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MODULI OF PARAHORIC $\mathcal{G}$-TORSORS ON A COMPACT RIEMANN SURFACE

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Dedicated to Professor K. Chandrasekharan in admiration

Abstract

Let X be an irreducible smooth projective algebraic curve of genus $g \geq 2$ over the ground field $\mathbb{C}$, and let G be a semisimple simply connected algebraic group. The aim of this paper is to introduce the notion of *semistable and stable parahoric* torsors under a certain Bruhat–Tits group scheme $\mathcal{G}$ and to construct the moduli space of semistable parahoric $\mathcal{G}$-torsors; we also identify the underlying topological space of this moduli space with certain spaces of homomorphisms of Fuchsian groups into a maximal compact subgroup of G. The results give a generalization of the earlier results of Mehta and Seshadri on parabolic vector bundles.

1. Introduction

Let X be a smooth projective curve defined over $\mathbb{C}$ of genus $g \geq 2$. Let $\mathcal{R} \subset X$ be a fixed set of points of X with $m = |\mathcal{R}|$, and let n_i be a set of positive integers attached to each of the points $x_i \in \mathcal{R}$. The uniformization theorem states that there exists a simply connected covering surface $q : \tilde{X} \to X$, unique up to isomorphism, subordinate to the given signature, i.e., ramified precisely over the points $\mathcal{R} = \{x_i\} \subset X$ together with ramification indices n_i at these points. Since $g \geq 2$, we may identify $\tilde{X}$ with $\mathbb{H}$ the *upper half space* (cf. [44, pp. 49–50]). Let π be the subgroup of the discontinuous group of automorphisms of $\mathbb{H}$ such that $X = \mathbb{H}/\pi$. Note that the action of π is *not* free. Let $q : \mathbb{H} \to X$ be the quotient projection. It is well known that the isotropy subgroups at the points $z_i \in q^{-1}(\mathcal{R})$ are cyclic of finite order. Let

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the isotropy subgroups be denoted by

$$\pi_{z_i} = \langle C_i \rangle$$

with C_i as generators. Thus each C_i is an element of order n_i.

We fix once and for all the set $\mathcal{R}$ and the indices n_i. Recall that the group π is a Fuchsian group generated by $2g+m$ elements $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m$, modulo the relations

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} \cdots C_1 \cdots C_m = I, \tag{1.0.0.1}$$

$$C_i^{n_i} = I, \ (i = 1, 2, \ldots, m). \tag{1.0.0.2}$$

Let G be a connected reductive algebraic group over $\mathbb{C}$, and let $K_G \subset G$ be a maximal compact subgroup of G.

1.0.1. Definition. The *type of a homomorphism* $\rho : \pi \to G$ is defined to be the set of conjugacy classes in G of the images $\rho(C_i)$ and is denoted by $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$. Equivalently, the type of ρ is the set of isomorphism classes of the local representations $\rho_{z_i} : \pi_{z_i} \to G, i = 1, \ldots, m$.

1.0.1.1. Notation. Let $R^{\boldsymbol{\tau}}(\pi, K_G)$ denote the space of homomorphisms $\rho : \pi \to K_G$ of type $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$.

1.0.2. Definition. A (π, G)-bundle on $\mathbb{H}$ is defined to be the trivial G-bundle $\mathbb{H} \times G$ on $\mathbb{H}$ with the π-structure given by $\gamma(z, g) = (z, \rho(\gamma).g)$, with ρ a homomorphism $\pi \to G$.

If $G = GL(n)$ is the full-linear group, the (π, G)-bundles on $\mathbb{H}$ have an equivalent description as π-vector bundles on $\mathbb{H}$. We recall ([36], [24]) that if $V \simeq \mathbb{H} \times \mathbb{C}^n$ is a π-vector bundle on $\mathbb{H}$, the vector bundle $W = q_*^\pi(V)$ (invariant direct image by q) on X acquires a *parabolic structure* which consists of the data assigning a *flag to the fiber of* W *at every ramification point in* X *for the covering* q together with a tuple of *weights*.

The invariant direct image functor $V \mapsto q_*^\pi(V)$ gives a fully faithful embedding of the category of π-vector bundles on $\mathbb{H}$ into the category of parabolic vector bundles on X (morphisms being taken as isomorphisms). Moreover, we can realize every parabolic bundle with rational weights as $q_*^\pi(V)$ for a suitable π and V (cf. [24]).

This translates easily into an equivalent description of $(\pi, GL(n))$-bundles on $\mathbb{H}$ as principal $GL(n)$-bundles on X with *parabolic structures*. One can define the concepts of *stability* (resp. *semistability*) for π-vector bundles (or equivalently parabolic bundles on X) and construct the corresponding moduli space of equivalence classes of semistable objects (fixing some invariants) as a normal projective variety. As topological spaces these moduli spaces can be identified with sets of equivalence classes of elements in $R^{\boldsymbol{\tau}}(\pi, U(n))$, i.e., *unitary representations of* π (see Mehta and Seshadri [24], Seshadri [36]), which generalize the results in Narasimhan and Seshadri [27] and Seshadri [35].

The purpose of this paper is to generalize the above results when the structure group G is no longer the full-linear group. Let us suppose hereafter that the group G is *semisimple and simply connected* (over $\mathbb{C}$) unless otherwise stated.

One can again give an equivalent description of (π, G)-bundles on $\mathbb{H}$ as certain intrinsically defined objects on X. However, the picture is more subtle than the case when G is the full-linear group. For instance, *it is not possible, in general, to associate in a natural manner a principal G-bundle on X to a (π, G)-bundle on* $\mathbb{H}$. The new objects on X, which give an equivalent description of (π, G)-bundles on $\mathbb{H}$, will be called parahoric bundles or *parahoric torsors*. These parahoric torsors are defined as pairs $(\mathscr{E}, \boldsymbol{\theta})$, where $\mathscr{E}$ is a torsor (i.e., principal homogeneous space) on X under a parahoric Bruhat–Tits group scheme $\mathcal{G}$, together with a prescription of *weights* $\boldsymbol{\theta}$, which are elements of the set of *rational* one-parameter subgroups of G (see the discussion below and Definition 6.1.1). We define notions of semistability and stability of such parahoric torsors and construct moduli spaces of these objects.

The torsors under parahoric group schemes that we consider here have been studied earlier by Pappas and Rapoport, without however the notion of weights (see [29] and [30]); in [30] they made some precise conjectures on the moduli stack of such torsors. Heinloth has since settled many of their conjectures (see [21]; we note that Heinloth works over arbitrary ground fields not just $\mathbb{C}$). We were led to the study of parahoric torsors in trying to interpret (π, G)-bundles on $\mathbb{H}$ as objects on X (inspired by A. Weil's work [44], as was the case in [24] and [36]). In Section 2 we link explicitly the ideas from the paper of Weil and Bruhat–Tits theory. This relationship plays a key role in the rest of the paper. We need to define a few technical terms before we can state the main results of our paper.

Let A_{x_i} be the completion of the local ring at x_i, and let K_{x_i} (or simply as K) be its quotient field, $x_i \in \mathcal{R}$. Let T be a maximal torus of G, and let $Y(T) := Hom(\mathbb{G}_m, T)$ be the group of one-parameter subgroups of T. Let $\mathbb{E} \simeq Y(T) \otimes \mathbb{R}$ and $\mathbb{E}_{\mathbb{Q}} \simeq Y(T) \otimes \mathbb{Q}$. By the general theory of Bruhat and Tits (see [9, Definition 5.2.6], and 2.1.2 below), one has certain collection of subsets $\{\Theta_i\} \subset \mathbb{E}_{\mathbb{Q}}^m$, where $m = |\mathcal{R}|$, and to each subset $\Theta_i \subset \mathbb{E}$, one can associate a parahoric subgroup $\mathcal{P}_{\Theta_i}(K) \subset G(K_{x_i})$, $i = 1, \ldots, m$, and furthermore, associated to each parahoric subgroup $\mathcal{P}_{\Theta_i}(K)$, there is a smooth group scheme $\mathcal{G}_{\Theta_i}$ over $D_{x_i} = Spec\ A_{x_i}$, known as a Bruhat–Tits group scheme.

More precisely (see 2.1.2), by fixing a root datum, the theory of buildings allows us to identify the vector space $\mathbb{E}$ with an *affine apartment* $App(G, K)$

in the Bruhat–Tits building, and each parahoric subgroup $\mathcal{P}_{\Omega_i}(K) \subset G(K)$ is precisely the stabilizer subgroup of a *facet* Ω_i of the affine apartment for the natural $G(K)$-action on the building. We will reserve the symbol Ω for a facet of the apartment.

It is explained in Section 5 as to how, given any finite subset of points $\mathcal{R} \subset X$ and a collection $\mathcal{G}_{\Theta_i}$ of *Bruhat–Tits group schemes* over D_{x_i}, one can construct a (global) group scheme $\mathcal{G}_{\Theta,X}$ over the projective curve X by *gluing* (see Lemma 5.2.2 and Definition 5.2.1) so that

$$(1.0.2.1) \qquad \mathcal{G}_{\Theta,X}|_{X-\mathcal{R}} \simeq G \times (X - \mathcal{R}),\ \mathcal{G}_{\Theta,X}|_{D_{x_i}} \simeq \mathcal{G}_{\Theta_i}, x_i \in \mathcal{R}.$$

We will call the set $\mathcal{R}$ the points of ramifications of $\mathcal{G}$.

Following Pappas and Rapoport ([30]), we will call $\mathcal{G}_{\Theta,X}$ the **parahoric Bruhat–Tits group schemes**. However, we wish to emphasize that both Pappas and Rapoport ([30]) and Heinloth [21] *do not* make the assumption that $\mathcal{G}_{\Theta,X}|_{X-\mathcal{R}} \simeq G \times (X - \mathcal{R})$, i.e., for them the group scheme need not be *generically split.*

It can be shown (see Remark 2.1.8) that to every set $\boldsymbol{\tau}$ of conjugacy classes and finite subset $\mathcal{R} \subset X$, we can associate a collection $\boldsymbol{\theta_\tau} = \{\theta_i\} \in \mathbb{E}_{\mathbb{Q}}^m$ of elements of $\mathbb{E}_{\mathbb{Q}}$ and also a parahoric group scheme $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$ on X such that the points of ramifications of $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$ is $\mathcal{R}$. The content of Theorem 1.0.3 below is that this correspondence $\boldsymbol{\tau} \mapsto \mathcal{G}_{\boldsymbol{\theta_\tau},X}$ extends precisely to give an identification of moduli spaces of representations with fixed conjugacy classes and that of torsors under $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$.

One of the key features of parahoric groups is that for any interior point θ of a facet Ω_i, we have an isomorphism $\mathcal{P}_{\Omega_i}(K) \simeq \mathcal{P}_{\theta}(K)$ (see the discussion in 2.1.2 below). In particular, any parahoric Bruhat–Tits group scheme $\mathcal{G}_{\Omega,X}$ associated to a collection of facets $\{\Omega_i\}$ is isomorphic to a $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$ for some $\boldsymbol{\tau}$.

Before going to the main results of this paper, we begin by observing that a collection $\boldsymbol{\theta_\tau} = \{\theta_i\}$ of rational weights entails a choice of ramification indices d_i at the points of $\mathcal{R}$ (see Remark 6.1.3). Since the genus $g \geq 2$, it is well known (see 2.2.1) that there exists a Galois cover $p : Y \to X$, with Galois group Γ, ramified precisely at $\mathcal{R}$ with the prescribed ramification indices d_i.

It is shown in Theorem 5.3.1 that there is an isomorphism between the moduli stack of (Γ, G)-bundles on Y of local type $\boldsymbol{\tau}$ and the stack of $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$-torsors on X.

We then define, in Section 6 of this paper, the concept of *semistable and stable* $\mathcal{G}$-torsors on X as well as the notion of S-equivalence. Our main results can be formulated as follows (see Theorem 8.1.11, Theorem 7.3.2, and Corollary 8.1.12 for notation and details):

1.0.3. Theorem. *Let $\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X}$ be a parahoric Bruhat–Tits group scheme associated to $\boldsymbol{\tau}$.*

(1) *The set $M_X(\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X})$ of S-equivalence classes of semistable $\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X}$-torsors on X gets a natural structure of an irreducible normal projective variety of dimension*

$$\dim_{\mathbb{C}}(G)(g-1) + \sum_{i=1}^{m} \frac{1}{2} e(\boldsymbol{\theta}_{\boldsymbol{\tau}}) \tag{1.0.3.1}$$

In fact, the variety $M_X(\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X})$ is the coarse moduli space for the functor of isomorphism classes of $\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X}$-torsors on X.

(2) *Let $\overline{K}_G = K_G/center$. There exists a Fuchsian group π and a bijective correspondence between the space $R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G$ of conjugacy classes of homomorphisms $\rho : \pi \to K_G$ of local type $\boldsymbol{\tau}$ and the set of S-equivalence classes of* semistable *$\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X}$-torsors.*

(3) *This correspondence induces a homeomorphism*

$$R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G \simeq M_X(\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X})$$

of the underlying topological spaces.

(4) *Under this correspondence, the subset of irreducible homomorphisms gets identified with isomorphism classes of* stable *$\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X}$-torsors.*

We make a few clarifying remarks on the paper.

1.0.4. Remark. (1) The moduli stack $\mathsf{Bun}_X(\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X})$ has been studied in detail by Heinloth ([21]).

(2) *Parabolic G-bundles.* If for a point $x \in \mathcal{R}$ the parahoric group $\mathcal{P}_{\Omega}(K_x)$ gets identified with the distinguished hyperspecial parahoric subgroup $G(A_x)$ (see 2.1.5 for the definition), the moduli space of parahoric torsors gets identified with the moduli space of principal G-bundles on X in the usual sense. In this case, the parahoric structure comes from the origin of $\mathbb{E}$ (see 2.1.2).

If $\mathcal{P}_{\Omega}(K_x)$ is a proper subgroup of $G(A_x)$, then under the evaluation map $ev : G(A_x) \to G(\mathbb{C})$, the subgroup $\mathcal{P}_{\Omega}(K_x)$ is the inverse image of a standard parabolic subgroup of G, so that in this case a quasi-parahoric torsor (see Definition 6.1.2) could indeed be called a *quasi-parabolic G-bundle* in the familiar sense of the term when $G = GL(n)$ is the full-linear group, i.e., the data consists of a principal G-bundle on X together with a parabolic subgroup of G (i.e., a "flag") for every $x \in \mathcal{R}$. This case corresponds to the situation when the parahoric subgroups defining the local Bruhat–Tits group schemes come from the interior of the Weyl alcove. Equivalently, the *weights* come from the interior of the Weyl alcove. These are the cases dealt with in Teleman and Woodward ([40]).

