris}}(X/K)$ respectively.

Lemma 1.3.20. *The Künneth components of the class $cl(u) \in H^4(X \times X)$ of any algebraic cycle on $X \times X$ are algebraic.*

Proof. We have that $H^1_{\mathrm{et}}(X, \mathbb{Q}_l) = H^3_{\mathrm{et}}(X, \mathbb{Q}_l) = 0$ and $H^1_{\mathrm{cris}}(X/W) = H^3_{\mathrm{cris}}(X/W) = 0$. Then the Künneth isomorphism reads

$$H^4(X \times X) = \big(H^4(X) \otimes H^0(X)\big) \oplus \big(H^2(X) \otimes H^2(X)\big) \oplus \big(H^0(X) \otimes H^4(X)\big).$$

Using this decomposition we write

$$cl(u) = u_0 \oplus u_2 \oplus u_4.$$

Every element of the one dimensional spaces $H^4(X) \otimes H^0(X)$ and $H^0(X) \otimes H^4(X)$ is algebraic. These are rational multiple of the classes of $\{pt\} \times X$ and $X \times \{pt\}$. Hence u_0 and u_4 are algebraic. It follows that u_2 is expressed as a linear combination of algebraic classes, hence it is algebraic. □

In particular, if $\Delta = \delta(X) \subset X \times X$ is the diagonal, then its Künneth components $cl(\Delta) = \pi_0 \oplus \pi_2 \oplus \pi_4 \in H^4(X \times X)$ are algebraic. Denote by $\langle \cdot, \cdot \rangle$ the intersection pairing on $\mathrm{CH}^2(X \times X)_{\mathbb{Q}}$.

Corollary 1.3.21. *Let $u \in \mathrm{CH}^2(X \times X)_{\mathbb{Q}}$ be a rational cycle and let $cl(u) \in H^4(X \times X)$ be its algebraic class. Then its characteristic polynomial $\det\big(1 - t \cdot cl(u)|H^2(X)\big)$ has rational coefficients which are independent of l and p (i.e., of $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ and $H^2_{\mathrm{cris}}(X/K)$). The coefficient in front of t^i is given by*

$$s_i = \langle u_i, \pi_2 \rangle$$

for $i = 1, \ldots, 22$.

Proof. The proof follows from the preceding lemma and by Theorem 3.1 in [Tat95]. □

Theorem 1.3.22 (Ogus). *If $p > 2$ then the natural morphism of groups*

$$\mathrm{Aut}_k(X)(k) \to \mathrm{Aut}\big(H^2_{\mathrm{cris}}(X/W)\big)$$

is injective.

Proof. This is a result of A. Ogus and can be found in his paper on Supersingular K3 crystals [Ogu79, §2, Cor. 2.5]. □

Proof of Proposition 1.3.18. Take an element $\alpha \in \mathrm{Aut}_k(X, \lambda)(k)$. According to Proposition 1.3.15 it has finite order. Denote by $u = \Gamma_\sigma \subset X \times X$ the graph of α. Then the automorphism of $H^2_{\mathrm{et}}(X, \mathbb{Q}_l)$ induced by $cl(u) \in H^4_{\mathrm{et}}(X \times X, \mathbb{Q}_l)$ is the one induced by α. By assumption it is the identity hence its characteristic polynomial is $(t-1)^{22}$. By Corollary 1.3.21 it is exactly the characteristic polynomial of the automorphism α^*_{cris} of $H^2_{\mathrm{cris}}(X/K)$ induced by α. Since α is an automorphism of finite order the induced map α^*_{cirs} on the crystalline cohomology is semi-simple (K has characteristic zero). Hence α^*_{cris} acts trivially on $H^2_{\mathrm{cris}}(X/K)$ and by Theorem 1.3.22 it is the identity automorphism as $H^2_{\mathrm{cris}}(X/W)$ is torsion free. □

Remark 1.3.23. Note that the only property of α which we used in the proof of Proposition 1.3.18 is that it has finite order. This is really essential as in general the characteristic polynomial of α^*_l will not give enough information to conclude that the action of α_{cris} on $H^2_{\mathrm{cris}}(X/W)$ is trivial. The proof given above shows actually that any automorphism of finite order α of X acting trivially on $H^2_{\mathrm{et}}(X, \mathbb{Z}_l)$ for some $l \neq p$ is the identity automorphism id_X.

1.4 The Moduli Stack of Polarized K3 Surfaces

We are ready to define moduli functors of (primitively) polarized K3 surfaces over $\mathrm{Spec}(\mathbb{Z})$. We will follow the line of thoughts in [DM68] in order to prove that these functors define Deligne-Mumford stacks. Shortly, this can be given in three steps.

1. Describe the deformations of primitively polarized K3 surfaces.
2. Construct a Hilbert scheme parameterizing K3 surfaces embedded in $\mathbb{P}^N$ for some appropriate $N \in \mathbb{N}$.
3. Construct a "Hilbert morphism" π_{Hilb} from the Hilbert scheme to the moduli stack which is surjective and smooth. Use this morphism to conclude that the moduli stack is a Deligne-Mumford stack.

These steps are spelled out in detail in Sections 1.4.1-1.4.3.

1.4.1 Deformations of K3 Surfaces

Let k be an algebraically closed field. Denote by W the ring of Witt vectors $W(k)$ in case $\mathrm{char}(k) = p > 0$ and $W = k$ otherwise. Let $\underline{A}$ be the category of local artinian W-algebras $(A, \mathfrak{m}_A)$ together with an isomorphism $A/\mathfrak{m}_A \cong k$ compatible with the isomorphism $W/pW \cong k$.

Let X_0 be a K3 surface over k. Consider the covariant functor

$$\mathrm{Def}_{\mathrm{Sch}}(X_0) \colon \underline{A} \to \mathrm{Sets}$$

given by

$$\begin{aligned}\mathrm{Def}_{\mathrm{Sch}}(X_0)(A) = \big\{\text{isom. classes of pairs } (X, \phi_0) \mid & \text{where } X \to \mathrm{Spec}(A) \\ & \text{is a K3 scheme and } \phi_0 \text{ is} \\ & \text{an isom. } \phi_0 \colon X \otimes_A k \cong X_0\big\}.\end{aligned}$$

Proposition 1.4.1. *The functor* $\mathrm{Def}_{\mathrm{Sch}}(X_0)$ *is pro-representable by a formal scheme* S *over* $\mathrm{Spf}(W)$ *which is formally smooth of relative dimension* 20 *i.e., it is (non-canonically) isomorphic to* $\mathrm{Spf}(W[[t_1, \dots, t_{20}]])$.

Proof. This is Corollary 1.2 in [Del81b] in case $\mathrm{char}(k) = p > 0$ and [LP81, Cor. 5.7] in case $\mathrm{char}(k) = 0$. $\square$

Let $\mathcal{L}_0$ be a line bundle on X_0. For moduli problems one should study the deformations of the pair $(X_0, \mathcal{L}_0)$. Define

$$\mathrm{Def}_{\mathrm{Sch}}(X_0, \mathcal{L}_0) \colon \underline{A} \to \mathrm{Sets}$$

to be the functor sending an object A of $\underline{A}$ to the isomorphism classes of triples $(X, \mathcal{L}, \phi_0)$ of flat deformations X of X_0 over A, an invertible sheaf $\mathcal{L}$ on X and an isomorphism $\phi_0 \colon (X, \mathcal{L}) \otimes_A k \cong (X_0, \mathcal{L}_0)$. We have a morphism

$$\mathrm{Def}_{\mathrm{Sch}}(X_0, \mathcal{L}_0) \to \mathrm{Def}_{\mathrm{Sch}}(X_0). \tag{1.5}$$

Theorem 1.4.2. *If the line bundle $\mathcal{L}_0$ is non-trivial, then the functor* $\mathrm{Def}_{\mathrm{Sch}}(X_0, \mathcal{L}_0)$ *is pro-representable by a formally flat scheme of relative dimension 19 over W and the morphism* (1.5) *is a closed immersion, defined by a single equation.*

Proof. See [Del81b, Prop. 1.5 and Thm. 1.6]. □

Deligne proves that if $\mathcal{L}_0$ is an ample line bundle over X_0 then one can find a discrete valuation ring R which is a finite W module and a lift $(X \to \mathrm{Spec}(R), \mathcal{L})$ of $(X_0, \mathcal{L}_0)$ over R. In general one needs ramified extensions of W in order to find a lift of $(X_0, \mathcal{L}_0)$. The next lemma shows that one can find a lift over W if the self-intersection of $\mathcal{L}_0$ is prime to the characteristic of k. More precisely one has:

Lemma 1.4.3. *Let $\mathcal{L}_0$ be an ample line bundle over X_0. If the polarization degree $(\mathcal{L}_0, \mathcal{L}_0)_{X_0} = 2d$ is prime to the characteristic of k, then* $\mathrm{Def}_{\mathrm{Sch}}(X_0, \mathcal{L}_0)$ *is formally smooth.*

Proof. According to [Ogu79, §2, Prop. 2.2] (see also Lemma 2.2.6 in [Del81a]) it is enough to see that $c_1(\mathcal{L}_0) \notin F^2 H^2_{DR}(X_0/k)$. Since we have that $(c_1(\mathcal{L}_0), c_1(\mathcal{L}_0)) = 2d \neq 0$ in k it follows that $c_1(\mathcal{L}_0) \notin F^2 H^2_{DR}(X_0/k)$. For the proof in the case k has characteristic zero we refer to [PSS72, §2, Thm. 1]. □

1.4.2 The Hilbert Scheme

Recall that if X is a K3 surface over a field k with an ample line bundle $\mathcal{L}$, then the Hilbert polynomial of $\mathcal{L}$ is $h_{\mathcal{L}}(x) = dx^2 + 2$, where $(\mathcal{L}, \mathcal{L}) = 2d$.

We fix two natural numbers n and d assuming that $n \geq 3$. Let $P_{d,n}(x)$ be the polynomial $n^2dx^2 + 2$ and let $N = P_{d,n}(1) - 1$. Denote by $\mathbf{Hilb}_N^{P_{d,n}}$ the *Hilbert scheme* over $\mathbb{Z}$ representing the subvarieties of $\mathbb{P}^N$ with Hilbert polynomial $P_{d,n}(x)$. Let

$$\pi \colon \mathcal{Z} \to \mathbf{Hilb}_N^{P_{d,n}}$$

be the universal family over the Hilbert scheme. For any morphism of schemes $f \colon S \to \mathbf{Hilb}_N^{P_{d,n}}$ we consider the following (Cartesian) diagram:

$$\begin{tikzcd} \mathcal{X} = S \times_{\left(\mathbf{Hilb}_N^{P_{d,n}}\right)} \mathcal{Z} \arrow[r, "f'"] \arrow[d, "\pi'"] & \mathcal{Z} \arrow[d, "\pi"] \\ S \arrow[r, "f"] & \mathbf{Hilb}_N^{P_{d,n}} \end{tikzcd} \tag{1.6}$$

Proposition 1.4.4. *There is a unique subscheme $H_{d,n}$ of $\mathbf{Hilb}_N^{P_{d,n}}$ with the property: A morphism of schemes $f\colon S \to \mathbf{Hilb}_N^{P_{d,n}}$ factors through $H_{d,n}$ if and only if the following conditions are satisfied.*

(i) The pull-back $\mathcal{X}$ of the universal family over $\mathbf{Hilb}_N^{P_{d,n}}$ is a K3 scheme over S (see Diagram (1.6) above),

(ii) the line bundle $f'^*\mathcal{O}_{\mathbb{P}^N}(1)$ is isomorphic to $\mathcal{L}^n \otimes \pi'^*\mathcal{M}$ for some ample line bundle $\mathcal{L}$ on $\mathcal{X}$ and some line bundle $\mathcal{M}$ on S,

(iii) for every geometric point $\bar{s}\colon \mathrm{Spec}(\Omega) \to S$ the natural homomorphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \otimes \Omega \to H^0(\mathcal{X}_{\bar{s}}, \mathcal{L}_{\bar{s}}^n)$$

is an isomorphism.

There exists an open subscheme $H_{d,n}^{pr}$ of $H_{d,n}$ such that: A morphism of schemes $f\colon S \to \mathbf{Hilb}_N^{P_{d,n}}$ factors through $H_{d,n}^{pr}$ if and only if conditions (i), (ii) and (iii) are satisfied and in addition for every geometric point $\bar{s}$ of S the line bundle $\mathcal{L}_{\bar{s}}$ from (ii) is primitive.

Proof. The proof of the proposition is standard and can be found in the case of curves in Mumford's book [Mum65, Ch. 5, §2, Prop. 5.1]. We shall sketch only the additional arguments needed in our situation.

There is a maximal open subscheme U_1 of $\mathbf{Hilb}_N^{P_{d,n}}$ such that every fiber of the pull-back $\mathcal{X}_1$ of the universal family $\mathcal{Z}$ over U_1 is a non-singular variety. Let U_2 be the open subscheme of U_1 consisting of the points s for which $H^1(\mathcal{X}_{1,s}, \mathcal{O}_{\mathcal{X}_{1,s}}) = 0$ (see [Har77, Ch. III, §12, Thm. 12.8]). Denote by $\mathcal{X}_2$ the pull-back of the universal family over U_2.

Let $\mathrm{Pic}_{\mathcal{X}_2/U_2}$ be the relative Picard scheme of $\mathcal{X}_2$ over U_2. The two line bundles $\Omega^2_{\mathcal{X}_2/U_2}$ and $\mathcal{O}_{\mathcal{X}_2}$ define two morphisms: $\omega, \lambda\colon U_2 \to \mathrm{Pic}_{\mathcal{X}_2/U_2}$. Define U_2 to be the fiber product:

$$\begin{tikzcd} U_3 \arrow[r] \arrow[d] & U_2 \arrow[d, "(\lambda,\omega)"] \\ \mathrm{Pic}_{\mathcal{X}_2/U_2} \arrow[r, "\Delta"] & \mathrm{Pic}_{\mathcal{X}_2/U_2} \times_{U_2} \mathrm{Pic}_{\mathcal{X}_2/U_2} \end{tikzcd}$$

where Δ is the diagonal morphism. Since $\mathrm{Pic}_{\mathcal{X}_2/U_2}$ is separated U_3 is a closed subscheme of U_2. The pull-back $\mathcal{X}_3 \to U_3$ of the universal family over $\mathbf{Hilb}_N^{P_{d,n}}$ is a K3 scheme.

Let $[n]\colon \mathrm{Pic}_{\mathcal{X}_3/U_3} \to \mathrm{Pic}_{\mathcal{X}_3/U_3}$ be the multiplication by-n-morphism. The pull-back of $\mathcal{O}_{\mathbb{P}^N}(1)$ over U_3 defines a morphism $\lambda\colon U_3 \to \mathrm{Pic}_{\mathcal{X}_3/U_3}$. Define U_4 to be the fiber product

$$\begin{tikzcd} U_4 \arrow[r] \arrow[d] & U_3 \arrow[d, "\lambda"] \\ \mathrm{Pic}_{\mathcal{X}_3/U_3} \arrow[r, "{[n]}"] & \mathrm{Pic}_{\mathcal{X}_3/U_3} . \end{tikzcd}$$

By Lemma 1.3.6 the morphism $[n]$ is a closed immersion hence U_4 is a closed subscheme of U_3. Clearly, U_4 is the subscheme of $\mathbf{Hilb}_N^{P_{d,n}}$ for which properties (i) and (ii) hold. One takes $H_{d,n}$ to be the (closed) subscheme of U_4 obtained as in the end of the proof of Proposition 5.1 in [Mum65, Ch. 5, §2] (where instead of Ω^1_{Γ/U_2} one works with the pull-back $\mathcal{L}'$ of the bundle $\mathcal{O}_{\mathbb{P}^N}(1)$). It satisfies all conditions of the proposition.

To show the existence of $H^{pr}_{n,d}$ one has to take the open subscheme U_4^0 of U_4 above corresponding to the points in U_4 over which the class of the pull-back of $\mathcal{O}_{\mathbb{P}^N}(1)$ in $\mathrm{Pic}_{\mathcal{X}_4/U_4}$ is only divisible by n. The existence of such a subscheme can be seen, in a way similar to the proof of Lemma 1.3.6 *(ii)*, using the fact that the homomorphisms $[n]\colon \mathrm{Pic}_{\mathcal{X}_4/U_4} \to \mathrm{Pic}_{\mathcal{X}_4/U_4}$ are closed immersions. $\square$

We will use the schemes $H_{d,n}$ and $H^{pr}_{d,n}$ to construct moduli stacks of polarized K3 surfaces over $\mathbb{Z}$.

1.4.3 The Moduli Stack

One way to construct the coarse moduli space of complex K3 surfaces with a primitive polarization of degree $2d$ is to use period maps. This approach is taken up in [BBD85, Exposé XIII, §3]. Here we will use rather different techniques to deal with this problem in positive and more generally in mixed characteristic.

Definition 1.4.5. Let d be a natural number. Consider the category $\mathcal{F}_{2d}$ defined in the following way:

Ob: The objects of $\mathcal{F}_{2d}$ are pairs $(\pi\colon X \to S, \lambda)$ consisting of a K3 space $\pi\colon X \to S$ with a primitive polarization λ of degree $2d$ over $S \in \mathrm{Sch}$.

Mor: For two objects $\mathcal{X}_1 = (\pi_1\colon X_1 \to S_1, \lambda_1)$ and $\mathcal{X}_2 = (\pi_2\colon X_2 \to S_2, \lambda_2)$ we define the morphisms to be

$$\begin{aligned}\mathrm{Hom}(\mathcal{X}_1, \mathcal{X}_2) = \big\{\text{pairs } (f_S, f) \mid & f_S\colon S_1 \to S_2 \text{ is a morph. of} \\ & \text{schemes and } f\colon X_1 \to X_2 \times_{S_2, f_S} S_1 \\ & \text{is an isom. over } S_1 \text{ with } f^*\lambda_2 = \lambda_1\big\}.\end{aligned}$$

The functor $p_{\mathcal{F}_{2d}}\colon \mathcal{F}_{2d} \to \mathrm{Sch}$ sending a pair $(\pi\colon X \to S, \lambda)$ to S makes $\mathcal{F}_{2d}$ into a category over Sch. We will denote by $\mathcal{F}_{2d,S}$ the full subcategory of $\mathcal{F}_{2d}$ consisting of the objects over S.

Definition 1.4.6. For a natural number d we define the category $\mathcal{M}_{2d}$ of K3 spaces with a polarization of degree $2d$ in the same way as in Definition 1.4.5 but taking as objects pairs of polarized K3 spaces $(\pi\colon X \to S, \lambda)$ over a scheme S.

We have that $\mathcal{F}_{2d}$ is a full subcategory of $\mathcal{M}_{2d}$. Those two categories are the same if and only if d is square-free.

Theorem 1.4.7. *The categories $\mathcal{F}_{2d}$ and $\mathcal{M}_{2d}$ are separated Deligne-Mumford stacks of finite type over $\mathbb{Z}$. The inclusion $\mathcal{F}_{2d} \hookrightarrow \mathcal{M}_{2d}$ is an open immersion.*

Definition 1.4.8. We will call $\mathcal{F}_{2d}$ *the moduli stack of primitively polarized K3 surfaces of degree* $2d$ and $\mathcal{M}_{2d}$ *the moduli stack of polarized K3 surfaces of degree* $2d$.

Remark 1.4.9. Let us explain first why we want to consider moduli of *primitively* polarized K3 surfaces. For various reasons we will have to work with algebraic spaces rather than with algebraic stacks. Just like in the case of abelian varieties one can introduce level structures on K3 surfaces and hope that the corresponding moduli problems are representable by algebraic spaces. We will define level structures on a polarized K3 surface (X, λ) using its primitive cohomology groups $P^2_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Z}_l(1))$ for certain primes l (see Section 1.5.1). To be able to do that we will need that $P^2_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Z}_l(1))$ belongs to a single isometry class of quadratic lattices, which is the case, if λ is primitive. We will use the moduli stack $\mathcal{M}_{2d}$ only in Chapter 2.

We will prove the theorem in a sequence of steps.

Lemma 1.4.10. *The categories $\mathcal{F}_{2d}$ and $\mathcal{M}_{2d}$ are groupoids.*

Proof. We have to check two axioms. See for instance [LMB00, Ch. 2, Def. 2.1] or p. 96 of [DM68]. One sees immediately that the usual notions of pull-backs satisfy these two axioms. □

Lemma 1.4.11. *The groupoids $\mathcal{F}_{2d}$ and $\mathcal{M}_{2d}$ are stacks for the étale topology.*

Proof. The proofs for $\mathcal{M}_{2d}$ and $\mathcal{F}_{2d}$ are exactly the same so we will prove the lemma for $\mathcal{F}_{2d}$. We have to check two properties. Namely, first we will show that for any scheme $S \in \mathrm{Sch}$ and any two objects $\mathcal{X}$ and $\mathcal{Y}$ over S the functor

$$\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y}) : (\mathrm{Sch}/S) \to \mathrm{Sets}$$

defined by

$$(\pi \colon S' \to S) \mapsto \mathrm{Hom}(\pi^*\mathcal{X}, \pi^*\mathcal{Y})$$

is a sheaf for the étale topology on S. Then we prove that descent data are effective (cf. [LMB00, Ch. 2, Def. 3.1] or Definition 4.1 in [DM68]).

The functor $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$ *is an étale sheaf:* Take two objects $\mathcal{X} = (X \to S, \lambda_X)$ and $\mathcal{Y} = (Y \to S, \lambda_Y)$ over S. Let S' be an S-scheme.

Let $\{S'_i\}_{i \in I}$ be an étale covering of S' and $f_j \in \mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})(S')$ for $j = 1, 2$ are two elements such that $f_1|_{S'_i} = f_2|_{S'_i}$. Then clearly $f_1 = f_2$ as isomorphisms of the pair $(\mathcal{X}_{S'}, \mathcal{Y}_{S'})$.

Let $\{S'_i\}_{i\in I}$ be an étale covering of S'. Suppose given elements $f_i \in \mathrm{Isom}_S(\mathcal{X},\mathcal{Y})(S'_i)$ such that $f_i|_{S'_{ij}} = f_j|_{S'_{ij}}$ where $S'_{ij} = S'_i \times_{S'} S'_j$. We have to show that those come from a global "isomorphism". Note that without loss of generality we may assume that $X_i \to S'_i$ are K3 schemes. Combining [Knu71, Ch. II, Prop. 1.4] and effectiveness of descent for morphisms of schemes (see [BLR90, Ch. 6, §1, Thm. 6(a)]) we conclude that f' descends to a morphism $f\colon X_{S'} \to Y_{S'}$ such that $f_{S'_i} = f_i$. Since $\mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{Y/S}$ are algebraic spaces (in particular sheaves for the étale topology on S) and $f^*\lambda_{Y_{S'}}|_{S'_i} = \lambda_{X_{S'}}|_{S'_i}$ we see that $f^*\lambda_{Y_{S'}} = \lambda_{X_{S'}}$. Hence we have that $f \in \mathrm{Isom}_S(\mathcal{X},\mathcal{Y})(S')$ and $f|_{S'_i} = f_i$. This shows that $\mathrm{Isom}_S(\mathcal{X},\mathcal{Y})$ is an étale sheaf.

Effectiveness of descent: Suppose given an étale cover S' of S and an object $\mathcal{X}' = (\pi'\colon X' \to S', \lambda')$ with descent datum over S. Without loss of generality we may assume that the algebraic space X' is actually a scheme (by refining the étale covering S' if needed). We have to show that $(\pi'\colon X' \to S', \lambda')$ descends to a polarized K3 space $(\pi\colon X \to S, \lambda)$ over S.

Denote by S'' the product $S' \times_S S'$ and let pr_i for $i = 1,2$ be the two projection maps. The descent datum on $X' \to S'$ over S identifies the two schemes pr_1^*X' and pr_2^*X'. Denote this scheme by R. Then we have two étale morphisms

$$R \rightrightarrows X'$$

which make $R \subset X' \times_S X'$ into an étale equivalence relation. Following the constructions of [Knu71, Ch. I, §5, 5.4] we obtain an algebraic space X over S such that $X \times_S S'$ is isomorphic to X'. Hence $\pi\colon X \to S$ is a K3 space.

Since $\mathrm{Pic}_{X/S}$ is an étale sheaf the local section λ' over S' together with descent datum over S give rise to a global section $\lambda \in \mathrm{Pic}_{X/S}(S)$ such that $\lambda_{S'} = \lambda'$. Clearly, λ is a polarization of $X \to S$. □

Next we deal with the representability of the isomorphism functors of polarized K3 surfaces. For two algebraic spaces X and Y over a base scheme S define the contravariant isomorphism functor

$$\mathrm{Isom}_S(X,Y)\colon (\mathrm{Sch}/S) \to \mathrm{Sets}$$

by

$$\mathrm{Isom}_S(X,Y)(T) = \{f\colon X_T \to Y_T \mid f \text{ is an isomorph. of alg. spaces over } T\}$$

for any S-scheme T.

Lemma 1.4.12. *For any $S \in \mathrm{Sch}$ and two objects $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{F}_{2d}$ (respectively $\mathcal{M}_{2d}$) over S, the functor $\mathrm{Isom}_S(\mathcal{X},\mathcal{Y})$ is representable by a separated scheme which is unramified and of finite type over S.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be the objects $(X \to S, \lambda_X)$ and $(Y \to S, \lambda_Y)$ respectively.

Step 1: We can find an étale cover S' of S such that $X' = X \times_S S'$ and $Y' = Y \times_S S'$ are K3 schemes over S'. By [Gro62, Exp. 221, §4.c]) the functors $\mathrm{Isom}_S(X', Y')$ and $\mathrm{Isom}_S(X' \times_X X', Y' \times_Y Y')$ are representable by schemes $\mathcal{U}$ and $\mathcal{V}$, locally of finite type over S. By Proposition 1.4 in [Knu71, Ch. II] one has an exact sequence of sets

$$0 \longrightarrow \mathrm{Isom}_S(X, Y)(T) \longrightarrow \mathrm{Isom}_S(X', Y')(T) \rightrightarrows \mathrm{Isom}_S(X' \times_X X', Y' \times_Y Y')(T).$$

Then we see that $\mathrm{Isom}_S(X, Y)$ is representable by the scheme defined by the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Isom}_S(X,Y) \arrow[r] \arrow[d] & \mathcal{V} \arrow[d, "\Delta"] \\ \mathcal{U} \arrow[r, "(pr_1^*, pr_2^*)"] & \mathcal{V} \times_S \mathcal{V} \end{tikzcd}$$

where $\Delta\colon \mathcal{V} \to \mathcal{V} \times_S \mathcal{V}$ is the diagonal morphism.

Step 2: By Step 1 the functor $\mathrm{Isom}_S(X, Y)$ is represented by a scheme locally of finite type over S. Then the functor $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$ is represented by the scheme defined by the following Cartesian diagram:

$$\begin{tikzcd} \mathrm{Isom}_S(\mathcal{X},\mathcal{Y}) \arrow[rr] \arrow[d] & & S \arrow[d, "\lambda_X"] \\ \mathrm{Isom}_S(X,Y) \times_S S \arrow[r, "(\mathrm{id}, \lambda_Y)"] & \mathrm{Isom}_S(X,Y) \times_S \mathrm{Pic}_{Y/S} \arrow[r] & \mathrm{Pic}_{X/S} \end{tikzcd}$$

where the bottom-right arrow is just the pull back morphism.

Step 3: We are left to show that $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$ is unramified over S. As in the proof of Theorem 1.3.13 it is enough to check the properties of $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$ when S is a spectrum of an algebraically closed field. In this case $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$ is either empty or it is isomorphic to $\mathrm{Aut}_k(X, \lambda)$. As the latter is separated, reduced and of finite type over k we conclude that the same holds for $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$. $\square$

Proof of Theorem 1.4.7: We will give the proof for $\mathcal{F}_{2d}$ in several steps. For the proof that $\mathcal{M}_{2d}$ is a Deligne-Mumford stack one should only replace $H^{pr}_{d,3}$ by $H_{d,3}$ below.

Step 1: We saw in Proposition 1.4.4 that there exists a Hilbert scheme $H^{pr}_{3,d}$, of finite type over $\mathbb{Z}$, classifying K3 surfaces with a polarization of degree $2d$ which are embedded in a projective space via the third power of the polarization. One has then the universal family $f\colon \mathcal{X} \to H^{pr}_{3,d}$ and we know that $\mathcal{O}_{\mathcal{X}}(1) \cong \mathcal{L}^3 \otimes f^*\mathcal{M}$ for some ample line bundle $\mathcal{L}$ on $\mathcal{X}$ of degree $2d$ and an invertible sheaf $\mathcal{M}$ on $H^{pr}_{3,d}$. Although the line bundle $\mathcal{L}$ with this property is not unique, its class $\lambda_{\mathcal{X}} = cl(\mathcal{L}) \in \mathrm{Pic}_{\mathcal{X}/H^{pr}_{3,d}}$ is uniquely determined as $\lambda^3_{\mathcal{X}} = cl(\mathcal{O}_{\mathcal{X}}(1))$. Define the morphism of stacks

$$\pi_{\mathrm{Hilb}} \colon H^{pr}_{3,d} \to \mathcal{F}_{2d}.$$

sending $H^{pr}_{3,d}$ to the pair $(f\colon \mathcal{X} \to H^{pr}_{3,d}, \lambda_{\mathcal{X}})$. By construction the self-intersection $(\lambda_{\mathcal{X},h}, \lambda_{\mathcal{X},h}) = 2d$ for any $h \in H^{pr}_{3,d}$ and $\lambda_{\mathcal{X}}$ is primitive so this morphism is correctly defined.

Step 2: *The morphism π_{Hilb} is surjective.* This follows form the definition (cf. [LMB00, Def. 3.6]) and Lemma 1.3.12. Indeed, for any $(\pi\colon X \to S, \lambda) \in \mathcal{F}_{2d}(S)$ one can find an étale cover $S' \to S$ such that $\pi_{S'}\colon X_{S'} \to S'$ is a K3 scheme and $\lambda_{S'}$ is equal to the class of a relatively ample line bundle $\mathcal{L}'$ on $X_{S'}$. By Lemma 1.1.10 the line bundle $\mathcal{L}'^3$ defines a closed immersion $X_{S'} \hookrightarrow \mathbb{P}(\pi_{S'*}\mathcal{L}'^3)$. Refining S' further if needed we may assume that $\mathbb{P}(\pi_{S'*}\mathcal{L}'^3)$ is isomorphic with $\mathbb{P}^{9d+1}_{S'}$. Then the inclusion $X_{S'} \hookrightarrow \mathbb{P}(\pi_{S'*}\mathcal{L}'^3)$ satisfies the conditions of Proposition 1.4.4 by construction. Hence it corresponds to a morphism $f_X\colon S' \to H^{pr}_{3,d}$ and we have that

$$\pi_{\mathrm{Hilb}}(f_X\colon S' \to H^{pr}_{3,d}) = (\pi_{S'}\colon X_{S'} \to S', \lambda_{S'}).$$

Step 3: *The morphism π_{Hilb} is representable and smooth.* Let S be a scheme and suppose given a morphism $S \to \mathcal{F}_{2d}$ corresponding to a primitively polarized K3 space $(\pi\colon X \to S, \lambda)$. We have to show that the product $S \times_{\mathcal{F}_{2d}} H^{pr}_{3d}$ is representable by an algebraic space which is smooth over S (via pr_1). By the surjectivity of π_{Hilb} one can find an étale cover S' of S and a projective embedding $X_{S'} \hookrightarrow \mathbb{P}^{9d+1}_{S'}$, defined by a very ample line bundle $\mathcal{L}^3$. It gives rise to a morphism $S' \to H^{pr}_{3,d}$ with

$$\pi_{\mathrm{Hilb}}(S' \to H^{pr}_{3,d}) = (X_{S'} \to S', \lambda_{S'}) \in \mathcal{F}_{2d}(S').$$

We claim that the product $S' \times_{\mathcal{F}_{2d}} H^{pr}_{3,d}$ is representable by a scheme isomorphic to $\mathrm{PGL}(9d+2)_{S'}$. For any S'-scheme U we have that

$$S' \times_{\mathcal{F}_{2d}} H^{pr}_{3,d}(U) = \left\{ \begin{array}{l} \big((U \to S'), (U \to H^{pr}_{3,d}), g\big) \quad | \\ \qquad g \in \mathrm{Hom}\big((X_U \to U, \lambda_U), \pi_{\mathrm{Hilb}}(U \to H^{pr}_{3,d})\big) \text{ in } \mathcal{F}_{2d} \end{array} \right\}$$

where $\pi_{\mathrm{Hilb}}(U \to H^{pr}_{3,d}) = (\mathcal{X}_U \to U, \lambda_{\mathcal{X}}|_U)$. Any such morphism g gives rise to an isomorphism $\mathcal{L}^3 \cong \mathcal{O}_{\mathcal{X}_U}(1) \otimes f_U^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on U and hence an isomorphism

$$\mathbb{P}(\pi_{U*}\mathcal{L}^3) \cong \mathbb{P}\big(f_{U*}\mathcal{O}_{\mathcal{X}_U}(1) \otimes \mathcal{M}\big).$$

But by condition (*iii*) of Proposition 1.4.4 we have an isomorphism

$$\mathbb{P}(f_{U*}\mathcal{O}_{\mathcal{X}_U}(1)\otimes\mathcal{M})\cong\mathbb{P}\big(pr_{U*}\mathcal{O}_{\mathbb{P}_U^{9d+1}}(1)\big)=\mathbb{P}_U^{9d+1}$$

and hence we obtain an isomorphism $\mathbb{P}(\pi_{U*}\mathcal{L}^3)\cong\mathbb{P}_U^{9d+1}$. This correspondence gives a bijection

$$S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}(U)\leftrightarrow\big\{\text{ isomorphisms }\mathbb{P}(\pi_{U*}\mathcal{L}^3)\cong\mathbb{P}_U^{9d+1}\ \big\}$$

and the right hand set can be identified with $\mathrm{PGL}(9d+1)_{S'}(U)$. For this we refer to the arguments given on pp. 101-103 in [Mum65]. Hence $S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}$ is representable by the scheme $\mathrm{PGL}(9d+1)_{S'}$ which is smooth over S'.

We will show next that $S\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}$ is a smooth algebraic space over S. We have a surjective map of étale sheaves

$$S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}\to S\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}.$$

The product

$$R:=\big(S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}\big)\times_{\left(S\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}\right)}\big(S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}\big)$$

can be identified with the smooth S-scheme $(S'\times_S S')\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}$. The natural morphism

$$R\to(S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d})\times(S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d})$$

is quasi-compact and the two projection maps

$$R\rightrightarrows S\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}$$

are étale as they correspond to the two étale projection morphisms $S'\times_S S'\rightrightarrows S$. Hence $S\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}$ is an algebraic space, which is moreover smooth over S as it possesses a smooth atlas $S'\times_{\mathcal{F}_{2d}}H^{pr}_{3,d}$ (over S).

Step 4: Using Remark 4.1.2 (i) in [LMB00, Ch. 4] (or Prop. 4.4 in [DM68]) and Lemma 1.4.12 we see that the diagonal morphism $\Delta\colon\mathcal{F}_{2d}\to\mathcal{F}_{2d}\times_{\mathbb{Z}}\mathcal{F}_{2d}$ is representable, separated and quasi-compact. Then we can apply Theorem 4.21 of [DM68] to the morphism $\pi_{\mathrm{Hilb}}\colon H^{pr}_{3,d}\to\mathcal{F}_{2d}$ and conclude that $\mathcal{F}_{2d}$ is a Deligne-Mumford stack of finite type over $\mathbb{Z}$.

Step 5: We will show that the algebraic stack $\mathcal{F}_{2d}$ is separated. As $\mathcal{F}_{2d}$ is of finite type over $\mathbb{Z}$ one can use the valuative criterion for separateness from [DM68, Thm. 4.18] (cf. [LMB00, Prop. 7.8 and Thm. 7.10]). It reduces to showing that if $(\pi_i\colon X_i\to S,\lambda_i)$, for $i=1,2$, are two primitively polarized K3 spaces over the spectrum S of a discrete valuation ring R with field of fractions K, then every isomorphism $f\colon(X_1\otimes K,\lambda_1\otimes K)\to(X_2\otimes K,\lambda_2\otimes K)$ extends to a S-isomorphism between (X_1,λ_1) and (X_2,λ_2). Note that after taking a finite étale covering of S we may assume that:

(a) X_i are schemes,

(b) $\lambda_i = c_1(\mathcal{L}_i)$ for some ample line bundle $\mathcal{L}_i$,

(c) f gives an isomorphism of pairs $f\colon (X_1 \otimes K, \mathcal{L}_1 \otimes K) \to (X_2 \otimes K, \mathcal{L}_2, \otimes K)$.

Then using [MM64, Thm. 2] (as a K3 surface is non-ruled) we see that f extends uniquely to an isomorphism between $(X_1, \mathcal{L}_1)$ and $(X_2, \mathcal{L}_2)$.

Step 6: We are left to show that the natural inclusion $\mathcal{F}_{2d} \hookrightarrow \mathcal{M}_{2d}$ is an open immersion. Take a noetherian scheme S and suppose given a morphism $S \to \mathcal{M}_{2d}$ corresponding to a polarized K3 space $(\pi\colon X \to S, \lambda)$. Let $f\colon S' \to S$ be an étale covering such that $\pi_{S'}\colon X_{S'} \to S'$ is strongly projective (cf. Step 2 in the proof of Theorem 1.4.7). According to Lemma 1.3.6 the set of points

$$S'^o = \{s \in S' \mid \text{such that } \lambda_{S',s} \text{ is primitive}\}$$

is an open subscheme of S'. The morphism f is étale and hence $f(S'^o) \subset S$ is also an open subscheme which represents $S \times_{\mathcal{M}_{2d}} \mathcal{F}_{2d}$. □

Remark 1.4.13. Another possible proof of Theorem 1.4.7 is to use Artin's criterion ([LMB00, Cor. 10.11]). This approach is taken up in [Ols04, Thm. 6.2] where M. Olsson constructs a compact stack of "polarized log K3 spaces" over $\mathbb{Q}$.

An immediate consequence of Theorem 1.4.7 is the existence of a coarse moduli space of polarized K3 surfaces. More precisely Corollary 1.3 in [KM97] says

Corollary 1.4.14. *The moduli stacks $\mathcal{F}_{2d}$ and $\mathcal{M}_{2d}$ have coarse moduli spaces which are separated algebraic spaces.*

Before going on we will shortly outline how one can obtain stronger results on coarse moduli schemes of polarized K3 surfaces in characteristic zero.

Approach via periods of K3 surfaces. As we mentioned in the beginning of this section one can use analytic methods to construct a coarse moduli scheme of primitively polarized K3 surfaces. Consider the complex space

$$\Omega^{\pm} = \{\omega \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) |\ \psi_{2d}(\omega, \omega) = 0 \text{ and } \psi_{2d}(\omega, \bar{\omega}) > 0\}$$

which consists of two connected components. It can be identified with the space

$$\mathrm{SO}(2,19)(\mathbb{R})/\big(\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R})\big).$$

Let Ω^+ denote one of its connected components, say corresponding to

$$\mathrm{SO}(2,19)(\mathbb{R})^+/\big(\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R})\big),$$

where $\mathrm{SO}(2,19)(\mathbb{R})^+$ is the connected component of $\mathrm{SO}(2,19)(\mathbb{R})$ containing the identity. It is a bounded symmetric domain of type IV and of dimension 19. Let Γ be the group $\{g \in \mathrm{O}(V_0)(\mathbb{Z}) \mid g(e_1 - df_1) = e_1 - df_1\}$ and denote by Γ^+ the subgroup of Γ of index 2 which consists of isometries preserving the connected components of $\Omega^{\pm}$. Then Γ^+ acts on Ω^+ properly discontinuously and the space Ω^+/Γ^+ is a coarse moduli scheme for primitively quasi-polarized complex K3 surfaces of degree $2d$. There is an open part Ω^0 of Ω^+ such that Ω^0/Γ^+ is a coarse moduli scheme for primitively polarized complex K3 surfaces of degree $2d$. For details and proofs we refer to [BBD85, Exp. XIII]. The existence of a coarse moduli scheme is Proposition 8 in *loc. cit.*.

Approach via geometric invariant theory. Let k be an algebraically closed field of characteristic zero. Then using the techniques of [Vie95, Ch. 8], and more precisely §8.2 (see Theorem 8.23), one can prove that the moduli functor $\mathcal{F}_{2d} \otimes k$ (respectively $\mathcal{M}_{2d} \otimes k$) has a quasi-projective coarse moduli scheme over k. Indeed, one has that Assumptions 8.22 in [Vie95, §8.2] are satisfied:

(i) The functor is locally closed. This follows from the proof of Proposition 1.4.4.

(ii) The separateness property is shown in Step 2 of the proof of Theorem 1.4.7.

(iii) The functor is bounded by Theorem 1.1.7. See also Remark 8.24 in *loc. cit.* and note that the condition 'ω^2 is trivial' is a locally closed condition.

One actually shows that the scheme in question is $H^{pr}_{3,d} \otimes k/\,\mathrm{PGL}(N)_k$ (respectively $H_{3,d} \otimes k/\,\mathrm{PGL}(N)_k$) for a suitable $N \in \mathbb{N}$.

Combining the approach to coarse moduli schemes via geometric invariant theory and Corollary 1.4.14 we conclude that $\mathcal{F}_{2d,\mathbb{Q}}$ (respectively $\mathcal{M}_{2d,\mathbb{Q}}$) has a quasi-projective coarse moduli scheme.

Proposition 1.4.15. *The moduli stacks $\mathcal{F}_{2d}$ and $\mathcal{M}_{2d}$ are smooth of relative dimension 19 over $\mathbb{Z}[\frac{1}{2d}]$.*

Proof. According to [LMB00, Prop. 4.15] we have to show that for any strictly henselian local ring R and surjection $\mathrm{Spec}(R) \to \mathrm{Spec}(R')$ defined by a nilpotent sheaf of ideals one has that the natural map

$$\mathrm{Hom}(\mathrm{Spec}(R'), \mathcal{F}_{2d,\mathbb{Z}[1/2d]}) \to \mathrm{Hom}(\mathrm{Spec}(R), \mathcal{F}_{2d,\mathbb{Z}[1/2d]})$$

is surjective. Since R is strictly henselian every K3 space over $\mathrm{Spec}(R)$ is a K3 scheme and the same holds for spaces over $\mathrm{Spec}(R')$ (see [GD67, EGA IV, 18.1.2]). Hence by Lemma 1.4.3 we conclude that $\mathcal{F}_{2d,\mathbb{Z}[1/2d]}$ is smooth over $\mathbb{Z}[1/2d]$.

The same argument applies also to the dimension claim. Since every K3 space over $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$ is a K3 scheme we conclude from Theorem 1.4.2 that the dimension of $\mathcal{F}_{2d,\mathbb{Z}[1/2d]}$ at every point is 19.

This proof also shows that $\mathcal{M}_{2d}$ is smooth of relative dimension 19. □

Remark 1.4.16. Since smoothness will be essential for all our further considerations, unless explicitly stated, by $\mathcal{F}_{2d}$ (respectively $\mathcal{M}_{2d}$) we will mean the smooth stack $\mathcal{F}_{2d} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2d}]$ (respectively $\mathcal{M}_{2d} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2d}]$) over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2d}])$.

We will end this section speculating about other possible moduli spaces and functors of polarized K3 surfaces. Note first that one could have started with a moduli functor $\mathcal{F}'_{2d}$ of (primitively) polarized K3 schemes of degree $2d$. The problem we came up with restricting only to schemes was proving effectiveness of descent for K3 schemes. For this reason one takes the "étale sheafification" of $\mathcal{F}'_{2d}$ considering (primitively) polarized K3 spaces. This makes the descent obstruction essentially trivial.

Next, one can consider deformations of polarized K3 surfaces as in Section 1.4.1 by algebraic spaces and not only schemes. For a polarized K3 surface (X_0, λ_0) over an algebraically closed field k define

$$\mathrm{Def}_{\mathrm{AlgSp}}(X_0, \lambda_0) \colon \underline{A} \to \mathrm{Sets}$$

to be the functor sending an object A of $\underline{A}$ to the isomorphism classes of triples $(\mathcal{X}, \lambda, \phi_0)$ where $(\mathcal{X} \to \mathrm{Spec}(A), \lambda)$ is a polarized K3 space and ϕ_0 is an isomorphism $\phi_0 \colon (\mathcal{X}, \mathcal{L}) \otimes_A k \cong (X_0, \mathcal{L}_0)$. Combining Theorem 1.4.7, Lemma 1.4.15 and [LMB00, Cor. 10.11] we conclude that $\mathrm{Def}_{\mathrm{AlgSp}}$ is pro-representable, formally smooth and of dimension 19.

1.5 Level Structures of Polarized K3 Surfaces

1.5.1 Level Structures

Recall that for an abelian scheme (A, λ) over a base scheme S and a natural number n which is invertible in S one defines a (Jacobi) level n-structure on A to be an isomorphism $\theta \colon A[n] \to (\mathbb{Z}/n\mathbb{Z})_S$ of étale sheaves on S satisfying some further properties. In other words, one uses the Tate module of an abelian variety in order to define level structures. For a K3 surface X we will use the same idea applied to $H^2_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Z}_l(1))$. More precisely, we will introduce the notion of level structures on primitively polarized K3 surfaces of degree $2d$ corresponding to open compact subgroups of $\mathrm{SO}(V_{2d}, \psi_{2d})(\mathbb{A}_f)$ (see below) and define moduli spaces of primitively polarized K3 surfaces with level structures. We set up some notations first.