(3) *Parahoric torsors which are not principal G-bundles.* We now consider parahoric subgroups of $G(K_x)$ which cannot be conjugated to subgroups of $G(A_x)$. For instance, barring $G(A_x)$, the rest of the maximal parahoric subgroups of $G(K_x)$ fall under this case (see [9]). The weights in these cases lie on the walls of the Weyl alcove (cf. Teleman [39, Section 9]).

It is this case which highlights one of the reasons why we need to give a subtler description of (Γ, G)-bundles on Y as parahoric torsors on X which do not support a principal G-bundle on X. Evidence to this effect was shown using Tannakian considerations in Balaji, Biswas, and Nagaraj [2], leading to the definition of a ramified bundle in [3]. More concrete examples were shown in [37] indicating what to expect in general.

(4) The striking cases which arise out of the present study are the non-hyperspecial maximal parahoric subgroups where a number of new phenomena show up. These correspond, on the side of the representations of the Fuchsian group (see (1.0.0.1)), to those maps $\rho : \pi \to K_G$ such that centralizers of the images of the elements $\rho(C_i)$ are *proper semisimple* subgroups of G (see Remark 2.1.8).

(5) After this paper was posted in the archives, we were informed by P. Boalch of his paper [6] where the parahoric structure is seen in the setting of regular singular connections.

2. Non-abelian functions and bounded groups

2.1. As the title suggests, the aim of this section is to tie up some ideas from the classical paper of A. Weil ([44]) and Seshadri ([36]) and Bruhat–Tits theory ([9]). This section is central to this paper.

2.1.1. Some preliminaries on root data. Let G be a semisimple, simply connected algebraic group defined over $\mathbb{C}$; we fix a maximal torus T of G. Let $X(T) := Hom(T, \mathbb{G}_m)$ be the character group and $Y(T) := Hom(\mathbb{G}_m, T)$ the group of one-parameter subgroups of T. Let $R = R(T, G) \subset X(T)$ be the root system associated to the adjoint representation of G, and let S be a system of simple roots.

Denote by $(\ ,\ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form. The set S determines a system of positive roots $R^+ \subset R$ and a Borel subgroup $B \subset G$ with unipotent radical U. We now order the set $R^+ = \{r_i\}, i = 1, \dots, q$. We then have a family $\{u_r : \mathbb{G}_a \to G \mid r \in R\}$ of *root homomorphisms* of groups such that one gets an isomorphism of varieties,

$$\prod_{r\in R^+} u_r : \prod_{r\in R^+} \mathbb{G}_a \to U. \tag{2.1.1.1}$$

For every root $r \in R$, we denote by $T_r = Ker(r)^0$, and $Z_r = Z_G(T_r)$, the centralizer of T_r in G. The derived group $[Z_r, Z_r]$ is of rank 1 and there exists a unique 1-PS, $r^\vee : \mathbb{G}_m \to T \cap [Z_r, Z_r]$, such that $T = Im(r^\vee).T_r$ and $(r^\vee, r) = 2$. The element $r^\vee$ is the coroot (or 1-PS) associated to r.

For each $r \in R$ the *root homomorphism*

$$u_r : \mathbb{G}_a \to G \tag{2.1.1.2}$$

is such that

$$t.u_r(a).t^{-1} = u_r(r(t).a) \tag{2.1.1.3}$$

for any $\mathbb{C}$-algebra A, $t \in T(A)$, $a \in A$, and such that the tangent map du_r induces an isomorphism

$$du_r : Lie(\mathbb{G}_a) \to (Lie\, G)_r$$

The functor $A \mapsto u_r(\mathbb{G}_a) = u_r(A)$ gives $U_r(A) \subset G(A)$. This determines a closed subgroup U_r of G and is called the *root group* corresponding to r.

Denote by $\{\alpha^* \mid \alpha \in S\}$ the coroots dual to $\{\alpha \in S\}$, i.e., $(\alpha^*, r) = \delta_{\alpha,r}$. Define

$$\mathbb{E} := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{2.1.1.4}$$

$$\mathbb{E}' := X(T) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.1.1.5}$$

Most often in fact we work with $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.1.2. Parahoric subgroups. Let K be the field $\mathbb{C}((z))$ of Laurent power series in z, and let $A = \mathbb{C}[[z]]$ be the ring of integers with residue field $\mathbb{C}$.

For the notion of Bruhat–Tits buildings and their behaviour under field extensions, see J. Tits [41, p. 43].

Once we fix a root datum for G, we see that we have a choice of an affine apartment; the choice of the maximal torus T then identifies $\mathbb{E}$ with an affine apartment $App(G, K)$ in the Bruhat–Tits building $\mathcal{B}(G, K)$.

A subset $M \subset G(K)$ is said to be *bounded* if for any regular function $f \in K[G]$, the values $v(f(m))$ for the valuation v on A, are bounded below when m runs over all elements of M. In particular, we may talk of *bounded subgroups*. A subgroup $M \subset G(K)$ is therefore bounded if the "order of poles" of elements of M is bounded. This can be made precise by taking a faithful representation $G \hookrightarrow GL(n)$ so that elements of M are represented by matrices with entries in K.

Let $\Theta \subset \mathbb{E}$ be a nonempty subset which is a facet. Denote by $\mathcal{P}_{\Theta}(K) \subset G(K)$ the subgroup generated by $T(A)$ and the groups $U_r(z^{m_r}A)$ for all the roots $r \in R$, where

$$m_r = m_r(\Theta) = -[\inf_{\theta \in \Theta}(\theta, r)], \tag{2.1.2.1}$$

where $[h]$ stands for the biggest integer smaller than or equal to h.

The group $\mathcal{P}_{\Theta}(K)$ is a bounded subgroup, more precisely it is a *parahoric subgroup* of $G(K)$ in the sense of Bruhat–Tits and conversely, any parahoric subgroup is bounded in the above sense (cf. Bruhat and Tits [9]).

The choice of a root datum identifies a parahoric subgroup $\mathcal{P}_{\Omega}(K) \subset G(K)$ as the stabilizer subgroup of $G(K)$ of a *facet* Ω of the affine apartment *App*(G, K) for the natural $G(K)$-action on $\mathcal{B}(G, K)$. By Tits [41, Section 3.1, p. 50], since we work with a semisimple and simply connected group G, we could in turn take any point in *general position*, i.e., an interior point in the facet, and consider the parahoric subgroup as the stabilizer of that point. Thus one can make an identification $\mathcal{P}_{\Omega}(K) \simeq \mathcal{P}_{\theta}(K)$ for an *interior point* θ in the facet Ω.

By the main theorem of Bruhat and Tits ([9]), there exist smooth group schemes $\mathcal{G}_{\Omega}$ over *Spec* A such that the group $\mathcal{G}_{\Omega}(A) = \mathcal{P}_{\Omega}(K)$ and, moreover, since A is a complete discrete valuation ring, the group scheme is uniquely determined up to unique isomorphism by its A-valued points (see [9, Section 1.7]).

Let $\theta \in \mathbb{E}$. Thus,

$$m_r = m_r(\theta) = -[(\theta, r)]. \tag{2.1.2.2}$$

In other words, we have

$$\mathcal{P}_{\theta}(K) = \langle T(A),\ U_r(z^{m_r(\theta)}A),\ r \in R \rangle. \tag{2.1.2.3}$$

To summarize, since we work with a semisimple and simply connected group G, all *parahoric groups* are, up to conjugacy by elements of $G(K)$, precisely the collection of groups $\{\mathcal{P}_{\theta}(K)\}_{\theta \in \mathbb{E}}$ (see [41, Section 3.1, p. 50]), and as such we will work with these groups. In fact, we may choose these θ to be in $\mathbb{E}_{\mathbb{Q}} = Y(T) \otimes \mathbb{Q}$. Again by [41, p. 51], the conjugacy classes of maximal parahoric subgroups of $G(K)$ are the stabilizers of the vertices of the building, and they are precisely $l + 1$ in number, where $l = rank(G)$. In particular, associated to the "origin" $0 \in \mathbb{E}$, we have the group $\mathcal{P}_0(K)$, which is nothing but the maximal bounded subgroup $G(A) \subset G(K)$.

Note that if θ lies in the lattice $Y(T)$ itself, then there exists $t \in T(K)$ such that

$$\mathcal{P}_{\theta}(K) = t.\mathcal{P}_0(K).t^{-1}. \tag{2.1.2.4}$$

2.1.3. Remark. Again we note that if $m_r(\theta) < 1$ for all $r \in R$, then $\mathcal{P}_{\theta}(K) \subset G(A)$. These parahoric subgroups then correspond to the standard parabolic subgroups of G.

2.1.4. Remark. In this remark we make some comments on the parahoric groups when we make the assumption that the group G is *simple*. Let $\alpha_{\max}$ denote the highest root. Then we can express it as

$$\alpha_{\max} = \sum_{\alpha \in S} c_\alpha \cdot \alpha \tag{2.1.4.1}$$

with $c_\alpha \in \mathbb{Z}^+$.

One can have a nicer choice of the points whose stabilizers give the maximal parahorics (see the last paragraph in [42, p. 662]), now that G is *simple*. For every $\alpha \in S$, if we define

$$\theta_\alpha = \frac{\alpha^*}{c_\alpha} \in \mathbb{E}, \tag{2.1.4.2}$$

then in fact, the groups $\{\mathcal{P}_{\theta_\alpha}(K)\}_{\alpha \in S} \cup \mathcal{P}_0(K)$ *represent the conjugacy classes under* $G(K)$ *of all* maximal parahoric subgroups *of* $G(K)$. In other words, these are indexed precisely by the vertices of the *extended Dynkin diagram.*

2.1.5. Hyperspecial parahorics. In Bruhat–Tits theory, we encounter the so-called *hyperspecial* maximal parahorics which have the following characterizing property: each parahoric group $\mathcal{P}_\Omega(K)$ is identified with $\mathcal{G}_\Omega(A)$, the A-valued points of a certain canonically defined smooth group scheme $\mathcal{G}_\Omega$ defined over A. It is a fact that the parahoric subgroup $\mathcal{P}_{\theta_\alpha}(K)$ is hyperspecial if and only if $c_\alpha = 1$ in the description of the long root $\alpha_{\max}$. The hyperspecial parahorics are listed at the end of [41].

2.1.6. The Weyl alcove. We now recall the description of the set of conjugacy classes in a compact semisimple and simply connected group in terms of the affine Weyl group W_{aff}.

Let $K_G \subset G$ be a maximal compact subgroup. For an arbitrary group S, let Torsion(S) denote the subset of elements of finite order in S. We then have the following identifications:

$$\text{Torsion}(K_G)/\text{conjugation} \simeq \text{Torsion}(T)/W,$$

$$(Y(T) \otimes \mathbb{Q}/\mathbb{Z})/W \simeq (Y(T) \otimes \mathbb{Q})/W_{\text{aff}}.$$

Further, if the group G is assumed to be *simple*, then $(Y(T) \otimes \mathbb{Q})/W_{\text{aff}}$ gets identified with the simplex (*the* (*rational*) *Weyl alcove*)

$$\mathcal{A} := \{x \in Y(T) \otimes \mathbb{Q} \mid (x, \alpha_{\max}) \leq 1, (x, \alpha_i) \geq 0, \forall \textit{ positive roots } \alpha_i\}.$$

2.1.7. Remark. In fact, the set of conjugacy classes of an element in K_G gets identified with T/W which is the Weyl alcove since any element of K_G is conjugate to an element in the maximal torus up to an element of the Weyl group (cf. [25, p. 151]).

2.1.8. Remark. Recall that vertices of the alcove $\mathcal{A}$ correspond to the vertices of the extended Dynkin diagram. Furthermore, to each point of $\mathcal{A}$ one can associate a *parahoric subgroup* of $G(K)$ and hence a canonically defined parahoric Bruhat–Tits group scheme. Thus, for each tuple $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}_{i=1}^m$ of conjugacy classes of elements of finite order in K_G, we have a point $\boldsymbol{\theta}_{\boldsymbol{\tau}} = \{\theta_i\}_{i=1}^m \in \mathcal{A}^m$, where $m = \#\{\text{of conjugacy classes}\}$ and hence an associated parahoric Bruhat–Tits group scheme $\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}}}$.

More can be said. Let $\alpha \in S$ be a simple root, and let $\mathcal{G}_{\theta_\alpha}$ be the Bruhat–Tits group scheme associated to the maximal parahoric $\mathcal{P}_{\theta_\alpha}(K)$. Let g_α be an element in K_G of finite order corresponding to θ_α. Then the centralizer $Z_G(g_\alpha)$ can be obtained from the *closed fiber of the group scheme* $\mathcal{G}_{\theta_\alpha}$; indeed, $Z_G(g_\alpha) \simeq \left(\mathcal{G}_{\theta_\alpha}\right)_x/\{\text{unipotent radical}\}$. When θ_α is *hyperspecial*, then $\mathcal{G}_{\theta_\alpha}$ is in fact a semisimple group scheme, and therefore $Z_G(g_\alpha) = G$. On the other hand, when θ_α is nonhyperspecial, the $Z_G(g_\alpha)$'s are precisely those subgroups of G which are proper semisimple subgroups of maximal rank in G corresponding to the classical Borel–de Siebenthal list (see [7]).

2.1.9. Remark. In the case when G is assumed to be simple and simply connected, by the description of the (rational) Weyl alcove $\mathcal{A}$ (see Definition 2.1.7) and the fact that the parahoric subgroups are determined by *interior points* of $\mathbb{E}$, it follows that up to conjugacy by $G(K)$, every parahoric subgroup of $G(K)$ can be identified with a $\mathcal{P}_\theta(K)$ for a suitable $\theta \in \mathcal{A}$. Moreover, by Remark 2.1.3, if $m_r(\theta) < 1$ for all $r \in R$, then $\mathcal{P}_\theta(K) \subset G(A)$.

2.1.10. Remark. We remark that even when G is semisimple, we still have the notion of an *alcove* $\mathcal{A}$, but it will no longer be a simplex as in the case when G is simple since there is no unique $\alpha_{\max}$, but $\mathcal{A}$ will now be a product of the Weyl alcoves associated to the simple factors of G. Again, parahoric subgroups will be parametrized by points of the alcove up to conjugacy by $G(K)$.

2.1.11. Standard parahorics (See Remarks 2.1.3, 2.3.2, and 2.1.9). Following the loop group terminology, the *standard parahoric subgroups* of $G(K)$ are parahoric subgroups of the distinguished hyperspecial parahoric subgroup $G(A)$. These are realized as inverse images under the evaluation map $ev : G(A) \to G(k)$ of standard parabolic subgroups of G. For any $I \subset S$, let R_I denote the set $R_I = R \cap \mathbb{Z}I$. Let $U_I := U((-R^+) \setminus R_I)$, and let $L_I := G(R_I)$. The standard parabolic $P_I \subset G$ is defined by $P_I := U_I L_I$.

In particular, the standard *Iwahori subgroup* $\mathfrak{I}$ is a standard parahoric, and indeed $\mathfrak{I} = ev^{-1}(B)$, with $B = U(-R^+)T$ being the standard Borel subgroup containing the fixed maximal torus T.

Since the *standard parahoric subgroups* of $G(A) = \mathcal{P}_0(K)$ are also indexed by the subsets of the set of simple roots, to avoid any confusion, we will denote the *standard parahoric subgroups* of $G(A)$ by $\mathcal{P}^{st}_I(K) := ev^{-1}(P_I)$ for every subset $I \subset S$.

For instance if $\alpha \in S$, let $S_\alpha := S \setminus \{\alpha\}$. Then $P_{S_\alpha} \subset G$ is a maximal parabolic subgroup and $ev^{-1}(P_{S_\alpha}) = \mathcal{P}^{st}_{S_\alpha}(K)$ is a standard parahoric which can be described as

$$\mathcal{P}^{st}_{S_\alpha}(K) = \langle T(A),\ U_r(A),\ r \in R_{S_\alpha} \cup (-R^+) \setminus R_{S_\alpha} \rangle. \tag{2.1.11.1}$$

Note that $ev^{-1}(L_{S_\alpha}) = \langle T(A),\ U_r(A), r \in R_{S_\alpha} \rangle$ and $ev^{-1}(U_{S_\alpha}) = \langle U_r(A), r \in (-R^+) \setminus R_{S_\alpha} \rangle$.

If $r \in R_{S_\alpha}$, the simple root α does not occur in r, in which case $(\theta_\alpha, r) = 0$. Hence $ev^{-1}(L_{S_\alpha}) \subset \mathcal{P}_{\theta_\alpha}(K) \cap \mathcal{P}_0(K)$.

Again if $r \in (-R^+) \setminus R_{S_\alpha}$, say $r = \sum a_\beta.\beta$, with $a_\beta \leq 0$ and $a_\alpha \neq 0$. By the definition of c_α, we have $-1 \leq \frac{a_\alpha}{c_\alpha} < 0$. It follows that $m_r(\theta_\alpha) = -[(\theta_\alpha, r)] = -[\frac{a_\alpha}{c_\alpha}] = 1$. Hence $ev^{-1}(U_{S_\alpha}) \subset \mathcal{P}_{\theta_\alpha}(K) \cap \mathcal{P}_0(K)$.

We therefore have the inclusions

$$\mathfrak{I} \subset \mathcal{P}^{st}_{S_\alpha}(K) \subset \mathcal{P}_{\theta_\alpha}(K) \cap \mathcal{P}_0(K). \tag{2.1.11.2}$$

These standard parahorics will play a role when we re-examine the Hecke correspondences.

2.2. Non-abelian functions and the unit group. For the purposes of working in the setting of algebraic curves instead of $\mathbb{H}$, we make a few observations. A result due to A. Selberg ([33]) states that if $A \subset GL(n, \mathbb{C})$ is a finitely generated subgroup, then A has a *normal subgroup* A_0 of finite index with no torsion. It follows from this that the discrete group $\pi \subset Aut(\mathbb{H})$ has a normal subgroup π_0 of finite index such that π_0 operates *freely* on $\mathbb{H}$. Let $Y = \mathbb{H}/\pi_0$ and $\Gamma = \pi/\pi_0$. Then there is a canonical action of Γ on Y such that $X = Y/\Gamma$. Let $p : Y \to X$ be the covering map, and note that $\Gamma = Gal(Y/X)$. Conversely, if Y is a Galois cover of X with the given signature (i.e., $\mathcal{R}$ and ramification indices), by the universality of $q : \mathbb{H} \to X$ for this signature, it follows that there is a $\pi_o \subset \pi$ acting freely on $\mathbb{H}$ such that $Y = \mathbb{H}/\pi_o$.

2.2.1. Remark. In other words, given a finite number of points $x_i \in X$ together with signatures or ramification indices n_i at these points, there exist ramified Galois covers $p : Y \to X$, albeit noncanonical, ramified precisely over the x_i with the given ramification indices.

Let $r : \mathbb{H} \to Y$ be the simply connected covering projection of Y. By definition, $\pi_o = \pi_1(Y)$, and we have the commutative diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\ r\ } & Y \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array} \tag{2.2.1.1}$$

with $q = p \circ r$. Let y_i be the image of z_i in Y, and let $\mathcal{R} = \{x_i\}$, with $\{x_i = p(y_i) \mid 1 \le i \le m\}$.

The map $r : \mathbb{H} \to Y$ is a local isomorphism. In fact, if $z \in \mathbb{H}$ maps to $y \in Y$, then r induces an isomorphism $\pi_z \xrightarrow{\sim} \Gamma_y$ of isotropy subgroups of π and Γ, respectively, as well as an isomorphism of a neighborhood of z onto that of y, respecting the actions of the isotropy groups.

Since the action of π_o is free on $\mathbb{H}$, by using the invariant direct image functor $r_*^{\pi_o}$, the study of (Γ, G)-bundles on Y reduces to the study of (π, G)-bundles on $\mathbb{H}$ and, thus, the study of (π, G)-bundles on $\mathbb{H}$ reduces to an algebraic problem since Y is a compact Riemann surface and hence a smooth projective curve.

2.2.2. Remark. By *local data* we mean, *the subset $\mathcal{R} \subset X$, together with the ramification indices or signatures n_i and local cyclic coverings of order n_i in formal neighborhoods of the ramification points.* It is obvious from the above discussion that constructions that involve only the local data are independent of the choice of Y, since in principle one could have used the universal cover $\mathbb{H}$. This could also be seen by using *orbifold stacks* constructed from the local data. In the course of this work, however, we will work with a fixed Y.

2.2.3. Definition. A (Γ, G)-bundle over Y is a principal G-bundle E (with a right G-action) together with a lift of the action of Γ on the total space of E as bundle automorphisms preserving the action of G.

2.2.4. Remark. Note that the actions of G and Γ on the total space of E commute, which is equivalent to the above condition that Γ acts as automorphisms preserving the action of G.

Now a (Γ, G)-bundle E on Y is *locally* a (Γ_y, G)-bundle at y. Recall that this (Γ_y, G)-bundle is *defined by a representation*; i.e., if N_y is a sufficiently small Γ_y-stable formal neighborhood of y, then this bundle is isomorphic to the (Γ_y, G)-bundle $N_y \times G$, for the *twisted* Γ_y-action on $E \times G$ given by a representation $\rho_y : \Gamma_y \longrightarrow G$, defined as

$$\gamma \cdot (u, g) = (\gamma u, \rho_y(\gamma) g), \ \ u \in N_y, \ \ \gamma \in \Gamma_y \tag{2.2.4.1}$$

(see for example Grothendieck [17, Proposition 1, p. 6], and in the setting of formal neighborhoods, see the more recent paper of Teleman and Woodward [40, Lemma 2.5]).

2.2.5. Observation. It is easily seen that these (Γ_y, G)-bundles given by representations are isomorphic as (Γ_y, G)-bundles if and only if the defining representations are equivalent. We call ρ_y the *local representations* associated to a (Γ, G)-bundle.

2.2.6. Definition. Let E be a (Γ, G)-bundle on Y. The *local type* of E at y is defined as the equivalence class of the local representation ρ_y and is denoted by τ_y.

By $\boldsymbol{\tau}$ we denote the set $\{\tau_y \mid y \in p^{-1}\mathcal{R})\}$ (see Definition 1.0.1). Let us denote by

$$\mathsf{Bun}_Y^{\boldsymbol{\tau}}(\Gamma, G) = \left\{ \begin{array}{l} \text{the set of isomorphism classes of} \\ (\Gamma, G) \text{ bundles with local type } \boldsymbol{\tau} \end{array} \right\}. \quad (2.2.6.1)$$

Let $D_x = Spec\ A$. Similarly, for $y \in p^{-1}(\mathcal{R})$, let $N_y = Spec\ B$, where B is the integral closure of A in $L = K(\omega)$, where ω is a primitive dth-root of z, $d = |\Gamma_y|$ and z is the uniformizer of A. Let $p : N_y \to D_x \simeq N_y/\Gamma_y$ be the totally ramified covering projection. Let E be the (Γ, G)-bundle on Y and $y \in p^{-1}(\mathcal{R})$. Consider the restriction of E to N_y. Then as we have seen above in (2.2.4.1), as a (Γ_y, G) bundle we can identify $E|_{N_y}$ with the trivial bundle $N_y \times G$ together with the twisted Γ_y-action.

2.2.7. Definition. Define $\bigcup_y$ to be the group

$$\bigcup_y = Aut_{(\Gamma_y, G)}(E|_{N_y}) \quad (2.2.7.1)$$

of (Γ_y, G) automorphisms of E over N_y. We call $\bigcup_y$ *the unit group* (or more precisely *the local unit group* at $y \in Y$) associated to E.

We work with notations fixed above. Let $\rho_y : \Gamma_y \to G$ be a representation. Let $\ell = rank(G)$, and we represent the maximal torus $T \subset G$ in the diagonal form as

$$T = \begin{bmatrix} t_1 & & 0 \\ & \cdot & \\ & & \cdot & \\ 0 & & t_\ell \end{bmatrix}, \quad (2.2.7.2)$$

where $\{t_1, \ldots, t_\ell\}$ is a basis of $X(T)$.

Since Γ_y is cyclic, we can suppose that the representation ρ_y of Γ_y in G factors through T (by a suitable conjugation).

The action of Γ_y on N_y canonically determines a character as follows. The action determines an action of Γ_y on the tangent space T_y to N_y at y. Since the tangent space to N_y is one-dimensional, this action is given by a character which we denote by χ_y (which is of order d). Fix a generator γ in Γ_y. The character χ_y is given by

$$\chi_y(\gamma).\omega = \zeta.\omega, \tag{2.2.7.3}$$

where ζ is a primitive dth-root of unity.

2.2.8. Lemma. *Let Γ_y be a cyclic group of order d acting on N_y as above. Then we have a canonical identification*

$$Hom(\Gamma_y, T) \simeq \frac{Y(T)}{d.Y(T)}. \tag{2.2.8.1}$$

Proof. This can easily be seen as follows. Observe that $X(\Gamma_y) \simeq \mathbb{Z}/d\mathbb{Z}$ by the canonical choice of the character χ_y as in (2.2.7.3). Then, we see that

$$Hom(\Gamma_y, T) = Hom(X(T), X(\Gamma_y)) = Hom(X(T), \mathbb{Z}/d\mathbb{Z}) = \frac{Y(T)}{d.Y(T)}. \quad \square$$

2.2.9. Remark. This lemma can be seen in the light of Remark 2.1.8. The equivalence class of a representation in $Hom(\Gamma_y, T)$ is given by the conjugacy class of the image of γ and hence a point of the Weyl alcove.

We now elaborate this identification for setting up the notations which play a key role in the next theorem.

Given a representation $\rho_y \in Hom(\Gamma_y, T)$, the image $\rho_y(\gamma)$ takes the form

$$\rho_y(\gamma) = \begin{bmatrix} \chi_y(\gamma)^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \chi_y(\gamma)^{a_\ell} \end{bmatrix}, \tag{2.2.9.1}$$

i.e., $\rho_y(\gamma)$ takes the form

$$\rho_y(\gamma) = \begin{bmatrix} \zeta^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \zeta^{a_\ell} \end{bmatrix} \text{ with } a_i \in \mathbb{Z}. \tag{2.2.9.2}$$

We can suppose that $|a_i| < d$ for all i (or even $0 \leq a_i < d$) and take

$$\eta_i = a_i/d, \text{ so that } |\eta_i| < 1. \tag{2.2.9.3}$$

Note that the numbers $\{a_1, a_2, \ldots, a_\ell\}$ are determined uniquely modulo d. Further, this is independent of the choice of ζ.

In terms of the local coordinates ω and z, we may identify the function ω^{a_i} with z^{η_i} where $z = \omega^d$. Define the rational map $\Delta : N_y \longrightarrow T$ or equivalently a morphism of the punctured disc $N_y - (0)$ as

$$\Delta = \Delta(\omega) = \begin{bmatrix} \omega^{a_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \omega^{a_\ell} \end{bmatrix} = \begin{bmatrix} z^{\eta_1} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & z^{\eta_\ell} \end{bmatrix}. \tag{2.2.9.4}$$

Then we have

$$\Delta(\gamma u) = \rho(\gamma)\Delta(u), \quad u \in N_y, \tag{2.2.9.5}$$

where Δ can be taken as a rational map $\Delta : N_y \longrightarrow G$ (through $T \hookrightarrow G$).

Consider the restriction of Δ to the *punctured disc*, and view it as a 1-PS i.e., $\Delta|_{Spec\ L} : \mathbb{G}_{m,L} \longrightarrow G$. This automatically gives a rational 1-PS of G, i.e., an element $\theta_{\tau_y} \in Y(T) \otimes \mathbb{Q}$, and the key point to note is that

$$d.\theta_{\tau_y} = \Delta, \qquad \text{i.e., } \theta_{\tau_y} \in \frac{Y(T)}{d.Y(T)}. \tag{2.2.9.6}$$

The association $\rho_y \mapsto \theta_{\tau_y}$ gives explicitly the identification obtained in Lemma 2.2.8. This is precisely what is described in terms of alcoves in Remark 2.1.8.