- All schemes in this section will be assumed to be locally noetherian.
- For a finite set of primes $\mathscr{B} = \{p_1, \ldots, p_r\}$ we denote by $\mathbb{Z}_{\mathscr{B}}$ the product $\prod_{p \in \mathscr{B}} \mathbb{Z}_p$ and by $N_{\mathscr{B}}$ the product of the primes in $\mathscr{B}$.
- We fix a natural number d. We shall use the notations $L_{2d,\mathscr{B}}$ and $L_{0,\mathscr{B}}$ for the quadratic lattices $L_{2d} \otimes \mathbb{Z}_{\mathscr{B}}$ and $L_0 \otimes \mathbb{Z}_{\mathscr{B}}$ (cf. Section 1.2.1).

- Let $\mathbb{K} \subset \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ be a subgroup of finite index and let $\mathscr{B} = \{p_1, \ldots, p_r\}$ be the set of prime divisors of $2d$ and primes p for which $\mathbb{K}_p \neq \mathrm{SO}(V_{2d})(\mathbb{Z}_p)$. We denote by $\mathbb{K}_{\mathscr{B}}$ the product $\prod_{p \in \mathscr{B}} \mathbb{K}_p$.

Level Structures. Let S be a connected scheme over $\mathbb{Z}[\frac{1}{p_1 \ldots p_r}]$ and suppose given a polarized K3 space $(\pi \colon X \to S, \lambda)$ of degree $2d$. Let $P^2_{\mathrm{et}}\pi_*\mathbb{Z}_{\mathscr{B}}(1)$ be the sheaf of primitive cohomology i.e., the orthogonal complement of $c_1(\lambda)$ in $R^2_{\mathrm{et}}\pi_*\mathbb{Z}_{\mathscr{B}}(1)$. Take a geometric point $\bar{b}$ of S and let $\bar{b} \colon \mathrm{Spec}(k(\bar{b})) \to S$ be the corresponding morphism of schemes. Consider the free $\mathbb{Z}_{\mathscr{B}}$-module of rank 21

$$P^2(\bar{b}) := \bar{b}^* P^2_{\mathrm{et}}\pi_*\mathbb{Z}_{\mathscr{B}}(1)$$

i.e., the fiber of $P^2_{\mathrm{et}}\pi_*\mathbb{Z}_{\mathscr{B}}(1)$ at $\bar{b}$ with its action of $\pi_1^{\mathrm{alg}}(S, \bar{b})$ and the bilinear form $\psi_{\lambda, \mathbb{Z}_{\mathscr{B}}}$.

Suppose given an class $\alpha_{\bar{b}}$ in the set

$$\Big\{ \mathbb{K}_{\mathscr{B}} \backslash \mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b})\big) \Big\}^{\pi_1^{\mathrm{alg}}(S,\bar{b})}$$

where $\mathbb{K}_{\mathscr{B}}$ acts on $\mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b})\big)$ on the right via its action on $L_{2d,\mathbb{Z}_{\mathscr{B}}}$ and $\pi_1^{\mathrm{alg}}(S, \bar{b})$ acts on the left via its action on $P^2(\bar{b})$. Let $\bar{b}'$ be another geometric point in S. The $\alpha_{\bar{b}}$ determines uniquely a class in

$$\Big\{ \mathbb{K}_{\mathscr{B}} \backslash \mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b}')\big) \Big\}^{\pi_1^{\mathrm{alg}}(S,\bar{b}')}$$

in the following way: One can find an isomorphism

$$\delta_\pi \colon \pi_1^{\mathrm{alg}}(S, \bar{b}) \cong \pi_1^{\mathrm{alg}}(S, \bar{b}') \tag{1.7}$$

and an isometry

$$\delta_{\mathrm{et}} \colon H^2_{\mathrm{et}}(X_{\bar{b}}, \mathbb{Z}_{\mathscr{B}}(1)) \to H^2_{\mathrm{et}}(X_{\bar{b}'}, \mathbb{Z}_{\mathscr{B}}(1))$$

determined uniquely by δ_π, mapping $c_1(\lambda_{\bar{b}})$ to $c_1(\lambda_{\bar{b}'})$, such that $\delta_{\mathrm{et}}(\gamma \cdot x) = \delta_\pi(\gamma) \cdot \delta_{\mathrm{et}}(x)$ for every $x \in H^2_{\mathrm{et}}(X_{\bar{b}}, \mathbb{Z}_{\mathscr{B}}(1))$ and $\gamma \in \pi_1^{\mathrm{alg}}(S, \bar{b})$. The isometry δ_{et} defines an isometry between $P^2(\bar{b})$ and $P^2(\bar{b}')$ which we will denote again by δ_{et}. Let $\tilde{\alpha}$ be a representative of the class $\alpha_{\bar{b}}$. Then the class $\alpha_{\bar{b}'}$ of $\delta_{\mathrm{et}} \circ \tilde{\alpha}$ in $\mathbb{K}_{\mathscr{B}} \backslash \mathrm{Isometry}\big(L_{2d,\mathscr{B}}, P^2(\bar{b}')\big)$ is $\pi_1^{\mathrm{alg}}(S, \bar{b}')$-invariant. Any other representative $\tilde{\alpha}_1$ of $\alpha_{\bar{b}}$ differs by an element in $\mathbb{K}_{\mathscr{B}}$ and hence gives rise to the same class $\alpha_{\bar{b}'}$ in $\mathbb{K}_{\mathscr{B}} \backslash \mathrm{Isometry}\big(L_{2d,\mathscr{B}}, P^2(\bar{b}')\big)$.

Any two isomorphisms (1.7) differ by an inner automorphism of $\pi_1^{\mathrm{alg}}(S, \bar{b})$ and therefore we see that that class of $\delta_{\mathrm{et}} \circ \tilde{\alpha}$ is independent of the choice of an isomorphism (1.7). This remark allows us to make the following definition.

Definition 1.5.1. A *level* $\mathbb{K}$*-structure* on a primitively polarized K3 space $(\pi\colon X \to S, \lambda)$ over a connected scheme $S \in (\mathrm{Sch}\,/\mathbb{Z}[1/p_1 \dots p_r])$ is an element of the set

$$\left\{\mathbb{K}_{\mathscr{B}}\backslash \mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b})\big)\right\}^{\pi_1^{\mathrm{alg}}(S,\bar{b})}.$$

The group $\mathbb{K}_{\mathscr{B}}$ acts on $\mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b})\big)$ on the right via its action on $L_{2d,\mathbb{Z}_{\mathscr{B}}}$ and $\pi_1^{\mathrm{alg}}(S,\bar{b})$ acts on the left via its action on $P^2(\bar{b})$. In general, a level $\mathbb{K}$-structure on $(\pi\colon X \to S, \lambda)$ is a level $\mathbb{K}$-structure on each connected component of S.

If $\tilde{\alpha}\colon L_{2d,\mathscr{B}} \to P^2_{et}(\bar{b})$ is a representative of the class α, then via the isomorphism

$$\tilde{\alpha}^{\mathrm{ad}}\colon \mathrm{O}(V_{2d})(\mathbb{Z}_{\mathscr{B}}) \cong \mathrm{O}(P^2(\bar{b}))(\mathbb{Z}_{\mathscr{B}})$$

the monodromy action

$$\rho\colon \pi_1^{\mathrm{alg}}(S,\bar{b}) \to \mathrm{O}(P^2(\bar{b}))(\mathbb{Z}_{\mathscr{B}})$$

factorizes through $\tilde{\alpha}^{\mathrm{ad}}(\mathbb{K}_{\mathscr{B}})$.

Remark 1.5.2. If all residue fields of the points in S in Definition 1.5.1 are zero, then one can define a level $\mathbb{K}$-structure to be an element of set

$$\left\{\mathbb{K}\backslash \mathrm{Isometry}\big(L_{2d,\hat{\mathbb{Z}}}, P^2(\bar{b})\big)\right\}^{\pi_1^{\mathrm{alg}}(S,\bar{b})}$$

where $P^2(\bar{b}) := \bar{b}^* P^2_{\mathrm{et}}\pi_*\hat{\mathbb{Z}}(1)$.

We will consider two important examples of level structures on primitively polarized K3 spaces.

Example 1.5.3. Fix a natural number n and consider the group

$$\mathbb{K}_n = \big\{\gamma \in \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})|\ \gamma \equiv 1 \pmod n\big\}.$$

Then the set $\mathscr{B}$ consists of the prime divisors of $2dn$. We will give a direct interpretation of level $\mathbb{K}_n$-structures.

Let S be a scheme over $\mathbb{Z}[1/2dn]$ and consider a primitively polarized K3 space $(\pi\colon X \to S, \lambda)$ of degree $2d$. As usual we denote by $P^2_{\mathrm{et}}\pi_*(\mathbb{Z}/n\mathbb{Z})(1)$ the orthogonal complement of $c_1(\lambda)$ in $R^2_{\mathrm{et}}\pi_*(\mathbb{Z}/n\mathbb{Z})(1)$ with respect to the bilinear form $\psi_n = \psi \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Then a level $\mathbb{K}_n$-structure amounts to giving an isomorphism

$$\alpha_N\colon \big(P^2_{\mathrm{et}}\pi_*(\mathbb{Z}/n\mathbb{Z})(1), \psi_{\mathcal{L},n}\big) \to (L_{2d,\mathbb{Z}/n\mathbb{Z}}, \psi_{2d,\mathbb{Z}/n\mathbb{Z}})_S$$

of étale sheaves on S, where $(L_{2d,\mathbb{Z}/n\mathbb{Z}}, \psi_{2d,\mathbb{Z}/n\mathbb{Z}})_S$ is the constant polarized étale sheaf over S with fibers $(L_{2d}, \psi_{2d}) \otimes \mathbb{Z}/n\mathbb{Z}$.

We will call level a $\mathbb{K}_n$-structure on X simply a *level* n*-structure*.

Example 1.5.4. Let G be the algebraic group $\mathrm{SO}(V_{2d})$ over $\mathbb{Q}$. Consider the *even Clifford algebra* $C^+(V_{2d}, \psi_{2d})$ over $\mathbb{Q}$ and let G_1 be the *even Clifford group* over $\mathbb{Q}$ (see [Lam73, Ch. V] and [Sch85, Ch. 9]). In other words we set

$$G_1 = \mathrm{CSpin}(V_{2d}) = \left\{ g \in C^+(V_{2d})^* \mid gV_{2d}g^{-1} = V_{2d} \right\}.$$

The natural homomorphism of linear algebraic groups $G_1 \to G$ given by $g \mapsto (v \mapsto gvg^{-1})$ fits into an exact sequence (see [Del72, §3.2])

$$0 \to \mathbb{G}_m \to G_1 \to G \to 0.$$

Set $G_1(\mathbb{Z})$ to be $G_1(\mathbb{Q}) \cap C^+(L_{2d})^*$. We have an exact sequence (see [And96a, §4.4])

$$0 \to \mathbb{Z}/2\mathbb{Z} \to G_1(\mathbb{Z}) \to G(\mathbb{Z}). \tag{1.8}$$

For a natural number n denote

$$\Gamma_n = \left\{ \gamma \in G(\mathbb{Z}) \mid \gamma \equiv 1 \pmod{n} \right\}$$

and

$$\Gamma_n^{\mathrm{sp}} = \left\{ \gamma \in G_1(\mathbb{Z}) \mid \gamma \equiv 1 \pmod{n} \text{ in } C^+(L_{2d}) \right\}.$$

If $n > 2$, then Γ_n and Γ_n^{sp} are torsion free. Hence one sees from the exact sequence (1.8) that Γ_n^{sp} is isomorphic with its image Γ_n^{a} in $G(\mathbb{Q})$. Note that, in general, Γ_n^{a} is not a congruence subgroup of $G(\mathbb{Z})$ (cf. footnote 10 in [And96a]).

Consider the group

$$\mathbb{K}_n^{\mathrm{sp}} = \left\{ \gamma \in G_1(\hat{\mathbb{Z}}) \mid \gamma \equiv 1 \pmod{n} \text{ in } C^+(L_{2d,\hat{\mathbb{Z}}}) \right\}.$$

We have that $\mathbb{K}_n^{\mathrm{sp}} \cap G_1(\mathbb{Q}) = \Gamma_n^{\mathrm{sp}}$. Moreover the image $\mathbb{K}_n^a$ of $\mathbb{K}_n^{\mathrm{sp}}$ in $G(\hat{\mathbb{Z}})$ is of finite index. Indeed, for every l not dividing $2nd$, the l-component of $\mathbb{K}_n^{\mathrm{a}}$ is $G(\mathbb{Z}_l)$ as shown in [And96a, §4.4]. Hence the set $\mathscr{B}$ for $\mathbb{K}_n^{\mathrm{a}}$ is the set of prime divisors of $2dn$.

We consider polarized K3 surfaces with level $\mathbb{K}_n^{\mathrm{a}}$-structure. Note that this level structure is in general finer than level $\mathbb{K}_n$-structure as $\mathbb{K}_n^{\mathrm{a}} \subset \mathbb{K}_n$ is of finite index. We will call it *spin level n-structure*.

Motivation. We will pause here and give a motivation for the rest of the definitions we make in this section. So far we have defined level $\mathbb{K}$-structures using the primitive second étale cohomology group of a polarized K3 surface. Using these level structures one can define moduli stacks $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ of primitively polarized K3 surfaces of degree $2d$ with a level $\mathbb{K}$-structure and show that they are algebraic spaces (cf. Theorem 1.5.11 below). We will see in Chapter 3 that, over $\mathbb{C}$, we can relate those spaces to the orthogonal Shimura variety associated to the group $\mathrm{SO}(2,19)$. More precisely we will define a period morphism

$$j_{d,\mathbb{K},\mathbb{C}} \colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(\mathrm{SO}(2,19), \Omega^{\pm})_{\mathbb{C}}$$

which is étale. This is similar to the case of moduli of abelian varieties where one can identify $\mathcal{A}_{g,1,n} \otimes \mathbb{C}$ with $Sh_{\Lambda_n}(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})_{\mathbb{C}}$. In general, due to the fact that the injective homomorphism (1.2)

$$i^{\mathrm{ad}} \colon \{g \in \mathrm{SO}(V_0)(\mathbb{Z}) \mid g(e_1 - df_1) = e_1 - df_1\} \hookrightarrow \mathrm{SO}(V_{2d})(\mathbb{Z})$$

defined in Section 1.2.1 is not surjective, the period map $j_{d,\mathbb{K},\mathbb{C}}$ need not be injective (cf. Remark 3.2.12). In order to construct an injective period morphism we will define level structures using the "full" second étale cohomology group of a K3 surface. We will use those full level structures in Chapter 3 to show that every complex K3 surface with complex multiplication by a CM-field E is defined over an abelian extension of E (Corollary 3.2.12).

Full Level Structures. The inclusion of lattices $i \colon L_{2d} \hookrightarrow L_0$ (see Section 1.2.1) defines injective homomorphisms of groups

$$i^{\mathrm{ad}} \colon \{g \in \mathrm{O}(V_0)(\hat{\mathbb{Z}}) \mid g(e_1 - df_1) = e_1 - df_1\} \hookrightarrow \mathrm{O}(V_{2d})(\hat{\mathbb{Z}})$$

and

$$i^{\mathrm{ad}} \colon \{g \in \mathrm{SO}(V_0)(\hat{\mathbb{Z}}) \mid g(e_1 - df_1) = e_1 - df_1\} \hookrightarrow \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}}).$$

Definition 1.5.5. A subgroup $\mathbb{K} \subset \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ of finite index is called *admissible* if it is contained in the image

$$i^{\mathrm{ad}}\big(\{g \in \mathrm{SO}(V_0)(\hat{\mathbb{Z}}) \mid g(e_1 - df_1) = e_1 - df_1\}\big) \subset \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}}).$$

If $\mathbb{K}$ is an admissible subgroup of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ then all its subgroups of finite index $\mathbb{K}' \subset \mathbb{K}$ are also admissible.

Example 1.5.6. The group $\mathbb{K}_{2d}$ is admissible. Hence all its subgroups of finite index are admissible, as well.

Example 1.5.7. If $d = 1$ then $\mathbb{K}_n$ is admissible for any $n \geq 2$.

Let $\mathbb{K}$ be an admissible subgroup of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ and let $\mathscr{B}$ be the set, consisting of all prime divisors of $2d$ and, of the primes p for which $\mathbb{K}_p \neq \mathrm{SO}(V_{2d})(\mathbb{Z}_p)$. Using the notations introduced before Definition 1.5.1 we set

$$H^2(\bar{b}) := \bar{b}^* R^2_{\mathrm{et}} \pi_* \mathbb{Z}_{\mathscr{B}}(1).$$

In order to simplify the notations we will identify a subgroup of $\{g \in \mathrm{SO}(V_0)(\hat{\mathbb{Z}}) \mid g(e_1 - df_1) = e_1 - df_1\}$ with its image in $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ under the injective homomorphism i^{ad}.

Definition 1.5.8. A *full level $\mathbb{K}$-structure* on a primitively polarized K3 space $(\pi\colon X \to S, \lambda)$ over a connected scheme $S \in (\mathrm{Sch}\,/\mathbb{Z}[1/p_1 \dots p_r])$ is an element of the set

$$\Big\{\mathbb{K}_{\mathscr{B}} \backslash \{g \in \mathrm{Isometry}\big(L_{0,\mathbb{Z}_{\mathscr{B}}}, H^2(\bar{b})\big) \mid g(e_1 - df_1) = c_1(\lambda_{\bar{b}})\}\Big\}^{\pi_1^{\mathrm{alg}}(S,\bar{b})}.$$

The group $\mathbb{K}_{\mathscr{B}}$ acts on $\{g \in \mathrm{Isometry}\big(L_{0,\mathbb{Z}_{\mathscr{B}}}, H^2(\bar{b})\big) \mid g(e_1 - df_1) = c_1(\lambda_{\bar{b}})\}$ on the right via its action on $L_{0,\mathbb{Z}_{\mathscr{B}}}$ and $\pi_1^{\mathrm{alg}}(S, \bar{b})$ acts on the left via its action on $H^2(\bar{b})$. A full level $\mathbb{K}$-structure on $(\pi\colon X \to S, \lambda)$ over a general base S is a full level $\mathbb{K}$-structure on each connected component of S.

Again, a class $\alpha_{\bar{b}}$ for a geometric point $\bar{b}$ as above determines uniquely a class $\alpha_{\bar{b}'}$ for any other geometric point $\bar{b}'$. If $\tilde{\alpha}\colon L_{0,\mathscr{B}} \to H^2(\bar{b})$ is a representative of the class α, then via the isomorphism

$$\tilde{\alpha}^{\mathrm{ad}}\colon \mathrm{O}(V_0)(\mathbb{Z}_{\mathscr{B}}) \cong \mathrm{O}(H^2(\bar{b}))(\mathbb{Z}_{\mathscr{B}})$$

the monodromy action $\rho\colon \pi_1^{\mathrm{alg}}(S, \bar{b}) \to \mathrm{O}(H^2(\bar{b}))(\mathbb{Z}_{\mathscr{B}})$ factorizes through $\tilde{\alpha}^{\mathrm{ad}}(\mathbb{K}_{\mathscr{B}})$.

Example 1.5.9. Let $n \geq 3$ be an integer. Define the group

$$\mathbb{K}_n^{\mathrm{full}} = \big\{g \in \mathrm{SO}(V_0)(\hat{\mathbb{Z}}) \mid g(e_1 - df_1) = e_1 - df_1 \text{ and } g \equiv 1 \pmod{n}\big\}.$$

By definition it is an admissible subgroup of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$. Let S be a scheme over $\mathbb{Z}[1/2dn]$ and consider a K3 space $(\pi\colon X \to S, \lambda)$ with a primitive polarization of degree $2d$. Then a full level $\mathbb{K}_n^{\mathrm{full}}$-structure amounts to giving an isomorphism

$$\alpha_N\colon \big(R^2_{\mathrm{et}}\pi_*(\mathbb{Z}/n\mathbb{Z})(1), \psi\big) \to (L_{0,\mathbb{Z}/n\mathbb{Z}}, \psi_{0,\mathbb{Z}/n\mathbb{Z}})_S$$

of étale sheaves on S, where $(L_{0,\mathbb{Z}/n\mathbb{Z}}, \psi_{0,\mathbb{Z}/n\mathbb{Z}})_S$ is the constant polarized étale sheaf over S with fibers $(L_0, \psi_0) \otimes \mathbb{Z}/n\mathbb{Z}$.

We will call a full level $\mathbb{K}_n^{\mathrm{full}}$-structure on X simply a *full level n-structure*.

1.5.2 Moduli Stacks of Polarized K3 Surfaces with Level Structure

In this section we will use the notion of a (full) level structure level structure to define moduli functors of primitively polarized K3 spaces with a (full) level structure. Using Artin's criterion and Proposition 1.3.18 we will show that these functors are representable by algebraic spaces over open parts of $\mathrm{Spec}(\mathbb{Z})$.

We shall be using the notations established in the beginning of Section 1.5.1. In particular we fix a natural number d. To a subgroup $\mathbb{K}$ of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ we associated a finite set of primes $\mathscr{B}$ and $N_{\mathscr{B}}$ will denote the product of these primes.

Moduli of K3 Surfaces with Level Structure. Let $\mathbb{K}$ be a subgroup of $\mathrm{SO}(\hat{\mathbb{Z}})$ of finite index. We will assume further that it is contained in $\mathbb{K}_n$ for some $n \geq 3$. Let $\mathcal{X}_1 = (\pi_1 \colon X_1 \to S_1, \lambda_1)$ and $\mathcal{X}_2 = (\pi_2 \colon X_2 \to S_2)$ be two objects of $\mathcal{F}_{2d}$. Suppose that S_1 and S_2 are connected and let $(f, f_S) \in \mathrm{Hom}(\mathcal{X}_1, \mathcal{X}_2)$ (in $\mathcal{F}_{2d}$). Let $\bar{b}_1$ and $\bar{b}_2$ be two geometric points of S_1 and S_2 such that $f_S(\bar{b}_1) = \bar{b}_2$. Then the morphism f defines a homomorphism $f^*_{\mathrm{et}} \colon P^2(\bar{b}_2) \to P^2(\bar{b}_1)$. Hence we obtain a map

$$f^\vee \colon \mathbb{K}_{\mathscr{B}} \backslash \mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{0,\mathscr{B}}}, P^2(\bar{b}_2)\big) \to \mathbb{K}_{\mathscr{B}} \backslash \mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b}_1)\big)$$

given by $\alpha \mapsto f^*_{\mathrm{et}} \circ \alpha$ and commuting with the monodromy actions on both sides.

Definition 1.5.10. For d and $\mathbb{K}$ as above consider the category $\mathcal{F}_{2d,\mathbb{K}}$ defined in the following way:

Ob: Triples $(\pi \colon X \to S, \lambda, \alpha)$ of a K3 space $\pi \colon X \to S$ with a primitive polarization λ of degree $2d$ and with a level $\mathbb{K}$-structure α on $(\pi \colon X \to S, \lambda)$.

Mor: Suppose given two triples $\mathcal{X}_1 = (\pi_1 \colon X_1 \to S_1, \lambda_1, \alpha_1)$ and $\mathcal{X}_2 = (\pi_2 \colon X_2 \to S_2, \lambda_2, \alpha_2)$. Let $f_S \colon S_1 \to S_2$ be a morphism of schemes. Choose base geometric points $\bar{b}'_1$ and $\bar{b}'_2$ on any two connected components S'_1 and S'_2 of S_1 and S_2 for which $f \colon S'_1 \to S'_2$ such that $f_S(\bar{b}'_1) = \bar{b}'_2$. Define the morphisms between $\mathcal{X}_1$ and $\mathcal{X}_2$ in the following way

$$\begin{aligned}\mathrm{Hom}(\mathcal{X}_1, \mathcal{X}_2) = \big\{\text{pairs } (f_S, f) \mid & f_S \colon S_1 \to S_2 \text{ is a morph. of spaces,} \\ & f \colon X_1 \to X_2 \times_{S_2, f_S} S_1 \text{ is an isom. of} \\ & S_1 - \text{spaces with } f^*\lambda_2 = \lambda_1 \text{ and} \\ & f^\vee(\alpha_1) = \alpha_2 \text{ on any conn. cmpt. of } S_1\big\}.\end{aligned}$$

Next we define three projection functors.

1. Consider the following forgetful functor

$$pr_{\mathcal{F}_{2d,\mathbb{K}}} \colon \mathcal{F}_{2d,\mathbb{K}} \to (\mathrm{Sch}\,/\mathbb{Z}[1/N_{\mathscr{B}}])$$

   sending a triple $(\pi \colon X \to S, \lambda, \alpha)$ to S. It makes $\mathcal{F}_{2d,\mathbb{K}}$ into a category over $(\mathrm{Sch}\,/\mathbb{Z}[1/N_{\mathscr{B}}])$.

2. For any $\mathbb{K}$, satisfying the assumptions of the beginning of the section, one has a projection functor

$$pr_{\mathbb{K}} \colon \mathcal{F}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]} \tag{1.9}$$

   sending a triple $(\pi \colon X \to S, \lambda, \alpha)$ to $(\pi \colon X \to S, \lambda)$ and an element $(f, f_S) \in \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ of $\mathcal{F}_{2d,\mathbb{K}}$ to (f, f_S).

3. For any two subgroups $\mathbb{K}_1 \subset \mathbb{K}_2$ of finite index in $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ (contained in some $\mathbb{K}_n$ for $n \geq 3$) one has a projection functor

$$pr_{(\mathbb{K}_1,\mathbb{K}_2)} \colon \mathcal{F}_{2d,\mathbb{K}_1,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]} \to \mathcal{F}_{2d,\mathbb{K}_2,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]}. \tag{1.10}$$

It sends an object $(X \to S, \lambda, \alpha_{\mathbb{K}_1})$ to $(X \to S, \lambda, \alpha_{\mathbb{K}_2})$ where $\alpha_{\mathbb{K}_2}$ is the class of $\alpha_{\mathbb{K}_1}$ in $\mathbb{K}_{2,\mathscr{B}}\backslash\mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b})\big)$. Morphism of $\mathcal{F}_{2d,\mathbb{K}_1,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]}$ are mapped to morphism of $\mathcal{F}_{2d,\mathbb{K}_2,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]}$ in the obvious way.

From the definitions of the functors we see that $pr_{\mathbb{K}_1} = pr_{(\mathbb{K}_1,\mathbb{K}_2)} \circ pr_{\mathbb{K}_2}$ over $\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]$.

Theorem 1.5.11. *The category $\mathcal{F}_{2d,\mathbb{K}}$ is a separated algebraic space over $\mathbb{Z}[1/N_{\mathscr{B}}]$. It is smooth of relative dimension 19 and the forgetful morphism* (1.9)

$$pr_{\mathbb{K}} \colon \mathcal{F}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]}$$

is finite and étale.

Proof. We divide the proof into several steps.
Step 1: The category $\mathcal{F}_{2d,\mathbb{K}}$ is a stack. The proof goes exactly in the same lines as the one of Lemma 1.4.11. We will use Artin's criterion (cf. [LMB00, Cor. 10.11]) to show that $\mathcal{F}_{2d,\mathbb{K}}$ is an algebraic space.

We claim that the diagonal morphism $\Delta\colon \mathcal{F}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{K}} \times_{\mathbb{Z}[1/N_{\mathscr{B}}]} \mathcal{F}_{2d,\mathbb{K}}$ is representable, separated and of finite type. By Remark 4.1.2 in [LMB00] it is equivalent to showing that for any two objects $\mathcal{X} = (X \to S, \lambda_X, \alpha_X)$ and $\mathcal{Y} = (Y \to S, \lambda_Y, \alpha_Y)$ the functor $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$ has these properties. We will prove first the following result.

Lemma 1.5.12. *For any object $\mathcal{X}$ of $\mathcal{F}_{2d,\mathbb{K}}$ we have that* $\mathrm{Aut}_S(\mathcal{X}) = \{\mathrm{id}_{\mathcal{X}}\}$.

Proof. By assumption the group $\mathbb{K}$ is contained in $\mathbb{K}_n$ for some $n \geq 3$. Hence a level $\mathbb{K}$-structure on a primitively polarized K3 space $(X \to S, \lambda)$ defines in a natural way (using the functor $pr_{(\mathbb{K},\mathbb{K}_n)}$) a level n-structure α_n on X. We have that

$$\mathrm{Aut}_S\big((X \to S, \lambda, \alpha)\big)(U) \subset \mathrm{Aut}_S\big((X \to S, \lambda, \alpha_n)\big)(U)$$

for an S-scheme U hence it is enough to prove the lemma assuming that $\mathbb{K} = \mathbb{K}_n$.

Let $\mathcal{X} = (X \to S, \lambda, \alpha)$ be an object in $\mathcal{F}_{2d,\mathbb{K}}$, let $f \in \mathrm{Aut}_S(\mathcal{X})(U)$ and assume that U is connected. Take a geometric point $\bar{b}\colon \mathrm{Spec}(\Omega) \to U$. Then for the finite set $\mathscr{B} = \{$the prime divisors of n$\}$ the morphism f induces an automorphism

$$f^*_{\mathrm{et}} \colon H^2_{\mathrm{et}}(X_{\bar{b}}, \mathbb{Z}_{\mathscr{B}}(1)) \to H^2_{\mathrm{et}}(X_{\bar{b}}, \mathbb{Z}_{\mathscr{B}}(1))$$

fixing $c_1(\lambda_{\bar{b}})$ and such that

$$f^*_{\mathrm{et}} \colon P^2_{\mathrm{et}}(X_{\bar{b}}, \mathbb{Z}/n\mathbb{Z}(1)) \to P^2_{\mathrm{et}}(X_{\bar{b}}, \mathbb{Z}/n\mathbb{Z}(1))$$

is the identity (cf. Example 1.5.3). As the automorphism f is of finite order we have that $f^*_{\text{et}} \in \mathrm{O}\big(P^2_{\text{et}}(X_{\bar{b}}, \mathbb{Z}_{\mathscr{B}}(1))\big)$ is semi-simple and its eigenvalues are roots of unity. We have further that $f^*_{\text{et}} \equiv 1 \pmod{n}$ so we conclude by [Mum74, Ch. IV, Application II, p. 207, Lemma] that f^*_{et} is the identity automorphism of $P^2_{\text{et}}(X_{\bar{b}}, \mathbb{Z}_{\mathscr{B}}(1))$. As it fixes $c_1(\lambda_{\bar{b}})$ we see that it acts as the identity on $H^2_{\text{et}}(X_{\bar{b}}, \mathbb{Z}_{\mathscr{B}}(1))$. Therefore by Proposition 1.3.18 we that $f = \mathrm{id}_{X_{\bar{b}}}$. As the geometric point $\bar{b}$ can be chosen arbitrary we have that $f = \mathrm{id}_{\mathcal{X}_U}$. □

We see from the lemma that for a S-scheme U the set $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})(U)$ is either empty or it consists of one element. Indeed, suppose that $f_i \in \mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})(U)$ for $i = 1, 2$. Then the composition $f_2^{-1} \circ f_1$ belongs to $\mathrm{Aut}_S(\mathcal{X})(U)$ and hence it is the identity. This shows that $\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})$ is representable and of finite type. The fact that it is unramified and separated over S follows from Lemma 1.4.12 as one has that

$$\mathrm{Isom}_S(\mathcal{X}, \mathcal{Y})(U) \subset \mathrm{Isom}_S\big((X \to S, \lambda_X), (Y \to S, \lambda_Y)\big)(U).$$

Next we claim that the stack $\mathcal{F}_{2d,\mathbb{K}}$ is locally of finite presentation. This follows from [AGV71, Exposé IX, 2.7.4] and the fact that $\mathcal{F}_{2d}$ is locally of finite presentation. Conditions *(iii)* and *(iv)* of [LMB00, Cor. 10.11] follow from the corresponding properties of $\mathcal{F}_{2d}$ and the fact that for any small surjection of rings $R \to R'$ the category of étale schemes over R is equivalent to the category of étale schemes over R' ([GD67, EGA IV, 18.1.2]).

Thus $\mathcal{F}_{2d,\mathbb{K}}$ is an algebraic stack. As $\mathrm{Aut}_S(\mathcal{X}) = \{\mathrm{id}_{\mathcal{X}}\}$ for any object we have that $\mathcal{F}_{2d,\mathbb{K}}$ is an algebraic space ([LMB00, Cor. 8.1.1]).

Step 2: We will show that the morphism of algebraic stacks $pr_{\mathbb{K}}\colon \mathcal{F}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]}$ is representable and étale. Indeed, let S be a connected scheme and suppose given a morphism $S \to \mathcal{F}_{2d}$ i.e., a polarized K3 space $(\pi\colon X \to S, \lambda)$ over S. Let $\bar{b}\colon \mathrm{Spec}(\Omega) \to S$ be a geometric point of S. Let $\rho\colon \pi^{\text{alg}}(S, \bar{b}) \to \mathrm{O}(P^2(\bar{b}))$ be the monodromy representation and let $\tilde{a}\colon L_{2d,\mathscr{B}} \to P^2(\bar{b})$ be an isometry. Then the preimage $\rho^{-1} \circ \alpha^{\text{ad}}(\mathbb{K}_{\mathscr{B}})$ is an open subgroup of $\pi_1^{\text{alg}}(S, \bar{b})$ (of finite index) and hence it defines an étale cover $S_{\tilde{\alpha}}$ of S. One has that the class α of $\tilde{\alpha}$ in $\mathbb{K}_{\mathscr{B}} \backslash \mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b})\big)$ is $\pi_1^{\text{alg}}(S_{\tilde{\alpha}}, \bar{b})$-invariant by construction (for a fixed geometric point $\bar{b} \in S_{\tilde{\alpha}}$ over $\bar{b}$). Therefore we obtain a primitively polarized K3 space $(X_{S_{\tilde{\alpha}}} \to S_{\tilde{\alpha}}, \lambda_{S_{\tilde{\alpha}}}, \alpha)$ with a level $\mathbb{K}$-structure α. For two markings $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ we have that $\tilde{\alpha}_1^{\text{ad}}(\mathbb{K}_{\mathscr{B}}) = \tilde{\alpha}_2^{\text{ad}}(\mathbb{K}_{\mathscr{B}})$ if and only if $\tilde{\alpha}_2^{-1} \circ \tilde{\alpha}_1$ is an element of the normalizer $N_{\mathrm{O}(V_{2d})(\mathbb{Z}_{\mathscr{B}})}(\mathbb{K}_{\mathscr{B}})$ of $\mathbb{K}_{\mathscr{B}}$ in $\mathrm{O}(V_{2d})(\mathbb{Z}_{\mathscr{B}})$.

Denote by S' the disjoint union of $S_{\tilde{\alpha}}$ where $\tilde{\alpha}$ runs over all (finitely many) classes in $\mathrm{O}(V_{2d})(\mathbb{Z}_{\mathscr{B}})/N_{\mathrm{O}(V_{2d})(\mathbb{Z}_{\mathscr{B}})}(\mathbb{K}_{\mathscr{B}})$. Let $(X' \to S', \lambda_{S'}, \alpha)$ be the primitively polarized K3 space with a level $\mathbb{K}$-structure given by the triple $(X_{S_{\tilde{\alpha}}} \to S_{\tilde{\alpha}}, \lambda_{S_{\tilde{\alpha}}}, \alpha)$ on the $\tilde{\alpha}$-*th* connected component $S_{\tilde{\alpha}}$ of S'. Then by construction we have a morphism of algebraic spaces

$$\pi\colon S' \to S \times_{\mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]}} \mathcal{F}_{2d,\mathbb{K}}$$

over S. This morphism is surjective. Indeed, by [LMB00, Prop. 5.4] this condition can be checked on points, in which case it is obvious by construction. The morphism $S' \to S$ is étale and therefore we conclude that $pr_{\mathbb{K},S}\colon S \times_{\mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]}} \mathcal{F}_{2d,\mathbb{K}} \to S$ and π are also étale. Hence $pr_{\mathbb{K}}$ is étale.

Step 3: By Step 2 and Theorem 1.4.7 the algebraic space $\mathcal{F}_{2d,\mathbb{K}}$ is smooth and of relative dimension 19 over $\mathbb{Z}[1/N_{\mathscr{B}}]$. □

Remark 1.5.13. Let $\mathbb{K}_1 \subset \mathbb{K}_2 \subset \mathbb{K}_n$ be subgroups of finite index in $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ and suppose that $n \geq 3$. Then the morphism (1.10) of algebraic spaces

$$pr_{(\mathbb{K}_1,\mathbb{K}_2)}\colon \mathcal{F}_{2d,\mathbb{K}_1,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]} \to \mathcal{F}_{2d,\mathbb{K}_2,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]}$$

is finite and étale. This follows from the theorem above and the relation $pr_{\mathbb{K}_1} = pr_{\mathbb{K}_2} \circ pr_{(\mathbb{K}_1,\mathbb{K}_2)}$.

Example 1.5.14. Let $n \geq 3$ be a natural number. Consider the group $\mathbb{K}_n$ defined in Example 1.5.3. We define $\mathcal{F}_{2d,n} = \mathcal{F}_{2d,\mathbb{K}_n}$ to be *the moduli space of primitively polarized K3 surfaces with level n-structure* over $\mathbb{Z}[1/2dn]$.

Example 1.5.15. Fix a natural number $n \geq 3$ and consider the group $\mathbb{K}_n^{\mathrm{a}}$ defined in Example 1.5.4. We define $\mathcal{F}_{2d,n^{\mathrm{sp}}} = \mathcal{F}_{2d,\mathbb{K}_n^{\mathrm{a}}}$ to be *the moduli space of polarized K3 surfaces with spin level n-structure* over $\mathbb{Z}[1/2dn]$.

Moduli K3 Spaces with Full Level Structures. Suppose that $\mathbb{K} \subset \mathbb{K}_n$ for some $n \geq 3$ is an admissible subgroup of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$. Let $\mathcal{X}_1 = (\pi_1\colon X_1 \to S_1, \lambda_1)$ and $\mathcal{X}_2 = (\pi_2\colon X_2 \to S_2)$ be two objects of $\mathcal{F}_{2d}$. Suppose that S_1 and S_2 are connected and let $(f, f_S) \in \mathrm{Hom}(\mathcal{X}_1, \mathcal{X}_2)$ (in $\mathcal{F}_{2d}$). Let $\bar{b}_1$ and $\bar{b}_2$ be two geometric points of S_1 and S_2 such that $f_S(\bar{b}_1) = \bar{b}_2$. Then the morphism f defines a homomorphism $f^*_{\mathrm{et}}\colon H^2(\bar{b}_2) \to H^2(\bar{b}_1)$ sending the class of $\lambda_{\bar{b}_2}$ to the class of $\lambda_{\bar{b}_1}$. Hence we obtain a map

$$\begin{aligned} f^{\vee}\colon \mathbb{K}_{\mathscr{B}}\backslash\big\{\ g \in \mathrm{Isometry}\big(L_{0,\mathbb{Z}_{\mathscr{B}}}, H^2(\bar{b}_2)\big) \mid g(e_1 - df_1) = c_1(\lambda_{2,\bar{b}_2})\ \big\} \to \\ \to \mathbb{K}_{\mathscr{B}}\backslash\big\{\ g \in \mathrm{Isometry}\big(L_{0,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b}_1)\big) \mid g(e_1 - df_1) = c_1(\lambda_{1,\bar{b}_1})\ \big\} \end{aligned}$$

given by $\alpha \mapsto f^*_{et} \circ \alpha$ and commuting with the monodromy actions on both sides.

Definition 1.5.16. For a natural number d and an admissible subgroup $\mathbb{K}$ of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ as above consider the category $\mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}}$ defined in the following way:

Ob: Triples $(\pi\colon X \to S, \lambda, \alpha)$ of a K3 space $\pi\colon X \to S$ over S with a primitive polarization λ of degree $2d$ and with a full level $\mathbb{K}$-structure α on $(\pi\colon X \to S, \lambda)$.

Mor: Suppose given two triples $\mathcal{X}_1 = (\pi_1\colon X_1 \to S_1, \lambda_1, \alpha_1)$ and $\mathcal{X}_2 = (\pi_2\colon X_2 \to S_2, \lambda_2, \alpha_2)$. Let $f_S\colon S_1 \to S_2$ be a morphism of schemes. Choose base geometric points $\bar{b}'_1$ and $\bar{b}'_2$ on any two connected components S'_1 and S'_2 of S_1 and S_2 for which $f\colon S'_1 \to S'_2$ such that $f_S(\bar{b}'_1) = \bar{b}'_2$. Define the morphisms between $\mathcal{X}_1$ and $\mathcal{X}_2$ in the following way

$$\begin{aligned}\mathrm{Hom}(\mathcal{X}_1, \mathcal{X}_2) = \big\{\text{pairs } (f_S, f) \mid & f_S\colon S_1 \to S_2 \text{ is a morph. of spaces,}\\ & f\colon X_1 \to X_2 \times_{S_2, f_S} S_1 \text{ is an isom. of}\\ & S_1-\text{spaces with } f^*\lambda_2 = \lambda_1 \text{ and}\\ & f^\vee(\alpha_1) = \alpha_2 \text{ on any conn. cmpt. of } S_1\big\}.\end{aligned}$$

A full level $\mathbb{K}$-structure α on a primitively polarized K3 space $(X \to S, \lambda)$ defines in a natural way a level $\mathbb{K}$-structure via the injective morphism

$$\begin{aligned}i^\vee_{\mathbb{Z}_{\mathscr{B}}}\colon \mathbb{K}_{\mathscr{B}}\backslash\{\ g \in \mathrm{Isometry}\big(L_{0,\mathbb{Z}_{\mathscr{B}}}, H^2(\bar{b})\big) \mid g(e_1 - df_1) = c_1(\lambda_{\bar{b}})\ \} \hookrightarrow\\ \hookrightarrow \mathbb{K}_{\mathscr{B}}\backslash\mathrm{Isometry}\big(L_{2d,\mathbb{Z}_{\mathscr{B}}}, P^2(\bar{b})\big)\end{aligned}$$

commuting with the monodromy action. This morphism is defined by the embedding of lattices $i\colon L_{2d} \hookrightarrow L_0$ (see (1.1) in Section 1.2.1). Using this, just like in the case of moduli of primitively polarized K3 surfaces with a level structure, we define natural functors.

4. Define a functor
$$i_{\mathbb{K}}\colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{K}}$$
sending $(X \to S, \lambda, \alpha)$ to $(X, \to S, \lambda, i^\vee(\alpha))$ which makes $\mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}}$ into a full subcategory of $\mathcal{F}_{2d,\mathbb{K}}$ over $(\mathrm{Sch}\,/\mathbb{Z}[1/N_{\mathscr{B}}])$.

5. One has the forgetful functor
$$pr_{\mathbb{K}}\colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]} \tag{1.11}$$
sending a triple $(\pi\colon X \to S, \lambda, \alpha)$ to $(\pi\colon X \to S, \lambda)$ and an element $(f, f_S) \in \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ of $\mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}}$ to (f, f_S).

6. For any two admissible subgroups $\mathbb{K}_1 \subset \mathbb{K}_2$ of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$, contained in some $\mathbb{K}_n$ for $n \geq 3$, one has a projection functor
$$pr_{(\mathbb{K}_1,\mathbb{K}_2)}\colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}_1,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]} \to \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}_2,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]} \tag{1.12}$$
defined in a similar way as the corresponding morphism (1.10) in 3.

The functors $pr_{\mathbb{K}}$ and $pr_{(\mathbb{K}_1,\mathbb{K}_2)}$ defined above are the restrictions of the corresponding functors (1.9) and (1.10) to the category of primitively polarized K3 surfaces with full level $\mathbb{K}_j$-structures via $i_{\mathbb{K}_j}$ for $j = 1, 2$.

Theorem 1.5.17. *Let $\mathbb{K}$ be an admissible subgroup of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ contained in $\mathbb{K}_n$ for some $n \geq 3$. The category $\mathcal{F}^{\text{full}}_{2d,\mathbb{K}}$ is a separated, smooth algebraic space of relative dimension 19 over $\mathbb{Z}[1/N_{\mathscr{B}}]$. The morphism $p_{2d,\mathbb{K}}\colon \mathcal{F}^{\text{full}}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]}$ is étale and the morphism $i_{\mathbb{K}}\colon \mathcal{F}^{\text{full}}_{2d,\mathbb{K}} \hookrightarrow \mathcal{F}_{2d,\mathbb{K}}$ is an open immersion.*

Proof. To prove that $\mathcal{F}^{\text{full}}_{2d,\mathbb{K}}$ is representable by an algebraic space of finite type over $\mathbb{Z}[1/N_{\mathscr{B}}]$ one follows the steps of the proof of Theorem 1.5.11. In this way we also see that the projection morphism $p_{2d,\mathbb{K}}\colon \mathcal{F}^{\text{full}}_{2d,\mathbb{K}} \to \mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]}$ is finite and étale. Therefore we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}^{\text{full}}_{2d,\mathbb{K}} \arrow[rr, "i_{\mathbb{K}}"] \arrow[dr, "p_{2d,\mathbb{K}}"'] & & \mathcal{F}_{2d,\mathbb{K}} \arrow[dl, "p_{2d,\mathbb{K}}"] \\ & \mathcal{F}_{2d,\mathbb{Z}[1/N_{\mathscr{B}}]} & \end{tikzcd}$$

where the two morphisms $p_{2d,\mathbb{K}}$ are étale and surjective. Hence $i_{\mathbb{K}}$ is also étale and therefore it is open. $\square$

Remark 1.5.18. Let $\mathbb{K}_1 \subset \mathbb{K}_2$ be two admissible subgroups of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$. Then the morphism of algebraic spaces

$$pr_{(\mathbb{K}_1,\mathbb{K}_2)}\colon \mathcal{F}^{\text{full}}_{2d,\mathbb{K}_1,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]} \longrightarrow \mathcal{F}^{\text{full}}_{2d,\mathbb{K}_2,\mathbb{Z}[1/N_{\mathscr{B}_1\cup\mathscr{B}_2}]}$$

is finite and étale. This follows from the theorem above and the relation $pr_{\mathbb{K}_1} = pr_{\mathbb{K}_2} \circ pr_{(\mathbb{K}_1,\mathbb{K}_2)}$.