2.2.10. Remark. Note that a choice of Δ is determined up to (right) multiplication by an element from $G(K)$.

2.3. The unit group and parahoric groups. The aim of this section is to prove the following:

2.3.1. Theorem. *The unit group* $\bigcup_y$ *(Definition* 2.2.7*) is isomorphic to a parahoric subgroup* $\mathcal{P}_{\theta_{\tau_y}}(K)$ *of* $G(K)$ *associated to the element* $\theta_{\tau_y} \in Y(T)\otimes\mathbb{Q}$. *Conversely, if* $\mathcal{P}_\theta(K)$ *is any parahoric subgroup of* $G(K)$*, then there exists a positive integer* d*, and a field extension* $L = K(\omega)$ *of degree* d *over* K *such that*

$$\mathcal{P}_\theta(K) \simeq \bigcup_y. \tag{2.3.1.1}$$

Proof. We first give a different description of the elements of $\bigcup_y$. By (2.2.4.1) a (Γ_y, G)-bundle on Y gets a Γ_y-equivariant trivialization; in other words, the Γ_y-action on $N_y \times G$ is given by a representation $\rho : \Gamma_y \longrightarrow G$

$$\gamma \cdot (u, g) = (\gamma u, \rho(\gamma)g), \ u \in N_y, \ \gamma \in \Gamma_y. \tag{2.3.1.2}$$

Let $\phi_0 \in \bigcup_y$, i.e., the map

$$\phi_0 : N_y \times G \longrightarrow N_y \times G \tag{2.3.1.3}$$

is equivariant for the Γ_y-action. Equivariance under G (by right multiplication) implies that

$$\phi_0(u, g) = (u, \phi(u)g),$$

where $\phi : N_{_y} \longrightarrow G$ is a regular map satisfying the Γ_y-equivariance

$$\phi(\gamma \cdot u) = \rho(\gamma)\phi(u)\rho(\gamma)^{-1}, u \in N_{_y}, \gamma \in \Gamma_y. \tag{2.3.1.4}$$

We may thus identify $\bigcup_y$ with

$$\bigcup_y = \{\phi : N_{_y} \to G \mid (2.3.1.4) \text{ holds}\} = Mor^{\Gamma_y}(N_y, G). \tag{2.3.1.5}$$

Since $N_y = Spec\ B$, we can view $\bigcup_y \subset G(B) \subset G(L)$.

Let Δ be as in (2.2.9.4). Consider the inner automorphism defined by Δ,

$$i_{_\Delta} : G(L) \to G(L) \tag{2.3.1.6}$$

given by $i_{_\Delta}(\eta) = \Delta^{-1}.\eta.\Delta$. Define

$$\bigcup\nolimits'_y := i_{_\Delta}\left(\bigcup\nolimits_y\right) \tag{2.3.1.7}$$

Let $\psi = i_{_\Delta}(\phi) = \Delta^{-1}.\phi.\Delta$ with $\phi \in \bigcup_y$. Then we observe that

$$\psi(\gamma u) = \psi(u)$$

so that $\psi \in G(L)^{\Gamma_y}$. That is, it *descends* to a rational function $\tilde{\psi} : D_{_x} \longrightarrow G$, where $\tilde{\psi}(z) := \psi(\omega)$. In other words, we get

$$\bigcup\nolimits'_y \subset G(K) = G(L)^{\Gamma_y}. \tag{2.3.1.8}$$

Note that $\bigcup'_y$ depends on the choice of Δ, and a different choice of Δ gives a subgroup which is a conjugate of $\bigcup'_y$ by an element of $G(K)$ (see Remark 2.2.10).

Then we *claim* the following:

$$\bigcup\nolimits'_y = \mathcal{P}_{_{\theta_{\tau_y}}}(K), \tag{2.3.1.9}$$

where $\theta_{\tau_y} \in Y(T) \otimes \mathbb{Q}$ is as in (2.2.9.6). Recall that

$$\mathcal{P}_{_{\theta_{\tau_y}}}(K) = \langle T(A),\ U_r(z^{m_r(\theta)}A),\ r \in R\rangle. \tag{2.3.1.10}$$

Let $\psi \in \bigcup'_y$, and let $\psi = i_{_\Delta}(\phi)$, with $\phi \in \bigcup_y$. Thus,

$$\phi = \Delta\psi\Delta^{-1}.$$

Consider the map $\phi : N_y \to G$. Let $G^o \subset G$ denote the *big cell* determined by the roots R (i.e., the inverse image in G of a dense B-orbit in G/B).

Let us assume for the moment that $\phi(N_y) \in G^o$. In other words, ϕ can be described *uniquely* as a tuple $\left(\{\phi_r\}_{r\in R}\ ,\ \phi_t\right)$, with $\phi_r(u) \in U_r$ and $\phi_t(u) \in T$ for $u \in N_y$.

We first consider the tuples $\big(\phi_r(u)\big)_{r\in R}$ and the corresponding tuple for ψ, namely, $\big(\psi_r(u)\big)_{r\in R}$, where the $\phi_t\ :\ N_y \to T$ and

$$\{\phi_r, \psi_r : \mathbb{G}_{a,L} \to G \mid r \in R\}.$$

The uniqueness of the decomposition of elements in the big cell and the invariance property of ϕ translates into invariance for each of the ϕ_r and ϕ_t. In other words, we have

$$\phi_r(\omega) = \Delta\psi_r(\omega)\Delta^{-1}, \tag{2.3.1.11}$$

i.e.,

$$\phi_r(\omega) = \psi_r(\omega)\omega^{r(\Delta)}. \tag{2.3.1.12}$$

In terms of $\tilde{\psi}$, this gives

$$\phi_r(\omega) = \tilde{\psi}_r(z) z^{\frac{r(\Delta)}{d}}. \tag{2.3.1.13}$$

Now interpreting the condition that the ϕ's are regular functions in the variable ω at $\omega = 0$, we see that the *order of pole* for $\psi_r(z)$ at $z = 0$, is bounded above by $[\frac{r(\Delta)}{d}]$ (the biggest integer smaller than or equal to $\frac{r(\Delta)}{d}$). In other words $\forall r \in R$,

$$\tilde{\psi}_r(z) \in U_r(z^{-[r(\theta_\Delta)]}A) = U_r(z^{m_r(\theta_\Delta)}A), \tag{2.3.1.14}$$

and hence $\tilde{\psi} \in \mathcal{P}_{\theta_{\tau_y}}(K)$.

Now, towards completing the proof of the claim (2.3.1.9), since $\phi_t(u) \in T$, by (2.3.1.4) it follows that ϕ_t is Γ_y-invariant and hence $\tilde{\psi}_t \in T(A)$.

We now take a closer look at the map $\phi : N_y \to G$. In general, the image $\phi(N_y)$ need not be contained in the big cell G^o. So we consider the image $\phi(y)$ of the point $y \in N_y$. Let $\phi(y) = g_o \in G$. Since the point $y \in N_y$ is Γ_y-fixed, it implies that $g_o \in G^{\Gamma_y}$. Thus, by (2.3.1.4), the point g_o lies in the centralizer $C_G(\rho(\gamma))$, of $\rho(\gamma)$ in G; the group $C_G(\rho(\gamma))$ is a Levi subgroup L_θ of the standard parabolic subgroup of G determined by the coroot $\theta = \theta_{\tau_y}$. The Levi subgroup can be described in terms of the $u_r : \mathbb{G}_a \to G$ given as in (2.1.1.2), namely $C_G(\rho(\gamma)) = L_\theta = \langle T,\ u_r(\mathbb{C}) \mid\ r \in R, \text{ and } m_r(\theta) = (\theta, r) = 0\rangle$ (see also section 2.1.11).

Furthermore, by the equation (2.2.9.4), which defines the function $\Delta : \mathbb{G}_m \to T$, it is immediate from (2.1.1.2) that $\Delta^{-1}.u_r.\Delta = u_r$ if $m_r(\theta) = (\theta, r) = 0$. The same obviously holds for the elements of the maximal torus. Hence the elements that commute with $\rho(\gamma)$ also commute with Δ. This implies immediately that $g_o = i_\Delta(g_o)$ and therefore g_o is an element of the parahoric subgroup $\mathcal{P}_{\theta_{\tau_y}}(K)$.

Now define $\phi_1 : N_y \to G$ by $\phi_1(u) = g_o^{-1}\phi(u)$. Then, $\phi_1(y) = 1$, and hence it lies in G^o. Hence by the openness of G^o and the fact that N_y is a formal neighborhood of y, it follows that $\phi_1(N_y) \subset G^o$. Also, clearly ϕ_1 satisfies (2.3.1.4) and hence by the earlier argument together with the fact that $i_\Delta(g_o) \in \mathcal{P}_{\theta_{\tau_y}}(K)$, we see that $i_\Delta(\phi) = \psi$ is an element in $\mathcal{P}_{\theta_{\tau_y}}(K)$. This completes the proof of the claim (2.3.1.9) without any assumptions.

Conversely, we show that any parahoric subgroup of $G(K)$ can be identified (up to conjugation by $G(K)$) with a *unit group* $\bigcup_y$. Let $\theta \in \mathbb{E}_{\mathbb{Q}}$, and let $\mathcal{P}_{\theta}(K)$ be a parahoric subgroup. We would like to modify θ to a θ_{τ_y} for a suitable $\Delta \in Y(T)$ (as in (2.2.9.6)) so that, interpreted as unit groups, we get $\mathcal{P}_{\theta}(K) \simeq \mathcal{P}_{\theta_{\tau_y}}(K) \simeq \bigcup_y$.

Expressing θ in terms of generators and clearing denominators, we see that there exists a positive integer d so that $d.\theta \in Y(T)$. Then the obvious choice for Δ is simply $d.\theta$, which therefore forces $\Delta \in Y(T)$. The choice of the *least* such d makes the choice of the local ramification index canonical.

Now we view Δ as a "rational" map $\Delta : N_y \to T$ and hence Δ can be expressed as in (2.2.9.4), the a_i's being determined by the following considerations: for $r \in R$ any root we define

$$r(\Delta) = d.(\theta, r).$$

By the discussion following Lemma 2.2.8, we have a $\theta = \theta_{\tau_y} \in \frac{Y(T)}{d.Y(T)}$, and the identification of Lemma 2.2.8 gives the representation $\rho : \Gamma_y \to T \subset G$. The representation ρ gives the action on the root groups $U_r(B) \subset G(B)$ which are given by (see (2.1.1.3))

$$\rho(\gamma).U_r(B).\rho(\gamma)^{-1} = U_r(\zeta^{r(\Delta)}B). \tag{2.3.1.15}$$

Retracing the steps in the first half of the proof, it is easy to see that $\mathcal{P}_{\theta}(K) \simeq \bigcup_y$, completing the proof of the theorem. □

2.3.1.1. Notation. Let $\theta \in Y(T) \otimes \mathbb{Q}$. Let $\Delta = d.\theta$ as above. Then we identify θ with θ_{τ_y} and denote by ρ_{θ} the homomorphism $\rho_y : \Gamma_y \to T$ associated to θ by Lemma 2.2.8. Note that ρ_{θ} acts on the root groups as in (2.3.1.15).

2.3.2. Remark. In the notations used above, if $m_r(\theta_{\tau_y}) < 1$ for all $r \in R$, such elements θ_{τ_y} in $\mathbb{E}_{\mathbb{Q}} = Y(T) \otimes \mathbb{Q}$ are precisely the points in the interior of the alcove $\mathcal{A}$ (see Remarks 2.1.3 and 2.1.10).

2.3.3. Remark. The first half of Theorem 2.3.1 can be seen as a consequence of general results on Galois fixed points in Bruhat–Tits buildings and a theorem of Rousseau ([41, 2.6.1], [16]; see also [29, Section 7]). For the converse in Theorem 2.3.1 considered in the general setting of Bruhat–Tits theory, we refer the reader to the papers by Gille ([14, Lemma I.1.3.2]), Larsen ([23, Lemma 2.4]), and Serre ([34, Proposition 8, p. 546]). The point of view presented here in terms of unit groups has its origins in the paper of Weil [44] and Seshadri [36], where completely analogous phenomena are studied in the setting of the general linear group. The striking fact is that when carried out for semisimple simply connected groups, they yield all parahoric groups when the residue field is of characteristic 0.

2.3.4. Example. Let us now take $G = GL(m)$. We invite the reader to compare this discussion with the one in Weil ([44, p. 56]). Then we can write $\phi = ||\phi_{ij}(\omega)||$, $\tilde{\psi} = ||\tilde{\psi_{ij}}(z)||$, $1 \leq i, j \leq m$ (as matrices). Then the equation (2.3.1.13) takes the form

$$\phi_{ij}(\omega) = \tilde{\psi_{ij}}(z) z^{\alpha_i - \alpha_j}. \tag{2.3.4.1}$$

We can suppose that $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m < 1$. Since $|\alpha_i - \alpha_j| < 1$, we deduce easily that $\tilde{\psi_{ij}}$ are regular i.e., $\bigcup_y \subset G(A)$. (To see this, suppose that $\tilde{\psi_{ij}}$ is *not* regular. Then considered as a function in ω ($z = \omega^d$), ψ_{ij} has a pole of order $\geq d$, whereas $z^{\alpha_i - \alpha_j}$ could have only a pole of order d (as a function in ω). But $\phi_{ij}(\omega)$ is regular, which leads to a contradiction).

2.3.5. Remark. It is remarked in [37, Case III, Page 8] that it was not clear whether the unit group in the situation considered there is a parahoric subgroup at all. In fact, this is indeed the case as can be seen from Theorem 2.3.1. Moreover, it is not too hard to check by some elementary computations that the unit group considered in [37, Case III, Page 8] does contain the standard Iwahori subgroup but only after a conjugation by a suitable element of $G(K)$.

3. The adèlic picture of (Γ, G)-bundles

3.1. We work with the notations of Section 2. In this section we give a description of (Γ, G) bundles analogous to the classical adèlic description as in Weil [44]; however, it plays no direct role in the subsequent sections.

Let E be a (Γ, G)-bundle on Y, and let $\mathsf{Bun}_Y^{\boldsymbol{\tau}}(\Gamma, G)$ be as in (2.2.6.1). Since the action of Γ on $Y - p^{-1}(\mathcal{R})$ is free, there is a principal G-bundle P on $X - \mathcal{R}$ such that then $E|_{Y-p^{-1}(\mathcal{R})} \simeq p^*(P)$. Since G is semisimple, by the theorem of Harder [20], P is *trivial*. Hence, $E|_{Y-p^{-1}(\mathcal{R})}$ is also *trivial* as a (Γ, G)-bundle.

Recall that around each point $y_i \in p^{-1}(\mathcal{R})$, we have formal neigbourhoods $N_{y_i} = Spec\ B_{y_i}$ with Γ_{y_i}-equivariant trivializations of $E|_{N_{y_i}}$ (see (2.2.4.1)). Note that by Beauville and Laszlo ([5]) any (Γ, G)-bundle of local type $\boldsymbol{\tau}$ can be obtained by patching $E|_{Y-p^{-1}(\mathcal{R})}$ with the $E|_{N_{y_i}}$'s (see Remark 5.2.6).

For simplicity of notation, we assume that $\mathcal{R} = \{x\}$. Two (Γ, G) bundles on Y are said to be *locally isomorphic at* x if they are isomorphic as (Γ, G)-bundles over $p^{-1}(D_x) = V_1$, D_x a formal neighborhood of x as above. We can suppose that V_1 is a disjoint union of Γ_y-invariant neighborhood N_y of y, with y being a point of Y lying over x. We see that two such bundles are locally isomorphic at x if and only if their restrictions to N_y are isomorphic as Γ_y-bundles. Observe that two (Γ, G)-bundles on Y are *locally isomorphic at x* if they are locally isomorphic in a formal neighborhood of any one point $y \in p^{-1}(x)$. Let

$$X_1 = X - x \quad \text{and} \quad Y_1 = p^{-1}(X_1). \tag{3.1.0.1}$$

Recall (2.2.4.1), the (Γ_y, G)-bundle $N_y \times G$ given by the twisted Γ_y action given by a representation $\rho_y : \Gamma_y \to G$.

$$\begin{gathered}\text{Let } E_1 := N_y \times G \text{ with the } (\Gamma_y, G)\text{-structure given by} \\ \gamma \cdot (u, g) = (\gamma u, \rho_y(\gamma) g),\ u \in N_y,\ \gamma \in \Gamma_y\end{gathered} \tag{3.1.0.2}$$

and

$$\begin{gathered}\text{Let } E_2 := Y_1 \times G \text{ with the } (\Gamma, G)\text{-structure given by} \\ \gamma \cdot (u, g) = (\gamma\, u, g),\ \gamma \in \Gamma \text{ and } u \in Y_1.\end{gathered} \tag{3.1.0.3}$$

Thus giving a (Γ, G)-bundle on Y of local type $\boldsymbol{\tau}$ (see Definition 2.2.6) is giving a *transition function*, i.e., a (Γ_y, G)-isomorphism

$$\Theta : E_2|_{N_y \cap Y_1} \longrightarrow E_1|_{N_y \cap Y_1}. \tag{3.1.0.4}$$

We denote by E_Θ the (Γ, G)-bundle given by the transition function Θ.

Observe that any transition function Θ can be viewed as a function Δ as in (2.2.9.5). In particular, if we take Δ as in (2.2.9.5), then viewed as a *transition function*, Δ defines a (Γ, G)-bundle, which we denote by E_Δ. We fix this bundle E_Δ as a *basepoint*.

By Theorem 2.3.1, this choice of Δ further identifies each unit group $\bigcup'_y$, $y \in p^{-1}(\mathcal{R})$, with a parahoric group $\mathcal{P}_{\theta_i}(K_{x_i})$, $x_i \in \mathcal{R}$. We fix such an identification.

3.1.1. Proposition. *For each $x_i \in \mathcal{R}$, fix a point $y_i \in p^{-1}(x_i)$. Further, fix at each y_i local data as in* (3.1.0.2) *and* (3.1.0.3) *and $\boldsymbol{\Delta} := \{\Delta_i\}$ as in* (2.2.9.5). *Let K_x be the quotient field of the complete local rings A_x at $x \in \mathcal{R}$, and let $k[X - \mathcal{R}]$ be the ring of functions on the affine curve $X - \mathcal{R}$. Then we have a well-defined set-theoretic identification*

$$\mathsf{Bun}^{\boldsymbol{\tau}}_Y(\Gamma, G) \simeq \Big[\prod_{x \in \mathcal{R}} \mathcal{P}_{\theta_i}(K_{x_i}) \backslash {\textstyle\prod_{x\in\mathcal{R}}} G(K_x) / G(k[X - \mathcal{R}])\Big], \tag{3.1.1.1}$$

where the basepoint $E_{\boldsymbol{\Delta}}$ given by the transition functions Δ_i's gets identified with the double coset represented by $1 \in \prod_{x\in\mathcal{R}} G(K_x)$.

Proof. For simplicity of notation, we assume again that $\mathcal{R} = \{x\}$. Let E_Θ and E_Υ be two (Γ, G)-bundles given by the transition functions Θ and Υ. Then E_Θ is (Γ, G)-isomorphic to E_Υ if and only if there exist a (Γ_y, G)-automorphism ϕ of E_1 and a (Γ, G)-automorphism μ of E_2 such that

$$\phi\ \Theta\ \mu = \Upsilon. \tag{3.1.1.2}$$

We now proceed to give a description of ϕ and μ which we base on the fixed choice of the function Δ.

Observe that by (3.1.0.3) the map μ is given by a morphism

$$\mu : Y_1 \times G \longrightarrow Y_1 \times G, \quad (u, g) \to (u, \mu(u)g), \tag{3.1.1.3}$$

where $\mu(\gamma \cdot u) = \mu(u), \gamma \in \Gamma$. In other words, the map μ goes down to a morphism $X_1 \longrightarrow G$, and we can view μ as an element in $G(X - x)$.

We now trace the various identifications by restricting the above picture to $N_y^* = N_y - (0)$; note that the (Γ, G)-isomorphism Θ is completely characterized by its restriction to N_y^*.

We observe by (3.1.0.3) that the restriction of E_2 to N_y^* is the (Γ_y, G)-bundle $N_y^* \times G$ over N_y^* with the action of Γ_y given by

$$\gamma : N_y^* \times G \longrightarrow N_y^* \times G,\ \gamma \in \Gamma_y, \quad \gamma(u, g) = (\gamma u, g). \tag{3.1.1.4}$$

The restriction of E_1 to N_y^* is the (Γ_y, G)-bundle $N_y^* \times G$ on N_y^* with the action of Γ_y given by

$$\gamma : N_y^* \times G \longrightarrow N_y^* \times G, \quad \gamma(u, g) = (\rho u, \rho(\gamma)g),\ \gamma \in \Gamma_y. \tag{3.1.1.5}$$

The restriction of $\Theta|_{N_y^*}$ of Θ to N_y^* (denoted again by Θ) is then a (Γ_y, G)-isomorphism of the bundle in (3.1.1.3) with the one of (3.1.1.2). We see easily that Θ is defined by the map

$$N_y^* \times G \longrightarrow N_y^* \times G, \quad (u, g) \longrightarrow (u, \Theta(u)g), \tag{3.1.1.6}$$

where $\Theta : N_y^* \to G$ is such that $\Theta(\gamma \cdot u) = \rho(\gamma)\Theta(u)$.

Recall that the map Δ as in (2.2.9.4) is a morphism $N_y^* \longrightarrow G$ and has similar properties. Thus, we can write

$$\Theta = \Delta\Theta_o \text{ such that } \Theta_o(\gamma u) = \Theta_o(u), \tag{3.1.1.7}$$

i.e., Θ_o descends to a regular map $D_x^* \longrightarrow G$, $D_x^* = D_x - (0)$.

The equivalence relation (3.1.1.2) therefore takes the form

$$\phi\ (\Delta\Theta_o)\ \mu = \Upsilon. \tag{3.1.1.8}$$

Multiplying on either side by Δ^{-1}, we get

$$(\Delta^{-1}\phi\Delta)\ \Theta_o\ \mu = \Delta^{-1}\Upsilon = \Upsilon_o. \tag{3.1.1.9}$$

By the proof of Theorem 2.3.1, ϕ identifies with an element $\psi(= i_\Delta(\phi))$ of the unit group $\bigcup'_y$, and we can write (3.1.1.9) as

$$\psi\ \Theta_o\ \mu = \Delta^{-1}\Upsilon = \Upsilon_o. \tag{3.1.1.10}$$

Therefore, $\Theta_o \in G(K_x)$ and $\psi \in \bigcup'_y = \mathcal{P}_\theta(K_x)$, and by (3.1.1.3) μ becomes a regular map $X_1 \longrightarrow G$ i.e., $\mu \in G(X - x)$.

From (3.1.1.10), we conclude that Θ and Υ give isomorphic (Γ, G)-bundles if and only if Θ_o and Υ_o are equivalent by the double coset relation; i.e., they give the same point in the double coset space which we denote by $[\Theta_o]$. If $\Theta_o \in G(K_x)$ gives $[\Theta_o]$ in the double coset space, by using (3.1.1.7) and the choice of Δ, we can reverse the process to get Θ and hence E_Θ. Thus, we get the following set-theoretic identification:

$$\mathsf{Bun}^{\boldsymbol{\tau}}_Y(\Gamma, G) \simeq \left[\mathcal{P}_\theta(K_x)\backslash^{G(K_x)}/G(X - x)\right], \tag{3.1.1.11}$$

where $E_\Theta \mapsto [\Theta_o]$.

Note that the basepoint E_Δ given by the transition functions Δ gets mapped to the *identity coset*; i.e., it is represented by $1 \in G(K_x)$ in the double coset space. □

4. Invariant direct image

4.1. In this section we study the torsor-analogue of the sheaf theoretic invariant direct image defined by Grothendieck [17]. The remarks in this section owe much to key inputs from Brian Conrad and Pramathanath Sastry.

4.1.1. Definition. Let $p : W \to T$ be a finite flat surjective morphism of normal, integral noetherian schemes such that the field extension of the function fields is Galois with Galois group $\Gamma := \mathrm{Gal}(k(W)/k(T))$. Observe that Γ acts on W as T-automorphisms and $T = W/\Gamma$. Such a morphism $p : W \to T$ is called a *Galois covering* with Galois group Γ.

4.1.2. Remark. The assumptions ensure that the Γ action extends to W. Observe that since W is normal, it is the integral closure of T in $k(W)$, the function field of W. Let $\phi \in \Gamma$, then ϕ acts as an automorphism of W over T. To see this, we may assume that $W = \mathit{Spec}\ B$ and $T = \mathit{Spec}\ A$. Then, B can be taken to be the integral closure of A in $k(W)$ and hence, $\phi(B) \subset B$, implying that ϕ is an automorphism of W over T. It also follows that $A = B^\Gamma$, since A is also normal. Hence, $T = W/\Gamma$ in the sense of Mumford.

Following [8], we can define the *direct image functor* p_* as the Weil restriction of scalars, i.e., we have a group functor $p_*(\mathscr{G}) := \mathrm{Res}_{W/T}(\mathscr{G})$ with the property that for any T-scheme S, we have a canonical bijection, i.e, the *adjunction*

$$Hom_T(S, p_*(\mathscr{G}) \simeq Hom_W(S \times_T W, \mathscr{G}), \tag{4.1.2.1}$$

which is functorial in S and $\mathscr{G}$.

We assume that $\mathscr{G}$ is an *affine group scheme* over W, so that $p_*(\mathscr{G})$ is representable by a group scheme (see [8, Theorem 4 and Proposition 6]). Suppose also that the Γ-action lifts to an action on the group scheme $\mathscr{G}$, in such a manner that the "multiplication map" and the "inverse map" on $\mathscr{G}$ are equivariant. We will term such a group scheme a Γ-*group scheme* on W.

Let S be a scheme over T, and $f \in p_*(\mathscr{G})(S) = Hom_W(S \times_T W, \mathscr{G})$, and let $\gamma \in \Gamma$.

There is a left action of Γ on $\mathscr{G}$ and a left action on $S \times_T W$ induced by its action on W. This induces a natural right action of Γ on $p_*(\mathscr{G})(S)$ given by

$$(f.\gamma)([s,w]) := \gamma^{-1}.f(\gamma.[s,w]), \;\; [s,w] \in S \times_T W. \tag{4.1.2.2}$$

We can now take the *fixed point* subscheme under the action of Γ. The general results on fixed point subschemes given in [13, Section 3] can be applied to our situation *since we are in characteristic* 0 and we get a canonically defined *smooth* closed X-subgroup scheme $p_*(\mathscr{G})^\Gamma \subset p_*(\mathscr{G})$. This is representable in our case since $p_*(\mathscr{G})$ is representable.

4.1.3. Definition (Invariant direct image)**.** Let $p : W \to T$ be as above, and let $\Gamma = \mathrm{Gal}(W/T)$. Let $\mathscr{G}$ be a smooth affine Γ-group scheme over W. We define the *invariant direct image* of $\mathscr{G}$ to be

$$p_*^\Gamma(\mathscr{G}) := p_*(\mathscr{G})^\Gamma, \tag{4.1.3.1}$$

i.e., for any T-scheme S, we have $p_*^\Gamma(\mathscr{G})(S) = \mathscr{G}(S \times_T W)^\Gamma$.

More generally, let E be any affine scheme over W with a lift of the Γ action. Then we define the *invariant direct image* of E to be $p_*^\Gamma(E) := p_*(E)^\Gamma$.

4.1.4. Lemma. *Let $p : W \to T$ be a finite flat surjective morphism of noetherien schemes. Let $\mathscr{G}$ be a smooth affine group scheme on W, and let E be a $\mathscr{G}$-torsor on W. Then $p_*(E)$ is a $p_*(\mathscr{G})$-torsor on T.*

Proof. The hypothesis implies that $p_*(\mathscr{G})$ and $p_*(E)$ exist as smooth schemes over T. The lemma follows immediately from the property that the direct image functor p_* respects fiber products (this is immediate from the functorial definition of restriction of scalars, see for example [11, Proposition A.5.2]). Applying p_* to the action map $\mathscr{G} \times_W E \to E$, it gives $p_*(\mathscr{G}) \times_T p_*(E) \to p_*(E)$. Moreover, we have an isomorphism,

$$\mathscr{G} \times_W E \simeq E \times_W E. \tag{4.1.4.1}$$

Now again apply p_* to get the desired isomorphism

$$p_*(\mathscr{G}) \times_T p_*(E) \simeq p_*(E) \times_T p_*(E). \tag{4.1.4.2}$$

This is also given in [11, Corollary A.5.4(3)] but with a more complicated proof. □

If the Γ-action lifts to an action on a W-group scheme so that the multiplication map and the inverse map are equivariant, then we will term such a group scheme a Γ-*group scheme* on W.

4.1.5. Lemma. *Suppose further that $p : W \to T$ is a Galois cover with Galois group Γ (Definition 4.1.1). Let $\mathscr{G}$ be a Γ-group scheme on W, and let E be a $(\Gamma, \mathscr{G})$-torsor on W. Let $p_*^\Gamma(\mathscr{G}) = \mathscr{H}$, and let $p_*^\Gamma(E) = F$. Then, F is a $\mathscr{H}$-torsor.*

Proof. For the first part, apply the *fixed point* functor to (4.1.4.2), i.e.,

$$(p_*(\mathscr{G}) \times_T p_*(E))^\Gamma \simeq (p_*(E) \times_T p_*(E))^\Gamma, \tag{4.1.5.1}$$

which gives

$$\mathscr{H} \times_T F \simeq F \times_T F, \tag{4.1.5.2}$$

proving that F is a $\mathscr{H}$-torsor on X. The smoothness of F over T holds as well since we work in char 0 (cf. [13, Section 3]). □

4.1.6. Theorem. *Let $\mathsf{Bun}_W(\Gamma, \mathscr{G})$ and $\mathsf{Bun}_T(\mathscr{H})$ denote the stacks of $(\Gamma, \mathscr{G})$-torsors on W and $\mathscr{H}$-torsors on T, respectively. Then the functor*

$$p_*^\Gamma : \mathsf{Bun}_W(\Gamma, \mathscr{G}) \to \mathsf{Bun}_T(\mathscr{H}) \tag{4.1.6.1}$$

is an isomorphism of stacks.