Example 1.5.19. Let $n \geq 3$ be a natural number. Consider the group $\mathbb{K}^{\text{full}}_n$ defined in Example 1.5.9. We define $\mathcal{F}^{\text{full}}_{2d,n} = \mathcal{F}^{\text{full}}_{2d,\mathbb{K}^{\text{full}}_n}$ to be *the moduli space of primitively polarized K3 surfaces with full level n-structure* over $\mathbb{Z}[1/2dn]$.

Chapter 2

Non-Emptiness of the Height Strata of $\mathcal{M}_{2d,\mathbb{F}_p}$

In this chapter we will consider the following problem: For a given natural number d and a prime number p determine all Newton polygons of K3 surfaces with a polarization of degree $2d$ over a field of characteristic p. This is an analogue of the Manin problem for Newton polygons of abelian varieties (cf. [Man63, Conjecture 2, p. 76]).

One can formulate this problem in terms of the height (Newton polygon) strata of $\mathcal{M}_{2d,\mathbb{F}_p}$. Determine the non-empty strata of $\mathcal{M}_{2d,\mathbb{F}_p}$. Constructing ample line bundles of appropriate degree on Kummer surfaces we will show that for any large enough d and any p, prime to $2d$, the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty. The proof is constructive and gives a bound for d. Thus we partially solve the following problem.

Question. *Are the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ non-empty for any d and any p not dividing $2d$?*

The organization of this chapter is the following. In Section 2.1 we recall some definitions and give an overview of some results on the height stratification of $\mathcal{M}_{2d,\mathbb{F}_p}$. Section 2.2 is devoted to Kummer surfaces. Starting with an ample line bundle on an abelian surface we describe a way of constructing ample line bundles on its associated Kummer surfaces. This allows us to find points in $\mathcal{M}_{2d,\mathbb{F}_p}(\bar{\mathbb{F}}_p)$ which belong to certain height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$. In Section 2.3 we take this idea one step further and construct Kummer morphisms from moduli stacks of polarized abelian surfaces to moduli stacks of polarized K3 surfaces. We use these morphisms to give an affirmative answer to the question posed above in case d is large enough.

2.1 The Height Stratification of $\mathcal{M}_{2d,\mathbb{F}_p}$

Let k be a perfect field of characteristic $p > 0$ and consider a K3 surface X over k. Consider the contravariant functor

$$\Phi^2 : \underline{\mathrm{Art}} \to \mathrm{Ab}$$

from the category of local artinian schemes to abelian groups defined by

$$\Phi^2(S) = \ker\big(H^2_{\mathrm{et}}(X \times S, \mathbb{G}_m) \to H^2_{\mathrm{et}}(X, \mathbb{G}_m)\big).$$

This functor is representable by a formal Lie group, denoted by $\widehat{Br}(X)$ and called the *formal Brauer group* of X.

Proposition 2.1.1. *The formal group $\widehat{Br}(X)$ is 1-dimensional and one has the following two possibilities for it:*

(a) The height of $\widehat{Br}(X)$ is infinite and then $\widehat{Br}(X) \cong \widehat{\mathbb{G}}_a$.

(b) The height is finite. Then $\widehat{Br}(X)$ is a p-divisible group. Moreover, its height satisfies $1 \leq h\big(\widehat{Br}(X)\big) \leq 10$.

For proofs we refer to [AM77]. From now on we will call the height of the formal Brauer group of X simply the height of X and denote it as $h(X)$.

The *Newton polygon* of X is the Newton polygon of the F-crystal $H^2_{\mathrm{cris}}(X/W)$ where W is the ring of Witt vectors $W(k)$ (see [Ill95, §1.3 (c)]). It is the lower convex polygon starting at $(0, 0)$ and ending at $(22, 22)$. The height of a K3 surface X can be read off from its Newton polygon. If α is the smallest slope of the Newton polygon of X, then $h(X) = 1/(1 - \alpha)$ if $\alpha \neq 1$ and infinity otherwise. This follows from Corollary 3.3 in [AM77, III].

The *Hodge polygon in degree m* of a non-singular projective variety X over k is defined as the increasing convex polygon starting at $(0, 0)$, having slope i with multiplicity $h^{i,m-i} = \dim_k H^{m-i}(X, \Omega^i_X)$. For a K3 surface X the Hodge polygon in degree 2 will be called the Hodge polygon of X. The Newton polygon of a K3 surface lies on or above its Hodge polygon ([Ill95, Thm. 1.3.9]).

Definition 2.1.2. A K3 surface X over k is called *ordinary* if any of the following equivalent conditions is satisfied:

(i) $h(X) = 1$.

(ii) The Newton and the Hodge polygon of X coincide i.e., the Newton slopes of X are 0 and 2 with multiplicity one, 1 with multiplicity 20.

Definition 2.1.3. A K3 surface X over k is called *supersingular* if any of the following equivalent conditions is satisfied:

(i) The height of X is infinite.

(ii) The Newton polygon is a straight line i.e., all Newton slopes of X are 1.

The fact that the two possible ways of defining ordinary and supersingular K3 surfaces are equivalent follows from [AM77, III, Cor. 3.3].

Definition 2.1.4. A K3 surface X over k is called *supersingular in the sense of Shioda* if the rank of $\mathrm{NS}(X)$ is 22.

One easily sees that if a K3 surface is supersingular in the sense of Shioda, then it is supersingular. It is a conjecture of M. Artin that, conversely, a supersingular K3 surface has Néron-Severi rank 22.

Example 2.1.5. Let $p \equiv 3 \pmod 4$ be a prime number. Then the Fermat K3 surface

$$x^4 + y^4 + z^4 + w^4 = 0$$

in $\mathbb{P}^3_{\mathbb{F}_p}$ is supersingular in the sense of Shioda (see [Shi75, Thm. 1]).

Let d be an integer and assume further that p does not divide $2d$. Consider the moduli stack $\mathcal{M}_{2d,\mathbb{F}_p} = \mathcal{M}_{2d} \otimes_{\mathbb{Z}} \mathbb{F}_p$ of K3 surfaces with a polarization of degree $2d$ over a basis in characteristic p. Define the height stratification of $\mathcal{M}_{2d,\mathbb{F}_p}$ as follows: For $h \geq 1$ let $\mathcal{M}^{(h)}_{2d,\mathbb{F}_p}$ be the full subcategory of $\mathcal{M}_{2d,\mathbb{F}_p}$

$$\mathcal{M}^{(h)}_{2d,\mathbb{F}_p}(S) = \big\{(X \to S, \lambda) \in \mathcal{M}_{2d,\mathbb{F}_p}(S) \mid h(X_{\bar{s}}) \geq h \text{ for every geometric point } \bar{s} \in S\big\}.$$

It is known that $\mathcal{M}^{(h)}_{2d,\mathbb{F}_p}$ is a closed substack of $\mathcal{M}_{2d,\mathbb{F}_p}$ of codimension at most $h-1$. One defines a stratification of $\mathcal{M}^{(11)}_{2d,\mathbb{F}_p}$ by the Artin invariant (see [Art74]). Let X be a supersingular K3 surface and let $\Delta(\mathrm{NS}(X))$ be the discriminant of the intersection pairing on $\mathrm{NS}(X)$

$$(\cdot,\cdot)\colon\ \mathrm{NS}(X) \times \mathrm{NS}(X) \to \mathbb{Z}.$$

One can show that $\mathrm{ord}_p(\Delta) = 2\sigma_0$ where σ_0 takes values $1, \dots, 10$. It is called the *Artin invariant* of X. Let Σ_i be the full subcategory of $\mathcal{M}^{(11)}_{2d,\mathbb{F}_p}$ defined by

$$\begin{aligned}\Sigma_i(S) = \big\{(X \to S, \lambda) \in \mathcal{M}_{2d,\mathbb{F}_p}(S) \mid\ & h(X_{\bar{s}}) = \infty \text{ and } \sigma_0(X_{\bar{s}}) \leq 11 - i \\ & \text{for every geometric point } \bar{s} \in S\big\}.\end{aligned}$$

In this way we obtain a filtration of the moduli space

$$\mathcal{M}_{2d,\mathbb{F}_p} = \mathcal{M}^{(1)}_{2d,\mathbb{F}_p} \supset \mathcal{M}^{(2)}_{2d,\mathbb{F}_p} \supset \cdots \supset \mathcal{M}^{(11)}_{2d,\mathbb{F}_p} = \Sigma_1 \supset \cdots \supset \Sigma_{10}. \tag{2.1}$$

This is a chain of 20 closed substacks and the dimension drops with at least one at each step.

Theorem 2.1.6. *For $h = 1, \ldots, 10, 11$ the locus $\mathcal{M}^{(h)}_{2d,\mathbb{F}_p}$, if non-empty, is of codimension $h-1$ and for $h \neq 11$ is a local complete intersection.*

Proof. We refer to [vdGK00, sect. 13,14 & 15]. The statement presented above is Theorem 15.1. □

Remark 2.1.7. B. Moonen and T. Wedhorn ([MW04]) have a theory of *F-zips* which gives a scheme-theoretic and uniform definition of the filtration (2.1). For details we refer to Example 7.4 in *loc. cit.*.

2.2 Kummer Surfaces

As we mentioned in the beginning of this chapter we will use Kummer surfaces to show that the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty for large enough d, prime to p. In Section 2.2.1 we will recall some basic facts about Kummer surfaces which we will need in the sequel. In the next section, starting with a polarized abelian surface (A, λ) we describe a way for constructing polarizations on its associated Kummer surface X. For this we make use of Seshadri constants.

2.2.1 Kummer Surfaces

Recall that to an abelian surface A over a field k of characteristic different from 2 we associated a K3 surface X, called the Kummer surface of A. Assume further that all points in $A[2](\bar{k})$ are k-rational. Using the notations established in Example 1.1.4 we see that the exceptional divisor $\tilde{E}$ on $\tilde{A}$ consists of 16 irreducible curves ${E'}_j$, $j = 1, \ldots, 16$, each corresponding to a point in $A[2](\bar{k})$. We have that $({E'}_j, {E'}_l)_{\tilde{A}} = \delta_{j,l}$ where $\delta_{j,l}$ is the Kronecker δ-function. Let us make the following notations:

$$\begin{aligned} {\mathcal{E}'}_j &:= \mathcal{O}_{\tilde{A}}({E'}_j) \text{ a line bundle on } \tilde{A}, \\ E_j &:= \iota(E'_j) \text{ a divisor on } X, \\ \mathcal{E}_j &:= \mathcal{O}_X(E'_j) \text{ the corresponding line bundle on } X. \end{aligned}$$

Then one has that

$$\iota^*\mathcal{E}_j \cong \mathcal{E}'^{\otimes 2}_j \quad \text{and} \quad (\mathcal{E}_j, \mathcal{E}_l)_X = 2\delta_{j,l}.$$

Moreover the line bundle $\bigotimes_{j=1}^{16} \mathcal{E}_j$ is divisible by 2 in $\mathrm{Pic}(X_{\bar{k}})$.

We turn next to some p-adic discrete invariants of Kummer surfaces. From now on we will assume that k is a field of positive characteristic different from 2. Then X is supersingular in the sense of Shioda if and only if A is supersingular. Indeed, as we have seen in Example 1.3.8 one has that $\mathrm{NS}(X_{\bar{k}})_{\mathbb{Q}} = \mathrm{NS}(A_{\bar{k}})_{\mathbb{Q}}^{[-1]_A} \bigoplus_{j=1}^{16} \mathbb{Q}$. Hence we have that $\mathrm{rk}_{\mathbb{Z}} \mathrm{NS}(X) = 22$ if and only if $\mathrm{rk}_{\mathbb{Z}} \mathrm{NS}(A) = 6$ which is equivalent to A being supersingular. We will determine the Newton polygon of X in term of the Newton polygon of A. To do that we shall need the following auxiliary result.

Lemma 2.2.1. *Let A be an abelian surface and X the associated Kummer surface over k. Then there is a natural isomorphism*

$$H^2_{\text{et}}(X_{\bar{k}}, \mathbb{Q}_l) \cong H^2_{\text{et}}(A_{\bar{k}}, \mathbb{Q}_l)^{[-1]_A} \oplus_1^{16} \mathbb{Q}_l(-1)$$

where $H^2_{\text{et}}(A_{\bar{k}}, \mathbb{Q}_l)^{[-1]_A}$ is the subspace of the elements invariant under $[-1]_A$.

Proof. This is a consequence of the following two lemmas.

Lemma 2.2.2. *Let $\pi\colon Z \to X$ be the blowing up of a non-singular variety X along a non-singular subvariety Y of codimension d in X. Then there is a natural isomorphism*

$$H^i_{\text{et}}(Z_{\bar{k}}, \mathbb{Q}_l) \cong H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Q}_l) \oplus \sum_{j=1}^{d-1} H^{i-2j}_{\text{et}}(Y_{\bar{k}}, \mathbb{Q}_l(-j))$$

Lemma 2.2.3. *Suppose that G is a finite group of automorphisms of a non-singular variety X and $Y = X/G$ is non-singular. Then $H^i_{\text{et}}(Y_{\bar{k}}, \mathbb{Q}_l)$ is isomorphic to the subspace of G-invariants in $H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Q}_l)$ i.e.,*

$$H^i_{\text{et}}(Y_{\bar{k}}, \mathbb{Q}_l) \cong H^i_{\text{et}}(X_{\bar{k}}, \mathbb{Q}_l)^G.$$

The proofs of those lemmas can be found in [HM78], [Ill77, Exp. VII] and [Gro57, Ch. V]. □

Lemma 2.2.4. *Let k be a finite field of characteristic different from 2. Then one has that*

(i) If A is ordinary, then the Newton polygon slopes of X are $\mu_1 = 0$; $\mu_2 = \cdots = \mu_5 = 1$; $\mu_6 = 2$; $\mu_j = 1$ for $j = 7, \ldots 22$. In this case X is ordinary i.e., its height is 1.

(ii) If the p-rank of A is 1, then the Newton polygon slopes of X are $\mu_1 = \mu_2 = 1/2$; $\mu_3 = \mu_4 = 1$; $\mu_5 = \mu_6 = 3/4$; $\mu_j = 1$ for $j = 7, \ldots 22$. In this case X has height is 2.

(iii) If A is supersingular, then X is supersingular and all its Newton polygon slopes are 1. In this case $\Delta(\mathrm{NS}(A))$ and $\Delta(\mathrm{NS}(X))$ differ only by a power of 2, hence the Artin invariant of X is 1 or 2. It is 1 if and only if A is a superspecial abelian surface.

Proof. As k is a finite field and X and A are projective varieties one can compute the Newton polygons of A and X using étale cohomology instead of crystalline cohomology. We refer to [Ill95, 1.3, Equality (1.3.5)] for an explanation and details. We will use the relation between the étale cohomology groups of A and X given in Lemma 2.2.1.

If the Newton polygon slopes of A are λ_j for $j = 1, \ldots, 4$, then those of X satisfy $\mu_1 = \lambda_1+\lambda_2$; $\mu_2 = \lambda_1+\lambda_3$; $\mu_3 = \lambda_1+\lambda_4$; $\mu_4 = \lambda_2+\lambda_3$; $\mu_5 = \lambda_2+\lambda_4$; $\mu_6 = \lambda_3+\lambda_4$; $\mu_i = 1$ for $i = 7, \ldots, 22$. The last statement follows from [Shi79, §3, Prop. 3.1]. □

2.2.2 Ample Line Bundles on Kummer Surfaces

Let k be an algebraically closed field of characteristic different from 2 and consider an abelian surface A over k. Denote by X the associated Kummer surface. In this section we will show how to construct ample line bundles on X starting with an ample bundle on A. This will allow us to give explicitly points in $\mathcal{M}_{2d,\mathbb{F}_p}$ for some d.

Let $\mathcal{L}$ be an ample line bundle on A with $\chi(\mathcal{L}) = d'$. Then by Riemann-Roch we have that $(\mathcal{L}, \mathcal{L})_A = 2d'$. Let $n \in \mathbb{N}$ and fix 16 positive integers n_j. Consider the line bundle $\mathcal{N}$ on $\tilde{A}$ given by

$$\mathcal{N} = \pi^*\big(\mathcal{L}^n \otimes [-1]_A^*\mathcal{L}^n\big) \otimes \left(\bigotimes_{j=1}^{16} \mathcal{E}_j'^{-2n_j}\right).$$

We will compute its self-intersection and show that $\mathcal{N}$ is the pull-back of a line bundle on X.

Lemma 2.2.5. *With the notations as above one has:*

(a) $(\mathcal{N}, \mathcal{N})_{\tilde{A}} = 8n^2d' - 4\sum_{j=1}^{16} n_j^2$*;*

(b) There exists a line bundle $\mathcal{M}$ *on* X *such that* $\iota^*\mathcal{M} \cong \mathcal{N}$*. The line bundle* $\mathcal{M}$ *is ample iff* $\mathcal{N}$ *is ample. Moreover if* $2d = (\mathcal{M}, \mathcal{M})_X$*, then we have that*

$$d = 2n^2d' - \sum_{j=1}^{16} n_j^2.$$

Proof. (a) Combining [Har77, Ch. V, §3, Prop. 3.2] and the fact that $[-1]_A^*\mathcal{L}$ and $\mathcal{L}$ are algebraically equivalent we get

$$\begin{aligned}(\mathcal{N}, \mathcal{N})_{\tilde{A}} &= 2n^2(\mathcal{L}, \mathcal{L})_A + 2n^2(\mathcal{L}, [-1]_A^*\mathcal{L})_A - 4\sum_{j=1} 16n_j^2 \\ &= 8n^2d' - 4\sum_{j=1} 16n_j^2.\end{aligned}$$

(b) Take a divisor $D \subset A \backslash A[2]$ such that $\mathcal{O}_A(D) = \mathcal{L}$. If $U := \tilde{A} \setminus \bigcup_{j=1}^{16} E_j'$ and $V = X - \bigcup_{j=1}^{16} E_j$, then the map $\iota\colon U \to V$ is étale. Consider the divisor

$$D_1 := \iota\big(n\pi^*(D) + n\pi^*([-1]_A^*D)\big)$$

on X. As $D_1 \subset V$ we see that $\iota^*D_1 = n\pi^*(D) + n\pi^*([-1]_A^*D)$. Hence if we set

$$\mathcal{P} := \mathcal{O}_X(D_1)$$

then we have that $\iota^*\mathcal{P} \cong \pi^*\big(\mathcal{L}^n \otimes [-1]_A^*\mathcal{L}^n\big)$ on $\tilde{A}$. Using the fact that $\iota^*\mathcal{E}_j \cong \mathcal{E}_j'^{\otimes 2}$ one sees that the line bundle

$$\mathcal{M} = \mathcal{P} \otimes \bigotimes_{j=1}^{16} \mathcal{E}_j^{\otimes -n_j}$$

satisfies $\iota^*\mathcal{M} \cong \mathcal{N}$ on $\tilde{A}$. Since ι is a finite morphism $\mathcal{M}$ is ample on X if and only if $\mathcal{N}$ is ample on $\tilde{A}$ ([Har77, Ch. III, Exercise 5.7 (d)]).

For the self-intersection number computation one has

$$2n^2(\mathcal{L},\mathcal{L})_A + 2n^2(\mathcal{L},[-1]_A^*\mathcal{L}) - 4\sum_{j=1}^{16} n_j^2 =$$
$$= (\mathcal{N},\mathcal{N})_{\tilde{A}} = (\iota^*\mathcal{M},\iota^*\mathcal{M})_{\tilde{A}} = \deg(\iota)(\mathcal{M},\mathcal{M})_X = 4d$$

which gives the formula from (b). □

Remark 2.2.6. Note that the line bundle $\mathcal{L}^n \otimes [-1]_A^*\mathcal{L}^n$ comes with a natural action of $[-1]_A$. Hence its pull-back $\pi^*\big(\mathcal{L}^n \otimes [-1]_A^*\mathcal{L}^n\big)$ comes equipped with an action of $[-1]_{\tilde{A}}$. Therefore one can apply [Mum74, Ch. III §10, Thm. 1(B)] to the morphism $\iota\colon \tilde{A} \to X$ and conclude that $\mathcal{P} = \iota_*\big(\pi^*(\mathcal{L}^n \otimes [-1]_A^*\mathcal{L}^n)\big)^{[-1]_{\tilde{A}}}$ is the line bundle described in the proof of part (b).

Lemma 2.2.5 suggests a way to construct ample line bundles on the Kummer surface X. We will give sufficient conditions under which $\mathcal{N}$ is ample on $\tilde{A}$. To do this we will make use of *multiple Seshadri constants*. We will recall the definition below. For details we refer to [Bau99].

Seshadri constants. Let $\mathcal{D}$ be an ample line bundle on A and let $x_1,\ldots,x_{16}$ be the points in $A[2](k)$ (recall that $k = \bar{k}$ and $\mathrm{char}(k) \neq 2$). We make this change of notations here to avoid any possible confusion as later we will compute Seshadri constants for the ample line bundle $\mathcal{D} = \mathcal{L} \otimes [-1]_A^*\mathcal{L}^{-1}$. Let $\mathrm{NS}(\tilde{A})_{\mathbb{R}}$ denote $\mathrm{NS}(\tilde{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $(\cdot,\cdot)_{\tilde{A},\mathbb{R}}$ be the induced bilinear form. We will call an element $\mathcal{R}$ of $\mathrm{NS}(\tilde{A})_{\mathbb{R}}$ *numerically effective*, or shortly *nef*, if for any irreducible curve Γ in $\tilde{A}$ we have that $(\mathcal{R},\mathcal{O}_{\tilde{A}}(\Gamma))_{\tilde{A},\mathbb{R}} \geq 0$. Further, for an element $\mathcal{R} \in \mathrm{NS}(\tilde{A})_{\mathbb{R}}$ and a real number ϵ we will denote by $\mathcal{R}^\epsilon$ the element $\epsilon \cdot \mathcal{R} \in \mathrm{NS}(\tilde{A})_{\mathbb{R}}$.

One shows that

$$\epsilon_{\mathcal{D}} = \sup\left\{\epsilon \in \mathbb{R} \;\middle|\; \pi^*\mathcal{D} \otimes \bigotimes_{i=1}^{16} \mathcal{E}_i'^{-\epsilon} \text{ is nef in } \mathrm{NS}(\tilde{A})_{\mathbb{R}}\right\}$$

exists. It is called the *multiple Seshadri constant* on A for $x_1,\ldots,x_{16}$. An equivalent definition of the Seshadri constant ϵ can be given in the following way:

$$\epsilon_{\mathcal{D}} = \inf \frac{(\mathcal{D},\mathcal{O}_A(C))_A}{\sum_{i=1}^{16} \mathrm{mult}_{x_i} C}$$

where $\mathrm{mult}_{x_i}C$ is the multiplicity of C at x_i and the infimum is taken over all irreducible curves C in A which pass through at least one x_i.

Remark 2.2.7. (1) If $0 < \delta < \epsilon_{\mathcal{D}}$, then the line bundle $\pi^*\mathcal{D} \otimes \bigotimes_{i=1}^{16} \mathcal{E}_i'^{-\delta}$ is nef. Moreover, one has the strict inequality

$$\big(\pi^*\mathcal{D} \otimes \bigotimes_{i=1}^{16} \mathcal{E}_i'^{-\delta}, \mathcal{O}_{\tilde{A}}(\Gamma)\big)_{\tilde{A},\mathbb{R}} > 0$$

for any irreducible curve Γ on $\tilde{A}$.

(2) If $0 < n_i < \epsilon_{\mathcal{D}}$, then $\pi^*\mathcal{D}\otimes\bigotimes_{i=1}^{16} \mathcal{E}_i'^{-n_i}$ is nef. Moreover, one has the strict inequality

$$\big(\pi^*\mathcal{D} \otimes \bigotimes_{i=1}^{16} \mathcal{E}_i'^{-n_i}, \mathcal{O}_{\tilde{A}}(\Gamma)\big)_{\tilde{A}} > 0$$

for any irreducible curve Γ on $\tilde{A}$.

These facts are clear from the second definition of ϵ.

Numerical estimates. We will apply the general results on Seshadri constants to our particular situation. To avoid confusion let us make the following convention: If A is an abelian surface, then by *an elliptic curve E in A* we shall mean an abelian subvariety E of A of dimension one.

Proposition 2.2.8. *Let A be an abelian surface over an algebraically closed field k of characteristic different from 2. Let $\{x_1, \dots, x_{16}\}$ be the set of two-torsion points on A. Then for an ample line bundle $\mathcal{D}$ on A we are in one of the following cases:*

(a) The Seshadri constant satisfies the inequality

$$\epsilon_{\mathcal{D}} \geq \frac{\sqrt{2(\mathcal{D},\mathcal{D})_A}}{16}.$$

(b) The abelian surface A contains a curve E of genus 1 such that

$$\epsilon_{\mathcal{D}} = \frac{(\mathcal{D}, \mathcal{O}_A(E))_A}{\#\{i|x_i \in E(k)\}}. \tag{2.2}$$

Proof. See [Bau99, Prop. 8.3]. Note that in this paper the assumption $k = \mathbb{C}$ is made. However, the proof of the above proposition uses only the Hodge index theorem, the Riemann-Roch theorem and some facts about blow-ups of curves. These results are valid over any algebraically closed field. □

Remark 2.2.9. Note that in (b) we may assume that E is an elliptic curve in A. Indeed, we have that E is a translate of an elliptic curve $E' \subset A$ by a point $a \in A$. Since $\mathcal{D}$ is ample, the line bundles $t_a^*\mathcal{D}$ and $\mathcal{D}$ are numerically equivalent. Therefore we have that

$$(\mathcal{D}, \mathcal{O}_A(E)) = (t_a^*\mathcal{D}, t_a^*\mathcal{O}_A(E)) = (\mathcal{D}, t_a^*\mathcal{O}_A(E)) = (\mathcal{D}, \mathcal{O}_A(E')).$$

We have further that $\#\{i | x_i \in E(k)\} \leq 4$. Indeed, all these points correspond to points $p_i = x_i - a \in E'(k)$ for which $[2]p_i = [-2]a$ is a fixed point in $E'(k)$. As the isogeny $[2]$ is of degree 4 (on E') there are at most four such points. So we have that

$$\begin{aligned} \epsilon_{\mathcal{D}} &= \frac{(\mathcal{D}, \mathcal{O}_A(E))_A}{\sum_{i=1}^{16} \mathrm{mult}_{x_i} E} = \frac{(\mathcal{D}, \mathcal{O}_A(E))_A}{\#\{i | x_i \in E(k)\}} = \\ &= \frac{(\mathcal{D}, \mathcal{O}_A(E'))_A}{\#\{i | x_i \in E(k)\}} \geq \frac{(\mathcal{D}, \mathcal{O}_A(E'))_A}{4} = \frac{(\mathcal{D}, \mathcal{O}_A(E'))_A}{\sum_{i=1}^{16} \mathrm{mult}_{x_i} E'} \geq \epsilon_{\mathcal{D}}. \end{aligned}$$

Therefore we have equalities and we conclude that $\#\{i | x_i \in E(k)\} = 4$. We also see that $a \in A[2](k)$.

In what follows we will try to avoid case (b) of Proposition 2.2.8 as much as possible. The reason is that one has little control over the intersection $(\mathcal{D}, \mathcal{O}_A(E))_A$ in terms of the degree of $\mathcal{D}$. The bound in (a) increases with $(\mathcal{D}, \mathcal{D})_A$, but A can contain curves of genus 1 of any given intersection index $(\mathcal{D}, \mathcal{O}_A(E))_A$, no matter how large $(\mathcal{D}, \mathcal{D})_A$ is.

We will need the following auxiliary result which we shall apply to a line bundle $\mathcal{L}$ defining the polarization λ on A (cf. the beginning of this section).

Lemma 2.2.10. *Let $\mathcal{L}$ be an ample line bundle on an abelian surface A and let $E \subset A$ be an elliptic curve.*

(a) Suppose that $(\mathcal{L}, \mathcal{O}_A(E))_A = 1$. Then there exists an elliptic curve $E' \subset A$ such that $A \cong E \times E'$. Moreover, if $\pi_1 \colon E \times E' \to E$ and $\pi_2 \colon E \times E' \to E'$ are the two projections, then there exists a point $P \in E$ and a line bundle $\mathcal{G}$ on E' such that

$$\mathcal{L} \cong \pi_1^*\mathcal{O}_E(P) \otimes \pi_2^*\mathcal{G}.$$

(b) Suppose that $(\mathcal{L}, \mathcal{O}_A(E))_A = m$ for some $m \in \mathbb{N}$. Then there exist an elliptic curve $E' \subset A$ and an isogeny $f \colon E \times E' \to A$ of degree at most m.

Proof. (a): The proof can be found in [Nak96, Lemma 2.6].

(b): Consider the homomorphism

$$\phi \colon A \xrightarrow{\varphi_{\mathcal{L}}} A^t \longrightarrow E^t$$

where $\varphi_{\mathcal{L}}$ is the map $a \mapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$ and the second map is the dual of the inclusion $E \to A$. Let E' be the reduced subscheme of the zero component of $\ker(\phi)$. Then E' is an elliptic curve in A. Note that $\mathcal{L}|_E$ is an invertible sheaf of degree at most m hence $(E, E')_A \leq m$. Define the homomorphism $E \times E' \to A$ to be $(P, P') \mapsto P + P'$. It is surjective and its kernel is a finite group scheme hence it is an isogeny. Moreover its degree is exactly $(E, E')_A \leq m$. □

To get explicit conditions under which $\mathcal{N}$ is ample on $\tilde{A}$, one has to give some explicit estimates for $\epsilon_{\mathcal{D}}$ for the ample line bundle $\mathcal{D} = \mathcal{L}^n \otimes [-1]_A^* \mathcal{L}^n$.

Lemma 2.2.11. *With the notations of Lemma 2.2.5 one has that*

(a) If $d', n, n_1, \ldots n_{16}$ satisfy the following three inequalities

$$2n^2 d' - \sum_{i=1}^{16} n_i > 0 \tag{2.3}$$

$$n_i < \frac{n}{4} \tag{2.4}$$

$$n_i < \frac{\sqrt{n^2 d'}}{8}, \tag{2.5}$$

then the line bundle $\mathcal{N}$ is ample on $\tilde{A}$.

(b) Assume further that $(A, \mathcal{L})$ is not isomorphic to a polarized product of elliptic curves, as in Lemma 2.2.10 (a). Then for the ampleness of $\mathcal{N}$ on $\tilde{A}$ it is enough to require $n_i < n/2$ instead of (2.4) *along with the other two inequalities* (2.3) *and* (2.5).

Proof. (a): Suppose that the inequalities (2.3), (2.4) and (2.5) are fulfilled. The first one simply says that $(\mathcal{N}, \mathcal{N})_{\tilde{A}} > 0$. The second two are exactly the ones obtained from the explicit estimates for $\epsilon_{\mathcal{D}}$.

Assume first that $\epsilon_{\mathcal{D}}$ is computed by an elliptic curve E. Since $\mathcal{L}$ is ample on A one has that $(\mathcal{L}, \mathcal{O}_A(E))_A \geq 1$. Hence by Proposition 2.2.8 we have that

$$\epsilon_{\mathcal{D}} = \frac{\big(\mathcal{L}^n \otimes [-1]_A^* \mathcal{L}^n, \mathcal{O}_A(E)\big)_A}{4} = \frac{2n\big(\mathcal{L}, \mathcal{O}_A(E)\big)_A}{4} \geq \frac{n}{2} \geq 2n_i$$

for every $i = 1, \ldots, 16$.

If $\epsilon_{\mathcal{D}}$ is not computed by by an elliptic curve, then case (a) of Proposition 2.2.8 and the fact that $(\mathcal{D}, \mathcal{D})_A = 8n^2 d'$ give the estimate

$$\epsilon_{\mathcal{D}} \geq \frac{\sqrt{n^2 d'}}{4} \geq 2n_i.$$

for every $i = 1, \ldots, 16$.

Thus if we impose these numerical conditions (2.3), (2.4) and (2.5) on n, d' and n_i, then by Proposition 2.2.8 we have that $2n_i < \epsilon_{\mathcal{D}}$. Hence by Remark 2.2.7 one has that $(\mathcal{N}, \mathcal{O}_{\tilde{A}}(\Gamma))_{\tilde{A}} > 0$ for any irreducible curve Γ on $\tilde{A}$. Therefore by the Nakai-Moishezon criterion ([Har77, Ch. V, §1, Thm. 1.10]) the line bundle $\mathcal{N}$ is ample.

(b): Suppose that $(A, \mathcal{L})$ is not isomorphic to a polarized product of elliptic curves, then $(\mathcal{L}, \mathcal{O}_A(E))_A \geq 2$. If $\epsilon_{\mathcal{D}}$ is computed by an elliptic curve E we have that

$$\epsilon_{\mathcal{D}} \geq n \geq 2n_i$$

for all $i = 1, \ldots, 16$. Otherwise, just like in (a) one has that

$$\epsilon_{\mathcal{D}} \geq \frac{\sqrt{n^2 d'}}{4} \geq 2n_i$$

for all i. Hence by the argument given in the proof of part (a) the line bundle $\mathcal{N}$ is ample. □

2.3 Kummer Maps and Non-Emptiness of the Height Strata

In the preceding section we gave a way to construct points in $\mathcal{M}_{2d}(\bar{\mathbb{F}}_p)$ starting with points in $\mathcal{A}_{2,d'}(\bar{\mathbb{F}}_p)$ for some well-chosen integers d and d'. Here we will show that this actually gives rise to morphisms between the stacks $\mathcal{A}_{2,d',2,\mathbb{F}_p}$ and $\mathcal{M}_{2d,\mathbb{F}_p}$. We call these maps Kummer morphisms and we give their construction in detail in Section 2.3.1. We will use them in Section 2.3.2 to produce supersingular points in $\mathcal{M}_{2d,\mathbb{F}_p}(\bar{\mathbb{F}}_p)$ for d large enough. In this way we will conclude that the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty for these d.

2.3.1 The Kummer Morphisms

We already saw that starting with an ample line bundle $\mathcal{L}$ on A with $\chi(\mathcal{L}) = d'$ and fixing integers $n, n_1, \ldots, n_{16} > 0$ one produces a K3 surface X and a line bundle $\mathcal{M}$ on it. This bundle is ample if further the numerical conditions from Lemma 2.2.11 are satisfied by $d', n, n_1, \ldots, n_{16}$. It turns out that the resulting line bundle $\mathcal{M}$ depends only on the class of $\mathcal{L}$ in $\mathrm{NS}(A)$. In other words, it depends only on the polarization $\lambda_{\mathcal{L}}$ defined by $\mathcal{L}$. Indeed, the construction

$$\mathcal{L} \mapsto \iota_*\big(\pi^*(\mathcal{L} \otimes [-1]_A^*\mathcal{L})\big)^{[-1]_{\tilde{A}}}$$

gives a homomorphism of group schemes $h\colon \mathrm{Pic}_{A/k} \to \mathrm{Pic}_{X/k}$ and since $\mathrm{Pic}^0_{X/k}$ is trivial we see that h vanishes on $\mathrm{Pic}^0_{A/k}$.

Suppose given numbers n, d' and $n_1, \ldots, n_{16}$ satisfying the inequalities from Lemma 2.2.11 (a). Then using the remark made above one shows that starting with a polarized abelian surface (A, λ) over an algebraically closed field k one gets a polarized K3 surface $(X, \mathcal{M})$. We will generalize this construction to a general base S. To do so let us first try to find a more intrinsic way of constructing the line bundle $\mathcal{M}$.

Let $\mathcal{L}$ be an ample line bundle on A and let $\lambda = \varphi_{\mathcal{L}}$. The polarization defined by $\mathcal{L} \otimes [-1]_A^*\mathcal{L}$ is 2λ. Let $\mathcal{P}$ be the Poincaré sheaf on $A \times A^t$, where A^t is the dual abelian surface. One has an isomorphism $[-1]_A^*\mathcal{P} \cong \mathcal{P}$. Then $\mathcal{D} = (\mathrm{id}_A \times \lambda)^*\mathcal{P}$ is a symmetric ample line bundle on A coming with an action of the group $\{\mathrm{id}_A, [-1]_A\}$. Moreover, the polarization $\varphi_{\mathcal{D}}$ is exactly 2λ.

The line bundles $\mathcal{D}$ and $\mathcal{L} \otimes [-1]_A^*\mathcal{L}$ are isomorphic. Indeed, consider the composition

$$A \xrightarrow{\Delta} A \times A \xrightarrow{\mathrm{id}_A \times \varphi_{\mathcal{L}}} A \times A^t.$$

where $\Delta\colon A \to A \times A$ is the diagonal. By construction, $(\mathrm{id}_A \times \varphi_{\mathcal{L}})^*\mathcal{P}$ is the Mumford bundle $\Lambda(\mathcal{L})$ on $A \times A$, which pulls-back to $[2]^*\mathcal{L} \otimes \mathcal{L}^{-2}$ under Δ. By Corollary 3 in [Mum74, Ch. II §6] we have that

$$[2]^*\mathcal{L} = \mathcal{L}^3 \otimes [-1]_A^*\mathcal{L}.$$

So we conclude that

$$\mathcal{L} \otimes [-1]_A^*\mathcal{L} = (\mathrm{id}_A \times \varphi_{\mathcal{L}})^*\mathcal{P}.$$

We will use the bundle $\mathcal{D}$ to generalize the construction given in Section 2.2.2 in relative settings.

We need to make another observation in order to be able to define Kummer morphisms. In the previous section we worked over an algebraically closed field k. Then we made use of points in $A[2](k)$ which give rise to some exceptional divisors on the blow-up surface $\tilde{A}$. We will carry out the same idea in the relative case. In order to be able to consider these exceptional divisors in general, for instance if the field k is not algebraically closed, we will be working with abelian surface with level 2-structure.

Let $(A \to S, \lambda, \alpha) \in \mathcal{A}_{2,d',2}(S)$ be a polarized abelian scheme over a base scheme S with a Jacobi level 2-structure α. Let $\mathcal{P}$ be the Poincaré bundle on $A \times_S A^t$ where A^t is the dual abelian scheme of A. Denote by $\mathcal{D}$ the symmetric relatively ample line bundle $(\mathrm{id}_A \times \lambda)^*\mathcal{P}$ on A (see [FC90, Ch. 1, §1, 1.6]). Consider the blow-up $\tilde{A}$ of A at $A[2]$. Then the automorphism $[-1]_A$ extends to an involution $[-1]_{\tilde{A}}$ on $\tilde{A}$. One forms the quotient X of $\tilde{A}$ by the finite automorphism group $\{\mathrm{id}_{\tilde{A}}, [-1]_{\tilde{A}}\}$. Further we use the sheaf $\mathcal{D}$ to construct a polarization on $\mathcal{X}$. We consider the sheaf

$$\mathcal{N} = \pi^*\mathcal{D}^n \otimes \left(\bigotimes_{j=1}^{16} \mathcal{E}_j'^{-2n_j} \right)$$

on $\tilde{A}$ where $\mathcal{E}'_j$ are the 16 exceptional sheaves. One uses then [Mum74, Ch. III §10, Thm. 1(B)] to conclude that $\mathcal{N}$ comes from a sheaf $\mathcal{M}$ on X as in Proposition 2.2.5 and Remark 2.2.6. Clearly this generalizes the construction we considered over an algebraically closed field k. The sheaf $\mathcal{M}$ is then fiberwise ample and hence S-ample by Lemma 1.1.10. This S-ample line bundle gives rise to a polarization of X. Isomorphisms of polarized abelian schemes with a Jacobi level 2-structure are sent to isomorphisms of polarized K3 schemes in a natural way. In this way we get a morphism of stacks

$$K_{n,n_1,\dots,n_{16}}\colon \mathcal{A}_{2,d',2} \to \mathcal{M}_{2d,\mathbb{Z}[1/2]}$$

sending an object $(A \to S, \lambda, \alpha) \in \mathcal{A}_{2,d',2}$ to the object $(X, \mathcal{M}) \in \mathcal{M}_{2d,\mathbb{Z}[1/2]}$. We summarize this in the theorem below.

Theorem 2.3.1. *Let $n, d', n_1, \dots, n_{16} \in \mathbb{N}$ and assume that they satisfy the numerical conditions* (2.3)*,* (2.4) *and* (2.5) *of Lemma 2.2.11. Then there exists a morphism of algebraic stacks*

$$K_{n,n_1,\dots,n_{16}}\colon \mathcal{A}_{2,d',2} \to \mathcal{M}_{2d,\mathbb{Z}[1/2]}$$

where $d = 2n^2d' - \sum_{j=1}^{16} n_j^2$. The morphism sends a polarized abelian surface, to its associated Kummer surface with an ample line bundle.

Definition 2.3.2. For any set of numbers $n, d', n_1, \dots, n_{16}$ satisfying the inequalities (2.3), (2.4) and (2.5) we will call the morphism $K_{n,n_1,\dots,n_{16}}\colon \mathcal{A}_{2,d',2} \to \mathcal{M}_{2d,\mathbb{Z}[1/2]}$ constructed in Proposition 2.3.1 the *Kummer morphism* (or *Kummer map*) defined by $n, d', n_1, \dots, n_{16}$.

Recall that there are some weaker conditions (Lemma 2.2.11 (b)) under which a polarized abelian surface, which is not isomorphic to a polarized product of elliptic curves, gives a polarized Kummer surface. We will deal with this case now. One has a natural map

$$p\colon \mathcal{A}_{1,1,2} \times \mathcal{A}_{1,d',2} \to \mathcal{A}_{2,d',2}$$

sending a pair of polarized elliptic curves to their polarized product as in Lemma 2.2.10. Consider the open substack

$$\mathcal{U}_{2,d',2} = \mathcal{A}_{2,d',2} \setminus p(\mathcal{A}_{1,1,2} \times \mathcal{A}_{1,d',2})$$

As we saw above one can construct a polarized Kummer surface out of any such abelian surface. In the same lines one gets

Proposition 2.3.3. *Let $n, d', n_1, \dots, n_{16} \in \mathbb{N}$ satisfy the conditions of Lemma 2.2.11 (b). Then there exists a morphism of stacks*

$$K_{n,n_1,\dots,n_{16}}\colon \mathcal{U}_{2,d',2} \to \mathcal{M}_{2d,\mathbb{Z}[1/2]}$$

as constructed in Theorem 2.3.1 where $d = 2n^2d' - \sum_{j=1}^{16} n_j^2$.

Proof. $K_{n,n_1,\ldots,n_{16}}$ maps a polarized abelian surface to the polarized Kummer surface and this time one has to impose the milder conditions of Lemma 2.2.11 due to the fact that the polarized products of elliptic curves are excluded. $\square$

Remark 2.3.4. Let (A, λ, α) be an a polarized abelian surface over a finite field k of characteristic $p > 2$. Then using Lemma 2.2.4 we see that the point $K_{n,n_1,\ldots,n_{16}}((A, \lambda, \alpha))$ in $\mathcal{M}_{2d,\mathbb{F}_p}(k)$ belongs to

(i) $\mathcal{M}^{(1)}_{2d,\mathbb{F}_p} \setminus \mathcal{M}^{(2)}_{2d,\mathbb{F}_p}$ if A is ordinary;

(ii) $\mathcal{M}^{(2)}_{2d,\mathbb{F}_p} \setminus \mathcal{M}^{(3)}_{2d,\mathbb{F}_p}$ if the p rank of A is 1;

(iii) $\Sigma_9 \setminus \Sigma_{10}$ if A is supersingular but not superspecial;

(iv) Σ_{10} if A is superspecial.

2.3.2 Non-Emptiness of the Height Strata

Fix a prime number $p > 2$. We will prove here that the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty for every large enough d prime to p. The idea is to use the Kummer maps and show that $\mathcal{M}_{2d,\mathbb{F}_p}$ contains a supersingular Kummer surface. Then by Theorem 2.1.6 all strata are non-empty and so one has the claimed dimensions.

Theorem 2.3.5. *For every large enough d prime to p the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty.*

We will need the following result first.

Lemma 2.3.6. *Every residue class modulo 2×9^2 can be represented by an integer of the form $\sum_{j=1}^{16} n_j^2$ with $1 \leq n_j \leq 4$ for all j.*

Proof. Explicit calculation. $\square$

Remark 2.3.7. We believe that the statement of the preceding lemma remains valid for all $n \geq 9$. In other words, all residues modulo $2n^2$ can be represented by an integer of the form $\sum_{j=1}^{16} n_j^2$ with $1 \leq n_j < \frac{n}{2}$. This is true for $n \in [9, 45]$.

Proof of Theorem 2.3.5. First note that if for a given d there exist numbers d' and $n_1, \ldots, n_{16}$ giving a Kummer map

$$K_{d',n_1,\ldots,n_{16}} : \mathcal{U}_{2,d',2,\mathbb{F}_p} \to \mathcal{M}_{2d,\mathbb{F}_p},$$

as in Proposition 2.3.3, then the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty. This follows from Remark 2.3.4 as one can find a supersingular point in $\mathcal{U}_{2,d',2,\mathbb{F}_p}$.