Proof. Lemma 4.1.5 shows that p_*^Γ gives a functor between the stacks. We now construct the candidate for the inverse.

Observe that the inclusion $p_*^\Gamma(\mathscr{G}) = \mathscr{H} \hookrightarrow p_*(\mathscr{G})$ gives by adjunction the morphism

$$p^*(\mathscr{H}) \to \mathscr{G} \tag{4.1.6.2}$$

of group schemes over W. Let $F \in \mathsf{Bun}_T(\mathscr{H})(S)$ be a $\mathscr{H}$-torsor on a T-scheme S. Let $p \times Id_S = q : W \times_T S \to S$ be the induced morphism. Observe that $q^*(F)$ becomes a $q^*(\mathscr{H})$-torsor on W, and via (4.1.6.2) we get the associated $\mathscr{G}$-torsor

$$q^*(F) \times^{q^*(\mathscr{H})} \mathscr{G}. \tag{4.1.6.3}$$

We observe that the $\mathscr{G}$-torsor $q^*(F) \times^{q^*(\mathscr{H})} \mathscr{G}$ is a $(\Gamma, \mathscr{G})$-torsor, where the Γ-action comes from the underlying Γ-action on $\mathscr{G}$. We also get the natural Γ-equivariant morphism,

$$q^*(F) \to q^*(F) \times^{q^*(\mathscr{H})} \mathscr{G}. \tag{4.1.6.4}$$

Now by pushing down this morphism using q_*^Γ, we get

$$F \to q_*^\Gamma(q^*(F) \times^{q^*(\mathscr{H})} \mathscr{G}). \tag{4.1.6.5}$$

To check that this last map is an isomorphism, we can restrict to étale neighborhoods on T where F is trivial (i.e., isomorphic to $\mathscr{H}$ as an $\mathscr{H}$-torsor), but this is obvious. This shows that $q_*^\Gamma(q^*(F) \times^{q^*(\mathscr{H})} \mathscr{G}) \simeq F$.

We need to check that this construction provides an equivalence of categories.

Suppose that E is a $(\Gamma, \mathscr{G})$-torsor on W and $F = p_*^\Gamma(E)$ is a $\mathscr{H}$-torsor. Now $p^*(F)$ is a $p^*(\mathscr{H})$-torsor. Therefore via (4.1.6.2), taking associated constructions, we get a $\mathscr{G}$-torsor $p^*(F) \times^{p^*(\mathscr{H})} \mathscr{G}$.

Again by adjunction applied to the inclusion $F = p_*^\Gamma(E) \hookrightarrow p_*(E)$, we get the morphism

$$p^*(F) \to E \tag{4.1.6.6}$$

and hence a morphism

$$p^*(F) \times^{p^*(\mathscr{H})} \mathscr{G} \to E \tag{4.1.6.7}$$

of $\mathscr{G}$-torsors.

Claim. The morphism (4.1.6.7) is an isomorphism of $(\Gamma, \mathscr{G})$-torsors.

Proof of Claim. Since the map p is finite, a cofinal system of étale neighborhoods of the fiber $p^{-1}(t)$ of a point $t \in T$ is given by pullbacks of étale neighborhoods of $t \in T$. This is a consequence of the compatibility of formation of strict henselization with respect to finite base change of algebras (cf. [19, EGA IV.4, 18.8.10]). This allows us to work étale locally on T.

Thus we may assume that F is trivial on T, and by the discussion above we may assume that E is also a trivial $\mathscr{G}$-torsor on W. This reduces to verifying that the map $p^*(\mathscr{H}) \times^{p^*(\mathscr{H})} \mathscr{G} \to \mathscr{G}$ is an isomorphism of trivial $\mathscr{G}$-torsors, which is obvious. This completes the proof of the theorem. □

4.1.7. Remark. As the referee pointed out to us, the equivalence can also be deduced from the observation that both the stacks in Theorem 4.1.6 are gerbes.

4.1.8. Remark. The notion of invariant direct image using Weil restriction of scalars is implicit in Edixhoven [13] and also in Pappas and Rapoport [29].

4.1.9. Remark. Let $\mathcal{O}_W(\mathscr{G})$ be the sheaf of groups on W for the étale topology associated to the group scheme $\mathscr{G}$. In fact, it is a Γ-sheaf of groups. The fact used in the argument above, namely étale trivializing neighborhoods in W, can be chosen as inverse images of étale opens from T shows firstly that any $\mathscr{G}$-torsor can be trivialized in such étale neighborhoods. In particular, taking $(\Gamma, \mathscr{G})$-torsors, this immediately gives a natural isomorphism of cohomology sets,

$$H^1_{\acute{e}t}(W, \Gamma, \mathcal{O}_W(\mathscr{G})) \simeq H^1_{\acute{e}t}(T, \mathcal{O}_T(p_*^\Gamma(\mathscr{G})). \tag{4.1.9.1}$$

This identifies the isomorphism classes of $(\Gamma, \mathscr{G})$-torsors on W with isomorphism classes of $p_*^\Gamma(\mathscr{G})$-torsors on T. The proof given above for the theorem gives a canonical identification and hence a stronger statement on stacks.

5. Bruhat–Tits group schemes and torsors

5.1. A Γ-group scheme on Y. We now revert to the notations in Section 2, where $p : Y \to X$ ramified over $\mathcal{R}$. The following construction plays an important role in the subsequent sections.

5.1.0.1. Notation. Fix a (Γ, G)-bundle F of local type $\boldsymbol{\tau}$. Let

$$\mathcal{G}_{F} := F \times^G G \tag{5.1.0.2}$$

denote the associated "adjoint" group scheme associated to F, G acting on itself by inner conjugation.

Recall that any (Γ, G)-bundle of local type $\boldsymbol{\tau}$ is locally isomorphic to any preassigned (Γ, G)-bundle of local type, in particular to the fixed bundle F. This therefore gives an identification of (Γ, G)-bundles of type $\boldsymbol{\tau}$ with those that are locally modeled after the fixed bundle F. We give a formal shape to this intuitive picture (see Grothendieck [18, Proposition 4.5.2]).

The bundle F can be viewed as a left $\mathcal{G}_{F}$-torsor, where the action is by automorphisms. In the sense of Giraud [15], F is a $(\mathcal{G}_{F}, G)$-bitorsor.

For any G-torsor E on Y coming with a right G-action, let E^{op} be the G-torsor with the induced left action $g.x := xg^{-1}$. We then have the "contracted product"

$$E \wedge^G F^{op} := \frac{E \times_Y F}{(xg, y) \sim (x, g.y)}. \tag{5.1.0.3}$$

It is a fact that the sheaf of local sections of $E \wedge^G F^{op}$ is the sheaf $Isom(E, F)$ of local isomorphisms of E with F. Since we work with affine group schemes, by usual descent for affine schemes, the contracted product is representable as a scheme.

Since F is a $(\mathcal{G}_{F}, G)$-bitorsor, it follows that F^{op} is a $(G, \mathcal{G}_{F})$-bitorsor. Thus the contracted product $E \wedge^G F^{op}$ is a right $\mathcal{G}_{F}$-torsor on Y; in fact we get the equivalence of stacks,

$$\mathsf{Bun}_{Y}(G) \simeq \mathsf{Bun}_{Y}(\mathcal{G}_{F}) \tag{5.1.0.4}$$

given by $E \mapsto E \wedge^G F^{op}$.

Now let E be a (Γ, G)-bundle of local type $\boldsymbol{\tau}$. Hence E is locally Γ-isomorphic to F. Since $\mathcal{G}_{F}$ is a Γ-group scheme, the association $E \mapsto E \wedge^{G} F^{op}$ in fact induces an identification

$$\mathsf{Bun}_{Y}^{\boldsymbol{\tau}}(\Gamma, G) \simeq \mathsf{Bun}_{Y}(\Gamma, \mathcal{G}_{F}), \tag{5.1.0.5}$$

the identification being obviously dependent on the choice of F.

5.1.1. We now return to the setting in Section 2, i.e., $p : N_y \to D_x$. Recall that $\Gamma_y = \mathrm{Gal}(N_y/D_x)$. Let F_y be any (Γ_y, G)-bundle of local type τ_y and therefore given as in (2.2.4.1). Since the underlying G-bundle is trivial, the associated adjoint group scheme $\mathcal{G}_{F_y}$ is isomorphic to the product $G \times N_y$. Hence the sections over $N_y = Spec\ B$ are given by $\mathcal{G}_{F_y}(B) \simeq G(B)$. As has been observed in (2.3.1.5), the local unit group of (Γ_y, G)-automorphisms $\bigcup_y$ is a subgroup of $G(B)$.

The content of the first half of Theorem 2.3.1 is that

$$\mathcal{G}_{F_y}(B)^{\Gamma_y} \simeq \mathcal{P}_{\theta_{\tau_y}}(K). \tag{5.1.1.1}$$

5.1.2. Proposition. *Let $\mathcal{G}_{\theta_{\tau_y}}$ be a Bruhat–Tits group scheme defined by the parahoric group $\mathcal{P}_{\theta_{\tau_y}}(K)$. Let $D_x = Spec\ A$ and $N_y = Spec\ B$. Let $\mathcal{G}_{F_y}$ be the Γ-group scheme on the fixed (Γ_y, G)-bundle F_y. Then*

$$\mathcal{G}_{\theta_{\tau_y}} \simeq p_*^{\Gamma_y}(\mathcal{G}_{F_y}). \tag{5.1.2.1}$$

In particular, if $\mathcal{G}_\theta$ is any Bruhat–Tits group scheme on $Spec\ A$, then by choosing θ_{τ_y} suitably, we can realize $\mathcal{G}_\theta$ as $p_*^{\Gamma_y}(\mathcal{G}_{F_y})$; for a scheme S over $\mathbb{C}$, we have the identification

$$Mor^{\Gamma_y}(N_y \times S, G) \simeq Mor(D_x \times S, \mathcal{G}_\theta). \tag{5.1.2.2}$$

Proof. First, since $\mathcal{G}_{F_y}$ is an affine group scheme, it follows that $p_*(\mathcal{G}_{F_y})$ and $p_*^{\Gamma_y}(\mathcal{G}_{F_y})$ are both representable as affine group schemes over A (see [8, Theorem 4 and Proposition 6]).

By Bruhat and Tits ([9, Section 1.7]), the smooth group scheme $\mathcal{G}_{\theta_{\tau_y}}$ on D_x is uniquely determined by its A-valued points which is the parahoric group $\mathcal{P}_{\theta_{\tau_y}}(K)$.

By the functorial property of the functor $p_*^{\Gamma_y}$, we see (by (2.2.9.1)) that

$$p_*^{\Gamma_y}(\mathcal{G}_{F_y})(A) = p_*(\mathcal{G}_{F_y})^{\Gamma_y} = \mathcal{G}_{F_y}(B)^{\Gamma_y} \simeq \mathcal{P}_{\theta_{\tau_y}}(K). \tag{5.1.2.3}$$

Thus, by the uniqueness of the Bruhat–Tits group scheme, we have an isomorphism of *Spec* A-group schemes: $\mathcal{G}_{\theta_{\tau_y}} \simeq p_*^{\Gamma_y}(\mathcal{G}_{F_y})$.

The identification (5.1.2.2) now follows from the functorial properties of restriction of scalars and fixed point schemes, since

$$Mor^{\Gamma_y}(N_y \times S, G) = p_*^{\Gamma_y}(\mathcal{G}_{F_y})(D_x \times S) = \mathcal{G}_\theta(D_x \times S). \tag{5.1.2.4}$$

The fact that any Bruhat–Tits group scheme can be realized this way follows from the converse in Theorem 2.3.1. □

5.2. Bruhat–Tits group schemes and patching. By the main theorem of Bruhat and Tits ([9]), there exist smooth group schemes $\mathcal{G}_{\Omega}$ over *Spec* A such that the group $\mathcal{G}_{\Omega}(A) = \mathcal{P}_{\Omega}(K)$.

5.2.1. Definition. A smooth group scheme $\mathcal{G}$ over X is called a parahoric Bruhat–Tits group scheme if there is a finite subset $\mathcal{R} = \{x_i\}$ of X and formal neigborhoods D_{x_i} at the x_i together with a collection of subset $\Theta \subset \mathbb{E}^m$ such that

$$\mathcal{G}|_{X-\mathcal{R}} \simeq G \times (X - \mathcal{R}),\ \mathcal{G}|_{D_{x_i}} \simeq \mathcal{G}_{\Theta_i}, x_i \in \mathcal{R}. \tag{5.2.1.1}$$

If $\Omega = \{\Omega_i\}$ is a collection of facets, then we denote a parahoric Bruhat–Tits group scheme defined by local group schemes $\mathcal{G}_{\Omega_i}$ by $\mathcal{G}_{\Omega,X}$. If $\boldsymbol{\theta} = \{\theta_i\} \in (Y(T) \otimes \mathbb{Q})^m$ are chosen in the interior of the facets Ω_i, then we have an isomorphism $\mathcal{G}_{\Omega,X} \simeq \mathcal{G}_{\boldsymbol{\theta},X}$

Conversely, given local Bruhat–Tits groups schemes $\mathcal{G}_{\Theta_i}$, one can construct a parahoric Bruhat–Tits group scheme using the following patching result from [10, Lemma 3.18] attributed to Raghunathan and Ramanathan.

5.2.2. Lemma. *Let X be a smooth projective curve, and let $k(X)$ be its function field. Let $x \in X$, and let A_x be the completion of $\mathcal{O}_{X,x}$ and K_x the completion of $k(X)$. Assume that we are given a triple (G_1, G_2, f) consisting of the following:*

(a) *An affine group scheme G_1 over $U = X - x$ of finite type.*
(b) *An affine and finitely presented group scheme G_2 over A_x.*
(c) *A K_x-group scheme isomorphism $f : G_1 \times_U K_x \simeq G_2 \times_U K_x$.*

Then there exists a group scheme $\mathcal{G}$, affine and of finite type over X such that $\mathcal{G} \times_X U \simeq G_1$ and $\mathcal{G} \times_X A_x \simeq G_2$, and both isomorphisms are compatible with f. Furthermore, if G_i are smooth, then so is $\mathcal{G}$.

5.2.3. Remark. The gluing result of Beauville and Laszlo ([5]), which is more general than the lemma above, shows that any $\mathcal{G}$-torsor E on X can be obtained by gluing the trivial torsor on some open subset $U \subset X$ and the trivial torsors on the formal completions at the points $\mathcal{R} = X - U$. Similarly, any (Γ, G)-bundle of local type $\boldsymbol{\tau}$ on Y is obtained by patching, as was explained in the beginning of Section 3.

5.2.4. Remark. The parahoric Bruhat–Tits group schemes defined above are a little more restrictive than the ones defined by Pappas and Rapoport [29]; they do not make the assumption that the group schemes are generically split.

5.2.5. Remark. Observe firstly that in Lemma 5.2.2 we can take a finite set of points $x_i \in X$ for the patching. It follows that given a finite $\mathcal{R} \subset X$, and a collection of subset $\Theta \subset \mathbb{E}^m$ together with *patching data* $f = \{f_i\}_{i=1}^m$ as in the lemma above, we have a parahoric Bruhat–Tits group scheme $\mathcal{G}_{\Theta,X}$ with $\mathcal{R}$ being the points of ramification of $\mathcal{G}_{\Theta,X}$.

5.2.6. Remark. Let F be a fixed (Γ, G)-bundle of local type $\boldsymbol{\tau}$. The group scheme $\mathcal{G}_F$ constructed in Notation 5.1.0.1 can be viewed as one obtained by gluing the local group schemes $\mathcal{G}_{F_y}$'s on $\{N_y\}_{y \in p^{-1}(\mathcal{R})}$ (see section 5.1.1) along with the constant group scheme $G \times (Y - p^{-1}(\mathcal{R}))$, the patching data coming from the transition functions of the bundle F.

5.2.7. Theorem. *Let F be a fixed (Γ, G)-bundle of local type $\boldsymbol{\tau}$ on Y. Let $\boldsymbol{\theta_\tau} = \{\theta_i\} \in (Y(T) \otimes \mathbb{Q})^m$ be the point associated to $\boldsymbol{\tau}$. Then the invariant direct image $p_*^\Gamma(\mathcal{G}_F)$ is a parahoric Bruhat–Tits group scheme of the form $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$. Conversely, let $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$ be any parahoric Bruhat–Tits group scheme on X, with $\mathcal{R}$ its set of ramifications. Then, there exists a Galois cover $p : Y \to X$ with Galois group Γ, and a (Γ, G)-bundle F of local type $\boldsymbol{\tau}$ with its adjoint Γ-group scheme $\mathcal{G}_F$ on Y, such that $p_*^\Gamma(\mathcal{G}_F) \simeq \mathcal{G}_{\boldsymbol{\theta_\tau},X}$.*

Proof. Since the group scheme $\mathcal{G}_F$ is affine over Y, $p_*^{\Gamma_y}(\mathcal{G}_F)$ is representable as a smooth affine group scheme over X (see [8, Theorem 4 and Proposition 6]).

Since the action of Γ on $Y - p^{-1}(\mathcal{R})$ is free, there is a principal G-bundle P on $X - \mathcal{R}$ such that then $F|_{Y - p^{-1}(\mathcal{R})} \simeq p^*(P)$. Since G is semisimple, by the theorem of Harder [20], P is *trivial.* Therefore, the (Γ, G)-bundle F when restricted to $Y - p^{-1}(\mathcal{R})$ is trivial as a (Γ, G)-bundle. Hence $\mathcal{G}_F$ is the split group scheme over $Y - p^{-1}(\mathcal{R})$. The result now follows from Proposition 5.1.2, the patching data f being the one pushed down from that of $\mathcal{G}_F$.

For the converse, observe that locally, the statement in the corollary is simply the converse in Proposition 5.1.2. The global statement now follows since the patching data f gives the gluing needed in Remark 5.2.3 which gives the recipe to construct F globally. □

5.2.8. Remark. An interesting consequence of Theorem 5.2.7 is that any parahoric Bruhat–Tits group scheme which is *generically split* is isomorphic to the invariant direct image of a group scheme $\mathcal{G}_F$ for a choice of (Γ, G)-bundle F. Moreover, this characterizes such group schemes. Observe that in the patching Lemma 5.2.2, one need not assume that the group scheme is generically split. Using this, one can show that the parahoric Bruhat–Tits group schemes considered by Pappas and Rapoport can also be realized as invariant direct images of Γ-group schemes, which however need not be of the form $\mathcal{G}_F$ for a (Γ, G)-bundle F of type $\boldsymbol{\tau}$.

5.3. Torsors under Bruhat–Tits group schemes. Let $\mathcal{G}_{\boldsymbol{\theta},X}$ be a Bruhat–Tits group scheme given by the local data $\boldsymbol{\theta} \in (Y(T) \otimes \mathbb{Q})^m$.

Let $\mathsf{Bun}_Y^{\boldsymbol{\tau}}(\Gamma, G)$ and $\mathsf{Bun}_X(\mathcal{G}_{\boldsymbol{\theta},X})$ be the moduli stacks of (Γ, G)-bundles of type $\boldsymbol{\tau}$ on Y and of $\mathcal{G}_{\boldsymbol{\theta},X}$-torsors on X, respectively. We now have the following key theorem.

5.3.1. Theorem. *Let $\mathcal{G}_{\boldsymbol{\theta_\tau},X}$ be as above. Let $p : Y \to X$ be as in Theorem* 5.2.7*. Then the stack $\mathsf{Bun}_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is isomorphic to the stack $\mathsf{Bun}_X(\mathcal{G}_{\boldsymbol{\theta_\tau},X})$.*

Proof. By Theorem 5.2.7, there exists a (Γ, G)-bundle F of local type $\boldsymbol{\tau}$ on Y such that $p_*^{\Gamma_y}(\mathcal{G}_F) \simeq \mathcal{G}_{\boldsymbol{\theta_\tau},X}$. By (5.1.0.5) we have an isomorphism $\mathsf{Bun}_Y^{\boldsymbol{\tau}}(\Gamma, G) \simeq \mathsf{Bun}_Y(\Gamma, \mathcal{G}_F)$. By Theorem 5.2.7 and Theorem 4.1.6, we get the isomorphism $\mathsf{Bun}_Y(\Gamma, \mathcal{G}_F) \xrightarrow{p_*^\Gamma} \mathsf{Bun}_X(\mathcal{G}_{\boldsymbol{\theta_\tau},X})$. This proves the theorem. □

5.3.2. Remark. By the above theorem, it follows that the stack $\mathsf{Bun}_X(\mathcal{G}_{\boldsymbol{\theta_\tau},X})$ depends on the local data $\boldsymbol{\theta_\tau}$ but not on the patching functions.

5.3.3. Remark. From the arguments and the results in the preceding pages, it would be clear to the reader that Theorems 5.2.7 and 5.3.1 also hold when X is $\mathbb{P}^1$ or an elliptic curve. Note however that Γ-covers Y will exist only when we assume that $|\mathcal{R}| \geq 3$ for $X = \mathbb{P}^1$, or $\mathcal{R} \neq \emptyset$, when X is an elliptic curve. This is so since the upper half space $\mathbb{H}$ is the universal ramified cover with the given signature even in these cases.

5.3.4. Remark. This theorem is the exact analogue of the fact that the *invariant direct image functor* p_*^Γ sets up an isomorphism between the functor of Γ-vector bundles and that of parabolic vector bundles. This is precisely the point of view in Grothendieck [17], Seshadri [36], and Mehta and Seshadri [24].

6. Stability and semistability

The aim of this section is to introduce the notion of semistability and stability of torsors under parahoric Bruhat–Tits group schemes introduced in the last section.

6.1. Parahoric torsors. As before, let $\mathcal{G}_{\Omega,X}$ be a Bruhat–Tits group scheme on the curve X associated to a collection of *facets* $\Omega = \{\Omega_i\}_{i=1}^m$, with $|\mathcal{R}| = m$.

6.1.1. Definition. A **quasi-parahoric** torsor $\mathscr{E}$ is a $\mathcal{G}_{\Omega,X}$-torsor on X.

6.1.2. Definition. A **parahoric torsor** is a pair $(\mathscr{E}, \boldsymbol{\theta})$ consisting of

(1) a $\mathcal{G}_{\Omega,X}$-torsor $\mathscr{E}$, i.e. a quasi-parahoric torsor on X, and
(2) weights, i.e., elements $\boldsymbol{\theta} = \{\theta_i\} \in (Y(T) \otimes \mathbb{Q})^m$ in the interior of the facets Ω_i.

6.1.3. Remark. Recall that choice of elements $\boldsymbol{\theta} = \{\theta_i\} \in (Y(T) \otimes \mathbb{Q})^m$ in the interior of the facets Ω_i identifies the group scheme $\mathcal{G}_{\Omega,X}$ with $\mathcal{G}_{\boldsymbol{\theta},X}$. Starting with a tuple of weights $\boldsymbol{\theta} \in (Y(T) \otimes \mathbb{Q})^m$, following the proof of the converse in Theorem 2.3.1, we get positive integers $d_1, d_2, \ldots, d_m$ such that $d_i.\theta_i \in Y(T)$. Fix $\mathcal{R} \subset X$ a finite subset with $|\mathcal{R}| = m$ where the group scheme $\mathcal{G}_{\boldsymbol{\theta},X}$ is the local Bruhat–Tits group scheme, with weights θ_i in the interior of Ω_i. By choosing the d_i to be the *least with this property*, we see that a choice of $\boldsymbol{\theta}$ entails a choice of ramification indices d_i at the points of $\mathcal{R}$. Then by generalities on ramified covers (see Remark 2.2.1), we can get a covering $p : Y \to X$, ramified over $\mathcal{R}$, with ramification indices d_i and with Galois group Γ. Note however, that the local data of $\{d_i\}$ and the ramification points associated to the weights are intrinsic, i.e., depends only on X (see Remark 2.2.1).

6.1.4. Remark. The weights $\boldsymbol{\theta}$ can always be chosen in $\mathcal{A}^m$, where $\mathcal{A}$ is the Weyl alcove.

6.1.5. Remark. The notion of weight defined above is the precise analogue of the classical weight for a parabolic vector bundle with *multiplicity* when the weights are rational (cf. [24, Definition 1.5, p. 211]). This can be seen by considering Example 2.3.4, which is in fact the original motivation for parabolic weights. In this context, we refer the reader to Boalch [6] where weights come up in a slightly different context.

6.2. Parabolic line bundles. Fix a finite subset $\mathcal{R} \subset X$ with $|\mathcal{R}| = m$.

6.2.1. Definition ([24, Definition 1.5, p. 211])**.** A *parabolic line bundle* on $(X, \mathcal{R})$ is a pair $(\mathcal{L}, \{\alpha_1, \ldots, \alpha_m\})$, where $\mathcal{L}$ is a line bundle on X together with an m-tuple of rational numbers $(\alpha_1, \ldots, \alpha_m)$ with $0 \leq \alpha_i \leq 1$. The *parabolic degree* of a parabolic line bundle is defined as

$$\operatorname{par}\deg(\mathcal{L}) = \deg(\mathcal{L}) + \sum_{i=1}^{m} \alpha_i.$$

6.2.2. Remark. Let $p : Y \to X$ be a Galois cover ramified over $\mathcal{R} \subset X$ with ramification indices $n_{y_i}, i = 1, \ldots, m$, at the points $y_i \in Y$ over $\mathcal{R}$, and let $\mathrm{Gal}(Y/X) = \Gamma$.

Let L be a Γ-line bundle on Y of local type $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$, where each $\boldsymbol{\tau}_i$ acts a character $\boldsymbol{\tau}_i(\zeta) = \zeta^{a_{y_i}}$ with $|a_{y_i}| < n_{y_i}, \forall i$. Then by [36] and [24], the *invariant direct image* $\mathcal{L} \simeq p_*^{\Gamma}(L)$ determines a parabolic line bundle on $(X, \mathcal{R})$ with parabolic weights $(\frac{a_{y_1}}{n_{y_1}}, \ldots, \frac{a_{y_m}}{n_{y_m}})$ and parabolic degree

$$\operatorname{par}\deg(p_*^{\Gamma}(L)) = \deg(p_*^{\Gamma}(L)) + \sum_{i=1}^{m} \frac{a_{y_i}}{n_{y_i}}.$$

6.2.3. Remark. In fact, all parabolic line bundles on $(X, \mathcal{R})$ can be realized in this manner, namely as *invariant direct images*; this is done by constructing a cover ramified over $\mathcal{R}$ with suitable ramification indices.

6.3. Semistability and stability of torsors. Let $\mathcal{G}_{\Omega,X}$ be a Bruhat–Tits group scheme on the curve X as in Definition 5.2.1. Let $(\mathscr{E}, \boldsymbol{\theta})$ be a parahoric torsor, i.e., the weights $\boldsymbol{\theta}$ are such that $\mathcal{G}_{\Omega,X} \simeq \mathcal{G}_{\boldsymbol{\theta},X}$.

Let $P_K \subset \mathcal{G}_K$ be a maximal parabolic subgroup of the generic fiber $\mathcal{G}_K$ of $\mathcal{G}_{\Omega,X}$. Let $\chi : P_K \to \mathbb{G}_{m,K}$ be a dominant character of the parabolic subgroup P_K. Then one knows that this defines an ample line bundle L_χ on $\mathcal{G}_K/P_K$. We see immediately that χ defines a line bundle L_χ on $\mathscr{E}_K(\mathcal{G}_K/P_K) \simeq \mathscr{E}_K/P_K$ as well, and by using a reduction section s_K, we therefore get a line bundle $s_K^*(L_\chi)$ on $X - \mathcal{R}$.

6.3.1. Proposition. *Let $\mathcal{G}_K$ be the generic fiber of the Bruhat–Tits group scheme $\mathcal{G}_{\Omega,X}$. Let $(\mathscr{E}, \boldsymbol{\theta})$ be a parahoric torsor. Let s_K be a generic reduction of structure group of $\mathscr{E}_K$ to P_K. Then the line bundle $s_K^*(L_\chi)$ on $X - \mathcal{R}$ has a canonical extension $L_\chi^{\boldsymbol{\theta}}$ to X as a parabolic line bundle.*

Proof. The choice of $\theta \in Y(T) \otimes \mathbb{Q}$ allows us to choose an integer d such that $d.\theta \in Y(T)$. Then we have a ramified cover $p : Y \to X$ with $\Gamma = \mathrm{Gal}(Y/X)$ ramified over x with ramification index d. By Theorem 5.3.1, the parahoric torsor $(\mathscr{E}, \boldsymbol{\theta})$, comes from a (Γ, G)-principal bundle E of local type $\boldsymbol{\tau}$ on Y; more precisely, $\mathscr{E} \simeq p_*^\Gamma(E \wedge^G F^{op})$ for a fixed (Γ, G)-bundle F.