Take $n = 9$ and let $d' \geq 26$ so that the conditions of Lemma 2.2.11 (b) give $n_j \in [1, 4]$. By Lemma 2.3.6 we can pick up 162 sets of numbers $(n_1, \dots, n_{16})$, $1 \leq n_j \leq 4$ which define Kummer maps as above and such that $F(n_1, \dots, n_{16}) = \sum_{j=1}^{16} n_j^2$ gives all possible resides modulo 2×9^2. Hence the images of $\mathcal{U}_{2,d',2,\mathbb{F}_p}$ under those Kummer maps land in $\mathcal{M}_{2d,\mathbb{F}_p}$ where $d = 2 \times 9^2 d' - \sum_{j=1}^{16} n_j^2$. Using this set of 162 sixteen-uples $(n_1, \dots, n_{16})$ and letting $d' \geq 26$ vary we can construct Kummer maps for $\mathcal{F}_{2d,\mathbb{F}_p}$ for all $d \geq 2 \times 9^2 \times 26 - 16 = 4196$. Therefore by the remark we started with the height strata of these moduli stacks are non-empty. This proves the assertion. $\square$

Using Kummer maps we saw that the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty if $d \geq 4196$. On the other hand one has that the Fermat K3 surface

$$x^4 + y^4 + z^4 + w^4 = 0$$

in $\mathbb{P}^3$, which is a Kummer surface by [Shi75, Thm. 1], is supersingular if $p \equiv 3 \pmod 4$. Hence using explicit Kummer surfaces one can show the non-emptiness of the height strata in lower polarization degrees. Using the same ideas as above we will "cut some more moduli points" of abelian surfaces in order to improve the estimates in Lemma 2.2.11. In this way we will lower the bound for d.

First we will settle the case when d is even. Let as before A be an abelian surface and let X be the associated Kummer surface. The invertible sheaf $\sum_{j=1}^{16} \mathcal{E}_j$ is divisible by 2 in $\mathrm{Pic}(X)$. Hence $\sum_{j=1}^{16} \mathcal{E}'_j$ comes from a line bundle on X modulo 2 torsion in $\mathrm{Pic}(\tilde{A})$. Note that this torsion has to come from $\mathrm{Pic}^0(A)$. So it does not change neither our constructions nor the intersection indexes we were dealing with. Consider the following subset of $\mathcal{A}_{2,d'}(\bar{\mathbb{F}}_p)$

$$U^3_{2,d'} = \big\{(A, \lambda) \in \mathcal{A}_{2,d'}(\bar{\mathbb{F}}_p) | \text{there does not exist an isogeny } E \times E \to A \text{ of degree } < 3\big\}.$$

For any d' the supersingular locus of $\mathcal{A}_{2,d',\mathbb{F}_p}$ remains non-empty because we exclude only finitely many points of it. Let $(A, \lambda) \in U^3_{2,d'}$ and let $\mathcal{L}$ be any ample line bundle on A defining the polarization λ. Then by Lemma 2.2.10 we have that $(\mathcal{L}, \mathcal{O}_A(E))_A \geq 3$ for every elliptic curve E in A. Taking this into account and following the proofs of Lemma 2.2.11 and Theorem 2.3.1 one constructs a Kummer map of sets

$$K_{d'} \colon U^3_{2,d'} \to \mathcal{M}_{2d,\mathbb{F}_p}(\bar{\mathbb{F}}_p)$$

where $n = n_1 = \dots = n_{16} = 1$, $d' \geq 32$ and $d = 2d' - 16$. Hence by Remark 2.3.4 we can conclude that

Corollary 2.3.8. *For all even $d \geq 48$ prime to p the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty.*

For the odd case we will construct Kummer maps with $n_1 = 1, n_2 = \cdots = n_{16} = 2$. Define as before the set

$$U^8_{2,d'} = \Big\{(A,\lambda) \in \mathcal{A}_{2,d'}(\bar{\mathbb{F}}_p) | \text{there does not exist an isogeny} \\ E \times E \to A \text{ of degree } < 8\Big\}$$

Take a point $(A,\lambda) \in U^8_{2,d'}$ and let $\mathcal{L}$ be any ample line bundle on A defining the polarization λ. Then according to Lemma 2.2.10 (b) we have that $(\mathcal{L}, \mathcal{O}_A(E))_A \geq 9$ for all elliptic curves E in A. Just as above one constructs a Kummer map of sets

$$K_{d'} : U^8_{2,d'} \to \mathcal{M}_{2d,\mathbb{F}_p}(\bar{\mathbb{F}}_p)$$

where $n = 1, n_1 = 1, n_2 = \ldots n_{16} = 2, d' \geq 512$ and $d = 2d' - 15 \times 4 - 1 = 2d' - 61$. Using these maps we obtain the following result.

Corollary 2.3.9. *For every odd $d \geq 963$ prime to p the height strata of $\mathcal{M}_{2d,\mathbb{F}_p}$ are non-empty.*

Chapter 3

Complex Multiplication for K3 Surfaces

The main theorem for complex multiplication of Shimura and Taniyama describes the action of the automorphisms in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, on the torsion points of an abelian variety A with complex multiplication. In [Del71] Deligne uses this description to define canonical models of Shimura varieties and proves that $\varprojlim \mathcal{A}_{g,1,n,\mathbb{Q}}$ is the canonical model of $Sh(\mathrm{CSp}_{2g}, \mathfrak{H}^{\pm})_{\mathbb{C}}$. See Théorème 4.21 in *loc. cit.*.

In this chapter we prove a similar result for moduli spaces of primitively polarized K3 surfaces. More precisely, for a certain class of compact open subgroups $\mathbb{K}$ of $\mathrm{SO}(V_{2d}, \psi_{2d})(\mathbb{A}_f)$ we define a period morphism

$$j_{d,\mathbb{K},\mathbb{C}} \colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(\mathrm{SO}(V_{2d}, \psi_{2d}), \Omega^{\pm})_{\mathbb{C}}.$$

We study the field of definition of this period map. Our main result is that $j_{d,\mathbb{K},\mathbb{C}}$ is defined over $\mathbb{Q}$. The proof of this result consists of two parts. We first prove a version of the main theorem of complex multiplication [Mil04, Thm. 11.2] for exceptional K3 surfaces. The proof is based on a construction due to Shioda and Inose. It occupies most of Sections 3.3.4-3.3.7. Then we show that the set of points in $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ corresponding to exceptional K3 surfaces with a given reflex field is dense for the Zariski topology. The proof of the fact that the field of definition of the period morphism $j_{d,\mathbb{K},\mathbb{C}}$ is $\mathbb{Q}$ is a formal consequence of those two results. As a corollary we obtain an analogue of the theorem of Shimura and Taniyama for all complex K3 surfaces with complex multiplication. Further, we prove that any such K3 surface can be defined over an abelian extension of its reflex field.

3.1 Hodge Structures of K3 Surfaces

3.1.1 Notations and Conventions

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ be the *Deligne torus* over $\mathbb{R}$. Consider the *weight character* $w\colon \mathbb{R}^\times = \mathbb{G}_m(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ given by $r \mapsto r^{-1}$. The *norm character* $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ is defined by $\mathrm{Nm}(z) = z\bar{z}$. The kernel of Nm is the *circle group* $\mathbb{U}_1 = \{z \in \mathbb{C}^* \mid |z| = 1\}$.

Definition 3.1.1. A $\mathbb{Q}$-*Hodge structure*, or shortly $\mathbb{Q}$-HS, is a finite dimensional vector space V over $\mathbb{Q}$ plus a homomorphism of algebraic groups

$$h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$$

such that $h \circ w$ is defined over $\mathbb{Q}$. A $\mathbb{Z}$-*Hodge structure* is a free $\mathbb{Z}$-module of finite rank V plus a homomorphism as above. We say that a $\mathbb{Q}$-HS ($\mathbb{Z}$-HS) V is *pure* of *weight* n for some $n \in \mathbb{Z}$ if

$$h \circ w\colon \mathbb{G}_m \to \mathrm{GL}(V)_\mathbb{R}$$

is multiplication by r^{-n}.

Example 3.1.2. For a compact Kähler manifold X the groups $H^i_B(X, \mathbb{Q})$ carry a natural $\mathbb{Q}$-HS of weight i.

Example 3.1.3. Consider the free $\mathbb{Z}$-module $\mathbb{Z}(n)$ of rank 1 and the homomorphism $h_{\mathbb{Z}(n)}\colon \mathbb{S} \to \mathrm{GL}(\mathbb{Z}(n)_\mathbb{R})$ such that $h_{\mathbb{Z}(n)}(z)$ acts as multiplication by $(z\bar{z})^n$. Then $\mathbb{Z}(n)$ is a pure $\mathbb{Z}$-HS of weight $-2n$ and type $\{-n, -n\}$.

If a vector space V carries a $\mathbb{Q}$-HS then one has a decomposition of $\mathbb{C}$ vector spaces

$$V_\mathbb{C} = \bigoplus V^{p,q}$$

such that $h_\mathbb{C}(z_1, z_2)(v) = z_1^{-p} z_2^{-q} v$ for every $v \in V^{p,q}$. This decomposition defines a *Hodge filtration*

$$F_h V\colon \cdots \supset F^p V \supset F^{p+1} V \supset \cdots$$

where $F^p V = \oplus_{r \geq p} V^{r,s}$.

We define μ_h to be the cocharacter of $\mathrm{GL}(V)$ given by

$$\mu_h(z) = h_\mathbb{C}(z, 1)$$

For a $\mathbb{Q}$-HS V we will denote by C the Weil operator $h(i)$.

Example 3.1.4. For the cocharacter $\mu_{\mathbb{Z}(n)}$ associated to $\mathbb{Z}(n)$ we have that $\mu_{\mathbb{Z}(n)}(z)$ acts as multiplication by z^n.

Example 3.1.5. Let V be a $\mathbb{Z}$-HS of type $\{(1,-1),(0,0),(-1,1)\}$. Then we have a decomposition $V_{\mathbb{C}} = V^{1,-1} \oplus V^{0,0} \oplus V^{-1,1}$ of $\mathbb{C}$ vector spaces. Then $\mu(z)$ acts on $V^{1,-1}$ as multiplication by z^{-1}, on $V^{-1,1}$ as multiplication by z and it acts as the identity on $V^{1,1}$.

For a $\mathbb{Q}$-HS V we will denote by $\mathrm{MT}(V)$ the *Mumford-Tate group* of V. Recall that it is the smallest algebraic subgroup of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that the homomorphism h defining the $\mathbb{Q}$-HS on V factorizes as $h\colon \mathbb{S} \to \mathrm{MT}(V)_{\mathbb{R}} \subset \mathrm{GL}(V)_{\mathbb{R}}$. We will denote by $\mathrm{Hg}(V)$ the *Hodge group* of the $\mathbb{Q}$-HS V. By definition it is the smallest algebraic subgroup $\mathrm{Hg}(V) \subset \mathrm{GL}(V)$ such that $h|_{\mathbb{U}_1} \to \mathrm{GL}(V_{\mathbb{R}})$ factorizes through $\mathrm{Hg}(V)_{\mathbb{R}}$.

Definition 3.1.6. Let V be a $\mathbb{Q}$-HS of weight n. A *polarization* ψ of V is a homomorphism of $\mathbb{Q}$-HS

$$\psi\colon V \times V \to \mathbb{Q}(-n)$$

such that $(2\pi i)^n \psi(x, Cy)$ is a positive definite symmetric form on $V_{\mathbb{R}}$.

One defines a polarization of $\mathbb{Z}$-HS using the same definition replacing $\mathbb{Q}$ by $\mathbb{Z}$.

3.1.2 Hodge Structures of K3-Type

In this section we will recall some facts concerning Hodge structures coming form cohomology of K3 surfaces. We also set up some notations.

Let X be a K3 surface or an abelian surface over $\mathbb{C}$. Then $H^2_B(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank either 22 or 6. It carries a $\mathbb{Z}$-HS

$$h\colon \mathbb{S} \to \mathrm{GL}(H^2_B(X,\mathbb{R}))$$

of weight 2 with Hodge numbers $h^{2,0} = h^{0,2} = 1$ and $h^{1,1} = 20$ or 4 respectively. Denote by h_X the morphism $h \otimes h_{\mathbb{Z}(1)}$ defining the $\mathbb{Z}$-HS on $H^2_B(X,\mathbb{Z}(1))$.

Assume now that X is a K3 surface and let λ be a primitive quasi-polarization on X (cf. Section 1.3.2). The orthogonal complement

$$P^2_B(X,\mathbb{Z}(1)) = c_1(\lambda)^{\perp} \subset H^2_B(X,\mathbb{Z}(1))$$

with respect to the Poincaré duality pairing (see Section 1.2.3 for details) carries a polarized $\mathbb{Z}$-HS. We have that $H^2_B(X,\mathbb{Q}(1)) = c_1(\lambda) \oplus P^2_B(X,\mathbb{Q}(1))$ as polarized $\mathbb{Q}$-HS. If we consider $\mathrm{SO}\big(P^2_B(X,\mathbb{Q}(1))\big)$ embedded in to $\mathrm{SO}\big(H^2_B(X,\mathbb{Q}(1))\big)$ by letting $g \in \mathrm{SO}\big(P^2_B(X,\mathbb{Q}(1))\big)$ act as the identity on the direct summand $c_1(\lambda)$ of $H^2_B(X,\mathbb{Q}(1))$, then we have that

$$h_X\colon \mathbb{S} \to \mathrm{SO}\big(P^2_B(X,\mathbb{R}(1))\big) \hookrightarrow \mathrm{SO}\big(H^2_B(X,\mathbb{R}(1))\big).$$

Hence the Mumford-Tate group $\mathrm{MT}\big(H^2_B(X,\mathbb{Q}(1))\big)$ is contained in $\mathrm{SO}(P^2_B(X,\mathbb{Q}(1)))$. The corresponding cocharacter μ_X, given by $\mu_X(z) = h_{X,\mathbb{C}}(z,1)$, acts as multiplication by z^{-1} on the $\{1,-1\}$ part of $P^2_B(X,\mathbb{C}(1))$, as the identity on the $\{0,0\}$ part and as multiplication by z on the $\{-1,1\}$ part.

Let X be a non-singular projective surface over $\mathbb{C}$ and let A_X be the image

$$A_X := c_1(\mathrm{Pic}(X)) \subset H^2_B(X,\mathbb{Z}(1)).$$

It is a polarized $\mathbb{Z}$-Hodge substructure of $H^2_B(X,\mathbb{Z}(1))$ of type $\{(0,0)\}$. Denote by T_X its complement

$$T_X := c_1(\mathrm{Pic}(X))^{\perp} \subset H^2_B(X,\mathbb{Z}(1))$$

with respect to the Poincaré duality pairing ψ_X. The bilinear form ψ_X restricts to a bilinear form on T_X.

Definition 3.1.7. For a non-singular complex projective surface X the lattice T_X together with the form $\psi_X|_{T_X}$ is called the *transcendental lattice* of X.

The lattice T_X carries a polarized $\mathbb{Z}$-HS (Hodge substructure of $H^2(X,\mathbb{Z}(1))$) of type $\{(-1,1),(0,0),(1,-1)\}$ and we have that $H^2_B(X,\mathbb{Q}(1)) = A_{X,\mathbb{Q}} \oplus T_{X,\mathbb{Q}}$ as polarized $\mathbb{Q}$-HS (see [Zar83, §1] for more details). If we consider $\mathrm{SO}(T_{X,\mathbb{Q}})$ embedded into $\mathrm{SO}\big(H^2_B(X,\mathbb{Q}(1))\big)$ by acting as the identity on the summand $A_{X,\mathbb{Q}}$ we see that

$$h_X\colon \mathbb{S} \to \mathrm{SO}(T_{X,\mathbb{R}}) \hookrightarrow \mathrm{SO}\big(H^2_B(X,\mathbb{R}(1))\big).$$

Hence we have that $\mathrm{MT}\big(H^2_B(X,\mathbb{Q}(1))\big) \subset \mathrm{SO}(T_{X,\mathbb{Q}})$. The cocharacter μ_X acts in the way described above.

We fix similar notations for étale cohomology groups. Denote by $A_{X,\hat{\mathbb{Z}}}$ the image $c_1(\mathrm{Pic}(X))_{\hat{\mathbb{Z}}} \subset H^2_{et}(X,\hat{\mathbb{Z}}(1))$ and let $A_{X,\mathbb{A}_f}$ be $A_{X,\hat{\mathbb{Z}}} \otimes \mathbb{A}_f$. We consider the *transcendental lattice*

$$T_{X,\hat{\mathbb{Z}}} := A^{\perp}_{X,\hat{\mathbb{Z}}} \subset H^2_{\mathrm{et}}(X,\hat{\mathbb{Z}}(1))$$

and let $T_{X,\mathbb{A}_f}$ be $T_{X,\hat{\mathbb{Z}}} \otimes \mathbb{A}_f$. Then by the comparison theorem between Betti and étale cohomology we have a natural isomorphism $T_{X,\hat{\mathbb{Z}}} \cong T_X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

Let X_1 and X_2 be two non-singular complex projective surfaces and let $f\colon X_1 \to X_2$ be a morphism. Then the induced homomorphism on Betti cohomology

$$f^*_B\colon H^2_B(X_2,\mathbb{Z}(1)) \to H^2_B(X_1,\mathbb{Z}(1))$$

restricts to a morphism of the corresponding transcendental lattices $f^*_B\colon T_{X_2} \to T_{X_1}$ (or respectively with $\mathbb{Q}$-coefficients). Similarly, we have an induced homomorphism $f^*_f\colon T_{X_2,\hat{\mathbb{Z}}} \to T_{X_1,\hat{\mathbb{Z}}}$ (or respectively with $\mathbb{A}_f$-coefficients) on étale cohomology. By the comparison theorem we know that $f^*_f = f^*_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

For any homomorphism $\alpha\colon T_{X_1} \to T_{X_2}$ of $\mathbb{Z}$-HS (or $\alpha\colon T_{X_1,\mathbb{Q}} \to T_{X_2,\mathbb{Q}}$ of $\mathbb{Q}$-HS respectively) we will denote by α_f the induced homomorphism

$$\alpha_f := \alpha \otimes \hat{\mathbb{Z}}\colon T_{X_1,\hat{\mathbb{Z}}} \to T_{X_2,\hat{\mathbb{Z}}}$$

(or $\alpha_f := \alpha \otimes \mathbb{A}_f\colon T_{X_1,\mathbb{A}_f} \to T_{X_2,\mathbb{A}_f}$, respectively).

Definition 3.1.8. Let $d \in \mathbb{N}$ and consider the vector space (V_{2d}, ψ_{2d}) defined in Section 1.2.1. A *polarized $\mathbb{Q}$-HS of K3-type of degree* $2d$ is a triple (V, ψ, h) where h is a homomorphism

$$h\colon \mathbb{S} \to \mathrm{SO}(V, \psi)_{\mathbb{R}}$$

such that

(a) h defines a $\mathbb{Q}$-HS of type $\{(-1,1),(0,0),(1,-1)\}$ on V and ψ is a polarization for h,

(b) The Hodge numbers of $V_{\mathbb{C}}$ are $h^{-1,1} = h^{1,-1} = 1$ and $h^{0,0} = 19$,

(c) (V, ψ), as an orthogonal space, is equivalent to (V_{2d}, ψ_{2d}).

Giving such h amounts to giving a 2-dimensional subspace $V_{\mathbb{R}}^{-}$ of $V_{\mathbb{R}}$ on which ψ is negative definite and an orientation of $V_{\mathbb{R}}^{-}$: For $z = \tau e^{i\theta} \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$, we have that $h(z)$ acts as the identity on the orthogonal complement $V_{\mathbb{R}}^{-,\perp}$ and as rotation on angle 2θ on $V_{\mathbb{R}}^{-}$ (see [Del72, §5.4 and 5.5]).

Note that as above one can see that for a polarized $\mathbb{Q}$-HS of K3-type h the corresponding cocharacter μ_h acts as in Example 3.1.5.

3.1.3 Hodge Groups of K3 Surfaces

We will give a short exposition of some results of Zarhin [Zar83] on Hodge groups of K3 surfaces which we will use in this chapter. Let X be a complex K3 surface and denote by $\mathrm{Hg}(X)$ and $\mathrm{MT}(X)$ the Hodge and the Mumford-Tate groups (which in our case are the same) of X associated to the $\mathbb{Z}$-HS on $H_B^2(X, \mathbb{Z}(1))$. We know that the homomorphism

$$h_X\colon \mathbb{S} \to \mathrm{SO}\big(H_B^2(X, \mathbb{R}(1))\big)$$

defining the $\mathbb{Z}$-HS on $H_B^2(X, \mathbb{Z}(1))$ factorizes thorough $\mathrm{SO}(T_{X,\mathbb{R}})$ and hence we have that $\mathrm{Hg}(X) \subset \mathrm{SO}(T_{X,\mathbb{Q}})$.

Theorem 3.1.9. *The vector space $T_{X,\mathbb{Q}}$ is a simple* $\mathrm{Hg}(X)$*-module.*

Proof. See Theorem 1.4.1 in [Zar83]. □

Definition 3.1.10. For a K3 surface X we define the *Hodge endomorphism algebra* of X to be

$$E_X := \mathrm{End}_{\mathrm{Hg}(X)}(T_{X,\mathbb{Q}}) = \mathrm{End}_{\mathrm{HS}}(T_{X,\mathbb{Q}}).$$

The Hodge endomorphism algebra of X is an analogue of the endomorphism algebra of an abelian surface.

Theorem 3.1.11. *The algebra E_X is a field which is either a totally real field or an imaginary quadratic extension of a totally real field i.e., a CM-field.*

Proof. For the first part of the theorem we refer to Theorem 1.6 in [Zar83]. It actually follows from the fact that one has a natural embedding

$$\epsilon_X \colon E_X \hookrightarrow \operatorname{End}_{\mathbb{C}} H^{2,0}(X) \cong \mathbb{C}.$$

For the second part see [Zar83, Thm. 1.5.1]. □

We shall now determine $\mathrm{Hg}(X)$ explicitly. For this purpose define

$$\phi \colon T_{X,\mathbb{Q}} \times T_{X,\mathbb{Q}} \to E_X$$

by $\phi(x, y) = \alpha$ where α is the unique element of E_X such that $\psi_X(ex, y) = \mathrm{tr}_{E_X/\mathbb{Q}}(\alpha e)$ for every $e \in E_X$. Such α exists (see [Zar83, §2.1]).

Theorem 3.1.12. *Let X be a complex K3 surface, let E_X be its Hodge endomorphism field and assume it is a CM-field. Then we have that*

$$\mathrm{Hg}(X) \cong \mathrm{U}(T_{X,\mathbb{Q}}, \phi)$$

i.e., the Hodge group of X is as big as possible.

Here $\mathrm{U}(T_{X,\mathbb{Q}}, \phi)$ is the unitary group of elements preserving the bilinear form ϕ viewed as a group over $\mathbb{Q}$.

Proof. See [Zar83, Thm 2.3.1]. □

If X is a K3 surface with a Hodge endomorphism algebra E_X which is a CM-field then E_X embedded via ϵ_X in $\mathbb{C}$ as above is the *reflex field* of the Mumford-Tate torus $\mathrm{MT}(X)$.

Definition 3.1.13. We say that a K3 surface X has *complex multiplication*, shortly *CM*, by E_X if X has a Hodge endomorphism algebra E_X which is a CM-field.

Definition 3.1.14. A complex K3 surface X is called *exceptional* if $\mathrm{rk}_{\mathbb{Z}} \operatorname{Pic}(X) = 20$.

If X is an exceptional K3 surface, then $T_{X,\mathbb{Q}}$ is a 2 dimensional $\mathbb{Q}$-vector space. From [Zar83, §2, 2.1 - 2.3] we see that the Hodge endomorphism field E_X is a quadratic imaginary field. Further, we have that $\mathrm{Hg}(X)$ (respectively $\mathrm{MT}(X)$) is a torus and therefore an exceptional K3 surface has CM by a quadratic imaginary field.

Definition 3.1.15. If X is an exceptional K3 surface over $\mathbb{C}$ we will say that it is of *CM-type* (E_X, ϵ_X) if E is its Hodge endomorphism field and $\epsilon_X \colon E_X \to \mathbb{C}$ is the natural embedding given by the action of E_X on the space of holomorphic forms on X.

3.2 The Shimura Variety $Sh(\mathrm{SO}(2,19),\Omega^{\pm})$

Over $\mathbb{C}$ the geometry of moduli spaces of primitively polarized K3 surfaces is connected to the geometry of the Shimura variety associated to $\mathrm{SO}(2,19)$. This is achieved via periods. In this chapter we will use this relation to prove the main theorems of complex multiplication for K3 surfaces. In Sections 3.2.1 - 3.2.4 we will give the needed preliminaries. More precisely, in Section 3.2.3 we describe a modular interpretation of the points in $Sh(\mathrm{SO}(2,19),\Omega^{\pm})_{\mathbb{C}}(\mathbb{C})$ in terms of periods of K3 surfaces. In the following section we define a period morphism $j_{d,\mathbb{K},\mathbb{C}}$ which is a slight modification of the period morphisms used in [PSS72], [Fri84], [BBD85] and others.

3.2.1 Special Orthogonal Groups

Let V be a 21 dimensional $\mathbb{Q}$ vector space and let ψ be a non-degenerate form on it of signature $(2-,19+)$. Then ψ is equivalent to the form

$$Q_d\colon\ -x_1^2-x_2^2+x_3^2+\cdots+x_{20}^2+dx_{21}^2$$

for some square free integer d. In general, if $d>1$ the two forms Q_d and Q_1 need not be equivalent over $\mathbb{Q}$. But the forms dQ_1 and Q_d are equivalent over $\mathbb{Q}$. One sees this using Theorem and the Corollary in [Ser73, Ch. IV, §3.3]. Therefore we have the following result.

Lemma 3.2.1. *The groups* $\mathrm{SO}(V,Q_1)$ *and* $\mathrm{SO}(V,Q_d)$ *are isomorphic over* $\mathbb{Q}$.

Notation 3.2.2. From now on we will denote by G or by $\mathrm{SO}(2,19)$ the algebraic group $\mathrm{SO}(V,\psi)$ over $\mathbb{Q}$.

3.2.2 The Shimura Datum $(G,\Omega^{\pm})$

Let V be a 21 dimensional vector space over $\mathbb{Q}$ and let ψ be a non-degenerate bilinear form on V of signature $(2-,19+)$. Define $\Omega^{\pm}$ to be the collection of all $\mathbb{Q}$-HS $h\colon\mathbb{S}\to G_{\mathbb{R}}=\mathrm{SO}(V_{\mathbb{R}})$ for which $\pm\psi$ is a polarization and the Hodge numbers are $h^{-1,1}=h^{1,-1}=1$ and $h^{0,0}=19$. Let $h_0\colon\mathbb{S}\to G_{\mathbb{R}}$ be an element of $\Omega^{\pm}$. Then $\Omega^{\pm}$ is equal to the $G(\mathbb{R})$-conjugacy class of the homomorphism h_0.

For an element $h\in\Omega^{\pm}$ let F_h be the associated Hodge filtration. Then the map $h\mapsto F_h^1$ identifies $\Omega^{\pm}$ with the space

$$\{\omega\in\mathbb{P}(V\otimes\mathbb{C})|\ \psi(\omega,\omega)=0 \text{ and } \psi(\omega,\bar{\omega})>0\}.$$

This gives $\Omega^{\pm}$ a complex structure for which the Hodge filtration F_h varies holomorphically with h. We also see that $\Omega^{\pm}$ consists of two connected components Ω^{+} and Ω^{-}

corresponding to the two possible orientations one can give to the space $V_{\mathbb{R}}^{-}$ corresponding to a morphism h. Further $\Omega^{\pm}$, being the $G(\mathbb{R})$-conjugacy class of a homomorphism $h_0\colon \mathbb{S} \to G_{\mathbb{R}}$ as above, can be identified with the space

$$\mathrm{SO}(2,19)(\mathbb{R})/\big(\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R})\big).$$

We choose Ω^{+} to be connected component corresponding to

$$\mathrm{SO}(2,19)(\mathbb{R})^{+}/\big(\,\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R})\big),$$

where $\mathrm{SO}(2,19)(\mathbb{R})^{+}$ is the connected component of $\mathrm{SO}(2,19)(\mathbb{R})$ containing the identity. This choice is non-canonical as it depends on the choice of h_0.

The pair $(G,\Omega^{\pm})$ is a Shimura datum with reflex field $\mathbb{Q}$. The last claim follows from [Del71, Prop. 3.8] and [And96a, Appendix 1, Lemma].

3.2.3 Modular Interpretation: Part I

Let $d \in \mathbb{N}$ and consider the quadratic space (V_{2d},ψ_{2d}) (see Section 1.2.1). As before we will denote by G the group $\mathrm{SO}(V_{2d},\psi_{2d})$. Let $\mathbb{K} \subset G(\mathbb{A}_f)$ be an compact open subgroup and consider the variety

$$Sh_{\mathbb{K}}(G,\Omega^{\pm}) = G(\mathbb{Q})\backslash\Omega^{\pm} \times G(\mathbb{A}_f)/\mathbb{K}$$

where $q(x,a)k = (qx,qak)$ for $q \in G(\mathbb{Q}), x \in \Omega^{\pm}, a \in G(\mathbb{A}_f)$ and $k \in \mathbb{K}$.

Define $\mathcal{H}_{\mathbb{K}}$ to be the set of 4-tuples

$$\big((W,h),s,\alpha\mathbb{K}\big)$$

where:

(i) $\big((W,h),s\big)$ is a polarized $\mathbb{Q}$-HS of K3 type (see Def. 3.1.8),

(ii) $\alpha\mathbb{K}$ is the $\mathbb{K}$-orbit of an $\mathbb{A}_f$-linear isomorphism

$$\alpha\colon V_{2d} \otimes \mathbb{A}_f \to W \otimes \mathbb{A}_f$$

such that $\psi_{2d}(v_1,v_2) = m_s \cdot s(\alpha(v_1),\alpha(v_2))$ for all $v_1,v_2 \in V \otimes \mathbb{A}_f$, where $m_s \in \mathbb{Q}_{+}$.

An isomorphism between $\big((W,h),s,\alpha\mathbb{K}\big)$ and $\big((W',h'),s',\alpha'\mathbb{K}\big)$ in $\mathcal{H}_{\mathbb{K}}$ is an isomorphism $b\colon (W,h) \to (W',h')$ of $\mathbb{Q}$-HS such that there exists $c \in \mathbb{Q}^{\times}$ for which $s = c\cdot s' \circ (b \times b)$ and $b \circ \alpha \equiv \alpha' \pmod{\mathbb{K}}$. From this we see that $c = m_s/m_{s'} \in \mathbb{Q}^{\times}/\mathbb{Q}^{\times 2}$.

Let $\big((W,h),s,\alpha\mathbb{K}\big)$ be an element of $\mathcal{H}_{\mathbb{K}}$. From (ii) and the fact that the signature of s is $(2-,19+)$ we conclude by the Hasse principle that there is an isomorphism $a\colon W \to V_{2d}$

with $s = m_s^{-1}\cdot\psi_{2d}\circ(a\times a)$. Further, the homomorphism $a\cdot h\colon \mathbb{S}\to G_{\mathbb{R}}$ defined by $z\mapsto a\circ h(z)\circ a^{-1}$ belongs to $\Omega^{\pm}$ and the composition

$$V_{2d,\mathbb{A}_f}\xrightarrow{\alpha} W_{\mathbb{A}_f}\xrightarrow{a} V_{2d,\mathbb{A}_f}$$

is an element of $G(\mathbb{A}_f)$. Indeed, we have that

$$\psi_{2d}\circ(a\times a)\circ(\alpha\times\alpha) = m_s\cdot s\circ(\alpha\times\alpha) = m_s m_s^{-1}\cdot\psi_{2d} = \psi_{2d}.$$

Another isomorphism $a'\colon W\to V_{2d}$, for which $s = m_s^{-1}\cdot\psi_{2d}\circ(a'\times a')$, differs from a by an element $q\in G(\mathbb{Q})$, say $a' = q\circ a$. Hence, replacing a by a' will change $[a\cdot h, a\circ\alpha]_{\mathbb{K}}$ with $[qa\cdot h, qa\circ\alpha]_{\mathbb{K}}$. Similarly, replacing α by αk for some $k\in\mathbb{K}$ will replace $[a\cdot h, a\circ\alpha]_{\mathbb{K}}$ with $[a\cdot h, a\circ\alpha k]_{\mathbb{K}}$. Therefore one has a well defined map

$$\mathcal{H}_{\mathbb{K}}\to G(\mathbb{Q})\backslash\Omega^{\pm}\times G(\mathbb{A}_f)/\mathbb{K}$$

given by

$$\big((W,h),s,\alpha\mathbb{K}\big)\mapsto[a\cdot h, a\circ\alpha]_{\mathbb{K}} \tag{3.1}$$

where $[a\cdot h, a\circ\alpha]_{\mathbb{K}}$ denotes the class of $(a\cdot h, a\circ\alpha)$.

Proposition 3.2.3. *The map defined by* (3.1) *gives a bijection between* $\mathcal{H}_{\mathbb{K}}/\{\text{isom.}\}$ *and* $Sh_{\mathbb{K}}(G,\Omega^{\pm})(\mathbb{C})$.

Proof. Suppose that $b\colon (W,h)\to(W',h')$ is an isomorphism of $\mathbb{Q}$-HS giving an isomorphism of the triples $\big((W,h),s,\alpha\mathbb{K}\big)$ and $\big((W',h'),s',\alpha'\mathbb{K}\big)$ in $\mathcal{H}_{\mathbb{K}}$. Then we have that $s = m_s/m_{s'}\cdot s'\circ(b\times b)$. Choose an isomorphism $a'\colon W'\to V_{2d}$ such that $\psi_{2d} = m_{s'}\cdot s'\circ(a'\times a')$. Then for the isomorphism $a\colon W\to V_{2d}$ defined by $a = a'\circ b$ we see that $\psi_{2d} = m_s\cdot s\circ(a\times a)$. Hence we have that $(a\cdot h, a\circ\alpha) = (a'\cdot h', a'\circ\alpha' k)$ where $b\circ\alpha = \alpha' k$.

Assume that $\big((W,h),s,\alpha\mathbb{K}\big)$ and $\big((W',h'),s',\alpha'\mathbb{K}\big)$ are mapped to the same point in $Sh_{\mathbb{K}}(G,\Omega^{\pm})(\mathbb{C})$. Choose two isomorphisms $a\colon W\to V_{2d}$ such that $\psi_{2d} = m_s\cdot s\circ(a\times a)$ and $\alpha'\colon W'\to V_{2d}$ for which $\psi_{2d} = m_{s'}\cdot s'\circ(a'\times a')$. We know that

$$(a\cdot h, a\circ\alpha) = (qa'\cdot h', qa'\circ\alpha' k)$$

for some $q\in G(\mathbb{Q})$ and $k\in\mathbb{K}$. After replacing a' by qa' we may suppose that $(a\cdot h, a\circ\alpha) = (a'\cdot h', a'\circ\alpha k)$. Then $b = a'\circ a^{-1}$ is an isomorphism of the triples $\big((W,h),s,\alpha\mathbb{K}\big)$ and $\big((W',h'),s',\alpha'\mathbb{K}\big)$. This shows that the map is injective. The surjectivity follows easily as any element $[h,g]_{\mathbb{K}}$ is the image of $\big((V_{2d},h),\psi_{2d},g\mathbb{K}\big)$. □

Remark 3.2.4. With this modular interpretation of $Sh_{\mathbb{K}}(G,\Omega^{\pm})(\mathbb{C})$ we may think of it as the set parameterizing 'isogeny' classes of polarized K3 surfaces with certain level structure up to an isomorphism. This is similar to the case of abelian varieties. See [Del71, §4, 4.11].

3.2.4 Modular Interpretation: Part II

Let d be a natural number and let $\mathbb{K} \subset \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ be a subgroup of finite index which is contained in some $\mathbb{K}_n$ for some $n \geq 3$. Our goal in this section is to construct a morphism

$$j_{d,\mathbb{K},\mathbb{C}} \colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}$$

mapping every primitively polarized complex K3 surface its periods (cf. Step 1 of the proof of Proposition 3.2.5 and Definition 3.2.8 below). We will use the notations established in Section 1.2.1.

Proposition 3.2.5. *For a natural number d and a group $\mathbb{K}$ as above one has an étale morphism of algebraic spaces*

$$j_{d,\mathbb{K},\mathbb{C}} \colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}.$$

Proof. We will divide the proof into several steps.
Step 1: We begin with a naive pointwise definition. Let (X, λ, α) be a complex K3 surface with a primitive polarization of degree $2d$ and a level $\mathbb{K}$-structure α. Let $\tilde{\alpha} \colon L_{2d,\hat{\mathbb{Z}}} \to P^2_{\mathrm{et}}(X, \hat{\mathbb{Z}}(1))$ be a representative of the class α. Choose an isometry $a \colon H^2_B(X, \mathbb{Z}(1)) \to L_0$ such that $a(c_1(\lambda)) = e_1 - df_1$ and $a \circ \tilde{\alpha} \colon L_{2d,\hat{\mathbb{Z}}} \to L_{2d,\hat{\mathbb{Z}}}$ is an element in $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$. Let $h_X \colon \mathbb{S} \to \mathrm{SO}(P^2_B(X, \mathbb{R}(1)))$ be the morphism defining the polarized $\mathbb{Z}$-HS on $P^2_B(X, \mathbb{Z}(1))$.

Claim 3.2.6. *The class $[a \circ h_X \circ a^{-1}, a \circ \tilde{\alpha}]_{\mathbb{K}}$ of the pair $(a \circ h_X \circ a^{-1}, a \circ \tilde{\alpha})$ in $Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}(\mathbb{C})$ is independent of a choice of the marking a and the lifting $\tilde{\alpha}$ of α.*

Proof. Indeed, any representative of the class of α is of the form $\tilde{\alpha} \circ \kappa$ for some $\kappa \in \mathbb{K}$ and any isometry $a' \colon H^2_B(X, \mathbb{Z}(1)) \to L_0$ such that $a(c_1(\lambda)) = e_1 - df_1$ (cf. Remark 1.2.6) and $a' \circ \tilde{\alpha} \circ \kappa \colon L_{2d,\hat{\mathbb{Z}}} \to L_{2d,\hat{\mathbb{Z}}}$ is equal to $g \circ a$ for some $g \in \mathrm{O}(V_0)(\mathbb{Z})$ with $g(e_1 - df_1) = e_1 - df_1$ and such that $g \in \mathrm{SO}(V_{2d})(\mathbb{Z})$. Hence we have that the new data produce a pair

$$(g \circ a \circ h_X \circ a^{-1} \circ g^{-1}, g \circ a \circ \tilde{\alpha} \circ \kappa) = \big(g \cdot (a \circ h_X \circ a^{-1}), g \circ a \circ \tilde{\alpha} \circ \kappa\big)$$

whose class in $Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}(\mathbb{C})$ is exactly $[a \circ h_X \circ a^{-1}, a \circ \tilde{\alpha}]_{\mathbb{K}}$. □

We will use this pointwise construction to define an algebraic morphism as claimed in the proposition.

Step 2: Let $U \to \mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ be a (smooth) atlas of $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ such that the pull-back of the universal family over $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ to U is a K3 scheme. Let V be a connected component of U and let $(\pi \colon X \to V, \lambda, \alpha)$ be the pull-back of the universal family to V. Define a map

$$j_{d,\mathbb{K},V} \colon V^{\mathrm{an}} \to Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}$$

by sending a point $s \in V^{\mathrm{an}}$ to the point associated to $(X_s, \lambda_s, \alpha_s)$ in Step 1. We will show that it is an algebraic morphism.

Step 3: We will show that $j_{d,\mathbb{K},V}$ is holomorphic and a local isomorphism. According to [Mil04, Lemma 5.13] the decomposition of $Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ into connected components is given in the following way:

$$Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}} = \coprod_{[g]\in\mathcal{C}} \Gamma_{[g]}\backslash\Omega^{+},$$

where $\mathcal{C} := G(\mathbb{Q})_{+}\backslash G(\mathbb{A}_f)/\mathbb{K}$ and $\Gamma_{[g]} = G(\mathbb{Q})_{+} \cap g\mathbb{K}g^{-1}$ for some representative g of $[g] \in \mathcal{C}$. We will first show that $j_{d,\mathbb{K},V}$ maps V^{an} into one connected component.

Suppose given two points s_1 and s_2 of V^{an}. One can find an isomorphism

$$\delta_\pi \colon \pi_1(V^{\mathrm{an}}, s_1) \cong \pi_1(V^{\mathrm{an}}, s_2)$$

and an isometry

$$\delta_B \colon H^2_B(X_{s_1}, \mathbb{Z}(1)) \to H^2_B(X_{s_2}, \mathbb{Z}(1))$$

mapping $c_1(\lambda_{s_1})$ to $c_1(\lambda_{s_2})$, such that $\delta_B(\gamma\cdot x) = \delta_\pi(\gamma)\cdot\delta_B(x)$ for every $x \in H^2_B(X_{s_1}, \mathbb{Z}(1))$ and $\gamma \in \pi_1(V^{\mathrm{an}}, s_1)$. The isometry δ_B defines thus an isometry between $P^2_B(X_{s_1}, \mathbb{Z}(1))$ and $P^2_B(X_{s_2}, \mathbb{Z}(1))$ which we will denote again by δ_B. If the level $\mathbb{K}$-structure on $\pi\colon X \to V$ is given by the class α in $\{\mathbb{K}\backslash\mathrm{Isometry}\big(L_{2d}, P^2(s_1)\big)\}^{\pi_1^{\mathrm{alg}}(V,s_1)}$ with respect to the geometric point s_1 and $\tilde{\alpha}$ is a representative of this class, then it is given at the point s_2 by the class of $\delta_B \circ \tilde{\alpha}$ in $\mathbb{K}\backslash\mathrm{Isometry}\big(L_{2d}, P^2(s_2)\big)$ which is $\pi_1^{\mathrm{alg}}(V, s_2)$-invariant (see the discussion before Definition 1.5.1 in Section 1.5.1). Hence if we take a marking $a\colon H^2_B(X_{s_1}, \mathbb{Z}(1)) \to L_0$ such that $a(c_1(\lambda_{s_1})) = e_1 - df_1$ then we can take a marking

$$a \circ \delta_B^{-1} \colon H^2_B(X_{s_2}, \mathbb{Z}(1)) \to L_0$$

for which we have that $a(c_1(\lambda_{s_2})) = e_1 - df_1$. So we see that

$$j_{d,\mathbb{K},V}(s_1) = [a \circ h_{s_1} \circ a^{-1}, a \circ \tilde{\alpha}]_{\mathbb{K}} \tag{3.2}$$

$$j_{d,\mathbb{K},V}(s_2) = [a \circ \delta_B^{-1} \circ h_{s_2} \circ \delta_B \circ a^{-1}, a \circ \delta_B^{-1} \circ \delta_B \circ \tilde{\alpha}]_{\mathbb{K}}. \tag{3.3}$$

The sheaf $R^2_B\pi_*\mathbb{Z}(1)$ is a local system on V^{an} for every point $s \in V^{\mathrm{an}}$ one can find an open neighborhood V_s of s in V^{an} such that the system $R^2_B\pi_*\mathbb{Z}(1)|_{V_s}$ is constant. We can find a marking $a\colon R^2_B\pi_*\mathbb{Z}(1)|_{V_s} \to (L_0)_{V_s}$ mapping $c_1(\lambda)$ to $e_1 - df_1$. According to [BBD85, §5, Theorem] (or [Gri71, 9.7]) the map

$$j\colon V_s \to \Omega^{\pm}$$

defined by $j(s) = a \circ h_{X_s^{\mathrm{an}}} \circ a^{-1}$ is holomorphic. As V_s is connected we may assume that its image in $\Omega^{\pm}$ under the morphism j is contained in Ω^{+}. Then we see from (3.2) and (3.3) that $j_{d,\mathbb{K},V}(V_s) \subset \Gamma_{[g]}\backslash\Omega^+$ where $g = a_s \circ \tilde{\alpha}_s$. Further, $pr\colon \Omega^+ \to \Gamma_{[g]}\backslash\Omega^+$ is holomorphic and we have that $j_{d,\mathbb{K},V}|_{V_s} = pr \circ j$. Hence $j_{d,\mathbb{K},V}|_{V_s}$ is holomorphic.

According to [And96a, §3.3, Prop. 3.3.1] applied to $X^{\mathrm{an}}_{V_s} \to V_s$, the holomorphic map j is a local isomorphism and therefore the same holds for $j_{d,\mathbb{K},\mathbb{C}}|_{V_s}$ as Ω^+ is the universal covering space of $\Gamma_{[g]}\backslash\Omega^+$. Those conclusions are valid for a neighborhood of any point s in V^{an} hence we see that $j_{d,\mathbb{K},V}\colon V^{\mathrm{an}} \to \Omega^+\backslash\Gamma$ is holomorphic and it is a local isomorphism.

Step 4: For every connected component V of U we defined a holomorphic morphism $j_{d,\mathbb{K},V}\colon V^{\mathrm{an}} \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ which is a local isomorphism. By Step 3 it factorizes though a connected component $\Gamma_{[g]}\backslash\Omega^+$ of $Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ for some $g \in G(\mathbb{A}_f)$ so using a result of A. Borel ([Del72, §5, Thm. 5.1]) we conclude that $j_{d,\mathbb{K},V}$ is an algebraic morphism. Indeed, we can apply Theorem 5.1 in *loc. cit.* because the group $\Gamma_{[g]}$ is torsion free as $\mathbb{K} \subset \mathbb{K}_n$ for some $n \geq 3$. We also have that $j_{d,\mathbb{K},V}$, being an analytic local isomorphism, is étale. Gluing the morphisms $j_{d,\mathbb{K},V}$ for all connected components V of U we obtain a morphism of $\mathbb{C}$-schemes

$$j_{2d,\mathbb{K},U}\colon U \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$$

which is étale.

Step 5: We will show that $j_{d,\mathbb{K},U}$ descends to a morphism of algebraic spaces

$$j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}.$$

We have to show that the two projection maps

$$j_{d,\mathbb{K},U} \circ pr_i\colon U \times_{\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}} U \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$$

for $i = 1, 2$ coincide (see [Knu71, Ch. II, §1, Prop. 1.4]). As $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ is a reduced algebraic space over $\mathbb{C}$ we have that $U \times_{\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}} U$ is a reduced $\mathbb{C}$-scheme ([Knu71, Ch. II, §1, Def. 1.1]). Hence we can check the equality of the two morphisms on $\mathbb{C}$-valued points.

Any $\mathbb{C}$-valued point on $U \times_{\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}} U$ is a pair $\big((X_1,\lambda_1,\alpha_1),(X_2,\lambda_2,\alpha_2),f\big)$ where f is an isomorphism of the objects (X_1,λ_1,α_1) and (X_2,λ_2,α_2) in $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$. Hence from the very definition of the morphism $j_{d,\mathbb{K},U}$ we easily see (just like in the proof of Proposition 3.2.3) that

$$j_{d,\mathbb{K},U} \circ pr_1\big((X_1,\lambda_1,\alpha_1),(X_2,\lambda_2,\alpha_2)\big) = j_{d,\mathbb{K},U} \circ pr_2\big((X_1,\lambda_1,\alpha_1),(X_2,\lambda_2,\alpha_2)\big).$$

Thus we have that $j_{d,\mathbb{K},U} \circ pr_1 = j_{d,\mathbb{K},U} \circ pr_2$ and therefore $j_{d,\mathbb{K},U}$ descends to a morphism $j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$. It is étale as $j_{d,\mathbb{K},U}$ is étale ([Knu71, Ch. II, §2, Def. 2.1]). $\square$

Corollary 3.2.7. *The algebraic space $\mathcal{F}_{2d,\mathbb{K},\mathbb{Q}}$ is a scheme.*

Proof. Combining the above proposition and [Knu71, Ch. II, §6, Cor. 6.17] we conclude that $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ is a scheme. Therefore $\mathcal{F}_{2d,\mathbb{K},\mathbb{Q}}$ is a scheme, as well. □

Definition 3.2.8. The map $j_{d,\mathbb{K},\mathbb{C}}$ is called the *period morphism* (or the *period map*) associated to d and $\mathbb{K}$. For every primitively polarized complex K3 surface (X,λ,α) of degree $2d$ and a level $\mathbb{K}$-structure α, the point $j_{d,\mathbb{K},\mathbb{C}}\big((X,\lambda,\alpha)\big) \in Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ is called the *period point* of (X,λ,α).

Remark 3.2.9. The period map $j_{d,\mathbb{K},\mathbb{C}}$ defined in the proof of Proposition 3.2.5 is a slight modification of the period maps used in [BBD85] to construct coarse moduli spaces of primitively polarized complex K3 surfaces. We consider moduli spaces over $\mathbb{Q}$ and these have more than one geometric connected component. The morphism constructed above takes this information in to account. We will see later that this is essential for having the period morphism defined over $\mathbb{Q}$.

In Section 3.3.10 we show that the image $j_{d,\mathbb{K},\mathbb{C}}(\mathcal{F}_{2d,\mathbb{K},\mathbb{C}})$ is dense in $Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ and that its complement is a divisor.

Suppose that $\mathbb{K}$ is an admissible subgroup of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ (see Definition 1.5.5). Then one can consider the moduli space $\mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K},\mathbb{C}}$ which is an open subspace of $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$. Hence we have a period map

$$j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$$

given by the restriction of $j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ to $\mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K},\mathbb{C}}$. We will show below that this period morphism is injective. This result is a direct consequence of the *global Torelli theorem for K3 surfaces*.

Theorem 3.2.10 (Piatetskij-Sapiro and Shafarevich). *Let X and X' be two K3 surfaces and let $\phi\colon H^2_B(X,\mathbb{Z}) \to H^2_B(X',\mathbb{Z})$ be a Hodge-effective isometry. Then there exists a unique isomorphism $u\colon X' \to X$ such that $u^*_B = \phi$.*

Proof. See [BBD85, Exp. VIII & IX] for the proof. Note that the statement of the theorem in [BBD85, Exposé VII, §3] is for Kähler K3 surfaces. Here we assume all surfaces to be algebraic so we have omitted this in the statement above. See also Corollary 11.2 in [BPvdV84] □

Proposition 3.2.11. *For an admissible subgroup $\mathbb{K}$ of $\mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ the period map*

$$j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$$

is an open immersion.

Proof. Note first that the algebraic space $\mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K},\mathbb{C}}$ is a scheme as we have an open immersion $i_{\mathbb{K}}\colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}} \hookrightarrow \mathcal{F}_{2d,\mathbb{K}}$ into a scheme (cf. Theorem 1.5.17). Further, by Proposition 3.2.5 the map $j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ is étale hence it is open. We have to show that it is an immersion. As both schemes are reduced it is enough to show that the morphism is injective on $\mathbb{C}$-valued points.

Suppose that $(X_i, \lambda_i, \alpha_i) \in \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K},\mathbb{C}}(\mathbb{C})$ for $i = 1, 2$ are two points such that

$$j_{d,\mathbb{K},\mathbb{C}}\big((X_1, \lambda_1, \alpha_1)\big) = j_{d,\mathbb{K},\mathbb{C}}\big((X_2, \lambda_2, \alpha_2)\big)$$

in $Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}$. Let $\tilde{\alpha}_i$ be two representatives of the classes α_i and let

$$a_i \colon H^2_B(X_i, \mathbb{Z}(1)) \to L_0$$

be two isometries as in the definition of the map $j_{d,\mathbb{K},\mathbb{C}}$. Then we have that

$$[a_1 \circ h_{X_1} \circ a_1^{-1}, a_1 \circ \tilde{\alpha}_1]_{\mathbb{K}} = [a_2 \circ h_{X_2} \circ a_2^{-1}, a_2 \circ \tilde{\alpha}_2]_{\mathbb{K}}.$$

Hence there are two elements $q \in G(\mathbb{Q})$ and $\kappa \in \mathbb{K}$ such that we have an equality

$$\big(q \cdot (a_1 \circ h_{X_1} \circ a_1^{-1}), q \circ a_1 \circ \tilde{\alpha}_1 \circ \kappa\big) = \big(a_2 \circ h_{X_1} \circ a_2^{-1}, a_2 \circ \tilde{\alpha}_2\big). \tag{3.4}$$

From the equality between the second elements in (3.4) we see that $q = a_2 \circ \tilde{\alpha}_2 \circ \kappa \circ \tilde{\alpha}_1^{-1} \circ a_1^{-1}$. Hence it belongs to $\{g \in \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}}) \mid g(e_1 - df_1) = e_1 - df_1\}$, by the very definition of a full level $\mathbb{K}$-structure, and being in $G(\mathbb{Q})$ we conclude that $q \in \{g \in \mathrm{SO}(V_{2d})(\mathbb{Z}) \mid g(e_1 - df_1) = e_1 - df_1\}$. The equality between the first elements in (3.4) shows that

$$a_2^{-1} \circ q \circ a_1 \colon H^2_B(X_2, \mathbb{Z}(1)) \to H^2_B(X_1, \mathbb{Z}(1))$$

is a Hodge isometry, mapping the class of λ_2 to the class of λ_1 and preserving the level structures. By the global Torelli theorem for K3 surfaces one concludes that it comes from an isomorphism of the triples $(X_1, \lambda_1, \alpha_1)$ and $(X_2, \lambda_2, \alpha_2)$. Therefore the morphism $j_{d,\mathbb{K},\mathbb{C}}$ is an immersion. □

Remark 3.2.12. In general, the morphism $j_{d,\mathbb{K},\mathbb{C}} \colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}$ need not be injective. We cannot apply the arguments of the proof of Proposition 3.2.11 as we only get a Hodge isometry

$$a_2^{-1} \circ q \circ a_1 \colon P^2_B(X_2, \mathbb{Z}(1)) \to P^2_B(X_1, \mathbb{Z}(1)).$$

As not every such isometry is induced by an isometry between the cohomology groups $H^2_B(X_2, \mathbb{Z}(1))$ and $H^2_B(X_1, \mathbb{Z}(1))$, mapping $c_1(\lambda_2)$ to $c_1(\lambda_1)$, we cannot conclude that (X_1, λ_1) and (X_2, λ_2) are isomorphic.

3.3 Complex Multiplication for K3 Surfaces

Here we will prove that the field of definition of $j_{d,\mathbb{K},\mathbb{C}}$ is $\mathbb{Q}$. This is an analogue of Theorem 4.21 in [Del71, §4] concerning periods of abelian varieties. We will do this first by proving a variant of the main theorem of complex multiplication for abelian varieties [Mil90, Ch. I, Thm. 5.3] in the case of exceptional K3 surfaces and then applying a density result for those surfaces. We will carry out this strategy in Sections 3.3.4-3.3.9. Before that we give a short review of some results from class field theory and canonical models of Shimura varieties.

We begin by making the following notation which will be used from now on.

Let $X/\mathbb{C}$ be a non-singular projective variety and consider an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$. The conjugate X^σ of X by σ is given by the Cartesian diagram

$$\begin{tikzcd} X^\sigma \arrow[r, "\beta"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\mathrm{Spec}(\sigma)"] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

and for any $n \in \mathbb{Z}$ we will denote by

$$\sigma_{X,f} \colon H^i_{\mathrm{et}}(X, \mathbb{A}_f(n)) \to H^i_{\mathrm{et}}(X^\sigma, \mathbb{A}_f(n))$$

or simply σ_f, the morphism on étale cohomology induced by β.

For a non-singular projective surface X the morphism β induces a morphism

$$\beta^* \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X^\sigma)$$

and we will denote it by σ_{Pic}. Recall that we have a decomposition

$$H^2_{\mathrm{et}}(X, \mathbb{A}_f(1)) = A_{X,\mathbb{A}_f} \oplus T_{X,\mathbb{A}_f}$$

where $A_{X,\mathbb{A}_f} = c_1(\mathrm{Pic}(X)) \otimes \mathbb{A}_f \subset H^2_{\mathrm{et}}(X, \mathbb{A}_f(1))$ and similarly for étale cohomology

$$H^2_{et}(X^\sigma, \mathbb{A}_f(1)) = A_{X^\sigma,\mathbb{A}_f} \oplus T_{X^\sigma,\mathbb{A}_f}.$$

We have that

$$\sigma_{X,f} = \sigma_{\mathrm{Pic},\mathbb{A}_f} \oplus \sigma_{X,f}|_{T_{X,\mathbb{A}_f}}$$

where $\sigma_{\mathrm{Pic},\mathbb{A}_f}$ is the morphism sending $c_1(\lambda)$ to $c_1(\lambda^\sigma)$ for any $\lambda \in \mathrm{Pic}(X)$. In the sequel we shall use the notation $\sigma_{X,f}$ for the morphism $\sigma_{X,f}|_{T_{X\mathbb{A}_f}} : T_{X,\mathbb{A}_f} \to T_{X^\sigma,f}$.

3.3.1 Class Field Theory

Let E be a number field and denote by E^{ab} the maximal abelian extension of E. Class field theory provides us with a description of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$. There exists a surjective homomorphism

$$\mathrm{rec}_E\colon \mathbb{A}_E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$$

such that $E^\times$ is in its kernel and for every finite abelian extension L of E the following diagram

$$\begin{tikzcd} E^\times\backslash\mathbb{A}_E^\times \arrow[r, "\mathrm{rec}_E", "\text{onto}"'] \arrow[d] & \mathrm{Gal}(E^{\mathrm{ab}}/E) \arrow[d, "\sigma\mapsto\sigma|_L"] \\ E^\times\backslash\mathbb{A}_E^\times/\,\mathrm{Nm}_{L/E}(\mathbb{A}_L^\times) \arrow[r, "\mathrm{rec}_{L/E}", "\cong"'] & \mathrm{Gal}(L/E) \end{tikzcd}$$

is commutative. We refer to [Neu86, Ch. 3] and [CF67, Ch. VII] for proofs and some properties of this homomorphism. If E is a quadratic imaginary field then rec_E gives rise to an isomorphism

$$\mathrm{rec}_E\colon E^\times\backslash\mathbb{A}_{E,f}^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E).$$

Indeed, one can use the description of the kernel of rec_E given in [CF67, Ch. VII, §5, 5.6] to see that.

To make notations easier when considering canonical models of Shimura varieties we define the map

$$\mathrm{art}_E\colon \mathbb{A}_E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$$

by $\mathrm{art}_E(\alpha) = \mathrm{rec}_E(\alpha)^{-1}$.

3.3.2 The Homomorphism r_h

Let V be a finite dimensional $\mathbb{Q}$-vector space and let

$$h\colon \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$$

be a $\mathbb{Q}$-HS on V. Let $T \subset \mathrm{GL}(V)$ be a $\mathbb{Q}$-torus and suppose that the homomorphism h factorizes thought $T_\mathbb{R}$. Then the same holds for the cocharacter μ_h (cf. Section 3.1.1) and we have that

$$\mu_h\colon \mathbb{G}_{m,\mathbb{C}} \to T_\mathbb{C}$$

is defined over $\bar{\mathbb{Q}}$. Let $E(h)$ be the field of definition of μ_h i.e., the reflex field of the pair (T, h). It is a number field. Composing $\mu_h\colon \mathbb{G}_{m,E(h)} \to T_{E(h)}$ with he norm morphism we obtain a homomorphism

$$r(T,h)\colon \mathrm{Res}_{E(h)/\mathbb{Q}}(\mathbb{G}_{m,E(h)}) \to T.$$

Explicitly, for an element $e \in E(h)^\times = \mathrm{Res}_{E(h)/\mathbb{Q}}(\mathbb{G}_{m,E(h)})(\mathbb{Q})$ we have

$$r(T,h)(e) = \prod_{\rho\colon E(h)\to\bar{\mathbb{Q}}} \rho(\mu_h(e))$$

where the sum runs over all different embeddings $\rho\colon E(h) \to \bar{\mathbb{Q}}$.

Definition 3.3.1. With notations as above define the homomorphism $r_h\colon \mathbb{A}^\times_{E(h)} \to T(\mathbb{A}_f)$ as being the composition

$$r_h\colon \mathbb{A}^\times_{E(h)} = \mathrm{Res}_{E(h)/\mathbb{Q}}(\mathbb{G}_{m,E(h)})(\mathbb{A}) \xrightarrow{\quad r(T,h) \quad} T(\mathbb{A}) \xrightarrow{\mathrm{proj}} T(\mathbb{A}_f).$$

We see that if $a \in \mathbb{A}^\times_{E(h)}$ and $a = (a_\infty, a_f) \in (E(h) \otimes \mathbb{R}) \oplus \mathbb{A}^\times_{E(h),f}$, then one has

$$r_h(a) = \prod_{\rho\colon E(h)\to\bar{\mathbb{Q}}} \rho(\mu_h(a_f)).$$

3.3.3 The Canonical Model of $Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$

Let $\mathbb{K}$ be a compact open subgroup of $G(\mathbb{A}_f)$. The canonical model $Sh_{\mathbb{K}}(G, \Omega^\pm)$ of $Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$ is scheme over $\mathbb{Q}$ (which is the reflex field of the Shimura datum $(G, \Omega^\pm)$) such that:

(i) one has an isomorphism $Sh_{\mathbb{K}}(G, \Omega^\pm) \otimes_{\mathbb{Q}} \mathbb{C} \to Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$;

(ii) $\mathrm{Aut}(\mathbb{C})$ acts on $Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$ via the isomorphism given by (i) as follows: For every special pair (T, x) of $(G, \Omega^\pm)$ one has that

$$\sigma[x, a]_{\mathbb{K}} = [x, r_x(s)a]_{\mathbb{K}}$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C}/E(x))$ and $s \in \mathbb{A}^\times_{E(x)}$ such that $\mathrm{art}_{E(x)}(s) = \sigma|_{E(x)^{\mathrm{ab}}}$. Here the morphism $r_x\colon \mathbb{A}^\times_{E(x)} \to T(\mathbb{A}_f)$ is the one associated to the pair (T, x) as in Definition 3.3.1.

These two properties determine the scheme $Sh_{\mathbb{K}}(G, \Omega^\pm)$ uniquely up to a unique isomorphism. For details concerning canonical models of Shimura varieties and their properties we refer to [Moo98, §2].

3.3.4 The Main Theorem of Complex Multiplication for Exceptional K3 Surfaces. Statement

Let X be an exceptional K3 surface of CM-type (E_X, ϵ_X) over $\mathbb{C}$. As in the case of abelian varieties with complex multiplication we are interested in a relation between the various cohomology groups of X and its conjugate X^σ by an automorphism σ of $\mathbb{C}$. In this section we will state the main results of complex multiplication for exceptional K3 surfaces. To make notations easier we will denote by

$$E := \epsilon_X(E_X) \subset \mathbb{C}$$

the reflex field of $\mathrm{Hg}(X)$.

Recall that the Hodge structure homomorphism $h_X : \mathbb{S} \to \mathrm{SO}(P_B^2(X, \mathbb{R}(1)))$ factorizes

$$h_X : \mathbb{S} \to \mathrm{Hg}(X)_\mathbb{R} \subset \mathrm{SO}(T_{X,\mathbb{R}}) \hookrightarrow \mathrm{SO}(P_B^2(X, \mathbb{R}(1))).$$

Let $\mu_X : \mathbb{G}_{m,E} \to \mathrm{Hg}(X)_E$ be the corresponding cocharacter and let

$$r_X : \mathbb{A}_E^\times \to \mathrm{Hg}(X)(\mathbb{A}_f) \subset \mathrm{SO}(T_{X,\mathbb{Q}})(\mathbb{A}_f)$$

be the morphism associated to $(\mathrm{Hg}(X), h_X)$ as in Definition 3.3.1.

Lemma 3.3.2. *Suppose given an exceptional K3 surface X of CM-type (E_X, ϵ_X). If $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$, then X^σ is an exceptional K3 surface and the reflex field of* $\mathrm{Hg}(X^\sigma)$ *is E.*

One can give a proof of the lemma using a 'a Hodge cycle is an absolute Hodge cycle' argument. We will give a proof in Section 3.3.7 using abelian surfaces.

Let X be an exceptional K3 surface of CM-type (E_X, ϵ_X) and let $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$. Then by a *$E_X$-linear isometry* $\eta : T_{X,\mathbb{Q}} \to T_{X^\sigma,\mathbb{Q}}$ we shall mean an isometry η such that

$$\eta(e \cdot t) = (\epsilon_{X^\sigma}^{-1} \circ \epsilon_X)(e) \cdot \eta(t)$$

for every $t \in T_{X,\mathbb{Q}}$ and $e \in E_X$.

Theorem 3.3.3 (Complex multiplication for exceptional K3 surfaces). *Let X be an exceptional K3 surface of CM-type (E_X, ϵ_X). Let $E = e_X(E_X) \subset \mathbb{C}$ be its reflex field and let $\sigma \in$* $\mathrm{Aut}(\mathbb{C}/E)$. *Then for any idèle $s \in \mathbb{A}_E^\times$ with* $\mathrm{art}_E(s) = \sigma|_{E^{\mathrm{ab}}}$ *there is a unique E_X-linear isomorphism of polarized $\mathbb{Q}$-HS*

$$\eta_X : T_{X,\mathbb{Q}} \to T_{X^\sigma,\mathbb{Q}}$$

such that $\eta_{X,f}(r_X(s)t) = \sigma_{X,f}(t)$ for every $t \in T_{X,\mathbb{A}_f}$.

If such η_X exists, then it is necessarily unique. Indeed, the condition imposed on $\eta_{X,f}$ determines it uniquely and hence η is also determined uniquely via the natural isomorphism $T_{X,\mathbb{A}_f} \cong T_{X,\mathbb{Q}} \otimes \mathbb{A}_f$.

We will give a proof of the theorem in Section 3.3.7. We will use first a geometric construction due to Shioda and Inose to reduce the problem to a similar statement for abelian surfaces. Then we will show that the corresponding statement for abelian surfaces follows from the main theorem of complex multiplication for abelian varieties. We present these results in the next two sections.

Remark 3.3.4. We wonder if one could give a 'direct' proof of Theorem 3.3.3 similar to the proof of Theorem 11.2 in [Mil04] using, for instance, arguments of the type 'a Hodge cycle is an absolute Hodge cycle' on a K3 surface. This can be done in the case (X, λ) is defined over an intermediate field $E \subset K \subset \mathbb{C}$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$.

3.3.5 The Results of Shioda and Inose

We shall describe a geometrical way for constructing exceptional K3 surfaces with given transcendental lattice using product abelian surfaces. We will follow the exposition of Shioda and Inose in their paper [SI77] with some notational differences.

Let $A = C_1 \times C_2$ be a product of two elliptic curves over $\mathbb{C}$ and let Y be the Kummer surface associated to A. Let $\pi\colon \tilde{A} \to A$ be the the blowing up of the 2-torsion of A and let $\iota\colon \tilde{A} \to Y$ be the morphism of degree 2 as in Example 1.1.4. Then we have a diagram

$$\begin{tikzcd} & \tilde{A} \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\ A & & Y. \end{tikzcd}$$

One has morphisms induced on Betti cohomology with $\mathbb{Z}$-coefficients and hence on the corresponding transcendental lattices

$$\pi^*\colon T_A \to T_{\tilde{A}} \quad \text{and} \quad \iota^*\colon T_Y \to T_{\tilde{A}}.$$

We know that π^* is an isomorphism of polarized $\mathbb{Z}$-HS and ι^* is an isomorphism of $\mathbb{Z}$-HS multiplying the intersection form by 2 i.e., $(\iota^* x, \iota^* y)_{\tilde{A}} = 2(x, y)_Y$.

Let $\{u_i\}_{i=1}^4$, $\{v_j\}_{j=1}^4$ be the four 2-torsion points on C_1 and C_2 respectively. Denote by E_{ij} the sixteen non-singular rational curves on Y corresponding to the points (u_i, v_j) on A. In other words we have that $E_{ij} = \iota\big(\pi^{-1}(u_i, v_j)\big)$. Following the notations of [SI77] we denote by F_i and G_j the non-singular rational curves $\iota\big(\pi^{-1}(u_i \times C_2)\big)$ and $\iota\big(\pi^{-1}(C_1 \times v_j)\big)$ on Y.

Consider the divisor

$$D = E_{21} + 2F_2 + 3E_{23} + 4G_3 + 5E_{13} + 6F1 + 3E_{12} + 4E_{14} + 2G_4$$

on Y. By Lemma 1.1 in [SI77] the linear system $|D|$ gives a morphism $\Phi\colon Y \to \mathbb{P}^1$ of which D is a singular fiber, say $D = \Phi^{-1}(t_0)$ for some $t_0 \in \mathbb{P}^1$. We look further at two divisors

$$B_1 = F_3 + E_{31} + E_{32} \quad \text{and} \quad B_2 = F_4 + E_{41} + E_{42}$$

on Y. One can see that they do not meet D and their supports are connected. Hence we conclude that the image $\Phi(B_i)$ is a point t_i in $\mathbb{P}^1$ and B_i is contained in the singular fiber $\Phi^{-1}(t_i)$ for $i = 1, 2$ (see the figure on page 122 of [SI77] and the comments following it). Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the finite morphism of degree 2 branched only at t_1 and t_2 and consider the fiber product $Y \times_{\Phi,\mathbb{P}^1,f} \mathbb{P}^1$.

Lemma 3.3.5. *The surface $Y \times_{\mathbb{P}^1} \mathbb{P}^1$ has a minimal model X which is a K3 surface (hence it is unique).*

Proof. See Lemma 3.1 in [SI77]. □

The elliptic pencil $\Phi\colon Y \to \mathbb{P}^1$ on Y induces an elliptic pencil $\Psi\colon X \to \mathbb{P}^1$ on X ([SI77, §3, p. 124]). The morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ induces an involution of the surface $Y \times_{\Phi,\mathbb{P}^1,f} \mathbb{P}^1$. Therefore it induces an involutive birational transformation of X, hence by the minimality of a K3 surface an automorphism a of X. It has 8 fixed points $\{p_i\}_{i=1}^{8}$ and Y is the minimal model of the quotient surface X/a. For details see [Kod63, §8, pp. 585-586, 591-592, 600-602] and the remarks after the proof of Lemma 3.1 on page 125 in [SI77].

Let $\beta\colon \tilde{X} \to X$ be the blow-up of the 8 points p_i, $i = 1, \ldots, 8$, on X. Then the involution a on X induces an involution $\tilde{a}$ on $\tilde{X}$. If we denote the quotient morphism $\tilde{X} \to \tilde{X}/\tilde{a}$ by γ then we have the following commutative diagram

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl, "\gamma"'] \arrow[dr, "\beta"] & \\
Y = \tilde{X}/\tilde{a} \arrow[dr] & & X. \arrow[dl] \\
 & X/a &
\end{tikzcd}$$

The degree of the morphism γ is 2. The map β induces an isomorphism of polarized $\mathbb{Z}$-HS $\beta^*\colon T_X \to T_{\tilde{X}}$. The main result of Section 2 of [SI77] is that $\gamma^*\colon T_Y \to T_{\tilde{X}}$ is an isomorphism of $\mathbb{Z}$-HS such that $(\gamma^* x, \gamma^* y)_{\tilde{X}} = 2(x, y)_Y$. Putting the preceding two diagrams together we obtain

$$\begin{tikzcd}
 & \tilde{A} \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & & \tilde{X} \arrow[dl, "\gamma"'] \arrow[dr, "\beta"] & \\
A & & Y & & X.
\end{tikzcd} \tag{3.5}$$

Shioda and Inose describe the relation between the transcendental lattices of A and X using those morphisms.

Theorem 3.3.6 (Shioda-Inose). *With the notations as above one has that the morphism*

$$\phi\colon T_X \to T_A.$$

defined as $\phi = \pi^{*-1} \circ \iota^* \circ \gamma^{*-1} \circ \beta^*$ induces an isomorphism of polarized $\mathbb{Z}$-HS.

Proof. The main difficulty is to prove that the map γ^* is an isomorphism. We refer to the proof of Theorem 2 in [SI77]. Note that Shioda and Inose use homology groups and we use cohomology groups. But in our case all those groups are free and we obtain the result using duality. □

Remark 3.3.7. Note that a priori the whole construction depends on choosing a numbering of $A[2](\mathbb{C})$. We shall be interested in constructing exceptional K3 surfaces. As we will see below for these surfaces the choices involved change only the morphisms β and γ but not the surface X itself.

Remark 3.3.8. Note that by the comparison theorem between Betti and étale cohomology the map $\phi_f = \pi_f^{*-1} \circ \iota_f^* \circ \gamma_f^{*-1} \circ \beta_f^*$ induces an isomorphism

$$\phi_f\colon T_{X,\mathbb{A}_f} \to T_{A,\mathbb{A}_f}.$$

Indeed, we have that $\phi_f = \phi \otimes_{\mathbb{Z}} \mathbb{A}_f$ and we know that ϕ is an isomorphism.

In order to explain the construction in the proof of the main result of [SI77] we will follow their notations working with homology instead of cohomology. If X is a nonsingular projective surface over $\mathbb{C}$ we will denote by T_X^{hom} the homological transcendental lattice. In other words we define $T_X^{\mathrm{hom}} = (\mathrm{Pic}(X))^\perp \subset H_2(X, \mathbb{Z}(-1))$.

Let X be an exceptional K3 surface over $\mathbb{C}$. Denote by p_X the period on T_X^{hom} i.e., the linear functional, determined up to a constant by

$$p_X(t) = \int_t \omega_X$$

for $t \in T_X^{\mathrm{hom}}$ and ω_X a non-vanishing holomorphic 2-form on X. We say that a basis $\{y_1, y_2\}$ is oriented if the imaginary part of $p_X(y_1)/p_X(y_2)$ is positive.

Let $\{y_1, y_2\}$ be an oriented basis of T_X^{hom}. In it the bilinear form on T_X^{hom} is given by a matrix

$$Q = \begin{pmatrix} \langle y_1, y_1\rangle & \langle y_1, y_2\rangle \\ \langle y_2, y_1\rangle & \langle y_2, y_2\rangle \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \tag{3.6}$$

for some $a, b, c \in \mathbb{Z}$ with $a, c > 0$ and $\Delta = b^2 - 4ac < 0$. Then we have that $E_X = \mathrm{End}_{\mathrm{HS}}(T_{X,\mathbb{Q}}) = \mathrm{End}_{\mathrm{HS}}(T_{X,\mathbb{Q}}^{\mathrm{hom}})$ is isomorphic to $\mathbb{Q}(\sqrt{\Delta}) \subset \mathbb{C}$. In our notations from Section 3.3.4 we have that $e_X(E) = E = \mathbb{Q}(\sqrt{\Delta})$.

Let $\tau_1 = (-b+\sqrt{\Delta})/2a$ and $\tau_2 = (b+\sqrt{\Delta})/2$ and consider the elliptic curve $C_i = \mathbb{C}/\Lambda_{\tau_i}$ where $\Lambda_{\tau_i} = \mathbb{Z} + \mathbb{Z}\tau_i$. These elliptic curves are isogenous and have complex multiplication by E.

Theorem 3.3.9 (Shioda-Inose). *Let X be an exceptional K3 surface over $\mathbb{C}$ and consider the product abelian surface $A = C_1 \times C_2$ where C_i for $i = 1,2$ are the CM elliptic curves defined above. Then the K3 surface X_A constructed in Theorem 3.3.6 is isomorphic to X.*

Proof. We refer to the proof of [SI77, Thm. 4]. The idea is to compare the lattices T_X^{hom} and $T_{X_A}^{\mathrm{hom}}$. Using Theorem 3.3.6 one sees that those two lattices are isometric. A result of Piatetskij-Shapiro and Shafarevich says that an exceptional K3 surface is uniquely determined by its transcendental lattice (see [PSS72, §8] and also the remarks made in [SM74]). Hence one concludes that X is isomorphic to X_A. □

Remark 3.3.10. Note that if X is an exceptional K3 surface then the construction described in Theorem 3.3.9 is independent of the numbering of $A[2](\mathbb{C})$. Indeed, starting with a numbering of $A[2](\mathbb{C})$ one can constructs an exceptional K3 surface X_1 and an isomorphism of polarized $\mathbb{Z}$-HS $\phi_1 \colon T_{X_1} \to T_A$. Starting with a different numbering and different f one constructs an exceptional K3 surface X_2 with an isomorphism of polarized $\mathbb{Z}$-HS $\phi_2 \colon T_{X_2} \to T_A$. Hence T_{X_1} and T_{X_2} are isometric and by the result of Piatetskij-Shapiro and Shafarevich X_1 and X_2 are isomorphic. Note that the morphisms involved in the construction might change.

3.3.6 Complex Multiplication for Product Abelian Surfaces

Let A be a complex abelian surface of CM type (E,Φ) and denote by E^* its reflex field. Let σ be an element of $\mathrm{Aut}(\mathbb{C}/E^*)$. The main theorem of complex multiplication gives a relation between Betti and étale cohomology of A and A^σ. We will need this in a special case. Before stating the result we introduce some notations.

Let $E \subset \mathbb{C}$ be a quadratic imaginary field and let C_1 and C_2 be two elliptic curves with CM by E. One should keep in mind here the data of Theorem 3.3.9. Then C_i is of CM-type $E \subset \mathbb{C}$. Let A be the product abelian surface $C_1 \times C_2$. Then the reflex field of the torus $\mathrm{MT}(A)$ is $E \subset \mathbb{C}$ (see [Shi98, Ch. IV, §18.7]). Consider the transcendental space $T_{A,\mathbb{Q}}$ and define

$$E_A := \mathrm{End}_{\mathrm{HS}}(T_{A,\mathbb{Q}}).$$

Then E_A is a quadratic imaginary field. The reflex field of the torus $\mathrm{MT}(T_{A,\mathbb{Q}})$ is $E \subset \mathbb{C}$. On the other hand if $\epsilon_A \colon E_A \to \mathrm{End}_{\mathbb{C}}(H^{2,0}(A)) \cong \mathbb{C}$ denote the action of E_A on the space of holomorphic two-forms on A, then just like in the case of K3 surfaces the field $\epsilon_A(E_A) \subset \mathbb{C}$ is the reflex field of $\mathrm{MT}(T_{A,\mathbb{Q}})$. Hence we have an isomorphism $\epsilon_A \colon E_A \to E$.

We have natural isomorphisms of cohomology groups

$$H^1_B(A,\mathbb{Z}) \cong H^1_B(C_1,\mathbb{Z}) \oplus H^1_B(C_2,\mathbb{Z}) \quad \text{and} \quad H^1_{\mathrm{et}}(A,\hat{\mathbb{Z}}) \cong H^1_{\mathrm{et}}(C_1,\hat{\mathbb{Z}}) \oplus H^1_{\mathrm{et}}(C_2,\hat{\mathbb{Z}}). \quad (3.7)$$

If $h\colon \mathbb{S} \to \mathrm{GL}(H^1_B(A,\mathbb{R}))$ and $h_i\colon \mathbb{S} \to \mathrm{GL}(H^1_B(C_i,\mathbb{R}))$, for $i=1,2$ are the corresponding homomorphisms defining the three $\mathbb{Z}$-HS, then we have that $h = h_1 \oplus h_2$. Hence we have that $\mu_h = \mu_{h_1} \oplus \mu_{h_2}$ where μ_h and μ_{h_i} are the cocharacters defined in Section 3.1.1. Further we know that

$$H^2_B(A,\mathbb{Z}(1)) \cong \big(\wedge^2 H^1_B(A,\mathbb{Z})\big) \otimes \mathbb{Z}(1) \quad \text{and} \quad H^2_{\mathrm{et}}(A,\hat{\mathbb{Z}}(1)) \cong \big(\wedge^2 H^1_{\mathrm{et}}(A,\hat{\mathbb{Z}})\big) \otimes \hat{\mathbb{Z}}(1) \tag{3.8}$$

and therefore combining (3.7) and (3.8) we have natural isomorphisms

$$\begin{gathered} H^2_B(A,\mathbb{Z}(1)) \cong \\ \Big(\wedge^2 H^1_B(C_1,\mathbb{Z}) \otimes \mathbb{Z}(1)\Big) \oplus \Big(\wedge^2 H^1_B(C_2,\mathbb{Z}) \otimes \mathbb{Z}(1)\Big) \oplus \Big(H^1_B(C_1,\mathbb{Z}) \otimes H^1_B(C_2,\mathbb{Z}) \otimes \mathbb{Z}(1)\Big) \end{gathered} \tag{3.9}$$

and

$$\begin{gathered} H^2_{\mathrm{et}}(A,\hat{\mathbb{Z}}(1)) \cong \\ \Big(\wedge^2 H^1_{\mathrm{et}}(C_1,\hat{\mathbb{Z}}) \otimes \hat{\mathbb{Z}}(1)\Big) \oplus \Big(\wedge^2 H^1_{\mathrm{et}}(C_2,\hat{\mathbb{Z}}) \otimes \hat{\mathbb{Z}}(1)\Big) \oplus \Big(H^1_{\mathrm{et}}(C_1,\hat{\mathbb{Z}}) \otimes H^1_{\mathrm{et}}(C_2,\hat{\mathbb{Z}}) \otimes \hat{\mathbb{Z}}(1)\Big). \end{gathered} \tag{3.10}$$

The spaces $\big(\wedge^2 H^1_B(C_1,\mathbb{Q})\big) \otimes \mathbb{Q}(1)$ and $\big(\wedge^2 H^1_B(C_1,\mathbb{Q})\big) \otimes \mathbb{Q}(1)$ (respectively with $\mathbb{A}_f$-coefficients) consist of algebraic classes. Hence for the homomorphism

$$h_A\colon \mathbb{S} \to \mathrm{GL}(H^2_B(A,\mathbb{R}(1)))$$

giving the $\mathbb{Z}$-HS on $H^2_B(A,\mathbb{Z}(1))$ we have

$$h_A = (\wedge^2 h) \otimes h_{\mathbb{Z}(1)} = \big(\wedge^2 h_1 \otimes h_{\mathbb{Z}(1)}\big) \oplus \big(\wedge^2 h_2 \otimes h_{\mathbb{Z}(1)}\big) \oplus \big(h_1 \otimes h_2 \otimes h_{\mathbb{Z}(1)}\big). \tag{3.11}$$

Then for the corresponding cocharacters one has

$$\mu_A = (\wedge^2 \mu_h) \otimes \mu_{\mathbb{Z}(1)} = \big(\wedge^2 \mu_1 \otimes \mu_{\mathbb{Z}(1)}\big) \oplus \big(\wedge^2 mu_2 \otimes \mu_{\mathbb{Z}(1)}\big) \oplus \big(\mu_1 \otimes \mu_2 \otimes \mu_{\mathbb{Z}(1)}\big). \tag{3.12}$$

As we explained in Section 3.1.1 the homomorphism h_A and the cocharacter μ_A factor trough $\mathrm{SO}(T_{A,\mathbb{Q}})$. The Mumford-Tate group $\mathrm{MT}\big(H^2_B(A,\mathbb{Q}(1))\big)$ is a torus and we have homomorphisms of algebraic groups

$$h_A\colon \mathbb{S} \to \mathrm{MT}\big(H^2_B(A,\mathbb{Q}(1))\big)_{\mathbb{R}} \subset \mathrm{SO}(T_{A,\mathbb{R}})$$

and

$$\mu_A\colon \mathbb{G}_{m,\mathbb{C}} \to \mathrm{MT}\big(H^2_B(A,\mathbb{Q}(1))\big)_{\mathbb{C}} \subset \mathrm{SO}(T_{A,\mathbb{C}}).$$

We have that $\mathrm{MT}(H^2_B(A,\mathbb{Q}(1))) = \mathrm{MT}(T_{X,\mathbb{Q}})$. The field of definition of μ_A is $E \subset \mathbb{C}$. Let

$$r_A\colon \mathbb{A}^\times_{E,f} \to \mathrm{MT}(T_{X,\mathbb{Q}})(\mathbb{A}_f) \subset \mathrm{SO}(T_{A,\mathbb{Q}})(\mathbb{A}_f)$$

be the morphism associated to $(\mathrm{MT}(T_{X,\mathbb{Q}}), h_A)$ as in Definition 3.3.1.

Let $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$. Then by a *$E_A$-linear isometry* $\eta\colon T_{A,\mathbb{Q}} \to T_{A^\sigma,\mathbb{Q}}$ we shall mean an isometry η such that

$$\eta(e \cdot t) = (\epsilon_{A^\sigma}^{-1} \circ \epsilon_A)(e) \cdot \eta(t)$$

for every $t \in T_{A,\mathbb{Q}}$ and $e \in E_X$. Note that this definition is correct as the reflex fields of $\mathrm{MT}(A)$ and $\mathrm{MT}(A^\sigma)$ are $E \subset \mathbb{C}$.

Proposition 3.3.11. *Let $A = C_1 \times C_2$ be a product of two elliptic curves with CM by a quadratic imaginary field E. Let σ be in $\mathrm{Aut}(\mathbb{C}/E)$ and let $s \in \mathbb{A}_E^\times$ be an idèle such that $\mathrm{art}_E(s) = \sigma|_{E^{\mathrm{ab}}}$. Then there exists an isogeny $\eta\colon A^\sigma \to A$ such that for the isometry $\eta_f^*\colon T_{A,\mathbb{A}_f} \to T_{A^\sigma,\mathbb{A}_f}$ induced by η acting on étale cohomology we have that $\eta_f^*(r_A(s)t) = \sigma_f(t)$ for every $t \in T_{A,\mathbb{A}_f}$.*

Proof. This formulation is in the spirit of [Mil04, Ch. 11, Thm. 11.2]. By Theorem 11.2 in *loc. cit.* we can find two isogenies $\eta_i\colon C_i^\sigma \to C_i$ for $i = 1, 2$ such that for the maps

$$\eta_{i,f}^*\colon H^1_{\mathrm{et}}(C_i, \mathbb{Q}) \to H^1_{\mathrm{et}}(C_i^\sigma, \mathbb{Q})$$

we have that $\eta_{i,f}^*(r_i(s)t) = \sigma_{C_i,f}(t)$ for every $t \in H^1_{\mathrm{et}}(C_i, \mathbb{A}_f)$, for $i = 1, 2$. Here

$$r_i\colon \mathbb{A}_E^\times \to \mathrm{MT}(C_i)(\mathbb{A}_f) \hookrightarrow \mathrm{GL}(H^1_B(C_i, \mathbb{Q}))(\mathbb{A}_f)$$

is the homomorphism associated to $(\mathrm{MT}(C_i), h_i)$.

Let $\eta\colon A^\sigma \to A$ be the product isogeny (η_1, η_2). It defines an E_A-linear isometry $\eta_{\mathbb{Q}}^*\colon T_{A,\mathbb{Q}} \to T_{A^\sigma,\mathbb{Q}}$. Using the decompositions

$$H^2_{\mathrm{et}}(A, \mathbb{A}_f(1)) = T_{A,\mathbb{A}_f} \oplus A_{A,\mathbb{A}_f} \quad \text{and} \quad H^2_{\mathrm{et}}(A^\sigma, \mathbb{A}_f(1)) = T_{A^\sigma,\mathbb{A}_f} \oplus A_{A^\sigma,\mathbb{A}_f}$$

we see that $\eta_f^*|_{A_{A,\mathbb{A}_f}}\colon A_{A,\mathbb{A}_f} \to A_{A^\sigma,\mathbb{A}_f}$ sends a class $c_1(\lambda)$ for $\lambda \in \mathrm{Pic}(A)$ to $c_1(\lambda^\sigma)$. Further, using the natural isomorphisms (3.7), (3.8), (3.9) and (3.10) we see that for

$$\eta_f^*\colon T_{A,\mathbb{A}_f} \to T_{A^\sigma,\mathbb{A}_f}$$

we have that $\eta_f^*(r(s)t) = \sigma_f(t)$. Here $r\colon \mathbb{A}_E^\times \to \mathrm{MT}(T_{X,\mathbb{Q}})(\mathbb{A}_f)$ is the morphism obtained as in Definition 3.3.1 using the cocharacter $\big(\wedge^2(\mu_1 \oplus \mu_2)\big) \otimes \mu_{\mathbb{Q}(1)}$ which is exactly μ_A. So we have that $r = r_A$ which finishes the proof. $\square$

3.3.7 Proof of the Main Theorem of Complex Multiplication for Exceptional K3 Surfaces

In this section we will give a proof of the claims announced in §3.3.4. We begin by putting together the results of the previous two sections.

Let X be an exceptional K3 surface over $\mathbb{C}$ of CM-type (E_X, ϵ_X). Let

$$E = \epsilon_X(E_X) = \mathbb{Q}(\sqrt{\Delta})$$

be the quadratic imaginary field defined by the discriminant of the form (3.6) in Section 3.3.5. Let A be the product abelian surface as in Theorem 3.3.9 associated to X. Let us further set $E_A = \mathrm{End}_{\mathrm{HS}}(T_{A,\mathbb{Q}})$. The two fields E_X and E_A are isomorphic as abstract fields. With the notations of the previous section (E_A, ϵ_A) is the reflex field of $\mathrm{MT}(T_{A,\mathbb{Q}})$. We have an isomorphism of polarized $\mathbb{Z}$-HS $\phi\colon T_X \to T_A$. We also look at the corresponding isomorphisms

$$\phi_{\mathbb{Q}}\colon T_{X,\mathbb{Q}} \to T_{A,\mathbb{Q}} \quad \text{and} \quad \phi_f\colon T_{X,\mathbb{A}_f} \to T_{A,\mathbb{A}_f}$$

induced by the actions of π, ι, γ and β on Betti cohomology with $\mathbb{Q}$ coefficients and on étale cohomology. We have that $\phi_{\mathbb{Q}} = \phi \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\phi_f = \phi \otimes_{\mathbb{Z}} \mathbb{A}_f$.

The morphism $\phi_{\mathbb{Q}}$ gives an isomorphism

$$\phi_{\mathbb{Q}}^{\mathrm{ad}}\colon E_X = \mathrm{End}_{\mathrm{HS}}(T_{X,\mathbb{Q}}) \to \mathrm{End}_{\mathrm{HS}}(T_{A,\mathbb{Q}}) = E_A.$$

We have further the two inclusions $\epsilon_X\colon E_X \to \mathrm{End}_{\mathbb{C}}(H^{2,0}(X)) \cong \mathbb{C}$ and $\epsilon_A\colon E_A \to \mathrm{End}_{\mathbb{C}}(H^{2,0}(A)) \cong \mathbb{C}$. The map $\phi_{\mathbb{Q}}$ is defined as a composition of algebraic morphisms and hence we have a commutative diagram

$$\begin{tikzcd} E_X \arrow[r, "\phi_{\mathbb{Q}}^{\mathrm{ad}}"] \arrow[d, "\epsilon_X"'] & E_A \arrow[d, "\epsilon_A"] \\ \mathrm{End}_{\mathbb{C}}(H^{2,0}(X)) \arrow[r, "(\pi_{DR}^{*-1}\circ\iota_{DR}^{*}\circ\gamma_{DR}^{*-1}\circ\beta_{DR}^{*})^{\mathrm{ad}}"] & \mathrm{End}_{\mathbb{C}}(H^{2,0}(A)). \end{tikzcd} \tag{3.13}$$

In other words $\phi_{\mathbb{Q}}^{\mathrm{ad}}$ gives an isomorphism of the CM-types (E_X, ϵ_X) and (E_A, ϵ_A). Therefore, with these identifications, $\phi_{\mathbb{Q}}$ commutes with the action of E on the two vector spaces $T_{X,\mathbb{Q}}$ and $T_{A,\mathbb{Q}}$ via the isomorphisms $e_X^{-1}\colon E \to E_X$ and $\epsilon_A^{-1}\colon E \to E_A$. Similarly ϕ_f is an $\mathbb{A}_{E,f}$-equivariant isomorphism via these actions. Further, via the isomorphism $\phi_{\mathbb{Q}}$, one can identify the cocharacters μ_X and μ_A and thus also the morphism r_X and r_A. Taking all these remarks in to account we see that for any idèle $s \in \mathbb{A}_E^{\times}$ the following diagram

$$\begin{tikzcd} T_{A,\mathbb{A}_f} \arrow[r, "r_A(s)"] \arrow[d, "\phi_f"] & T_{A,\mathbb{A}_f} \arrow[d, "\phi_f"] \\ T_{X,\mathbb{A}_f} \arrow[r, "r_X(s)"] & T_{X,\mathbb{A}_f} \end{tikzcd} \tag{3.14}$$

is commutative.

Let σ be an element of $\mathrm{Aut}(\mathbb{C}/E)$. Making a base change $\mathrm{Spec}(\sigma)\colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$ of Diagram (3.5) we obtain a diagram

$$\begin{tikzcd} & \tilde{A}^\sigma \arrow[dl, "\pi^\sigma"'] \arrow[dr, "\iota^\sigma"] & & \tilde{X}^\sigma \arrow[dl, "\gamma^\sigma"'] \arrow[dr, "\beta^\sigma"] & \\ A^\sigma & & Y^\sigma & & X^\sigma. \end{tikzcd} \tag{3.15}$$

Denote by $\phi^\sigma\colon T_{X^\sigma} \to T_{A^\sigma}$ the isomorphism of polarized $\mathbb{Z}$-HS, defined as in Theorem 3.3.6, by $(\pi^\sigma)^{*-1} \circ (\iota^\sigma)^* \circ (\gamma^\sigma)^{*-1} \circ (\beta^\sigma)^*$. Consider the isomorphisms induced on Betti cohomology with $\mathbb{Q}$ coefficients and étale cohomology

$$\phi^\sigma_{\mathbb{Q}}\colon T_{X^\sigma,\mathbb{Q}} \to T_{A^\sigma,\mathbb{Q}} \quad \text{and} \quad \phi^\sigma_f\colon T_{X^\sigma,\mathbb{A}_f} \to T_{A^\sigma,\mathbb{A}_f}$$

These isomorphism are defined by algebraic morphisms and hence just as above we conclude that $\phi^\sigma_{\mathbb{Q}}$ defines an isomorphism of the reflex fields $(E_{X^\sigma}, \epsilon_{X^\sigma})$ and $(E_{A^\sigma}, \epsilon_{A^\sigma})$.

Proof of Lemma 3.3.2. For an element $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ the reflex fields of $\mathrm{Hg}(T_{A,\mathbb{Q}})$ and $\mathrm{Hg}(T_{A^\sigma,\mathbb{Q}})$ are E. Hence we have an isomorphism $e_{A^\sigma} \circ (\phi^\sigma_{\mathbb{Q}})^{\mathrm{ad}}\colon E_{X^\sigma} \to E$ and therefore the reflex field of $\mathrm{Hg}(X^\sigma)$ is $E \subset \mathbb{C}$. □

Before giving the proof of Theorem 3.3.3 we shall make a final remark. The map ϕ_f is defined as a composition of the algebraic maps $\pi, \iota, \gamma, \beta$ and their inverses acting on étale cohomology. The isomorphism ϕ^σ_f is defined in the same way using the conjugates $\pi^\sigma, \iota^\sigma, \gamma^\sigma, \beta^\sigma$. Hence we see that we have the following commutative diagram of étale transcendental spaces:

$$\begin{tikzcd} T_{X,\mathbb{A}_f} \arrow[r, "\sigma_{X,f}"] \arrow[d, "\phi_f"'] & T_{X^\sigma,\mathbb{A}_f} \arrow[d, "\phi^\sigma_f"'] \\ T_{A,\mathbb{A}_f} \arrow[r, "\sigma_{A,f}"] & T_{A^\sigma,\mathbb{A}_f}. \end{tikzcd} \tag{3.16}$$

Proof of Theorem 3.3.3. Let $s \in \mathbb{A}_E^\times$ be an idèle such that $\mathrm{art}_E(s) = \sigma|_{E^{\mathrm{ab}}}$. By Proposition 3.3.11 we have an isogeny

$$\eta\colon A^\sigma \to A$$

such that for the induced isomorphism $\eta^*_f\colon T_{A,\mathbb{A}_f} \to T_{A^\sigma,\mathbb{A}_f}$ on étale cohomology with $\mathbb{A}_f$-coefficients we have $\eta^*_f(r_A(s)t) = \sigma_{A,f}(t)$ for every $t \in T_{A,\mathbb{A}_f}$.

Using the isomorphisms of $\mathbb{Q}$-HS $\phi_{\mathbb{Q}}$ and $\phi^\sigma_{\mathbb{Q}}$ we obtain an isomorphism of $\mathbb{Q}$-HS

$$\eta_X\colon T_{X,\mathbb{Q}} \to T_{X^\sigma,\mathbb{Q}}$$

defined as $\eta_X = \phi^\sigma_{\mathbb{Q}} \circ \eta^*_{\mathbb{Q}} \circ \phi_{\mathbb{Q}}^{-1}$. In other words we define η_X by completing the diagram

$$\begin{tikzcd} T_{X,\mathbb{Q}} \arrow[r, dashed, "\eta_X"] \arrow[d, "\phi_{\mathbb{Q}}"] & T_{X^\sigma,\mathbb{Q}} \arrow[d, "\phi^\sigma_{\mathbb{Q}}"] \\ T_{A,\mathbb{Q}} \arrow[r, "\eta^*_{\mathbb{Q}}"] & T_{A^\sigma,\mathbb{Q}} \end{tikzcd} \tag{3.17}$$

where $\eta_{\mathbb{Q}}^*$ is the $\mathbb{Q}$-HS morphism induced by η on Betti cohomology. Note that from the remarks made above the isomorphism η_X is E_X-equivariant. We get an $\mathbb{A}_{E,f}$-linear isomorphism $\eta_{X,f}\colon T_{X,\mathbb{A}_f} \to T_{X^\sigma,\mathbb{A}_f}$ by tensoring η_X with $\mathbb{A}_f$. By the commutativity Diagrams (3.14) and (3.16), and using the fact that $\phi_f = \phi_{\mathbb{Q}} \otimes \mathbb{A}_f$ and $\phi_f^\sigma = \phi_{\mathbb{Q}}^\sigma \otimes \mathbb{A}_f$ we see that the following diagram is commutative:

$$\begin{tikzcd}
T_{A,\mathbb{A}_f} \arrow[dr, "r_A(s)"] \arrow[drr, "\sigma_{A,f}"] \arrow[dd, "\phi_f"] & & \\
& T_{A,\mathbb{A}_f} \arrow[r, "\eta_f^*"] \arrow[dd, "\phi_f"] & T_{A^\sigma,\mathbb{A}_f} \arrow[dd, "\phi_f^\sigma"] \\
T_{X,\mathbb{A}_f} \arrow[dr, "r_X(s)"] \arrow[drr, "\sigma_{X,f}"] & & \\
& T_{X,\mathbb{A}_f} \arrow[r, "\eta_{X,f}"] & T_{X^\sigma,\mathbb{A}_f}.
\end{tikzcd} \tag{3.18}$$

Hence we have a $\mathbb{Q}$-HS isomorphism $\eta_X\colon T_{X,\mathbb{Q}} \to T_{X^\sigma,\mathbb{Q}}$ such that $\eta_{X,f}(r_X(s)t) = \sigma_{X,f}(t)$ for every $t \in T_{X,\mathbb{A}_f}$. □

Remark 3.3.12. Note that the morphisms η_X and $\eta_{X,f}$ are induced by a cycle in $X^\sigma \times X$. Indeed, if $\Gamma_\eta \subset A^\sigma \times A$ is the graph of η, then the isomorphisms η_X and $\eta_{X,f}$ are given by the cycle

$$Z = \big((\beta^\sigma, \beta) \circ (\gamma^\sigma, \gamma)^{-1} \circ (\iota^\sigma, \iota) \circ (\pi^\sigma, \pi)^{-1}\big)(\Gamma_\eta) \subset X^\sigma \times X$$

as in [Kle95, §3].

3.3.8 Some Special Points on $Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$

Let $d \in \mathbb{N}$ and let $\mathbb{K} \subset \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ be a subgroup of finite index such that $\mathbb{K} \subset \mathbb{K}_n$ for some $n \geq 3$. In order to carry out our strategy for proving that the morphism $j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$ is defined over $\mathbb{Q}$ we need to find enough special points on $Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$ for which we can control the Galois action.

Proposition 3.3.13. *Let $E \subset \mathbb{C}$ be a quadratic imaginary field. Then the set of special points $[x, a]_{\mathbb{K}} \in Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}(\mathbb{C})$ with reflex field E is dense for the Zariski topology.*

Proof. Let C be an elliptic curve over $\mathbb{C}$ with CM by E and consider the product abelian surface $A = C \times C$. Let P be a point of infinite order in C and consider the divisor

$$D_1 = P \times C + C \times P$$

on A. As $D_1 = pr_1^*P + pr_2^*P$, where $pr_i\colon A \to C$ is the projection morphism onto the i-th factor, we have that it is an ample divisor on A. Its self-intersection number $(D_1, D_1)_A$ is 2.

Let X be the Kummer surface associated to A. Then X is an exceptional K3 surface and the reflex field of $\mathrm{MT}(X)$ is exactly E. Let $\pi\colon \tilde{A} \to A$ be the blowing-up of $A[2]$ and $\iota\colon \tilde{A} \to X$ be the morphism of degree 2 (cf. Example 1.1.4). Then the line bundle

$$\mathcal{L} := \mathcal{O}_X\big(\iota(\pi^*(D_1))\big)$$

defines a quasi-polarization on X and one easily computes that $(\mathcal{L}, \mathcal{L})_X = 2$. Hence $\mathcal{L}$ is primitive.

Let $P_B^2(X, \mathbb{Z}(1))$ be the primitive Betti cohomology group with respect to $c_1(\mathcal{L})$. Fix an isometry $a\colon P_B^2(X, \mathbb{Z}(1)) \to L_2$. Then we have a point $x := a \circ h_X \circ a^{-1}$ in $\Omega^\pm$. The Mumford-Tate group of the $\mathbb{Q}$-HS x induced on V_2 is $a^{\mathrm{ad}}(\mathrm{MT}(X))$ and hence its reflex field is E. By the strong approximation theorem the orbit $G(\mathbb{Q}) \cdot x$ is dense in $\Omega^\pm$ hence the set of points $\{[x, a]_{\mathbb{K}} \mid a \in G(\mathbb{A}_f)\}$ is dense in

$$Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}} = G(\mathbb{Q})\backslash\Omega^\pm \times G(\mathbb{A}_f)/\mathbb{K}.$$

□

Remark 3.3.14. Similar density results appear in various papers on the Torelli theorem for K3 surfaces. The difference with our situation is that in those papers one mainly works with the full period domain of dimension 20. We also mention Lemma 7.1.2 in [And96b] which almost gives the result we need.

Corollary 3.3.15. *Let $d \in \mathbb{N}$ and let $E \subset \mathbb{C}$ be a quadratic imaginary field. The set of points $x \in \mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ corresponding to exceptional K3 surfaces X of CM-type (E_X, ϵ_X) such that $\epsilon_X(E_X) = E$ is dense for the Zariski topology in $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$.*

Proof. We have an étale morphism $j_{d,\mathbb{K},\mathbb{C}}\colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$. According to the preceding proposition the set of points $[x, a]_{\mathbb{K}} \in Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}(\mathbb{C})$ with reflex field E is dense in $Sh_{\mathbb{K}}(G, \Omega^\pm)_{\mathbb{C}}$. Therefore the preimage of this set under $j_{d,\mathbb{K},\mathbb{C}}$ in $\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ is also dense. It consists exactly of the exceptional K3 surfaces X of CM-type (E_X, ϵ_X) such that $\epsilon_X(E_X) = E$, with a polarization of degree $2d$ and a level $\mathbb{K}$-structure. □

3.3.9 Complex Multiplication for K3 Surfaces

We will prove here that the field of definition of the morphism $j_{d,\mathbb{K},\mathbb{C}}$ is $\mathbb{Q}$. To do that we will use the density result for exceptional polarized K3 surfaces and Theorem 3.3.3 which establishes a relation between the Galois action on such a surface and its periods.

Theorem 3.3.16. *Let $d \in \mathbb{N}$ and let $\mathbb{K} \subset \mathrm{SO}(V_{2d})(\hat{\mathbb{Z}})$ be a subgroup of finite index such that $\mathbb{K} \subset \mathbb{K}_n$ for some $n \geq 3$. Then the morphism $j_{d,\mathbb{K},\mathbb{C}} \colon \mathcal{F}_{2d,\mathbb{K},\mathbb{C}} \to Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}$ is defined over $\mathbb{Q}$. In other words one has an étale morphism*

$$j_{d,\mathbb{K}} \colon \mathcal{F}_{2d,\mathbb{K},\mathbb{Q}} \to Sh_{\mathbb{K}}(G, \Omega^{\pm})$$

such that $j_{d,\mathbb{K}} \otimes \mathbb{C} = j_{d,\mathbb{K},\mathbb{C}}$.

Proof. We will divide the proof into three steps.
Step 1. Let $x \in \mathcal{F}_{2d,\mathbb{K},\mathbb{C}}$ be a point corresponding to an exceptional K3 surface with CM by E. We will show first that for every $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ we have $j_{d,\mathbb{K},\mathbb{C}}(\sigma(x)) = \sigma(j_{d,\mathbb{K},\mathbb{C}}(x))$.

Let $E \subset \mathbb{C}$ be a quadratic imaginary field, let (X, λ, α) be a polarized exceptional K3 surface of CM-type (E_X, ϵ_X) with a level $\mathbb{K}$-structure α such that $e_X(E_X) = E$. Then we have the triple

$$\big((P_B^2(X, \mathbb{Z}(1)), h_X), \psi_X, \tilde{\alpha}\mathbb{K}\big)$$

where $\tilde{\alpha}$ is a representative of the class α.

Let $\tilde{\alpha}$ be a representative of the class α and let $a_X \colon P_B^2(X, \mathbb{Z}(1)) \to L_{2d}$ be an isometry as in the definition of the morphism $j_{d,\mathbb{K},\mathbb{C}}$ (see Step 1 of the proof of Proposition 3.2.5). Via this isometry we have an inclusion of algebraic groups $a_X^{\mathrm{ad}} \colon \mathrm{MT}(X) \hookrightarrow G$. By the modular description of $Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}}$ and the definition of $j_{d,\mathbb{K},\mathbb{C}}$ we have

$$j_{d,\mathbb{K},\mathbb{C}}\big((X, \lambda, \alpha)\big) = [a_X \circ h_X \circ a_X^{-1},\ a_X \circ \tilde{\alpha}]_{\mathbb{K}}.$$

We have that $P_B^2(X, \mathbb{Q}(1)) = T_{X,\mathbb{Q}} \oplus A_X^{\lambda}$ where

$$A_X^{\lambda} := c_1(\lambda)^{\perp} \subset A_{X,\mathbb{Q}} \subset H_B^2(X, \mathbb{Q}(1))$$

as polarized $\mathbb{Q}$-HS. By definition the action of E_X on A_X^{λ} is trivial. The same decomposition holds for étale cohomology $P_{\mathrm{et}}^2(X, \mathbb{A}_f(1)) = T_{X,\mathbb{A}_f} \oplus A_{X,\mathbb{A}_f}^{\lambda}$ where

$$A_{X,\mathbb{A}_f}^{\lambda} := c_1(\lambda)^{\perp} \subset A_{X,\mathbb{A}_f} \subset H_{\mathrm{et}}^2(X, \mathbb{A}_f(1)).$$

By the comparison theorem between Betti and étale cohomology these two decompositions are compatible with tensoring with $\mathbb{A}_f$.

Let $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ and consider the conjugate $\sigma(X, \lambda, \alpha) = (X^{\sigma}, \lambda^{\sigma}, \alpha^{\sigma})$ defined by the base change $\mathrm{Spec}(\sigma) \colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$. The surface X^{σ} is also exceptional and for it we have similar decompositions of $P_B^2(X^{\sigma}, \mathbb{Q}(1))$ and $P_{\mathrm{et}}^2(X^{\sigma}, \mathbb{A}_f(1))$. The base change morphism $\mathrm{Spec}(\sigma)$ induces a morphism $\sigma_{\mathrm{Pic}} \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X^{\sigma})$, preserving the intersection forms on both spaces and sending λ to λ^{σ}. Hence we obtain an isomorphism of polarized $\mathbb{Q}$-HS $\sigma_{\mathrm{Pic},\mathbb{Q}} \colon A_{X,\mathbb{Q}}^{\lambda} \to A_{X^{\sigma},\mathbb{Q}}^{\lambda}$ and an isomorphism $\sigma_{\mathrm{Pic},f} \colon : A_{X,\mathbb{A}_f}^{\lambda} \to A_{X^{\sigma},\mathbb{A}_f}^{\lambda}$ such that $\sigma_{\mathrm{Pic},\mathbb{Q}} \otimes \mathbb{A}_f = \sigma_{\mathrm{Pic},f}$. Note that by its very definition $\sigma_{\mathrm{Pic},f}$ is nothing else but σ_f restricted to A_{X,A_f}^{λ}.

By Theorem 3.3.3 there exists an isomorphism of polarized $\mathbb{Q}$-HS $\eta_X\colon T_{X,\mathbb{Q}} \to T_{X^\sigma,\mathbb{Q}}$ such that $\eta_X^{-1} \circ \sigma_f(t) = r_X(s)(t)$ for every $t \in T_{X,\mathbb{A}_f}$. Define the isomorphism of polarized $\mathbb{Q}$-HS

$$\eta = \eta_X \oplus \sigma_{\mathrm{Pic},\mathbb{Q}}\colon P_B^2(X,\mathbb{Q}(1)) \to P_B^2(X^\sigma,\mathbb{Q}(1)).$$

Then we obtain an isomorphism of primitive étale cohomology

$$\eta_f := \eta \otimes \mathbb{A}_f\colon P_{\mathrm{et}}^2(X,\mathbb{A}_f(1)) \to P_{\mathrm{et}}^2(X^\sigma,\mathbb{A}_f(1))$$

for which, by the remarks made above, we have $\eta^{-1} \circ \sigma_f(t) = r_X(s)(t)$ for every $t \in P_{\mathrm{et}}^2(X,\mathbb{A}_f(1))$.

Consider the isometry

$$a_{X^\sigma} = a_X \circ \eta^{-1}\colon P_B^2(X^\sigma,\mathbb{Q}(1)) \to V_{2d}.$$

We have that $\tilde{\alpha}^\sigma = \sigma_f \circ \tilde{\alpha}$ and we will see that $a_{X^\sigma} \circ \tilde{\alpha}^\sigma \in G(\mathbb{A}_f)$ i.e., that we can use the marking a_{X^σ} to compute the periods of $(X^\sigma,\lambda^\sigma,\alpha^\sigma)$. We compute

$$\begin{aligned} a_{X^\sigma} \circ \tilde{\alpha}^\sigma &= a_X \circ \eta^{-1} \circ \tilde{\alpha} = a_X \circ r_X(s) \circ \tilde{\alpha} \\ &= a_X \circ a_X^{-1} \circ r_{(a_X \circ h_X \circ a_X^{-1})}(s) \circ a_X \circ \tilde{\alpha} \\ &= r_{(a_X \circ h_X \circ a_X^{-1})}(s) \circ a_X \circ \tilde{\alpha} \end{aligned} \tag{3.19}$$

and hence $a_{X^\sigma} \circ \tilde{\alpha}^\sigma$ belongs to $G(\mathbb{A}_f)$. Here the morphism

$$r_{(a_X \circ h_X \circ a_X^{-1})}\colon \mathbb{A}_{E,f}^\times \to a_X^{\mathrm{ad}}(\mathrm{MT}(X))(\mathbb{A}_f)$$

is the homomorphism associated to the special pair $(a_X^{\mathrm{ad}}(\mathrm{MT}(X)), a_X \circ h_X \circ a_X^{-1})$ of $(G,\Omega^\pm)$ as in Definition 3.3.1. From the modular description of $Sh_{\mathbb{K}}(G,\Omega^\pm)_{\mathbb{C}}(\mathbb{C})$ given in Proposition 3.2.3 we see that

$$j_{d,\mathbb{K},\mathbb{C}}\big((X^\sigma,\lambda^\sigma,\alpha^\sigma)\big) = [a_{X^\sigma} \circ h_{X^\sigma} \circ a_{X^\sigma}^{-1},\ a_{X^\sigma} \circ \tilde{\alpha}^\sigma]_{\mathbb{K}}.$$

Then using (3.19) we compute

$$\begin{aligned} j_{d,\mathbb{K},\mathbb{C}}\big(\sigma(X,\lambda,\alpha)\big) &= [a_{X^\sigma} \circ h_{X^\sigma} \circ a_{X^\sigma}^{-1},\ a_{X^\sigma} \circ \tilde{\alpha}^\sigma]_{\mathbb{K}} \\ &= [a_X \circ (\eta^{-1} \circ h_{X^\sigma} \circ \eta) \circ a_X^{-1},\ a_X \circ \alpha^{-1} \circ \sigma_f \circ \tilde{\alpha}]_{\mathbb{K}} \\ &= [a_X \circ h_X \circ a_X^{-1},\ r_{(a_X \circ h_X \circ a_X^{-1})}(s) \circ a_X \circ \tilde{\alpha}]_{\mathbb{K}} \\ &= \sigma\big(j_{d,\mathbb{K},\mathbb{C}}((X,\lambda,\alpha))\big). \end{aligned}$$

Hence the action of $\mathrm{Aut}(\mathbb{C}/E)$ on the point (X,λ,α) commutes with $j_{d,\mathbb{K},\mathbb{C}}$.

Step 2. For a fixed quadratic field $E \subset \mathbb{C}$ the set of polarized exceptional K3 surfaces X of CM-type (E_X,ϵ_X) with $\epsilon_X(E_X) = E$ is Zariski dense in $\mathcal{F}_{d,\mathbb{K},\mathbb{C}}$ (see Corollary

3.3.15). According to Step 1 the action of $\mathrm{Aut}(\mathbb{C}/E)$ commutes with $j_{d,\mathbb{K},\mathbb{C}}$ on that set. Hence it commutes with $j_{d,\mathbb{K},\mathbb{C}}$ and by Proposition 13.1 in [Mil04] we conclude that $j_{d,\mathbb{K},\mathbb{C}}$ is defined over E.

Step 3. Choose two quadratic imaginary fields $E_1 \subset \mathbb{C}$ and $E_2 \subset \mathbb{C}$ such that $E_1 \cap E_2 = \mathbb{Q}$. By the previous step we know that $j_{d,\mathbb{K},\mathbb{C}}$ is defined over E_1 and E_2. Hence it is defined over their intersection $\mathbb{Q}$ which is the reflex field of $Sh(G,\Omega^{\pm})_{\mathbb{C}}$. □

As a corollary of the preceding theorem one can obtain an analogue of the main theorem for complex multiplication for abelian varieties [Mil04, Ch. 11, Thm. 11.2] for CM K3 surfaces. Let X be a K3 surface over $\mathbb{C}$ with CM by $E = \epsilon_X(E_X)$. We denote by $r_X\colon \mathbb{A}_E^{\times} \to \mathrm{MT}(X)(\mathbb{A}_f)$ the morphism associated to the pair $(\mathrm{MT}(X), h_X)$.

Corollary 3.3.17 (Complex multiplication for K3 surfaces). *Let X be a polarized K3 surface over $\mathbb{C}$ of degree $2d$ with CM by a field E. Recall that we consider E embedded in to $\mathbb{C}$ via ϵ_X. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ and an idèle $s \in \mathbb{A}_E^{\times}$ such that $\mathrm{art}_E(s) = \sigma|_{E^{\mathrm{ab}}}$ there is an isomorphism of polarized $\mathbb{Q}$-HS*

$$\eta\colon P_B^2(X,\mathbb{Q}(1)) \to P_B^2(X^{\sigma},\mathbb{Q}(1))$$

such that for $\eta_f = \eta \otimes \mathbb{A}_f\colon P_{\mathrm{et}}^2(X,\mathbb{A}_f(1)) \to P_{\mathrm{et}}^2(X^{\sigma},\mathbb{A}_f(1))$ we have $\eta_f(r_X(s)t) = \sigma_f(t)$ for every $t \in P_{\mathrm{et}}^2(X,\mathbb{A}_f(1))$.

Proof. Let λ be the polarization on X. We can introduce a level 3-structure α on (X,λ) so that $(X,\lambda,\alpha) \in \mathcal{F}_{2d,3,\mathbb{C}}(\mathbb{C})$. Let

$$a_X\colon P_B^2(X,\mathbb{Z}(1)) \to L_{2d}$$

and

$$a_{X^{\sigma}}\colon P_B^2(X^{\sigma},\mathbb{Z}(1)) \to L_{2d}$$

be two markings as in the construction of the morphism $j_{d,\mathbb{K}_3,\mathbb{C}}$ (cf. Step 1 of the proof of Proposition 3.2.5). Then we have that

$$j_{d,\mathbb{K}_3,\mathbb{C}}\big((X,\lambda,\alpha)\big) = [a_X \circ h_X \circ a_X^{-1},\ a_X \circ \tilde{\alpha}]_{\mathbb{K}_3}$$

and

$$j_{d,\mathbb{K}_3,\mathbb{C}}\big((X,\lambda,\alpha)\big) = [a_{X^{\sigma}} \circ h_{X^{\sigma}} \circ a_{X^{\sigma}}^{-1},\ a_{X^{\sigma}} \circ \tilde{\alpha}^{\sigma}]_{\mathbb{K}_3}.$$

Using Theorem 3.3.16 and the definition of a canonical model (cf. Section 3.3.3) we see that

$$\big(q \cdot (a_X \circ h_X \circ a_X^{-1}), q a_X \circ r_X(s) \circ \tilde{\alpha}\big) = (a_{X^{\sigma}} \circ h_{X^{\sigma}} \circ a_{X^{\sigma}}^{-1},\ a_{X^{\sigma}} \circ \sigma_f \circ \tilde{\alpha}) \tag{3.20}$$

for some $q \in G(\mathbb{Q})$. Hence comparing the first terms in (3.20) we see that

$$\eta := a_{X^\sigma}^{-1} \circ q \circ a_X \colon : P_B^2(X, \mathbb{Q}(1)) \to P_B^2(X^\sigma, \mathbb{Q}(1))$$

defines an isometry of $\mathbb{Q}$-HS. Form the equality between the second terms we see that $qa_X \circ r_X(s) = a_{X^\sigma} \circ \sigma_f$ i.e., that $\eta_f \circ r_X(s) = \sigma_f$. □

Before stating our final result we will point out a difference between the approach to the theory of complex multiplication for abelian varieties given, for instance, in [Mil04, Ch. 10, 11 and 12] and the one for K3 surfaces given in this chapter. In the case of abelian varieties one first proves an analogue of Corollary 3.3.17 and then derives an analogue of Theorem 3.3.16 from it (cf. [Del71, §4]). Here we do the opposite as we do not see a way to prove directly Corollary 3.3.17. The reason is the following: For a K3 surface X with CM by E_X and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/E_X)$ one has little control over the transcendental lattice T_{X^σ} of X^σ, unless X is exceptional.

Remark 3.3.18. The statement in Corollary 3.3.17 can be given in a form not including any polarizations. With the notations as above for a K3 surface with CM by E one simply gets a $\mathbb{Q}$-HS isometry $\alpha \colon H_B^2(X, \mathbb{Q}(1)) \to H_B^2(X^\sigma, \mathbb{Q}(1))$ such that for the induced isomorphism $\eta_f \colon H^2_{\text{et}}(X, \mathbb{A}_f(1)) \to H^2_{\text{et}}(X^\sigma, \mathbb{A}_f(1))$ on étale cohomology one has that $\eta_f(r_X(s)t) = \sigma_f(t)$ for any $t \in H^2_{\text{et}}(X, \mathbb{A}_f(1))$.

Corollary 3.3.19. *Every complex K3 surface with CM can be defined over a number field which is an abelian extension of its Hodge endomorphism field.*

Proof. Let (X, λ) be a polarized K3 surface of degree $2d$ with CM by E_X and choose a full level 3-structure α on (X, λ). We have an open embedding

$$j_{d,\mathbb{K}_3^{\text{full}}} \colon \mathcal{F}_{2d,3,\mathbb{Q}}^{\text{full}} \hookrightarrow Sh_{\mathbb{K}_3^{\text{full}}}(G, \Omega^\pm)$$

of schemes over $\mathbb{Q}$. The point (X, λ, α) maps to a special point in $Sh_{\mathbb{K}_3^{\text{full}}}(G, \Omega^\pm)_{\mathbb{C}}$ which according to [Del71, §3, 3.15] can be defined over an abelian extension of its reflex field $e_X(E_X)$. Therefore X can be defined over an abelian extension of E_X. Note that one can give a description of the corresponding extension in terms of the reciprocity law as explained in *loc. cit.*. □

Remark 3.3.20. Let us mention that a similar result can be found in the literature. Shioda and Inose ([SI77, Thm. 6]) prove that any exceptional K3 surface can be defined over a number field. As we shall see below one can actually give such a number field explicitly and conclude that it is an abelian extension of the Hodge endomorphism field of the exceptional K3 surface. Piatetskij-Shapiro and Shafarevich prove, using the Torelli theorem for K3 surfaces, that every K3 surface with CM can be defined over a number field. We refer to Theorem 4 in [PSS75]

Example 3.3.21. Let X be an exceptional K3 surface with CM by E. Let C_1 and C_2 be the two elliptic curves from Theorem 3.3.9 and denote by j_1 and j_2 their j-invariants. By the theory of complex multiplication for elliptic curves we know that $K_1 = E(j_1, C_1[2])$ and $K_2 = E(j_2, C_2[2])$ are abelian extensions of E. We can see that all morphisms and surfaces involved in the construction described in Theorems 3.3.6 and 3.3.9 are defined over the composite K_1K_2. Hence X is defined over K_1K_2 which is an abelian extension of E.

3.3.10 Final Comments

Complex multiplication. Corollaries 3.3.17 and 3.3.19 are analogues to two of the main theorems of the theory of complex multiplication for abelian varieties ([Mil90, Ch. I, §5, Cor. 5.5]). Another important result of that theory is that every abelian variety with CM defined over a number field K has potentially good reduction at every prime ideal of K. We wonder if a similar result holds for K3 surfaces with CM.

Question. *Let K be a number field and suppose given a K3 surface with CM over K. Does X have potentially good reduction at every prime ideal of K?*

One could follow the line of thoughts of [Mil04, Ch. 10, Prop. 10.5]. In this way we can see that for a prime $\mathfrak{p}$ of K the inertia action of $I_{\mathfrak{p}}$ on $H^2_{\text{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)$ factorizes through a finite group. To finish "the proof" we need a Néron-Ogg-Shafarevich-type criterion for potentially good reduction of K3 surfaces. To our knowledge, in general, this is an open problem. Such a criterion exists for discrete valuation rings of characteristic zero. This follows form the degeneration result of Kulikov ([Kul77, Thm. II and Thm. 2.7] and [PP81]).

The period morphism. One knows that the period morphisms used in [BBD85] and [Fri84] are dominant. Further, the complement of their images are divisors. We will show here that the same holds for $j_{d,\mathbb{K},\mathbb{C}}$.

Recall that we have a decomposition

$$Sh_{\mathbb{K}}(G, \Omega^{\pm})_{\mathbb{C}} = \coprod_{[g]\in\mathcal{C}} \Gamma_{[g]}\backslash\Omega^{+},$$

where $\mathcal{C} := G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/\mathbb{K}$ and $\Gamma_{[g]} = G(\mathbb{Q})_+ \cap g\mathbb{K}g^{-1}$ for some representative g of $[g] \in \mathcal{C}$. Let X denote the geometric connected component of the canonical model $Sh_{\mathbb{K}}(G, \Omega^{\pm})$ corresponding to $\Gamma_{[1]}\backslash\Omega^{+}$. Denote by $E_X \subset \mathbb{C}$ its field of definition. It is an abelian extension of $\mathbb{Q}$ and one can see that

$$Sh_{\mathbb{K}}(G, \Omega^{\pm}) = \coprod_{\sigma\in\mathrm{Gal}(E_X/\mathbb{Q})} X^{\sigma}. \tag{3.21}$$

According to Propositions 7 and 8 in [BBD85, Exp. XIII, p. 150] we have that $j_{d,\mathbb{K},\mathbb{C}}(\mathcal{F}_{2d,\mathbb{K},\mathbb{C}}) \cap X$ is dense in X and its complement in X is a divisor. The points in the complement correspond to quasi-polarized K3 surfaces. Hence using Theorem 3.3.16 and (3.21) we conclude that $j_{d,\mathbb{K},\mathbb{C}}(\mathcal{F}_{2d,\mathbb{K},\mathbb{C}})$ is dense in $Sh_{\mathbb{K}}(G,\Omega^{\pm})_{\mathbb{C}}$ and its complement is a divisor.

Chapter 4

Kuga-Satake Morphisms

In [KS67] M. Kuga and I. Satake associate to every complex polarized K3 surface $(X, \mathcal{L})$ an abelian variety A using a transcendental construction involving the second primitive Betti cohomology group of X. This construction gives a relation between the Betti cohomology groups of X and of A. P. Deligne ([Del72]) shows, among other things, that a similar relation holds for étale cohomology groups and uses it to prove the Weil conjecture for K3 surfaces. In [And96a] Y. André studies the rationality properties of the Kuga-Satake construction, describing the motive of a K3 surface and computing the motivic Galois group.

We see that, using the Kuga-Satake construction, one can deduce some properties of K3 surfaces, mostly of motivic nature, from the corresponding properties of abelian varieties. Those results concern mainly K3 surfaces in characteristic zero as we possess a transcendental way for constructing Kuga-Satake varieties. We come across the following problem.

Question. *Is there a* geometric *Kuga-Satake construction?*

Of course, we have to explain what we mean by a 'geometric' construction. We are interested in any construction of a well-defined Kuga-Satake abelian variety, giving rise to the étale cohomology relation in [Del72, (6.6.1)] and in [And96a, Def. 4.5.1]. We also require that this construction can be carried out over any base field of characteristic $p \geq 0$, possibly excluding some finite number of primes p.

In this chapter we suggest a partial solution to the question taking up a zig-zag way. Our strategy is to give an interpretation of the Kuga-Satake construction in characteristic zero as a morphism between $\mathcal{F}_{2d,\mathbb{K},K}$ and $\mathcal{A}_{g,d',n,K}$ for some compact open subgroup $\mathbb{K} \subset G(\mathbb{A}_f)$ and a number field K. Then we extend this morphism over an open part of $\mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in K. We explain this in more detail below.

Here is the organization of the chapter. In the first few sections we give some background material which will be used in the sequel and we outline the Kuga-Satake construction. In Section 4.2.5, using the results of Chapter 3, we give two constructions of Kuga-Satake morphisms depending on some choices. For any $n \geq 3$ we construct a Kuga-Satake morphism

$$f^{ks}_{d,a,n,\mathbb{Q}} \colon \mathcal{F}_{2d,n^{\mathrm{sp}},\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}$$

mapping every primitively polarized complex K3 surface (X, λ, ν) with a spin level n-structure ν to its associated Kuga-Satake abelian variety A with a certain polarization of degree ${d'}^2$. Making some further choices we define a Kuga-Satake morphism

$$f^{ks}_{d,a,n,\gamma,E_n} \colon \mathcal{F}_{2d,n^{\mathrm{sp}},E_n} \to \mathcal{A}_{g,d',n,E_n}$$

over a finite abelian extension E_n of $\mathbb{Q}$ having the same property.

Section 4.3.1 is the core of the matter discussed here. Suppose given two numbers d, n with $n \geq 3$ and a smooth scheme U over a discrete valuation ring R, of mixed characteristic $(0, p)$. Denote by K the field of fractions of R. Assume further that p does not divide dn. We give sufficient conditions under which one can extend a morphism $f_K \colon U \otimes K \to \mathcal{A}_{g,d',n,K}$ over $\mathrm{Spec}(R)$. Using this result we show that the Kuga-Satake morphism $f^{ks}_{d,a,n,\gamma,E_n}$ extends over an open part of $\mathrm{Spec}(\mathcal{O}_{E_n})$, where $\mathcal{O}_{E_n}$ is the ring of integers in E_n. We end the chapter with some applications of the existence of Kuga-Satake morphisms in mixed characteristic to canonical lifts of ordinary K3 surfaces and abelian varieties.

4.1 Extension of Polarizations of Abelian Schemes

In this section we give some results on extension of polarizations of abelian schemes. We will use them to extend Kuga-Satake morphisms in positive characteristic. We fix a discrete valuation ring R with field of fractions K and residue field k.

Lemma 4.1.1. *Let A be an abelian scheme over a discrete valuation ring R and let λ_K be a polarization of the generic fiber A_K of A. Then λ_K extends uniquely to a polarization of A.*

Proof. By [FC90, Ch. 1, Prop. 2.7] λ_K extends uniquely to a homomorphism $\lambda \colon A \to A^t$ over R. It suffices to show that $2 \cdot \lambda$ is a polarization. But $2 \cdot \lambda = \varphi_{\mathcal{M}}$ where $\mathcal{M} = (\mathrm{id}_A, \lambda)^* \mathcal{P}_A$ and $\mathcal{P}_A$ is the Poincaré bundle on $A \times A^t$. We conclude by [Ray70, Cor. VIII 7] that $\mathcal{M}$ is relatively ample, hence $2 \cdot \lambda$ is a polarization. □

Lemma 4.1.2. *Suppose given a locally noetherian, regular scheme U and a dense open subscheme $V \subset U$ such that the codimension of $U \setminus V$ in U is at least 2. Let $A \to U$ be an abelian scheme and let λ_V be a polarization of $A_V \to V$. Then λ_V extends uniquely to a polarization λ of $A \to U$.*

Proof. Applying [FC90, Ch. 1, Prop. 2.7] as in the proof of the previous lemma we see that λ_V extends uniquely to an isogeny $\lambda \colon A \to A^t$ over U.

By assumption there is an étale covering $\pi_V\colon \tilde{V} \to V$ such that the pull-back $\lambda_{\tilde{V}}\colon A_{\tilde{V}} \to A^t_{\tilde{V}}$ of λ_V is equal to $\varphi_{\mathcal{L}_{\tilde{V}}}$ for an ample line bundle $\mathcal{L}_{\tilde{V}}$ on $A_{\tilde{V}}$. By the Zariski-Nagata purity theorem (see [Gro71, Exp. X, Cor. 3.3]), the morphism π_V extends to an étale covering $\pi\colon \tilde{U} \to U$. Let $j\colon \tilde{V} \to \tilde{U}$ be the inclusion. Then the sheaf $\mathcal{L} := j_*\mathcal{L}_{\tilde{V}}$ is a line bundle (cf. [FC90, Ch. V, Lemma 6.2]). The isogenies $\lambda_{\tilde{V}}$ and $\varphi_{\mathcal{L}_{\tilde{V}}}$ coincide so by the unicity part of [FC90, Ch. 1, Prop. 2.7] we see that $\lambda_{\tilde{U}} = \varphi_{\mathcal{L}}$.

To show that $\lambda_{\tilde{U}}$ is a polarization we apply Corollary VIII 7 of [Ray70] as in the proof of the preceding lemma. □

4.2 Kuga-Satake Morphisms Over Fields of Characteristic Zero

In the following sections we will recall the construction of Kuga-Satake abelian varieties associated to polarized K3 surfaces. In our exposition we will follow [Del72] and [And96a]. In Section 4.2.5 we will use these ideas and the results of Chapter 3 to define Kuga-Satake morphisms over number fields.

4.2.1 Clifford Groups

Clifford groups will play an essential rôle in the construction of the Kuga-Satake morphisms and in this section we will give a short review of the results we will use later on. For details we refer to [Lam73, Ch. V] and [Sch85, Ch. 9].

Let d be a natural number. For simplification we change the notations of Section 1.2.1 by setting (L, ψ) to be the lattice (L_{2d}, ψ_{2d}) and (V, ψ) to be the quadratic space $(L_{2d}, \psi_{2d}) \otimes \mathbb{Q}$.

Denote by G the algebraic group $\mathrm{SO}(V, \psi) \cong \mathrm{SO}(2, 19)$ over $\mathbb{Q}$ (cf Section 3.2.1). Further, following the notations of Example 1.5.4 we consider the *even Clifford algebra* $C^+(V)$ and the *Clifford group* $G_1 := \mathrm{CSpin}(V)$ of (V, ψ). Recall that one has a homomorphism of algebraic groups

$$\alpha\colon \mathrm{CSpin}(V) \to \mathrm{SO}(V, \psi) \tag{4.1}$$

defined by

$$\alpha(g) = (v \mapsto gvg^{-1})$$

which is called the *adjoint representation* of $\mathrm{CSpin}(V)$ on V. The kernel of the adjoint representation of $\mathrm{CSpin}(V)$ is $\mathbb{G}_m$ and one has a short exact sequence

$$1 \to \mathbb{G}_m \to G_1 \to G \to 1. \tag{4.2}$$

Then $G = G_1^{\mathrm{ad}}$ and we have that the center $Z(G_1)$ of G_1 is $\mathbb{G}_m$.

There is a canonical involution

$$\iota\colon C^+(L) \to C^+(L)$$

which acts trivially on the constants $\mathbb{G}_m \hookrightarrow G_1$. We define the *spinorial norm*

$$\mathrm{N}\colon G_1 \to \mathbb{G}_m \tag{4.3}$$

by setting

$$\mathrm{N}(g) = \iota(g)g.$$

It is a surjective homomorphism and we denote its kernel by $\mathrm{Spin}(V)$. The spinorial norm gives rise to a short exact sequence

$$1 \to \mathrm{Spin}(V) \to G_1 \to \mathbb{G}_m \to 1.$$

One has that $G_1^{\mathrm{der}} = \mathrm{Spin}(V)$ is the derived group of G and $\mathrm{N}\colon G_1 \to \mathbb{G}_m = G_1^{\mathrm{ab}}$ is the maximal abelian quotient of G_1. The group $\mathrm{Spin}(V)$ is simply connected.

Further, we dispose of an embedding $\mathrm{CSpin}(V) \hookrightarrow C^+(V)^*$ and left multiplication by elements of $\mathrm{CSpin}(V)$ on $C^+(V)$ gives an inclusion of algebraic groups

$$\beta\colon\ \mathrm{CSpin}(V) \hookrightarrow \mathrm{GL}(C^+(V)). \tag{4.4}$$

See [Del72, §3.2]. It is called the *spin representation* of $\mathrm{CSpin}(V)$ on $C^+(V)$.

4.2.2 Kuga-Satake Abelian Varieties Associated to Polarized K3 Surfaces

In this section we recall the construction of Kuga-Satake abelian varieties. We will follow closely [Del72] and [And96a].

Let $(X, \mathcal{L})$ be a primitively polarized complex K3 surface of degree $2d$. Fix a marking $m\colon H^2_B(X, \mathbb{Z}(1)) \to L_0$ such that $m(c_1(\mathcal{L})) = e_1 - df_1$ (cf. Section 1.2.1 and Remark 1.2.6). Then we obtain an isometry $m\colon P^2_B(X, \mathbb{Z}(1)) \to L$ and hence the homomorphism $h_X\colon \mathbb{S} \to \mathrm{SO}\big(P^2_B(X, \mathbb{Z}(1))\big)$ defines an element

$$h_m := m \circ h_X \circ m^{-1}\colon \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$$

of $\Omega^{\pm}$. There is a unique homomorphism

$$\tilde{h}_m\colon \mathbb{S} \to G_{1,\mathbb{R}}$$

such that $h_m = \alpha \circ \tilde{h}_m$, where $\alpha\colon G_1 \to G$ is the adjoint representation homomorphism (see [Del72, §4.2]). Let W denote the $\mathbb{Z}$-module $C^+(L)$. The composition of the homomorphism $\tilde{h}_m$ with the spin representation $\beta\colon G_1 \hookrightarrow \mathrm{GL}(W_{\mathbb{R}})$

$$\beta \circ \tilde{h}_m\colon \mathbb{S} \to \mathrm{GL}(W_{\mathbb{R}})$$

gives rise to a polarizable $\mathbb{Z}$-HS of type $\{(1,0),(0,1)\}$ on W. We refer to [Del72, Prop. 4.5] for a proof. Hence $\beta \circ \tilde{h}_m$ defines a complex abelian variety $A = A(L,h)$, given by the condition that $H^1_B(A,\mathbb{Z}) = W$ as $\mathbb{Z}$-HS. Its dimension is $g = 2^{19}$.

If we take a different marking $m' \colon H^2_B(X,\mathbb{Z}(1)) \to L_0$ with $m'(c_1(\lambda)) = e_1 - df_1$, then we have that $m' \circ m^{-1} \circ h_m(z) = h_{m'}(z) \circ m' \circ m^{-1}$ for all $z \in \mathbb{S}$. Therefore $C^+(m' \circ m) \colon W \to W$ defines an isomorphism between the $\mathbb{Z}$-HS on W induced by $\beta \circ \tilde{h}_m$ and $\beta \circ \tilde{h}_{m'}$. Hence we obtain an isomorphism between the abelian varieties associated to $(W, \beta \circ \tilde{h}_m)$ and $(W, \beta \circ \tilde{h}_{m'})$. Thus we see that the construction described above associates to a polarized K3 surface $(X,\mathcal{L})$ an abelian variety A, which does not depend on the choice of a marking m.

Definition 4.2.1. The abelian variety A is called the *Kuga-Satake abelian variety* associated to $(X,\mathcal{L})$.

We will see in Section 4.2.4 how to give explicitly polarizations of A.

Example 4.2.2. We shall describe explicitly how to obtain the Hodge structure on $C^+(V)$ in terms of the one on V. Choose an orthonormal basis (e_1,e_2) of $V_+ = V_\mathbb{R} \cap (V^{-1,1} \oplus V^{1,-1})$ and let $e_+ = e_1e_2$. Choose an orientation of (e_1,e_2) such that $e_1 - ie_2$ spans $V^{1,-1}$. Then multiplication by e_+

$$x \mapsto e_+x$$

defines a complex structure on $C^+(V_\mathbb{R})$ which corresponds to the morphism $\tilde{h} \colon \mathbb{S} \to \mathrm{GL}(C^+(V_\mathbb{R}))$ defined above. The Kuga-Satake abelian variety A associated to $(X,\mathcal{L})$ is exactly the complex torus $C^+(V_\mathbb{R})/C^+(L)$ where $C^+(V_\mathbb{R})$ is considered as a complex vector space via the complex structure given by multiplication by e_+. For further details we refer to the articles of Satake [Sat66], Kuga and Satake [KS67] and van Geemen [vG00].

Example 4.2.3. As before X will be a complex K3 surface. Instead of taking the orthogonal complement of an ample line bundle one can consider a subgroup $N \subset c_1(\mathrm{Pic}(X)) \subset H^2_B(X,\mathbb{Z}(1))$ and its complement

$$L_N = N^\perp \subset H^2_B(X,\mathbb{Z}(1))$$

with respect to the bilinear form ψ. Then L_N carries a natural polarized $\mathbb{Z}$-HS of type $\{(1,-1),(0,0),(-1,1)\}$ and one can consider again $C^+(L_N)$ and give it a polarized $\mathbb{Z}$-HS of type $\{(1,0),(0,1)\}$ as above. It gives rise to a complex abelian variety A_N associated to the pair (X,N). We refer to [Mor85, §4.1] for further comments.

For two subgroups $N \subset N' \subset \mathrm{NS}(X)$ with $d = \dim_\mathbb{Q}(N'_\mathbb{Q}/N_\mathbb{Q})$ one has that A_N is isogenous to a product of 2^d copies of $A_{N'}$. For a proof see [Mor85, §4.4].

Example 4.2.4. Let X be an exceptional K3 surface. Then the transcendental space $T_{X,\mathbb{Q}} = c_1(\mathrm{Pic}(X)_{\mathbb{Q}})^{\perp}$ is of dimension 2 over $\mathbb{Q}$. By the preceding remarks we conclude that A is isogenous to a product of 2^{19} copies of an elliptic curve E which has complex multiplication. See also [KS67, pp. 241-242].

Remark 4.2.5. Note that from the very construction of A we have that the Mumford-Tate group $\mathrm{MT}(A)$ is contained in G_1 viewed as a subgroup of $\mathrm{GL}(C^+(V))$ via the spin representation (4.4). Moreover, from the short exact sequence (4.2) we see that $\mathbb{G}_m = \ker(\alpha)$ is contained in $\mathrm{MT}(A)$ for weight reasons and hence we have an exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{MT}(A) \to \mathrm{MT}(X) \to 1.$$

We also conclude that $\mathrm{Hg}(A)$ is an extension of $\mathrm{Hg}(X)$ by $\mathbb{Z}/2\mathbb{Z}$.

4.2.3 Endomorphisms

Denote by C^+ the opposite ring $C^+(L)^{\mathrm{op}}$. It is non-canonically isomorphic to $C^+(L)$. Let $(X, \mathcal{L})$ be a primitively polarized K3 surface. Fix a marking $m\colon P^2_B(X, \mathbb{Z}(1)) \to L$ as in Section 4.2.2 and let $h_m := m \circ h_X \circ m^{-1} \colon \mathbb{S} \to G_{\mathbb{R}}$ be the homomorphism defining the $\mathbb{Z}$-HS on L. The right action of C^+ on $W := C^+(L)$ commutes with the morphism $\beta \circ \tilde{h}_m$ so the Kuga-Satake abelian variety A has complex multiplication by C^+ (cf. [And96a, §4.2] and [Del72, §3.3]). In other words there is an injection

$$\gamma\colon C^+ \hookrightarrow \mathrm{End}(A). \tag{4.5}$$

In fact one can see that there is an isomorphism of $\mathbb{Z}$-HS of type $\{(-1, 1), (0, 0), (1, -1)\}$

$$\phi_{\mathbb{Z}}\colon C^+(L)_{\mathrm{ad}} \to \mathrm{End}_{C^+}(W)$$

where $C^+(L)_{\mathrm{ad}}$ is the $\mathbb{Z}$-HS obtained from (L, h) using the tensor construction $C^+(\ )$.

4.2.4 Polarizations

Let $(X, \mathcal{L})$ be a complex K3 surface with a primitive polarization λ of degree $2d$ and let $m\colon H^2_B(X, \mathbb{Z}(1)) \to L_0$ be a marking such that $m(c_1(\mathcal{L})) = e_1 - df_1$. Let $h_m\colon \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$ be the $\mathbb{Z}$-HS induced on L by h_X and let let A be its associated Kuga-Satake abelian variety. We will show how to give explicitly a polarization of the $\mathbb{Z}$-HS W $(= C^+(L) = H^1_B(A, \mathbb{Z}))$.

Let $\iota\colon C^+(L) \to C^+(L)$ be the canonical involution of the even Clifford algebra. Fix a non-zero element $a \in C^+$ such that $\iota(a) = -a$. Then the skew-symmetric form

$$\varphi_a\colon W \otimes W \to \mathbb{Z}(-1) \tag{4.6}$$

given by

$$\varphi_a(x, y) = \operatorname{tr}(\iota(x)ya)$$

defines a polarization for the $\mathbb{Z}$-HS on W if and only if the symmetric bilinear form $i\varphi_a(x, \tilde{h}_m(i)y)$ is positive definite (here $i = \sqrt{-1}$). Lemma 4.3 in [Del72] (see also Example 4.2.6) guarantees the existence of an element $a \in C^+$ for which $\pm\varphi_a$ is a polarization.

Example 4.2.6. Let $e_1, \ldots, e_{21}$ be an orthogonal basis of (V, φ) such that $\psi(e_i, e_i) < 0$ for $i = 1, 2$. Let $m \neq 0$ be an integer such that $me_1e_2 \in C^+(L)$. One has that $\iota(me_1e_2) = -me_1e_2$ and if $h \in \Omega^\pm$, then either $\varphi_{me_1e_2}$ or $-\varphi_{me_1e_2}$ is a polarization for $\tilde{h}$. For a proof we refer to [vG00, Prop. 5.9].

Remark 4.2.7. Note that the degree of the polarization φ_a depends only on a and d and can be computed explicitly.

Remark 4.2.8. Let $a \in C^+$ be an element such that $\iota(a) = -a$ and, say φ_a is a polarization the $\mathbb{Z}$-HS on W induced by $\tilde{h}$. Then φ_a defines a polarization $\mu\colon A \to A^t$ which gives rise to a Rosati involution $\dagger$ on $\operatorname{End}^0(A) = \operatorname{End}(A) \otimes \mathbb{Q}$. One can see that the restriction of the Rosati involution to $C^+ \otimes \mathbb{Q} \hookrightarrow \operatorname{End}^0(A)$ (cf. (4.5)) is given by

$$f^\dagger = a^{-1}\iota(f)a$$

for all $f \in C^+(V)$. Hence $C^+(V)$ is stable under $\dagger$.

Remark 4.2.9. Note that we make some non-canonical choices to define a polarization on A. For instance, it is not clear if two different markings $m_i\colon H^2_B(X, \mathbb{Z}(1)) \to L_0$ for which $m_i(c_1(\lambda)) = e_1 - df_1$ give rise to two isomorphic *polarized* abelian varieties (A, μ_1) and (A, μ_2).

4.2.5 Kuga-Satake Morphisms over Fields of Characteristic Zero

Recall that we associated to every polarized complex K3 surface a complex abelian variety. We will explain here how to do this in families. Following the line of thoughts in [Del72] and [And96a] we define Kuga-Satake morphisms from the moduli spaces of polarized K3 surfaces with certain level structures to moduli stacks of polarized abelian varieties. We shall keep the notations from the previous sections.

Consider the Shimura datum $(G, \Omega^\pm)$ (cf. Section 3.2.2) and let $h_0\colon \mathbb{S} \to G_R$ be an element of $\Omega^\pm$. Let $\tilde{h}_0\colon \mathbb{S} \to G_{1,\mathbb{R}}$ be the unique homomorphism such that $h_0 = \alpha \circ \tilde{h}_0$ (cf. Section 4.2.2). Define $\Omega_1^\pm$ to be the $G_1(\mathbb{R})$-conjugacy class of $\tilde{h}_0$. The pair $(G_1, \Omega_1^\pm)$ defines a Shimura datum with reflex field $\mathbb{Q}$. We refer to [And96a, Appendix 1] for a proof.

The adjoint representation (4.1) defines a morphism of Shimura data

$$\alpha\colon (G_1, \Omega_1^{\pm}) \to (G, \Omega^{\pm})$$

in the following way: $\alpha_{gr}\colon G_1 \to G$ is the adjoint representation homomorphism and $\alpha_{HS}\colon \Omega_1^{\pm} \to \Omega^{\pm}$ the the morphism sending $\tilde{h}$ to $\alpha \circ \tilde{h}$. The morphism α_{HS} is well-defined as $h = g \circ \tilde{h}_0 \circ g^{-1}$ for some $g \in G_1(\mathbb{R})$ and hence $\alpha \circ \tilde{h} = \alpha(g) \circ h_0 \circ \alpha(g)^{-1} \in \Omega^{\pm}$. Moreover $\alpha_{HS}\colon \Omega_1^{\pm} \to \Omega^{\pm}$ is an analytic isomorphism ([Del72, §4.2] or [Mil92, Lemma 4.11]).

Fix a natural number $n \geq 3$. Let $\mathbb{K}^{\mathrm{sp}} \subset G_1(\mathbb{A}_f)$ be a subgroup of finite index in $\mathbb{K}_n^{\mathrm{sp}}$ and denote by $\mathbb{K}^{\mathrm{a}}$ the image $\alpha(\mathbb{K}^{\mathrm{sp}}) \subset G(\mathbb{A}_f)$ which is a subgroup of finite index in $\mathbb{K}_n^{\mathrm{a}}$ (cf. Example 1.5.4). Then one has a morphism of quasi-projective $\mathbb{Q}$-schemes

$$\alpha_{(\mathbb{K}^{\mathrm{sp}}, \mathbb{K}^{\mathrm{a}})}\colon Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1, \Omega_1^{\pm}) \to Sh_{\mathbb{K}^{\mathrm{a}}}(G, \Omega^{\pm}). \tag{4.7}$$

Consider the group $C = \mathbb{G}_m(\mathbb{Q}) \backslash \alpha^{-1}(\mathbb{K}^{\mathrm{a}}) / \mathbb{K}^{\mathrm{sp}}$. We have that

$$\begin{aligned} C &= \mathbb{G}_m(\mathbb{Q}) \backslash \alpha^{-1}(\mathbb{K}^{\mathrm{a}}) / \mathbb{K}^{\mathrm{sp}} = \mathbb{G}_m(\mathbb{Q}) \backslash \mathbb{G}_m(\mathbb{A}_f) \mathbb{K}^{\mathrm{sp}} / \mathbb{K}^{\mathrm{sp}} \\ &= \mathbb{G}_m(\mathbb{Q}) \backslash \mathbb{G}_m(\mathbb{Q}) \mathbb{G}_m(\hat{\mathbb{Z}}) \mathbb{K}^{\mathrm{sp}} / \mathbb{K}^{\mathrm{sp}} \cong \mathbb{G}_m(\hat{\mathbb{Z}}) / (\mathbb{G}_m(\hat{\mathbb{Z}}) \cap \mathbb{K}^{\mathrm{sp}}). \end{aligned} \tag{4.8}$$

The group C acts on $Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1, \Omega_1^{\pm})_{\mathbb{C}}$ via right multiplication. We have that $Z(G_1) = \mathbb{G}_m$ and $G = G_1/Z(G_1)$. Further, by Hilbert's Theorem 90, $H^1(k, \mathbb{G}_m) = 0$ for all fields of characteristic zero, hence we can apply Lemma 4.13 in [Mil92] and conclude that the morphism $\alpha_{(\mathbb{K}^{\mathrm{sp}}, \mathbb{K}^{\mathrm{a}})} \otimes \mathbb{C}$ is a Galois cover with a Galois group C. As C acts on $Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1, \Omega_1^{\pm})_{\mathbb{C}}$ via Hecke correspondences we see that these automorphisms are defined over $\mathbb{Q}$. Therefore the morphism (4.7) is a Galois cover with a Galois group C.

We will describe more explicitly the relation between these two Shimura varieties over $\mathbb{C}$. Consider the finite sets $\mathcal{C}_{G_1} := G_1(\mathbb{Q})_+ \backslash G_1(\mathbb{A}_f) / \mathbb{K}^{\mathrm{sp}}$ and $\mathcal{C}_G := G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f) / \mathbb{K}^{\mathrm{a}}$. The homomorphism α defines a surjective map of sets $\alpha\colon \mathcal{C}_{G_1} \to \mathcal{C}_G$ (cf. (4.2)). Note that C naturally acts on $\mathcal{C}_{G_1}$ from the right. With this action the map α makes $\mathcal{C}_{G_1}$ into a C-torsor over $\mathcal{C}_G$ (in the sense of sets); in other words, if $[g] \in \mathcal{C}_G$ and $g_1 \in G_1(\mathbb{A}_f)$ is an element with $\alpha([g_1]) = [g]$, then the map $c \to \alpha^{-1}([g])$ given by $u \mapsto [g_1 u]$ is a bijection.

One has that the decomposition of $Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1, \Omega_1^{\pm})_{\mathbb{C}}$ into connected components is

$$Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1, \Omega_1^{\pm})_{\mathbb{C}} = \coprod_{[g] \in \mathcal{C}_{G_1}} \Gamma'_{[g]} \backslash \Omega_1^{+}$$

where $\Gamma'_{[g]} = G_1(\mathbb{Q})_+ \cap g \mathbb{K}^{\mathrm{sp}} g^{-1}$, for some representative g of the class $[g]$ (see [Mil04, §5, Lemma 5.13]. Similarly, we have that

$$Sh_{\mathbb{K}^{\mathrm{a}}}(G, \Omega^{\pm})_{\mathbb{C}} = \coprod_{[g] \in \mathcal{C}_G} \Gamma_{[g]} \backslash \Omega^{+}$$

where $\Gamma_{[g]} = G(\mathbb{Q})_+ \cap g\mathbb{K}^{\mathrm{a}}g^{-1}$ for some representative g of $[g]$.

The morphism $\alpha_{(\mathbb{K}^{\mathrm{sp}},\mathbb{K}^{\mathrm{a}})}$ maps the connected component $\Gamma'_{[g]}\backslash\Omega_1^+$ to $\Gamma_{[\alpha(g)]}\backslash\Omega^+$ sending the class $[\tilde{h}]$ to the class $[h]$ (cf. §4.2.2). The restriction

$$\alpha_{(\mathbb{K}^{\mathrm{sp}},\mathbb{K}^{\mathrm{a}})} \colon \Gamma'_{[g]}\backslash\Omega_1^+ \to \Gamma_{[\alpha(g)]}\backslash\Omega^+ \tag{4.9}$$

is an isomorphism of complex quasi-projective varieties. Indeed, α maps $\Gamma'_{[g]}$ surjectively onto $\Gamma_{[\alpha(g)]}$ and as $-1 \notin \Gamma'_{[g]}$ (because $-1 \notin g\mathbb{K}_n^{\mathrm{sp}}g^{-1} \supset g\mathbb{K}^{\mathrm{sp}}g^{-1}$) one concludes from (1.8) that $\Gamma'_{[g]}$ is mapped isomorphically onto $\Gamma_{[\alpha(g)]}$. The morphism $\alpha_{HS} \colon \Omega_1^+ \to \Omega^+$ is an isomorphism so we see that (4.9) is an isomorphism as well. Further, we have that

$$\alpha^{-1}_{(\mathbb{K}^{\mathrm{sp}},\mathbb{K}^{\mathrm{a}})}(\Gamma_{[g]}\backslash\Omega^+) = \coprod_{u\in C} \Gamma'_{[g_1 u]}\backslash\Omega_1^+ \tag{4.10}$$

where $g_1 \in G(\mathbb{A}_f)$ with $\alpha(g_1) = g$.

Denote by W the $\mathbb{Z}$-module $C^+(L)$ and choose an element $a \in C^+$ such that $\iota(a) = -a$. Recall that for such an element we have defined a bilinear form $\varphi_a \colon W\otimes W \to \mathbb{Z}(-1)$ (see (4.6)). The image of G_1 under the spin representation $\beta \colon G_1 \hookrightarrow \mathrm{GL}(W_{\mathbb{Q}})$ is actually contained in $\mathrm{CSp}(W_{\mathbb{Q}}, \varphi_a)$. Indeed, for any element $\gamma \in G_1$ we have that

$$\begin{aligned} \varphi_a(\gamma x, \gamma y) &= \mathrm{tr}(\iota(\gamma x)\gamma y a) = \mathrm{tr}(\iota(x)\iota(\gamma)\gamma y a) \\ &= \mathrm{tr}\big(\iota(x)\mathrm{N}(\gamma) y a\big) = \mathrm{N}(\gamma)\mathrm{tr}(\iota(x) y a) \\ &= \mathrm{N}(\gamma)\varphi_a(x, y) \end{aligned}$$

hence $\beta(\gamma) \in \mathrm{CSp}(W_{\mathbb{Q}}, \varphi_a)$. Further, if the bilinear form φ_a defines a polarization for a Hodge structure $\beta \circ \tilde{h}_1$ on W, then it defines a polarization for all Hodge structures $\beta \circ \tilde{h}$ on W, for which $\tilde{h}$ belongs to the connected component of $\Omega_1^{\pm}$ of $\tilde{h}$. If φ_a is a polarization for those $\tilde{h}$ coming from the elements in Ω^+, then $-\varphi_a$ is a polarization for the $\tilde{h}$ coming from the elements in Ω^-.

Assumption 4.2.10. *We assume that $a \in C^+$ is such that $\iota(a) = -a$ and that φ_a or $-\varphi_a$ defines a polarization for the $\mathbb{Z}$-Hodge structures induced on W by $\beta \circ \tilde{h}$ for any $\tilde{h} \in \Omega_1^{\pm}$.*

Define the inclusion of Shimura data

$$\beta \colon (G_1, \Omega^{\pm}) \hookrightarrow (\mathrm{CSp}(W_{\mathbb{Q}}, \varphi_a), \mathfrak{H}^{\pm}).$$

as $\beta_{gr} \colon G_1 \hookrightarrow \mathrm{CSp}(W_{\mathbb{Q}}, \varphi_a)$ being the spin representation (4.4) and $\beta_{HS} \colon \Omega_1^{\pm} \hookrightarrow \mathfrak{H}^{\pm}$ mapping $\tilde{h}$ to $\beta \circ \tilde{h}$.

Let Λ_n be the congruence level n-subgroup of $\mathrm{CSp}(W_{\mathbb{Q}}, \varphi_a)(\mathbb{A}_f)$ corresponding to the lattice W of $W_{\mathbb{Q}}$. In other words we take

$$\Lambda_n = \{g \in \mathrm{CSp}(W_{\mathbb{Q}}, \varphi_a)(\mathbb{A}_f) \mid gW_{\hat{\mathbb{Z}}} = W_{\hat{\mathbb{Z}}} \text{ and } g \equiv 1 \pmod{n}\}.$$

It is clear from the definitions that $\beta(\mathbb{K}^{\mathrm{sp}}) \subset \beta(\mathbb{K}^{\mathrm{sp}}_n) \subset \Lambda_n$ hence we obtain a morphism of quasi-projective $\mathbb{Q}$-schemes

$$\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)} \colon Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1,\Omega^{\pm}) \to Sh_{\mathbb{K}^{\mathrm{sp}}_n}(G_1,\Omega^{\pm}) \to Sh_{\Lambda_n}(\mathrm{CSp}(W_{\mathbb{Q}},\varphi_a),\mathfrak{H}^{\pm}).$$

Note that fixing the lattice W, respectively the arithmetic group Λ_n, one has an immersion $Sh_{\Lambda_n}(\mathrm{CSp}(W_{\mathbb{Q}},\varphi_a),\mathfrak{H}^{\pm}) \hookrightarrow \mathcal{A}_{g,d',n,\mathbb{Q}}$ where d' is explicitly computed in terms of d and a (cf. Remark 4.2.7). It is given by the identification of $Sh_{\Lambda_n}(\mathrm{CSp}(W_{\mathbb{Q}},\varphi_a),\mathfrak{H}^{\pm})$ with a some component $\mathcal{A}_{g,\delta,n,\mathbb{Q}}$ of $\mathcal{A}_{g,d',n,\mathbb{Q}}$ corresponding to an elementary divisor sequence $\delta = (d_1,\dots,d_r)$, uniquely determined by φ_a, with $d_1\cdots d_r = d'$ (cf. Definition 1.3 in [dJ93]). We can put all morphisms considered so far in the following diagram

$$\begin{tikzcd}
Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1,\Omega_1^{\pm}) \arrow[r, "\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}"] \arrow[d, "\alpha_{(\mathbb{K}^{\mathrm{sp}},\mathbb{K}^{\mathrm{a}})}"] & \mathcal{A}_{g,d',n,\mathbb{Q}} \arrow[d, "pr_n"] \\
Sh_{\mathbb{K}^{\mathrm{a}}}(G,\Omega^{\pm}) & \mathcal{A}_{g,d',\mathbb{Q}}.
\end{tikzcd} \tag{4.11}$$

First Construction of Kuga-Satake morphisms. Both morphisms $\alpha_{(\mathbb{K}^{\mathrm{sp}},\mathbb{K}^{\mathrm{a}})}$ and pr_n are quotient morphisms as $\mathcal{A}_{g,d',\mathbb{Z}[1/n]}$ is the quotient stack $[\mathcal{A}_{g,d',n}/\,\mathrm{GL}(W_{\mathbb{Q}})(\mathbb{Z}/n\mathbb{Z})]$ (cf. [MB85, Ch VII, 4.3.4]). Moreover, as C acts freely on $Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1,\Omega^{\pm})$ we have that the stack $[Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1,\Omega_1^{\pm})/C]$ is represented by the quotient scheme $Sh_{\mathbb{K}^{\mathrm{a}}}(G_1,\Omega^{\pm}) \cong Sh_{\mathbb{K}^{\mathrm{sp}}}(G_1,\Omega_1^{\pm})/C$. The spin representation defines a homomorphism (see (4.8))

$$\beta \colon C = \mathbb{G}_m(\hat{\mathbb{Z}})\mathbb{K}^{\mathrm{sp}}/\mathbb{K}^{\mathrm{sp}} \to \mathrm{GL}(W_{\mathbb{Q}})(\mathbb{Z}/n\mathbb{Z}). \tag{4.12}$$

We will show that $\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}$ descends to a morphism $\beta_{\mathbb{K}^{\mathrm{a}}} \colon Sh_{\mathbb{K}^{\mathrm{a}}}(G,\Omega^{\pm}) \to \mathcal{A}_{g,d',\mathbb{Q}}$. To do this we have to check that $\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}$ is equivariant with respect to the homomorphism (4.12). Both $Sh_{\mathbb{K}^{\mathrm{a}}}(G,\Omega^{\pm})$ and $\mathcal{A}_{g,d',n,\mathbb{Q}}$ are reduced schemes over $\mathbb{Q}$ so we can check the statement on $\mathbb{C}$-valued points. In other words we have to show that

$$\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}\big(g\cdot[\tilde{h},r]_{\mathbb{K}^{\mathrm{sp}}}\big) = \beta(g)\cdot\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}\big([\tilde{h},r]_{\mathbb{K}^{\mathrm{sp}}}\big)$$

for any $g \in C, \tilde{h} \in \Omega^{\pm}$ and $r \in G_1(\mathbb{A}_f)$. But this is tautology as from the definitions we see that

$$\begin{aligned}
\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}\big(g\cdot[\tilde{h},r]_{\mathbb{K}^{\mathrm{sp}}}\big) &= \beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}\big([\tilde{h},rg]_{\mathbb{K}^{\mathrm{sp}}}\big) = [\beta\circ\tilde{h},\beta(rg)]_{\Lambda_n} \\
&= \beta(g)\cdot[\beta\circ\tilde{h},\beta(r)]_{\Lambda_n} = \beta(g)\cdot\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}\big([\tilde{h},r]_{\mathbb{K}^{\mathrm{sp}}}\big).
\end{aligned}$$

Hence $\beta_{(\mathbb{K}^{\mathrm{sp}},\Lambda_n)}$ descends to a morphism of algebraic stacks

$$\beta_{\mathbb{K}^{\mathrm{a}}} \colon Sh_{\mathbb{K}^{\mathrm{a}}}(G,\Omega^{\pm}) \to \mathcal{A}_{g,d',\mathbb{Q}}.$$

Recall that we have a period morphism $j_{d,\mathbb{K}^{\mathrm{a}}} \colon \mathcal{F}_{2d,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \to Sh_{\mathbb{K}^{\mathrm{a}}}(G,\Omega^{\pm})$ which sends any complex polarized K3 surface with level $\mathbb{K}^{\mathrm{a}}$-structure to its period point (cf. Proposition 3.2.5 and Theorem 3.3.16).

Definition 4.2.11. Define the *Kuga-Satake morphism* associated to d, a and $\mathbb{K}^{\mathrm{a}}$

$$f^{ks}_{d,a,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \colon \mathcal{F}_{2d,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}$$

to be the composite $f^{ks}_{d,a,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} = \beta_{\mathbb{K}^{\mathrm{a}}} \circ j_{d,\mathbb{K}^{\mathrm{a}}}$.

Thus we have proved the following statement.

Proposition 4.2.12. *Let $d, n \in \mathbb{N}$ with $n \geq 3$ and let $\mathbb{K}^{\mathrm{sp}} \subset \mathbb{K}^{\mathrm{sp}}_n$ be a subgroup of finite index. Fix a non-zero element $a \in C^+$ which satisfies Assumption 4.2.10. Then one has a Kuga-Satake morphism*

$$f^{ks}_{d,a,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \colon \mathcal{F}_{2d,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}$$

where $g = 2^{19}$ and d' depends explicitly on a and d. It maps every primitively polarized complex K3 surface (X, λ, ν) with a level $\mathbb{K}^{\mathrm{a}}$-structure ν to its associated Kuga-Satake abelian variety A with a certain polarization of degree d'^2.

Remark 4.2.13. Note that if $\mathbb{K}_1$ is a subgroup of $G_1(\hat{\mathbb{Z}})$ of finite index contained in $\mathbb{K}^{\mathrm{sp}}_n$ and such that $\alpha(\mathbb{K}_1) = \mathbb{K}^{\mathrm{a}}$, then the morphism $\beta_{(\mathbb{K}_1,\Lambda_n)} \colon Sh_{\mathbb{K}_1}(G_1, \Omega_1^{\pm}) \to \mathcal{A}_{g,d',n,\mathbb{Q}}$ also descends to the morphism $\beta_{\mathbb{K}^{\mathrm{a}}} \colon Sh_{\mathbb{K}^{\mathrm{a}}}(G, \Omega^{\pm}) \to \mathcal{A}_{g,d',\mathbb{Q}}$.

Example 4.2.14. Take $\mathbb{K}^{\mathrm{sp}}$ to be the group $\mathbb{K}^{\mathrm{sp}}_n$. Then $\mathbb{K}^{\mathrm{a}} = \mathbb{K}^{\mathrm{a}}_n$ and we obtain a Kuga-Satake morphism

$$f^{ks}_{d,a,n,\mathbb{Q}} \colon \mathcal{F}_{2d,n^{\mathrm{sp}},\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}.$$

Remark 4.2.15. If $\mathbb{K}^{\mathrm{a}}$ is an admissible subgroup of $\mathrm{SO}(V)(\hat{\mathbb{Z}})$ (see Definition 1.5.5), then we have an open immersion $j_{d,\mathbb{K}^{\mathrm{a}}} \colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \hookrightarrow Sh_{\mathbb{K}^{\mathrm{a}}}(G, \Omega^{\pm})$ and therefore we obtain a Kuga-Satake morphism

$$f^{ks}_{d,a,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \colon \mathcal{F}^{\mathrm{full}}_{2d,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}$$

defined by $f^{ks}_{d,a,\mathbb{K}^{\mathrm{a}},\mathbb{Q}} = \beta_{\mathbb{K}^{\mathrm{a}}} \circ j_{d,\mathbb{K}^{\mathrm{a}}}$.

Remark 4.2.16. One might want to descend the Kuga-Satake morphism defined in Proposition 4.2.12 to a morphism $\mathcal{F}_{2d,\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}$. The essence of the problem is that the Kuga-Satake construction described above requires a non-canonical choice of an element $a \in C^+$ to define a polarization. One can show that the obstruction for descending the Kuga-Satake morphism to a map $\mathcal{F}_{2d,\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}$ is equivalent to the problem posed in Remark 4.2.9.

Our main goal in this chapter is to define Kuga-Satake morphisms in mixed characteristic. As we will see later (Remark 4.3.8) there are problems extending the morphism $f^{ks}_{d,a,n} \colon \mathcal{F}_{2d,n^{\mathrm{sp}},\mathbb{Q}} \to \mathcal{A}_{g,d',\mathbb{Q}}$ due to the fact that $\mathcal{A}_{g,d'}$ is an algebraic stack. We will give a second construction of Kuga-Satake morphisms below to which we can apply the extension result of Section 4.3.1.

Second Construction of Kuga-Satake Morphisms. We will construct a morphism $f^{ks}_{d,a,\gamma,n,E}\colon \mathcal{F}_{2d,n^{\mathrm{sp}},E} \to \mathcal{A}_{g,d',n,E}$ for a number field E which can be determined via class field theory from the data d, a, γ, n (see below). To do that we will first determine the fields of definition of the geometric connected components of $Sh_{\mathbb{K}_n^{\mathrm{sp}}}(G_1, \Omega_1^{\pm})$ and $Sh_{\mathbb{K}_n^{\mathrm{a}}}(G, \Omega^{\pm})$.

We have that

$$\pi_0\big(Sh_{\mathbb{K}_n^{\mathrm{sp}}}(G_1, \Omega_1^{\pm})_{\mathbb{C}}\big) \cong G_1(\mathbb{Q})_+\backslash G_1(\mathbb{A}_f)/\mathbb{K}_n^{\mathrm{sp}} \cong \mathbb{G}_m(\mathbb{A})/\big(\mathbb{Q}^{\times}\mathbb{R}_{>0}\mathrm{N}(\mathbb{K}_n^{\mathrm{sp}})\big)$$

where $\mathrm{N}\colon G_1 \to G_1^{\mathrm{ab}} = \mathbb{G}_m$ is the spinorial norm homomorphism (see (4.3)). Denote by E_n the subfield of $\mathbb{Q}^{\mathrm{ab}}$ corresponding to the group $\mathbb{Q}^{\times}\mathbb{R}_{>0}\mathrm{N}(\mathbb{K}_n^{\mathrm{sp}})$ via class field theory (cf. Section 3.3.1). Then we have an isomorphism

$$\mathrm{art}_{E_n/\mathbb{Q}}\colon \mathbb{G}_m(\mathbb{A})/\big(\mathbb{Q}^{\times}\mathbb{R}_{>0}\mathrm{N}(\mathbb{K}_n^{\mathrm{sp}})\big) \to \mathrm{Gal}(E_n/\mathbb{Q}).$$

The Galois action on the geometric connected components of $Sh_{\mathbb{K}_n^{\mathrm{sp}}}(G_1, \Omega_1^{\pm})$ is given as follows: Let Y be the connected component $\Omega_1^{+}/\Gamma'_{[1]}$. It is defined over E_n and if $\sigma \in \mathrm{Gal}(E_n/\mathbb{Q})$ is an automorphism such that $\mathrm{art}_{E_n/\mathbb{Q}}(\sigma) = \mathrm{N}(g)$ for some $g \in G_1(\mathbb{A}_f)$, then $Y^{\sigma} = \Omega_1^{+}/\Gamma'_{[g]}$. We have that

$$Sh_{\mathbb{K}_n^{\mathrm{sp}}}(G_1, \Omega_1^{\pm}) = \bigcup_{\sigma \in \mathrm{Gal}(E_n/\mathbb{Q})} Y^{\sigma}. \tag{4.13}$$

For details see [Kud97, §2]. Further, if we denote by X the connected component $\Omega^{+}/\Gamma_{[1]}$ of $Sh_{\mathbb{K}_n^{\mathrm{a}}}(G, \Omega^{\pm})_{\mathbb{C}}$, then its field of definition E_X is a subfield of E_n and hence it is an abelian extension of $\mathbb{Q}$. We have that $[E_n : E_X] = \#C = \varphi(n)$ (cf. (4.8)) and

$$Sh_{\mathbb{K}_n^{\mathrm{a}}}(G, \Omega^{\pm}) = \bigcup_{\sigma \in \mathrm{Gal}(E_X/\mathbb{Q})} X^{\sigma}. \tag{4.14}$$

In order to define a Kuga-Satake morphism $f^{ks}_{d,a,n,\gamma,E}\colon \mathcal{F}_{2d,n^{\mathrm{sp}},E} \to \mathcal{A}_{g,d',n,E}$ we will give a a section of $\alpha_{(\mathbb{K}_n^{\mathrm{sp}},\mathbb{K}_n^{\mathrm{a}})}$ and use (4.11). We see from (4.9), (4.10), (4.13) and (4.14), that giving such a section is equivalent to giving a set-theoretic section of the homomorphism $\mathrm{Gal}(E_n/\mathbb{Q}) \to \mathrm{Gal}(E_X/\mathbb{Q})$. For any such (set-theoretic) section $\gamma\colon \mathrm{Gal}(E_X/\mathbb{Q}) \to \mathrm{Gal}(E_n/\mathbb{Q})$ one has a morphism

$$\delta_{\gamma}\colon Sh_{\mathbb{K}_n^{\mathrm{a}}}(G, \Omega^{\pm}) = \bigcup_{\sigma \in \mathrm{Gal}(E_X/\mathbb{Q})} X^{\sigma} \cong \bigcup_{\sigma \in \mathrm{Gal}(E_X/\mathbb{Q})} Y^{\gamma(\sigma)} \subset Sh_{\mathbb{K}_n^{\mathrm{sp}}}(G_1, \Omega_1^{\pm}) \tag{4.15}$$

which is defined over E_n.

Definition 4.2.17. Define the *Kuga-Satake morphism* associated to d, a, n and γ

$$f^{ks}_{d,a,n,\gamma,E_n}\colon \mathcal{F}_{2d,n^{\mathrm{sp}},E_n} \to \mathcal{A}_{g,d',n,E_n}$$

to be the composite $f^{ks}_{d,a,n,\gamma,E_n} = \beta_{(\mathbb{K}_n^{\mathrm{sp}},\Lambda_n)} \circ \delta_{\gamma} \circ j_{d,\mathbb{K}_n^{\mathrm{a}},E_n}$ defined over E_n.

Thus we have proved the following statement.

Proposition 4.2.18. *Let $d, n \in \mathbb{N}$ with $n \geq 3$. Let E_n and E_X be as above and suppose given a set-theoretic section γ of the homomorphism* $\mathrm{Gal}(E_n/\mathbb{Q}) \to \mathrm{Gal}(E_X/\mathbb{Q})$. *Fix a non-zero element $a \in C^+$ which satisfies Assumption 4.2.10. Then one has a Kuga-Satake morphism*

$$f^{ks}_{d,a,n,\gamma,E_n} : \mathcal{F}_{2d,n^{\mathrm{sp}},E_n} \to \mathcal{A}_{g,d',n,E_n}$$

where $g = 2^{19}$ and d' depends explicitly on a and d. It maps every primitively polarized complex K3 surface (X, λ, ν) with a spin level n-structure ν to its associated Kuga-Satake abelian variety A with a certain polarization of degree ${d'}^2$ and a certain level n-structure. Further, by construction, for any choice of a section γ we have that $f^{ks}_{d,a,n} \otimes E_n = pr_n \circ f^{ks}_{d,a,n,\gamma,E_n}$.

As we have seen, there are many possible ways of defining Kuga-Satake morphisms. In general, one has to make non-canonical choices in order to find a section of $\alpha_{(\mathbb{K}^{\mathrm{sp}}_n, \mathbb{K}^{\mathrm{a}}_n)}$ in (4.11) and define a morphism $\mathcal{F}_{2d,n^{sp},\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}$.

Below we will explain the relative Kuga-Satake construction of Deligne ([Del72, §5]) in our framework. Consider the diagram

$$\begin{tikzcd} \Gamma'_{[1]}\backslash\Omega^+_1 \arrow[r, "\beta_{(\mathbb{K}^{\mathrm{sp}}_n, \Lambda_n)}"] \arrow[d, "\alpha_{(\mathbb{K}^{\mathrm{sp}}_n, \mathbb{K}^{\mathrm{a}}_n)}", "\cong"'] & \mathcal{A}_{g,d',n,\mathbb{C}} \\ \Gamma_{[1]}\backslash\Omega^+. & \end{tikzcd}$$

Over $\mathbb{C}$ one can define a morphism $\mathcal{F}_{2d,n^{sp},\mathbb{C}} \to \Gamma_{[1]}\backslash\Omega^+$ by mapping all connected components of $\mathcal{F}_{2d,n^{sp},\mathbb{C}}$ to $\Gamma_{[1]}\backslash\Omega^+$. See the proof of Proposition 5.7 in [Del72]. Composing these two maps we obtain a morphism

$$f_n : \mathcal{F}_{2d,n^{sp},\mathbb{C}} \to \Gamma_{[1]}\backslash\Omega^+ \cong \Gamma'_{[1]}\backslash\Omega^+_1 \to \mathcal{A}_{g,d',n,\mathbb{C}}$$

which is the relative Kuga-Satake construction described in [Del72, §5] and [And96a, §5]. One can show that this morphism is defined over a number field. Suppose further, that $n = 3$ or 4. Here is a possible way to study the field of definition of f_n. Combining Proposition 8.3.5 and Theorem 8.4.3 in [And96a] one can see that f_n is defined over the composite of E_n with any field $K \subset \mathbb{C}$ for which $\mathcal{F}_{2d,n^{\mathrm{sp}},\mathbb{Q}}$ has a K-valued point. Then by Theorem 7 in [Riz04] the morphism f_n is defined over the composite of E_n with the fields of definition of the geometric connected components of $\mathcal{F}_{2d,n^{\mathrm{sp}},\mathbb{Q}}$. In general, this field can be a non-trivial extension of E_n.

Remark 4.2.19. The construction of Kuga-Satake morphisms described in this section and the one given in [Del72] and [And96a] differ in the choice of a period morphism. We use the "modified" period map $j_{d,\mathbb{K},\mathbb{C}}$ in order to be able to apply the results of Chapter 3. In this way we can control explicitly the fields of definition of the morphisms involved in the relative Kuga-Satake construction and therefore the field of definition of $f^{ks}_{d,a,n,\gamma,\mathbb{C}}$.

We will end this section with a result comparing the étale cohomology of a K3 surface and its associated Kuga-Satake abelian variety. Let U^i be a geometric connected component of $\mathcal{F}_{2d,n^{sp},E_n}$ which is defined over a field $i\colon K \hookrightarrow \mathbb{C}$. Let $(\pi_{X^i}\colon X^i \to U^i, \lambda^i, \nu^i)$ be the pull-back of the universal family to U^i. Denote by $(\pi_{A^i}\colon A^i \to U^i, \mu^i, \epsilon^i)$ the polarized abelian scheme with level n-structure $f^{ks}_{d,a,n,\gamma,E_n}((\pi_{X^i}\colon X^i \to U^i, \lambda^i, \nu^i))$.

Taking a base change $i\colon K \to \mathbb{C}$ we have an abelian scheme $(A^i_{\mathbb{C}} \to U^i_{\mathbb{C}}, \mu^i_{\mathbb{C}}, \epsilon^i_{\mathbb{C}})$ which is exactly $f^{ks}_{d,a,n,\gamma,\mathbb{C}}\big((X^i_{\mathbb{C}} \to U^i_{\mathbb{C}}, \lambda^i_{\mathbb{C}}, \nu^i_{\mathbb{C}})\big)$ and which, by construction, has multiplication by C^+. Further, we know that $\mathrm{End}_{U^i_{\mathbb{C}}}(A^i_{\mathbb{C}}) = C^+$ (see the beginning of §8 in [And96a]) and one has further that $\mathrm{End}_{U^i}(A^i) = C^+$.

Lemma 4.2.20. *There is a unique isomorphism of $\mathbb{Z}_l$-sheaves*

$$C^+(P^2_{\mathrm{et}}\pi_{X^i,*}\mathbb{Z}_l(1)) \cong \mathrm{End}_{C^+}(R^1_{\mathrm{et}}\pi_{A^i,*}\mathbb{Z}_l).$$

Proof. One repeats step by step the proof of Lemma 6.5.13 in [Del72]. □

Corollary 4.2.21. *Let K be a field of characteristic zero and suppose given a K-valued point $(X, \lambda, \nu) \in \mathcal{F}_{2d,n^{\mathrm{sp}},E_n}(K)$. If (A, μ, ϵ) is the corresponding Kuga-Satake abelian variety $f^{ks}_{d,a,n,\gamma,E_n}\big((X, \lambda, \nu)\big)$, then one has an isomorphism of* $\mathrm{Gal}(\bar{K}/K)$*-modules*

$$C^+\big(P^2_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Z}_l(1))\big) \cong \mathrm{End}_{C^+}\big(H^1_{\mathrm{et}}(A_{\bar{K}}, \mathbb{Z}_l)\big)$$

for any prime number l.

Proof. It follows from the preceding lemma. □

Remark 4.2.22. Note that if R is a discrete valuation ring with a maximal ideal $\mathfrak{p}$ and field of fractions K of characteristic zero, containing E_n. Suppose given a polarized K3 surface (X, λ, α) with spin level n-structure over K and let (A, μ, β) be the corresponding Kuga-Satake abelian variety. Suppose further that X has good reduction modulo $\mathfrak{p}$. Then the inertia subgroup $I_{\mathfrak{p}}$ acts trivially on $P^2_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Z}_l(1))$ for every l different from the characteristic of $R/\mathfrak{p}$. As shown in [Del72, 6.6] and [And96a, §9, Lemma 9.3.1] this implies that $I_{\mathfrak{p}}$ acts via a finite group on $H^1_{\mathrm{et}}(A_{\bar{K}}, \mathbb{Z}_l)$ i.e., that A has potentially good reduction at $\mathfrak{p}$. Since the n-torsion is rational over K we conclude (as in [And96a, Lemma 9.3.1]) that A has good reduction at $\mathfrak{p}$.

4.3 Extension of the Kuga-Satake Morphisms in Positive Characteristic

The following two sections contain the main results of Chapter 4. We show that the Kuga-Satake morphism from Definition 4.2.17 extends in positive characteristic. In this way we give a partial answer to the question posed in the beginning of the chapter.

In Section 4.3.1 we prove an abstract extension result concerning morphisms from smooth schemes into $\mathcal{A}_{g,d',n}$. Then we use this in the next section to show that $f^{ks}_{d,a,n,\gamma,E_n}$ extends over an open part of $\mathrm{Spec}(\mathcal{O}_{E_n})$.

4.3.1 The Extension Result

Let us fix the following notations we will use in this section:

- R will be a discrete valuation ring of mixed characteristic $(0,p)$ where $p > 2$. Denote by η and s the generic and the special points of $\mathrm{Spec}(R)$, respectively. Further, let K be the fraction field of R and k will denote the residue field of R;
- U will be a smooth scheme over R;
- We fix three natural numbers g, d' and $n \geq 3$ and denote by $\mathcal{A}$ the moduli stack $\mathcal{A}_{g,d',n,R}$ of g-dimensional abelian varieties with polarization of degree d'^2 and Jacobi level n-structure over R. We will assume that p does not divide $d'n$.
- Assume given a morphism $f_\eta \colon U_\eta \to \mathcal{A}_\eta$.

We are interested in extending the morphism f_η over R. Of course, in general one cannot expect to be able to do this without further assumptions on f_η and U. We will list some conditions below which, if satisfied, will guarantee the existence of an extension of f_η.

Assumption 4.3.1. *Let x be a point on the special fiber U_s of U, let $\mathcal{O}_{U,x}$ be its local ring and denote by L the field of fractions of $\mathcal{O}_{U,x}$. Then the morphism $f_\eta \colon \mathrm{Spec}(L) \to \mathcal{A}_\eta$ extends to a morphism $\tilde{f} \colon \mathrm{Spec}(\mathcal{O}_{U,x}) \to \mathcal{A}$.*

We will show that if this assumption is fulfilled for certain points of U then the morphism does extend over R. More precisely we have

Proposition 4.3.2. *Let U be a smooth scheme over R and let $f_\eta \colon U_\eta \to \mathcal{A}_\eta$ be a morphism. Assume that the total ramification index e of R satisfies $e < p-1$ and that all generic points of the special fiber U_s of U satisfy Assumption 4.3.1. Then f_η extends uniquely to a morphism $f \colon U \to \mathcal{A}$ over R.*

Proof. We will divide the proof into several steps.

Step 1: We will prove first that if the morphism extends then the extension is unique.

Lemma 4.3.3. *Let V be a scheme over R and suppose given a morphism $f_\eta \colon V_\eta \to \mathcal{A}_\eta$. Assume that it extends to a morphism $f_{V'} \colon V' \to \mathcal{A}$ over a dense open subscheme V' of V. Then this extension is unique.*

Proof. This boils down to the fact that $\mathcal{A}$ is separated over R. Assume that there exist two morphisms, say F_1 and F_2 extending f_η over V. Consider the morphism $(F_1, F_2) \colon V \to \mathcal{A} \times_R \mathcal{A}$. The locus where $F_1 = F_2$ is the pull-back $(F_1, F_2)^{-1}\Delta_{\mathcal{A}}$ of the diagonal $\Delta_{\mathcal{A}} \subset \mathcal{A} \times_R \mathcal{A}$ which is closed as $\mathcal{A}$ is separated over R. Hence we conclude that $F_1 = F_2$ on V'. ☐

In particular, we conclude that if f extends over U, then this extension is unique.

Step 2: There exists a maximal open subscheme V of U such that f_η extends to a morphism $f_V\colon V \to \mathcal{A}$. Indeed, if f extends over two open subschemes V_1 and V_2 of U, then by Lemma 4.3.3 above those two extensions agree on $V_1 \cap V_2$. Hence we can glue the two morphism and get a morphism $f_{V_1\cup V_2}\colon V_1 \cup V_2 \to \mathcal{A}$. This shows that one can take V to be the union of all open subschemes V' of U such that f_η extends to a morphism $f_{V'}\colon V' \to \mathcal{A}$.

Step 3: Consider the graph Γ_η of $f_\eta\colon U_\eta \to \mathcal{A}_\eta$ in $U_\eta \times \mathcal{A}_\eta$. Take the flat extension $\bar{\Gamma}$ of Γ_η over R i.e., the closure of Γ_η in $U \times \mathcal{A}$. We have the projection map $pr_1\colon \bar{\Gamma} \to U$. Let $\{U_s^i\}_{\{i\in I\}}$ be the set of connected components of the special fiber U_s and for each i let $U^{(i)}$ be the open subscheme $U \setminus \left(\cup_{j\neq i} U_s^j\right)$. We look at the set

$$U_{\mathrm{ft}}^{(i)} := \{p \in U^{(i)} \mid pr_1 \text{ is flat at all points of } pr_1(p)\}.$$

which is open in $U^{(i)}$ ([GD67, EGA IV, §8, Prop. 8.9.4]). We will call this set, with its induced scheme structure the maximal subscheme of $U^{(i)}$ over which this projection map is flat. Let V_{ft} be the open subscheme $\bigcup_i U_{\mathrm{ft}}^{(i)}$ of U.

Lemma 4.3.4. *The maximal open subscheme V given in Step 2 over which f_η extends is equal the open subscheme V_{ft} of U over which the morphism $pr_1\colon \bar{\Gamma} \to U$ is flat.*

Proof. Note first that scheme V from Step 2 is contained in V_{ft}. Indeed, the morphism $pr_1\colon pr_1^{-1}(V) \to V$ is an isomorphism, as $pr_1^{-1}(V) \subset U \times \mathcal{A}$ is the graph of f_V, and therefore it is flat.

Since $pr_1\colon pr_1^{-1}(V_{\mathrm{ft}}) \to V_{\mathrm{ft}}$ is flat the dimension of the fibers is constant. It is zero on the generic fiber hence this morphism is quasi-finite. Using Zariski's Main Theorem ([GD67, EGA IV, §8, Thm. 8.12.6]) one can factor $pr_1|_{V_{\mathrm{ft}}}$ as an open immersion and a finite morphism $pr_1^{-1}(V_{\mathrm{ft}}) \to \tilde{V} \to V_{\mathrm{ft}}$. But generically, one every connected component of V_{ft}, the degree of the finite morphism is 1, hence it is 1 everywhere. This means that $pr_1\colon pr_1^{-1}(V_{\mathrm{ft}}) \to V_{\mathrm{ft}}$ is an isomorphism. Hence one can extend f_η on V_{ft} using the second projection map $pr_2\colon \bar{\Gamma} \to \mathcal{A}$. Therefore we get the other inclusion $V_{\mathrm{ft}} \subset V$. $\square$

Step 4: We will show that the open subscheme V given is Step 2 contains all generic points of U_s. We need some auxiliary results.

Lemma 4.3.5. *If $x \in U$ satisfies Assumption 4.3.1, then $x \in V$.*

Proof. According to Step 3 we have to show that $pr_1\colon \bar{\Gamma} \to U$ is flat at x.

Claim 4.3.6. *Let X be a scheme over R and let $\Gamma_\eta \subset X_\eta$ be a closed subscheme. Take the flat extension $\bar{\Gamma}$ of Γ_η. Let $i\colon Y \to X$ be a flat morphism over R and set $\Delta_\eta\colon = i^*\Gamma_\eta$. Let $\bar{\Delta}$ be the flat extension of Δ_η in Y. Then one has that $\bar{\Delta} = i^*\bar{\Gamma}$.*

Proof. Since flatness is stable under base change we have that $i^*\bar{\Gamma} \to \bar{\Gamma}$ is flat. Hence $i^*\bar{\Gamma} \to R$ being the composite $i^*\bar{\Gamma} \to \bar{\Gamma} \to R$ of two flat maps is also flat. Moreover, by the very definitions we have that $(i^*\bar{\Gamma})_\eta = i^*\Gamma_\eta = \Delta_\eta$. Therefore by uniqueness of the flat extension (see [GD67, EGA IV, §2, Prop. 2.8.5]) we conclude that $\bar{\Delta} = i^*\bar{\Gamma}$. $\square$

For a scheme X denote by $|X|$ its underlying topological space.

Claim 4.3.7. *Let X and T be two schemes and $h\colon T \to X$ be a morphism. Take a point $x \in |X|$, let $i\colon \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ be the morphism associated to x and let $i^*h\colon i^*T \to \operatorname{Spec}(\mathcal{O}_{X,x})$ be the pull-back map. If i^*h is flat, then h is flat above x i.e., it is flat at all points $t \in h^{-1}(x)$.*

Proof. If $t \in h^{-1}(x)$, then $\mathcal{O}_{T,t} \cong \mathcal{O}_{i^*T,t}$ as $\mathcal{O}_{X,x}$-algebras and hence we obtain the result in the claim. $\square$

Let us go back to the proof of the Lemma 4.3.5. We have the following diagram:

$$\begin{tikzcd}
i^*\bar{\Gamma} \arrow[rr] \arrow[d] & & \Gamma \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_{U,x}) \times \mathcal{A} \arrow[rr, "\text{flat}"] \arrow[d] & & U \times \mathcal{A} \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_{U,x}) \arrow[rr, "\text{flat}"] \arrow[dr, "\text{flat}"'] & & U \arrow[dl, "\text{smooth}"] \\
& \operatorname{Spec}(R). &
\end{tikzcd}$$

Let $\alpha = \operatorname{Spec}(L)$ where L is the field of fractions of $\mathcal{O}_{U,x}$. Consider the point $\Delta_\eta = (\alpha, f_\eta(\alpha))$ on $U_\eta \times \mathcal{A}_\eta$. Then by Claim 4.3.6 applied to

$$\begin{aligned}
X &= \operatorname{Spec}(\mathcal{O}_{U,x}) \times \mathcal{A} \\
Y &= U \times \mathcal{A} \text{ over } R \\
\Gamma_\eta &= \text{the graph of } f_\eta
\end{aligned}$$

we conclude that $i^*\bar{\Gamma} = \bar{\Delta}$, where $\bar{\Delta}$ is the flat closure over R.

Since Assumption 4.3.1 holds for the point x the map $f_\eta\colon \operatorname{Spec}(L) \to \mathcal{A}_\eta$ extends to a morphism $\tilde{f}\colon \operatorname{Spec}(\mathcal{O}_{U,x}) \to \mathcal{A}$ and we see that $\bar{\Delta}$ is the graph of $\tilde{f}$. In particular we have that $\bar{\Delta} \cong \operatorname{Spec}(\mathcal{O}_{U,x})$ hence it is flat over $\operatorname{Spec}(\mathcal{O}_{U,x})$. If we apply Claim 4.3.7 with $T = \bar{\Gamma}$ and $X = U$ we get that $pr_1\colon \bar{\Gamma} \to U$ is flat at x. Therefore by Step 3 we conclude that $x \in V$. This finishes the proof of the Lemma. $\square$

Since all generic points of the special fiber U_s satisfy Assumption 4.3.1 we conclude that they are contained in V.

Step 5: By Step 4 there exists an open dense subscheme V of U, containing the generic points of the connected components of the special fiber U_s over which the morphism $f_\eta\colon U_\eta \to \mathcal{A}_\eta$ extends to a morphism $f_V\colon V \to \mathcal{A}$ over R. It corresponds to a polarized abelian scheme with level n-structure $(A_V, \lambda_V, \alpha_V)$ over V.

As V contains strictly the generic fiber U_η and all generic points of the special fiber U_s we have that $\operatorname{codim}_U U \setminus V \geq 2$. Since U is smooth over R and by assumption $e < p-1$ then by a result of Faltings (Lemma 3.6 in [Moo98]) one concludes that $A_V \to V$ extends to an abelian scheme $A \to U$.

Further, by Lemma 4.1.2 the polarization λ_V extends to a polarization $\lambda\colon A \to A^t$. Since p does not divide n, the level n-structure α_V extends uniquely to a level n-structure α on (A, λ). Hence we get a polarized abelian scheme (A, λ, α) extending $(A_V, \lambda_V, \alpha_V)$. This corresponds to a morphism $f\colon U \to \mathcal{A}$ extending f_η. □

Remark 4.3.8. We will apply Proposition 4.3.2 to show that the Kuga-Satake morphism constructed in Proposition 4.2.18 extends over an open part of $\operatorname{Spec}(\mathcal{O}_{E_n})$, where $\mathcal{O}_{E_n}$ is the ring of integers in E_n. One might want to use the same line of thoughts and try to extend the Kuga-Satake morphism $f^{ks}_{d,a,\mathbb{K}^{\mathrm{a}},\mathbb{Q}}$ defined in Proposition 4.2.12 over an open part of $\operatorname{Spec}(\mathbb{Z})$. The problem which one comes up with is to carry on Step 3 in this situation. One can define an equivalence of the closure $\bar{\Gamma}$ of Γ. In general, the morphism $\bar{\Gamma}_{\mathrm{ft}} := pr^{-1}(V_{\mathrm{ft}}) \to V_{\mathrm{ft}}$ might not be representable so one cannot use Zariski's Main Theorem ([LMB00, Thm. 16.5]).

4.3.2 Extension of the Kuga-Satake Morphisms

In this section we will use the notations established in §4.2.5. In particular, we fix two natural numbers d and n and let us suppose that $n \geq 3$. Let γ be a set-theoretic section of the homomorphism $\mathrm{Gal}(E_n/\mathbb{Q}) \to \mathrm{Gal}(E_X/\mathbb{Q})$. We will show below that the Kuga-Satake morphism $f^{ks}_{d,a,n,\gamma,E_n}$ extends over an open part of $\operatorname{Spec}(\mathcal{O}_E)$ where $\mathcal{O}_{E_n}$ the ring of integers in E_n.

Theorem 4.3.9. *Let $d, n \in \mathbb{N}$, $n \geq 3$ and suppose that $a \in C^+$ satisfies Assumption 4.2.10. Then the Kuga-Satake morphism $f^{ks}_{d,n,a,\gamma,E_n}\colon \mathcal{F}_{2d,n^{\mathrm{sp}},E_n} \to \mathcal{A}_{g,d',n,E_n}$ extends uniquely to a morphism*

$$f^{ks}_{d,a,n,\gamma}\colon \mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]} \to \mathcal{A}_{g,d',n,\mathcal{O}_{E_n}[1/N]}$$

where $N = 2dd'nl$ and l is the product of the prime numbers p whose ramification index e_p in E_n is $\geq p-1$.

Proof. Let us first shorten the notations a bit by setting $\mathcal{F}$ to be $\mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]}$ and $\mathcal{A}$ to be $\mathcal{A}_{g,d',n,\mathcal{O}_{E_n}[1/N]}$. Let $\pi\colon U \to \mathcal{F}$ be an atlas of $\mathcal{F}$ (i.e., an étale surjective morphism) over $\mathcal{O}_{E_n}[1/N]$. We may assume that the pull-back of the universal family of polarized K3 surfaces to U is a K3 scheme. The map $f^{ks}_{d,a,n,\gamma,E_n}\colon \mathcal{F}_{E_n} \to \mathcal{A}_{E_n}$ defines a morphism $f_{E_n} = f^{ks}_{d,a,n,\gamma,E_n} \circ \pi_{E_n}\colon U_{E_n} \to \mathcal{A}_{E_n}$. We will fist extend f_{E_n} to a morphism over $\mathcal{O}_{E_n}[1/N]$ and then using a descent argument show that it comes from a morphism $f^{ks}_{d,a,n,\gamma}\colon \mathcal{F}_{d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]} \to \mathcal{A}_{g,d',n,\mathcal{O}_{E_n}[1/N]}$.

Let $\mathfrak{p}$ be a prime ideal of E_n not dividing N and let $R = \mathcal{O}_{E_n,(\mathfrak{p})}$ be the localization of $\mathcal{O}_{E_n}$ at $\mathfrak{p}$. As before $\{s,\eta\}$ will be the special and the generic points of $\mathrm{Spec}(R)$. In order to apply Proposition 4.3.2 to f_{E_n} and U_R, which we will denote by $U^{\mathfrak{p}}$, we have to show that all generic points of the special fiber $U^{\mathfrak{p}}_s$ of $U^{\mathfrak{p}}$ satisfy Assumption 4.3.1.

Let $x \in |U^{\mathfrak{p}}_s|$ be a generic point. Then $\mathcal{O}_{U^{\mathfrak{p}},x}$ is a discrete valuation ring with a maximal ideal $\mathfrak{m}_x$. Let us denote its field of fractions by L. Taking the pull-back of the universal family of polarized K3 surfaces with spin level n-structures via the canonical morphism $\mathrm{Spec}(\mathcal{O}_{U^{\mathfrak{p}},x}) \to U^{\mathfrak{p}}$ we obtain a K3 scheme $(X \to \mathrm{Spec}(\mathcal{O}_{U^{\mathfrak{p}},x}),\lambda,\nu)$. Then f_{E_n} gives a morphism $\mathrm{Spec}(L) \to \mathcal{A}_{E_n}$ and let (A,μ,ϵ) be the corresponding abelian variety over L. It is the Kuga-Satake abelian variety associated to the generic fiber of $(X \to \mathrm{Spec}(\mathcal{O}_{U^{\mathfrak{p}},x}),\lambda,\nu)$. We can apply Remark 4.2.22 (or alternatively by Lemma 9.3.1 in [And96a] we conclude that A has potentially good reduction and as the n-torsion points are L-rational, then A has good reduction) to see that the abelian variety A has good reduction at $\mathfrak{m}_x$. In other words the Néron model of A over $\mathcal{O}_{U^{\mathfrak{p}},x}$ is an abelian scheme. By Lemma 4.1.1, the polarization λ extends uniquely over $\mathcal{O}_{U^{\mathfrak{p}},x}$ and as $\mathfrak{p}$ does not divide n, the level n-structure extends uniquely, as well. Hence the morphism $\mathrm{Spec}(L) \to \mathcal{A}_{E_n}$ extends to a morphism $\mathrm{Spec}(\mathcal{O}_{U^{\mathfrak{p}},x}) \to \mathcal{A}_R$. Therefore, by Proposition 4.3.2 applied to $U^{\mathfrak{p}}$, f_{E_n} and $R = \mathcal{O}_{E_n,(\mathfrak{p})}$ one can extend $f_{E_n}\colon U_{E_n} \to \mathcal{A}_{E_n}$ to a morphism $f_{\mathfrak{p}}\colon U^{\mathfrak{p}} \to \mathcal{A}_R$.

The morphism f_{E_n} can be extended uniquely over $\mathcal{O}_{E_n,(\mathfrak{p})}$ for any p not dividing N. Hence we conclude that it extends uniquely to a morphism $f\colon U \to \mathcal{A}$ over $\mathbb{Z}[1/N]$.

We are left to show that f descends to $\mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]}$. By [Knu71, Ch. II, §1, Prop. 1.4] one has the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_S(\mathcal{F},\mathcal{A}) \xrightarrow{\alpha^*} \mathrm{Hom}_S(U,\mathcal{A}) \overset{pr_1^*}{\underset{pr_2^*}{\rightrightarrows}} \mathrm{Hom}_S(U',\mathcal{A})$$

where $U' = U \times_{\mathcal{F}} U$. Note that both $pr_1^*(f)$ and $pr_2^*(f)$ are extensions of the morphism $pr_1^* \circ \pi^*(f^{ks}_{d,a,n,\gamma,E_n}) = pr_2^* \circ \pi^*(f^{ks}_{d,a,n,\gamma,E_n})$ over $\mathcal{O}_{E_n}[1/N]$. Since U' is a smooth scheme over $\mathcal{O}_{E_n}[1/N]$ ([Knu71, Def. 1.1]) and $\mathcal{A}$ is separated just like in Lemma 4.3.3 we conclude that such an extension is unique. Hence we one has that $pr_1^*(f) = pr_2^*(f)$ and therefore by the above exact sequence f comes from a morphism

$$f^{ks}_{d,a,n,\gamma}\colon \mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]} \to \mathcal{A}_{g,d',n,\mathcal{O}_{E_n}[1/N]}$$

over $\mathcal{O}_{E_n}[1/N]$. □

We end this section with a few remarks concerning the Kuga-Satake morphism in mixed characteristic.

Remark 4.3.10. In the proof of Proposition 4.3.2 we used a result of Faltings to show that certain morphisms extend in positive characteristic. This is really an essential step of our strategy for defining Kuga-Satake abelian varieties in positive characteristic. In Theorem 4.3.9 we have to exclude the primes p for which the ramification index e_p is $\geq p-1$, as Lemma 3.6 in [Moo98] does not hold for these primes. See Section 6 in [dJO97] and Section 3.4 in [Moo98].

Remark 4.3.11. We use the notations of Theorem 4.3.9. Suppose k is a field of characteristic p such that p does not divide N and let R be a discrete valuation ring of mixed characteristic $(0,p)$ with field of fractions k. Then to every primitively polarized K3 surface with a spin level n-structure (X,λ,ν) over k we associate via $f^{ks}_{d,a,n,\gamma}$ a polarized abelian variety with level n-structure (A,μ,ϵ) over k. We will call A the *Kuga-Satake abelian variety* associated to (X,λ,ν). Further, if (X_1,λ_1,ν_1) and (X_2,λ_2,ν_2) are two lifts of (X,λ,ν) over R, then the special fibers of $(A_i,\mu_i,\epsilon_i) := f^{ks}_{d,a,n,\gamma}\big((X_i,\lambda_i,\nu_i)\big)$, for $i=1,2$ are the same.

Remark 4.3.12. In characteristic zero one can show that the image $f^{ks}_{d,n,a,\gamma,E_n}(\mathcal{F}_{2d,n^{\mathrm{sp}},E_n})$ in $\mathcal{A}_{g,d',n,E_n}$ is locally closed. Indeed, as we saw in Proposition 3.2.5 the period map is open and the morphisms $\beta_{(\mathbb{K}^{\mathrm{sp}}_n,\Lambda_n)}$ and δ_γ involved in the construction of $f^{ks}_{d,n,a,\gamma,E_n}$ are finite (see Definition 4.2.17). It is interesting to know if the same holds in mixed characteristic. This question is directly connected to the existence of an analogue of the Néron-Ogg-Shafarevich criterion for potentially good reduction of K3 surfaces. As we already mentioned in Section 3.3.10, in general, this is still an open problem.

4.4 Applications

We end Chapter 4 with some applications of the existence of Kuga-Satake morphisms in mixed characteristic. In Section 4.4.1 we show that the étale cohomology relations [Del72, (6.6.1)] and in [And96a, Def. 4.5.1] hold for the Kuga-Satake abelian varieties defined in §4.3.2. Then, in Section 4.4.2, we study the behavior of $f^{ks}_{d,a,n,\gamma}$ at ordinary points. Suppose that k is a finite field of characteristic p where p does not divide N (cf. Theorem 4.3.9) and let $(X,\mathcal{L},\nu)\in\mathcal{F}_{2d,n^{\mathrm{sp}},\mathbb{F}_p}(k)$ be an ordinary point. We will prove that the canonical lift $(X^{\mathrm{can}},\mathcal{L},\nu)$ over $W(k)$ is mapped to the canonical lift $(A^{\mathrm{can}},\mu,\epsilon)$ of $(A,\mu,\epsilon)=f^{ks}_{d,a,n,\gamma}\big((X,\mathcal{L},\nu)\big)$.

4.4.1 Cohomology Groups

Let d and n be two natural numbers and suppose further that $n\geq 3$. With the notations as in Sections 4.2.5 and 4.3.2 let $a\in C^+$ be an element satisfying Assumption 4.2.10 and let γ be a set-theoretic section of the homomorphism $\mathrm{Gal}(E_n/\mathbb{Q}) \to \mathrm{Gal}(E_X, \mathbb{Q})$. Then we have a Kuga-Satake morphism

$$f^{ks}_{d,a,n,\gamma} \colon \mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]} \to \mathcal{A}_{g,d',n,\mathcal{O}_{E_n}[1/N]}$$

where $N = 2dd'nl$ and l is the product of the prime numbers p whose ramification index e_p in E_n is $\geq p-1$.

Let k be a field of characteristic p and suppose given a k-valued point $(X, \lambda, \nu) \in \mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]}$. Denote by (A, μ, β_n) the polarized Kuga-Satake abelian variety with level n-structure $f^{ks}_{d,a,n,\gamma}((X, \lambda, \nu))$.

Lemma 4.4.1. *With the notations as above one has and isomorphism of* $\mathrm{Gal}(\bar{k}/k)$*-modules*

$$C^+\big(P^2_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Z}_l(1))\big) \cong \mathrm{End}_{C^+}(H^1_{\mathrm{et}}(A_{\bar{k}}, \mathbb{Z}_l))$$

for any $l \neq p$.

Proof. Let $(\mathcal{X}, \lambda, \nu)$ be a lift of (X, λ, ν) over $W(k)$ (which exists because $\mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]}$ is smooth over $\mathcal{O}_{E_n}[1/N]$) and let $(\mathcal{A}, \mu, \epsilon)$ be the Kuga-Satake variety $f^{ks}_{d,a,n,\gamma}\big((\mathcal{X}, \lambda, \nu)\big)$. By Corollary 4.2.21 we have an isomorphism of $\mathrm{Gal}(\bar{K}/K)$-modules

$$C^+\big(P^2_{\mathrm{et}}(\mathcal{X}_{\bar{K}}, \mathbb{Z}_l(1))\big) \cong \mathrm{End}_{C^+}(H^1_{\mathrm{et}}(\mathcal{A}_{\bar{K}}, \mathbb{Z}_l))$$

for any l. Hence if $l \neq p$ one can apply the smooth base change theorem for étale cohomology to prove the claimed isomorphism. $\square$

Remark 4.4.2. Note that one can use this isomorphism in case $k = \mathbb{F}_q$ to compute the Newton polygon of A in terms of the Newton polygon of X. For instance one can see that if X is ordinary then A is also ordinary. We refer to [Nyg83, Prop. 2.5] for a proof.

4.4.2 Canonical Lifts of Ordinary K3 Surfaces

Let k be a perfect field of characteristic $p > 0$ and let $W(k)$ be the ring of Witt vectors. Suppose given an ordinary K3 surface X_0 over k. Denote by $\mathcal{X}/S$ the universal deformation of X_0 over $W(k)$. We know that S is formally smooth of dimension 20 (cf. Proposition 1.4.1).

In [AM77, IV] Artin and Mazur define the *enlarged formal Brauer group* Ψ_{X_0} of X_0 which is a p-divisible group over k and such that its connected component $\Psi^0_{X_0}$ is $\hat{Br}(X_0)$. With the notations of Section 1.4.1 let $R \in \underline{A}$ be a local artinian ring with residue field k and let $(X \to \mathrm{Spec}(R), \phi)$ be a deformation of X_0 over R. Then the enlarged formal Brauer group Ψ_X over $\mathrm{Spec}(R)$ exists. It is a p-divisible group over $\mathrm{Spec}(R)$ and the isomorphism ϕ induces an isomorphism of p-divisible groups over $\mathrm{Spec}(k)$

$$\phi_{Br} \colon \Psi_X \otimes_R k \to \Psi_{X_0}.$$

In other words Ψ_X is a lifting of Ψ_{X_0} over R. Let

$$\mathrm{Def}_{\mathrm{Sch}}(X_0)\colon \underline{A} \to \mathrm{Sets}$$

be the covariant deformation functor defined in Section 1.4.1 and let

$$\mathrm{DefBr}_{X_0}\colon \underline{A} \to \mathrm{Sets}$$

be the covariant functor

$$\mathrm{DefBr}_{X_0}(R) = \begin{array}{l}\big\{\text{isomorphism classes of pairs } (G,\phi) \text{ where } G \text{ is a} \\ p-\text{divisible group over } R \text{ and } \phi\colon G\otimes_R k \cong \Psi_{X_0}\big\}.\end{array}$$

We have the following Serre-Tate theory for ordinary K3 surfaces:

Theorem 4.4.3 (Nygaard). *For any $R \in \underline{A}$ the map*

$$\mathrm{Def}_{\mathrm{Sch}}(X_0)(R) \to \mathrm{DefBr}_{X_0}(R)$$

defined by

$$(X \to \mathrm{Spec}(R), \phi) \mapsto (\Psi_X, \phi_{Br})$$

is a bijection.

Proof. For a proof we refer to [Nyg83, Thm 1.1]. □

Let G be a lifting of Ψ_{X_0} over R. Since height one groups are rigid, we have precisely one lifting G_R^0 of $\hat{Br}(X_0) = \Psi_{X_0}^0$ to $\mathrm{Spec}(R)$. Similarly, étale groups are also rigid so there is a unique lift G_R^{et} of $\Psi_{X_0}^{\mathrm{et}}$ to $\mathrm{Spec}(R)$. So we for any lifting G of Ψ_{X_0} to $\mathrm{Spec}(R)$ we have en exact sequence

$$0 \to G_R^0 \to G \to G_R^{\mathrm{et}} \to 0$$

lifting

$$0 \to \hat{Br}(X_0) \to \Psi_{X_0} \to \Psi_{X_0}^{\mathrm{et}} \to 0$$

over $\mathrm{Spec}(R)$.

If we consider the trivial extension $G = G_R^0 \times G_R^{\mathrm{et}}$, then by Theorem 4.4.3 above there is a unique lifting X_R^{can} of X_0 over $\mathrm{Spec}(R)$ such that $\Psi_{X_R^{\mathrm{can}}} = G_R^0 \times G_R^{\mathrm{et}}$. For any $n \in N$ taking $R = W_n$ we obtain a lifting $X_n = X_{W_n}^{\mathrm{can}}$. The projective system $\{X_n\}$ defines a proper flat formal scheme $\{X_n\}$ over $\mathrm{Spf}(W)$. It is algebrizable and defines a K3 scheme X^{can} over $\mathrm{Spec}(W)$ which we will call the *canonical lift* of X_0. Every line bundle of X_0 lifts uniquely to a line bundle on X^{can}. For a proof of these facts we refer to [Nyg83, Prop. 1.8].

With the notations established in Section 4.2.5 let d and $n \geq 3$ be two natural numbers, and let $a \in C^+$ be an element satisfying Assumption 4.2.10. Choose a set-theoretic section γ of the homomorphism $\mathrm{Gal}(E_n/\mathbb{Q}) \to \mathrm{Gal}(E_X/\mathbb{Q})$ so that we have a Kuga-Satake morphism

$$f^{ks}_{d,a,n,\gamma} \colon \mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]} \to \mathcal{A}_{g,d',n,\mathcal{O}_{E_n}[1/N]}$$

where $N = 2dd'nl$ and l is the product of the prime numbers p whose ramification index e_p in E_n is $\geq p-1$. Let $k = \mathbb{F}_q$ be a finite field and suppose given an ordinary point $(X_0, \mathcal{L}_0, \nu_0) \in \mathcal{F}_{2d,n^{\mathrm{sp}},\mathcal{O}_{E_n}[1/N]}(k)$ (in particular p does not divide N). Denote by $(X^{\mathrm{can}}, \mathcal{L})$ the canonical lift of X_0 to W. The spin level n-structure ν_0 also lifts uniquely to a spin level n-structure on X^{can} as p does not divide N. Denote by (A^{ks}, μ, ϵ) the abelian scheme $f^{ks}_{d,a,n,\gamma}\big((X^{\mathrm{can}}, \mathcal{L}, \nu)\big)$ over $\mathrm{Spec}(W)$ and let (A_0, μ_0, ϵ_0) be the triple $(A^{ks}, \mu, \epsilon) \otimes k$ over k.

The following result was suggested to us by B. Moonen.

Proposition 4.4.4. *The abelian scheme A^{ks} is the canonical lift of A_0 over* $\mathrm{Spec}(W)$.

Proof. By Theorem 2.7 in [Nyg83] we know that, after a base change $R' \to R$, the abelian scheme A^{ks} is isogenous to the canonical lift A^{can} of A_0. Hence we conclude that A^{ks} is a quasi-canonical lift.

Let

$$\mathrm{Def}_{(A_0,\mu_0)} \colon \underline{A} \to \mathrm{Sets}$$

be the covariant functor

$$\mathrm{Def}_{(A_0,\mu_0)}(R) = \begin{array}{l}\{\text{isom. classes of polarized abelian schemes } (\mathrm{A}, \mu, \phi) \text{ over R} \\ \qquad \text{and an isomorphism } \phi \colon (A, \mu) \otimes_R k \to (A_0, \mu_0)\}.\end{array}$$

This functor is representable by a formal smooth scheme $\mathfrak{A}_{(A_0,\mu_0)}$ which has a structure of a formal torus. For details we refer to [Kat81, Thm. 1.2.1 and Thm. 2.1] and [Moo95, Ch. III, §1].

The lift A^{ks}/W defines a point $s \in \mathfrak{A}_{(A_0,\mu_0)}(W)$. Just like in Lemma 1.5 in [Moo95, Ch. III, §1] we conclude that since A^{ks} is a quasi-canonical lift then s is a torsion point. As $\mathfrak{A}_{(A_0,\mu_0)}(W)$ is l-divisible for all $l \neq p$ we have that $s^{p^m} = 1 \in \mathfrak{A}_{(A_0,\mu_0)}(W)$. But s is defined over W which is unramified and any p^m torsion point is defined over ramified rings unless $m = 0$. Hence $s = 1$ which corresponds to the canonical lift in the *Serre-Tate coordinates*. Therefore A^{ks} is the canonical lift of its special fiber. □

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_{E_n}$ which does not divide N and let $k := \mathcal{O}_{E_n}/\mathfrak{p}$ be its residue field. It is a finite field and let p be its characteristic. Let R be the localization of $\mathcal{O}_{E_n}$ at $\mathfrak{p}$. It is a discrete valuation ring with a residue field k. Following the notations of Section 2.1 we set $\mathcal{F}^{(2)}_{2d,n^{\mathrm{sp}},k}$ to be the non-ordinary locus of $\mathcal{F}_{2d,n^{\mathrm{sp}},k}$. It is a closed subspace of $\mathcal{F}_{2d,n^{\mathrm{sp}},k}$ and we consider

$$\mathcal{F}^{\mathrm{ord}}_{2d,n^{\mathrm{sp}},R} := \mathcal{F}_{2d,n^{\mathrm{sp}},R} \setminus \mathcal{F}^{(2)}_{2d,n^{\mathrm{sp}},k}$$

which is an open subspace of $\mathcal{F}_{2d,n^{\mathrm{sp}},R}$.

Corollary 4.4.5. *The restriction of the Kuga-Satake morphism*

$$f^{ks}_{d,a,n,\gamma,R}\colon \mathcal{F}^{\mathrm{ord}}_{2d,n^{\mathrm{sp}},R} \to \mathcal{A}_{g,d',n,R}$$

is quasi-finite.

Proof. Note first that $f^{ks}_{d,a,n,\gamma,E_n}$ is a quasi-finite morphism. Indeed, by construction (see Definition 4.2.17) we have that $f^{ks}_{d,a,n,\gamma,E_n} = \beta_{(\mathbb{K}^{\mathrm{sp}}_n,\Lambda_n)} \circ \delta_\gamma \circ j_{d,\mathbb{K}^{\mathrm{a}},E_n}$ where

$$j_{d,\mathbb{K}^{\mathrm{a}}_n,E_n}\colon \mathcal{F}_{2d,n^{\mathrm{sp}},E_n} \to Sh_{\mathbb{K}^{\mathrm{sp}}_n}(G,\Omega^{\pm})_{E_n}$$

is an étale morphism of noetherian schemes and

$$\beta_{(\mathbb{K}^{\mathrm{sp}}_n,\Lambda_n)} \circ \delta_\gamma\colon Sh_{\mathbb{K}^{\mathrm{sp}}_n}(G,\Omega^{\pm})_{E_n} \to \mathcal{A}_{g,d',n,E_n}$$

is a quasi-finite morphism. Therefore $f^{ks}_{d,a,n,\gamma,E_n}$ is a quasi-finite morphism. To finish the proof we have to show that for any $\bar{k}$-valued point $y \in \mathcal{A}_{g,d',n,k}(\bar{k})$ there are only finitely may $\bar{k}$-valued points $x \in \mathcal{F}^{\mathrm{ord}}_{2d,n^{\mathrm{sp}},k}(\bar{k})$ such that $f^{ks}_{d,a,n,\gamma,k}(x) = y$.

Suppose that (X_1,λ_1,ν_1) and (X_2,λ_2,ν_2) are two ordinary K3 surfaces over a finite field $L \subset k$ in $\mathcal{F}_{2d,n^{\mathrm{sp}},k}(L)$ such that

$$f^{ks}_{d,a,n,\gamma,k}\big((X_1,\lambda_1,\nu_1)\big) = f^{ks}_{d,a,n,\gamma,k}\big((X_2,\lambda_2,\nu_2)\big) = (A,\mu,\epsilon).$$

Taking a finite extension of L, if needed, we may assume that λ_1 and λ_2 are classes of ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X_1 and X_2, respectively. Let $(X_1^{\mathrm{can}},\mathcal{L}_1,\nu_1)$ and $(X_2^{\mathrm{can}},\mathcal{L}_2,\nu_2)$ be the two canonical lifts over $W(L)$. Denote the field of fractions of $W(L)$ by K. We have that $(X_1,\mathcal{L}_1,\nu_1) \cong (X_2,\mathcal{L}_2,\nu_2)$ if and only if $(X_1^{\mathrm{can}},\mathcal{L}_1,\nu_1)\otimes K \cong (X_2^{\mathrm{can}},\mathcal{L}_2,\nu_2)\otimes K$. By Proposition 4.4.4 we have that

$$f^{ks}_{d,a,n,\gamma,E_n}\big((X_1^{\mathrm{can}},\mathcal{L}_1,\nu_1)\otimes K\big) = f^{ks}_{d,a,n,\gamma,k}\big((X_2^{\mathrm{can}},\mathcal{L}_2,\nu_2)\otimes K\big) = (A^{\mathrm{can}},\mu,\epsilon)\otimes K$$

hence we conclude the $f^{ks}_{d,a,n,\gamma,k}$ is quasi-finite from the fact that $f^{ks}_{d,a,n,\gamma,E_n}$ is quasi-finite. $\square$

Combining Théorème 16.5 in [LMB00] and Corollary 6.16 in [Knu71, Ch. II, §6] with the preceding corollary we obtain the following result.

Corollary 4.4.6. *There exists a scheme Z over R, a finite morphism $\pi\colon Z \to \mathcal{A}_{g,d',n,R}$ and an open immersion $i\colon \mathcal{F}^{\mathrm{ord}}_{2d,n^{\mathrm{sp}},R} \hookrightarrow Z$ such that $f^{ks}_{d,a,n,\gamma,R} = \pi \circ i$. Therefore the algebraic space $\mathcal{F}^{\mathrm{ord}}_{2d,n^{\mathrm{sp}},R}$ is a scheme.*

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Index of Notations

Index of Terminology

Samenvatting

Een K3 oppervlak X is een algebraïsch oppervlak (variëteit van dimensie twee) zodanig dat:

X is niet-singulier en projectief,

de *kanonieke divisor is triviaal*, $\Omega_X^2 \cong \mathcal{O}_X$, en

en *de irregulariteit is nul*, $H^1(X, \mathcal{O}_X) = 0$.

Om een klasse van algebraïsche variëteiten te bestuderen construeren we *moduliruimtes*. Een punt van een dergelijke ruimte correspondeert met een isomorfieklasse van de variëteiten die we bestuderen: we "parametrizeren zulke isomorfieklassen". Eigenschappen van een moduliruimte geven informatie over de manier waarop zulke variëteiten onder deformaties in elkaar kunnen overgaan. Zulke ruimtes werden klassiek over $\mathbb{C}$ geconstrueerd, maar in de moderne algebraïsche meetkunde geven we de voorkeur aan moduliruimtes over een willekeurige basis, bij voorbeeld over $\mathbb{Z}$.

We beschouwen de moduliruimte van K3 oppervlakken met een *polarizatie*. Over een algebraïsch gesloten lichaam k, definiëren we een polarizatie op X als een ampele lijnbundel $\mathcal{L}$ op X. De zelf-intersectie index noemen we de polarisatiegraad van $\mathcal{L}$.

Over $\mathbb{C}$ construeren we moduliruimtes met behulp van periode-afbeeldingen. Voor gepolarizeerde K3 oppervlakken met een niveau structuur geeft deze transcendente methode een moduliruimte die een open deelruimte is van de Shimura variëteit geassocieerd met de groep $\mathrm{SO}(2, 19)$ (zie Hoofdstuk 1, p. 28 en Hoofdstuk 3, §3.2.4). Over $\mathbb{Z}$ gebruiken we algebraïsch-meetkundige methoden zoals ontwikkeld door Artin (§§1.4.3 en 1.5.2).

Als we algebraïsche krommen bestuderen, in plaats van K3 oppervlakken, dan construeren we een abelse variëteit, genaamd de *Jacobiaan* van de kromme. De meetkunde van die abelse variëteit beschrijft eigenschappen van die kromme. In dit proefschrift gebruiken we een analoge constructie voor K3 oppervlakken. Met elke gepolariseerd K3 oppervlak associëren we een abelse variëteit.

1. *Kuga-Satake morfismen in karakteristiek nul.*
   In [KS67] construeren M. Kuga en I. Satake bij elk complex, gepolariseerd K3 oppervlak $(X, \mathcal{L})$ een complexe abelse variëteit A. Ze maken gebruik van een

transcendente constructie. De variëteit A heet de Kuga-Satake abelse variëteit van $(X, \mathcal{L})$. Met behulp van deze constructie definiëren we een morfisme van een moduliruimte van K3 oppervlakken naar een moduliruimte van gepolarizeerde abelse variëteiten over $\mathbb{C}$. Vervolgens bewijzen we dat deze afbeelding gedefiniëerd is over een eindige uitbreiding van $\mathbb{Q}$ (zie §4.2.5).

2. *Kuga-Satake abelse variëteiten over een willekeurige basis.*
We bewijzen dat de Kuga-Satake morfismen gedefiniëerd in punt 1 uitgebreid kunnen worden over de ring van gehelen in het betreffende getallenlichaam. Op deze manier krijgen we een definitie van een Kuga-Satake morfisme over een willekeurige basis, en dus een "constructie" van Kuga-Satake abelse variëteiten in positieve karakteristiek.

3. *Kan deze constructie van Kuga-Satake abelse variëteiten gegeven worden met methoden uit de algebraïsche meetkunde ?*
Dit is de voornaamste motivatie van dit werk. De methode van punten 1 en 2 is een algebraïsch-meetkundige uitbreiding van een transcendente constructie. De constructie van Kuga-Satake abelse variëteiten in het algemeen is in zekere zin indirect. Een antwoord op de interessante vraag in punt 3 hebben we (nog) niet kunnen geven.

Moduliruimtes.
In Hoofdstuk 1 definiëren we moduliruimtes $\mathcal{M}_{2d}$ (resp. $\mathcal{F}_{2d}$) van K3 oppervlakken met een (primitieve) polarizatie van graad $2d$. We definiëren een niveau structuur op K3 oppervlakken over een willekeurige basis en we construeren moduliruimten $\mathcal{F}_{2d,\mathbb{K}}$ van gepolarizeerde K3 oppervlakken met een niveau structuur.

Strata.
In positieve karakteristiek vinden we interessante deelvariëteiten van moduliruimtes van gepolarizeerde abelse variëteiten en van krommen. Een dergelijke locus kan worden gegeven door het vastleggen van een discrete invariant, zoals bijvoorbeeld een filtratie op BT_1-groepen of een Newton polygoon ([Oor01a] en [Oor01b]). Een analoge methode kan gevolgd worden voor K3 oppervlakken.

Voor de moduliruimte $\mathcal{M}_{2d}\otimes\mathbb{F}_p$ van gepolarizeerde K3 oppervlakken in karakteristiek p bestuderen we in Hoofdstuk 2 de deelruimtes $\mathcal{M}^{(h)}_{2d,\mathbb{F}_p}$ als de verzamelingen van de K3 oppervlakken met hoogte ten minst h voor $h \in [1, 11]$. De verzameling van deze 11 ruimtes heet "de hoogte stratificatie" van deze moduliruimte. Een interessante vraag is of voor een gegeven d en p de deelruimtes $\mathcal{M}^{(h)}_{2d,\mathbb{F}_p}$ niet leeg zijn. Dit is het analogon voor K3 oppervlakken van het vermoeden van Manin voor Newton polygonen van abelse variëteiten ([Man63, Conj. 2, p. 76]). In Hoofdstuk 2 geven we een gedeeltelijk antwoord op deze vraag. We bewijzen, dat als d groot genoeg is en p niet $2d$ deelt, dan is elk stratum in de "hoogte stratificatie" van $\mathcal{M}_{2d}\otimes\mathbb{F}_p$ niet leeg. We geven ook een expliciete grens voor d aan (zie §2.3.2).

Complexe vermenigvuldiging voor K3 *oppervlakken.*
Zij A een abelse variëteit met complexe vermenigvuldiging. De hoofdstelling voor complexe vermeningvuldiging van Shimura en Taniyama beschrijft de actie op de torsiepunten van A van de automorfismen in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ die het reflexlichaam van A invariant laten. Deligne ([Del71]) gebruikt deze theorie als startpunt voor de definitie van canonieke modellen van Shimura variëteiten.

In Hoofdstuk 3 doen we iets dergelijks voor moduliruimtes van K3 oppervlakken. We bewijzen dat $\mathbb{Q}$ het lichaam is van definitie van de periode-afbeeldingen $j_{d,\mathbb{K},\mathbb{C}}$. Deze stelling noemen we “de hoofdstelling van de theorie van complexe vermeningvuldiging voor K3 oppervlakken” (zie §3.3.9). Als een gevolg bewijzen we voor K3 oppervlakken met complexe vermeningvuldiging het analogon van de stelling van Shimura en Taniyama.

Uitbreiden van het Kuga-Satake morfisme.
In Hoofdstuk 4 construeren we de Kuga-Satake abelse variëteit van een K3 oppervlak over een willekeurige basis: voor elke $d \in \mathbb{N}$ en $n \in \mathbb{N}$, $n \geq 3$ definiëren we een Kuga-Satake afbeelding $f^{ks}_{d,a,n,\gamma,E_n}$ over een abelse uitbreiding E_n van $\mathbb{Q}$ (zie §4.2.5). Dit morfisme associëert aan een complex K3 oppervlak met “spin niveau n-structuur” een abelse variëteit met extra structuur (een polarizatie en een niveau n-structuur). We bewijzen dat het Kuga-Satake morfisme uitbreidt tot een morfisme over een open deel $\mathrm{Spec}(\mathcal{O}_{E_n}[1/N])$ van $\mathrm{Spec}(\mathcal{O}_{E_n})$, waar $\mathcal{O}_{E_n}$ is de ring van gehelen in E_n en $N \in \mathbb{N}$ een natuurlijk getal dat op een expliciete manier afhangt van d, d', n en E_n.

Verder bestuderen we een paar eigenschappen van de Kuga-Satake morfismen. Stel dat p een priemgetal is dat N niet deelt en dat k een eindig lichaam van karakteristiek p is. Als x een punt is in de moduliruimte van gepolariseerde K3 oppervlakken corresponderend met een gewoon K3 oppervlak over k, dan is het punt $y := f^{ks}_{d,a,n,\gamma}(x)$ ook gewoon. We bewijzen dat het Kuga-Satake morfisme aan de canonieke lift x^{can} van x over $W(k)$ toevoegt de canonieke lift y^{can} van y (§4.4.2). Als een gevolg zien we dat de restrictie van $f^{ks}_{d,a,n,\gamma}$ tot de gewone locus van de moduliruimte van gepolarizeerde K3 oppervlakken over $\mathbb{F}_p$ een quasi-eindig morfisme is.

Curriculum Vitae

Jordan Rizov was born on 4 January 1977 in Sofia, Bulgaria. He grew up in Sofia and he attended secondary school there.

In the period 1991–1995 he was a student at the National School for Mathematics and Natural Sciences in Sofia. It was a time full of mathematical competitions, olympiads, table tennis games and trips, which he enjoyed a lot. A great success was a radio he assembled himself.

In October 1995 he started as a student of mathematics at Sofia University. In the last year of his studies he went to the Netherlands to attend the MRI Master class on Arithmetic Algebraic Geometry at the University of Utrecht. In July 2000 he graduated from Sofia University and the University of Groningen under the supervision of Ivan Chipchakov and Jaap Top.

In October 2000 he started as an a.i.o. (a PhD student) at the University of Utrecht. One year later he switched to a N.W.O. PhD position under the supervision of Ben Moonen and Frans Oort. During the spring of 2002 he was a visiting student at the Massachusetts Institute of Technology in Cambridge, U.S.A.

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