The maximal parabolic subgroup $P_K \subset \mathcal{G}_K$ immediately gives a maximal parabolic $Q \subset G$, and the reduction s_K gives in turn a Γ-equivariant reduction of structure group t_L, i.e., a section of E_L/Q_L, where L denotes the quotient field of B the local ring in Y over $x \in X$. By virtue of the projectivity of Y, the reduction section t_L extends to a Γ-equivariant reduction of structure group t as a section of E/Q. The dominant character χ gives a dominant character η of Q and the section t gives a Γ-line bundle $t^*(L_\eta)$ on Y.

We observe that the line bundle $L_\chi^{\boldsymbol{\theta}} := p_*^\Gamma(t^*(L_\eta))$ gives the required extension of $s_K^*(L_\chi)$. By the very definition of the invariant direct image (see Remark 6.2.2), we see that $L_\chi^{\boldsymbol{\theta}} = p_*^\Gamma(t^*(L_\eta))$ gets the natural structure of a *parabolic line bundle.*

Note that the parabolic line bundle extension $L_\chi^{\boldsymbol{\theta}}$ obtained above depends only on local data coming from $\boldsymbol{\theta}$ (see Remark 6.1.3) and hence is intrinsic on X, i.e., it does not depend on the choice of Y (see Remark 2.2.2). □

We have the following general definition of stability and semistability for (Γ, G)-bundles following A. Ramanathan [32, Lemma 2.1].

6.3.2. Definition (Semistability and stability)**.** Let G be a reductive algebraic group. A (Γ, G)-bundle E on Y is called Γ-*semistable* (resp. Γ-*stable*) if for every maximal parabolic subgroup $P \subset G$, every Γ-invariant reduction of structure group $\sigma : Y \to E(G/P)$, and for every dominant character $\chi : P \to \mathbb{G}_m$, we have $\deg \sigma^*(L_\chi) \leq 0$. (resp. < 0).

6.3.3. Remark ([31, Definition 1.1])**.** Equivalently, E is called Γ-*semistable* (resp. Γ-*stable*) if for every maximal parabolic subgroup $P \subset G$ and every Γ-invariant reduction of structure group $\sigma : Y \to E(G/P)$, we have $\deg \sigma^*(E(\mathfrak{g}/\mathfrak{p})) \geq 0$ (resp. > 0), where $\mathfrak{g}$ (resp. $\mathfrak{p}$) is the Lie algebra of G (resp. P). Note that $E(\mathfrak{g}/\mathfrak{p})$ can be identified with $E(T_{G/P})$, where $T_{G/P}$ is the relative tangent sheaf to the morphism $E(G/P) \to X$.

We therefore have the following analogous definition:

6.3.4. Definition. Let $\mathcal{G} = \mathcal{G}_{\Omega,X}$. A parahoric torsor $(\mathscr{E}, \boldsymbol{\theta})$ is called *stable (resp. semistable)* if for every maximal parabolic $\mathcal{P}_K \subset \mathcal{G}_K$, for every dominant character χ as above and for every reduction of structure group s_K, we have

$$\operatorname{par}\deg(L_\chi^{\boldsymbol{\theta}}) < 0 \text{ (resp. } \leq 0).$$

The following theorem is immediate from the above discussions together with Definition 6.3.2.

6.3.5. Theorem. *The isomorphism*

$$p_*^\Gamma : \mathsf{Bun}_Y^{\boldsymbol{\tau}}(\Gamma, G) \xrightarrow{\sim} \mathsf{Bun}_X(\mathcal{G}_{\boldsymbol{\theta},X})$$

given by Theorem 5.3.1 *identifies the substacks of stable* (*resp. semistable*) *parahoric torsors with stable* (*resp. semistable*) (Γ, G)*-bundles of local type* $\boldsymbol{\tau}$ *on the ramified cover* Y*.*

6.3.6. Remark. Recall the classical definition of a stable parabolic vector bundle as given in [24]. Note that the definition in [24, Definition 1.13] is the one which arises out of interpreting the π-stability of the π-vector for the invariant direct image. By Remark 6.1.5 the notion of parabolic weights defined in [24] is the same as the one given here when $G = GL(n)$. Our definition of Γ-stability for (Γ, G)-bundles generalizes the one given by A.Ramanathan for G-bundles, which generalizes the usual notion of stability of vector bundles.

6.3.7. Remark. Following [21, Definition 17] one can define a parabolic subgroup $\mathcal{P} \subset \mathcal{G}_{\theta,X}$ of the group scheme $\mathcal{G}_{\theta,X}$ as the flat closure of a parabolic subgroup of the generic fiber $\mathcal{G}_K$ of $\mathcal{G}_{\theta,X}$. Let E be a $\mathcal{G}_{\theta,X}$-torsor on X. Then as in ([21, Lemma 23]) one can show that if $\mathcal{P}_K \subset \mathcal{G}_K$ is a parabolic subgroup and E is a $\mathcal{G}_{\theta,X}$-torsor on X, then any choice of a reduction section $s_K \in E_K(\mathcal{G}_K/\mathcal{P}_K)$ defines a parabolic subgroup $\mathcal{P}' \subset \mathcal{G}_{\theta,X}$ together with a reduction s' of E to $\mathcal{P}'$.

In fact, these results in [21] can be deduced by using the invariant direct image concept. Let $H \subset G$ be a closed subgroup, and let F be a fixed (Γ, G)-bundle of type $\boldsymbol{\tau}$ with a Γ-invariant reduction of structure to H. Let the induced (Γ, H)-bundle obtained from this reduction be denoted by F_H. We consider the adjoint group scheme $\mathcal{G}_{F_H}$ as defined in Notation 5.1.0.1. Then, $\mathcal{G}_{F_H} \subset \mathcal{G}_F$ is a closed subgroup scheme. By [8, Proposition 2, p. 192] and taking Γ-invariants, it follows immediately by [13, Proposition 3.4] that $p_*^\Gamma(\mathcal{G}_{F_H}) \subset p_*^\Gamma(\mathcal{G}_F) \simeq \mathcal{G}_{\theta,X}$ is a closed smooth X-subgroup scheme. In particular, if P is a parabolic subgroup of G, the invariant direct image $p_*^\Gamma(\mathcal{G}_{F_P})$ gives the flat closure $\mathcal{P} \subset \mathcal{G}_{\theta,X}$.

6.3.8. Remark (Harder–Narasimhan reduction)**.** With the definition of semistability in place, it is routine now to define the Harder–Narasimhan reduction for a $\mathcal{G}_{\theta,X}$-torsor by using the identification of Theorem 6.3.5. The existence of a parahoric Harder–Narasimhan reduction follows from the existence of a Γ-equivariant parabolic Harder–Narasimhan reduction for a (Γ, G)-bundle together with Remark 6.3.7. In other words, the canonical Harder–Narasimhan parabolic subgroup scheme of the parahoric group scheme will be the invariant direct image of the Γ-invariant Harder–Narasimhan parabolic subgroup P of G. The well-definedness follows since it is the flat closure of the generic Harder–Narasimhan parabolic $P_K \subset \mathcal{G}_K$. The Harder–Narasimhan reduction of structure group for the torsor will be realized as the invariant direct image of the corresponding (Γ, P)-reduction on Y. The uniqueness of the Harder–Narasimhan reduction for (Γ, G)-bundles shows the uniqueness of the Harder–Narasimhan reduction of a parahoric torsor as well.

7. Unitary representations of π

7.1. Manifold of irreducible unitary representations of π. Notations in this section are as in the Introduction.

Let ρ be a representation of π on a vector spaceband V (over $\mathbb{R}$) such that $d = \dim V$, and let ρ act unitarily. We now recall the following result from Weil [45, p. 156], noting that since π acts unitarily, it leaves a nondegenerate form on V invariant and therefore in Weil's notation, $i = i' = \dim_{\mathbb{R}} H^0(\pi, \rho)$.

7.1.1. Proposition. *We have the following equality of dimensions:*

$$\dim_{\mathbb{R}} H^1(\pi, \rho) = 2d(g-1) + 2\dim_{\mathbb{R}} H^0(\pi, \rho) + \sum_{\nu=1}^{m} e_\nu, \tag{7.1.1.1}$$

where e_ν is the rank of the endomorphism $(I - \rho(C_\nu))$ of V.

Let K_G be a maximal compact subgroup of G and $Lie(K_G)$ denote *the Lie algebra* of K_G, which is a real vector space of dimension d, where $d = \dim(G)$.

As in the Introduction, we assume that $X = \mathbb{H}/\pi$, with $x \in X$ corresponding to $z \in \mathbb{H}$. Let π_z be the stabilizer at z (cyclic of order n_x), and let γ be a generator of π_z. Now let $\rho : \pi \to K_G$ be a unitary representation of π (see Definition 1.0.1).

7.2. Explicit computation when G is simple. Let $\alpha \in S$, and let θ_α be as in (2.1.4.2). Let ρ_{θ_α} be the local representation as in Notation 2.3.1.1. Let $\rho_{\theta_\alpha}(\gamma) \in K_G$ be the image of the generator γ of π_z. Note that the choice of the simple root α and the identification of the representation ρ with ρ_{θ_α} amounts to *fixing the local type* of the representation $\rho : \pi \to K_G$, i.e., the *conjugacy class* of $\rho(\gamma)$ in K_G.

We denote by Ad ρ_{θ_α}, the adjoint transformation on $Lie(K_G)$, namely if $M \in Lie(K_G)$, $M \mapsto \rho_{\theta_\alpha}(\gamma) M \rho_{\theta_\alpha}(\gamma)^{-1}$, then we have

7.2.1. Proposition. *Let $e(\theta_\alpha)$ denote the rank of $(Id - \mathrm{Ad}\, \rho_{\theta_\alpha})$ on $Lie(K_G)$. Then*

(7.2.1.1) $$e(\theta_\alpha) = \dim_{\mathbb{R}}(K_G) - 2\mu(\alpha) - 2\nu(\alpha) - \ell = 2.(\dim_{\mathbb{C}}(G/P_\alpha)) - \mu(\alpha)),$$

where P_α is the maximal parabolic subgroup of G associated to α and

(7.2.1.2) $$\mu(\alpha) = \#\{r \in R^+ \mid r = c_\alpha.\alpha + \sum_{\beta \neq \alpha} x_\beta.\beta\},$$

(7.2.1.3) $$\nu(\alpha) = \#\{r \in R^- \mid r \textit{ involves simple roots} \neq \alpha\},$$

and $\ell = \mid S \mid$.

Proof. Make K_G operate on itself by inner conjugation. Then the rank of $(Id - \mathrm{Ad}\, \rho_{\theta_\alpha})$ acting on the Lie algebra $Lie(K_G)$ equals *the dimension of the orbit through $\rho_{\theta_\alpha}(\gamma)$ for the action of K_G on itself by inner conjugation.*

We may assume for the purpose of this computation that $\rho_{\theta_\alpha}(\gamma)$ lies in the maximal torus. First we compute the number of roots $r \in R$ so that the corresponding root group $U_r(B)$ is centralized by $\rho_{\theta_\alpha}(\gamma)$. Recall from Notation 2.3.1.1 that the action of $\rho_{\theta_\alpha}(\gamma)$ on U_r is given as

(7.2.1.4) $$\rho_{\theta_\alpha}(\gamma).U_r(B).\rho_{\theta_\alpha}(\gamma)^{-1} = U_r(\zeta^{r(\Delta_\alpha)} B),$$

where as seen earlier, $r(\Delta_\alpha) = d.(\theta_\alpha, r)$. Since ζ is a primitive dth-root of unity, we need to compute the # $\{r \in R \mid (\theta_\alpha, r) = \pm 1 \textit{ or } 0\}$. It is easy to see that

(7.2.1.5) $$\{r \in R \mid (\theta_\alpha, r) = \pm 1 \textit{ or } 0\} = \bigcup_{i=1}^{4} A_i(\alpha),$$

where for $i = 1, 2$,

(7.2.1.6) $$A_i(\alpha) = \{r \in R^{\pm} \mid r = \pm c_\alpha.\alpha + \sum_{\beta \neq \alpha} \pm x_\beta.\beta\},$$

(7.2.1.7) $$A_3(\alpha) = \{r \in R^- \mid r \textit{ involves simple roots} \neq \alpha\},$$

and

(7.2.1.8) $$A_4(\alpha) = \{r \in R^+ \mid r \textit{ involves simple roots} \neq \alpha\}.$$

Since the maximal torus centralizes $\rho_{\theta_\alpha}(\gamma)$, we see that the dimension of the centralizer of $\rho_{\theta_\alpha}(\gamma)$ is

(7.2.1.9) $$\#\{r \in R \mid (\theta_\alpha, r) = \pm 1 \textit{ or } 0\} + \mid S \mid .$$

Observe that $\mid A_4 \mid = \mid A_3 \mid$ and $\mid A_1 \mid = \mid A_2 \mid$. To compute the rank of $(Id - \mathrm{Ad}\, \rho_{\theta_\alpha})$, we simply subtract the above number (7.2.1.9) from $\dim_{\mathbb{R}}(K_G)$ to get the first expression for $e(\alpha)$. We see that

(7.2.1.10) $$\nu(\alpha) = \dim_{\mathbb{C}}(P_\alpha/B),$$

where P_α is the maximal parabolic subgroup of G defined by the simple root $\alpha \in S$. Thus,

$$\dim_{\mathbb{R}}(K_G) - 2.\nu(\alpha) - \ell = \dim_{\mathbb{C}}(G) - 2.\nu(\alpha) - \ell = 2.\dim_{\mathbb{C}}(G/P_\alpha),$$

since $2.\dim(B) - \ell = \dim(G)$. Hence, $e(\theta_\alpha) = 2.(\dim_{\mathbb{C}}(G/P_\alpha)) - \mu(\alpha))$, and the proposition now follows. □

7.2.2. Corollary. *Let $\alpha \in S$ be such that $\mathcal{P}_{\theta_\alpha}(K)^{hs}$ is a maximal parahoric subgroup in $G(K)$* which is hyperspecial. *Then $e(\theta_\alpha) = 0$ and conversely.*

Proof. By Bruhat–Tits theory, the hyperspecial parahorics are simply the maximal parahorics $\{\mathcal{P}_{\theta_\alpha}(K) \mid \forall \alpha \in S, \text{with } c_\alpha = 1\}$ up to conjugacy by $G(K)$. In these cases, the number $\mu(\alpha)$ will now be

$$\mu(\alpha) = \#\{r \in R^+ \mid r \text{ involves } \alpha\}$$

since the largest possible coefficient for such an α in any positive root is 1. Hence α is hyperspecial if and only if $\mu(\alpha) = \dim(G/P_\alpha)$ and we are through by Proposition 7.2.1. □

7.3. The moduli dimension. Let G be semisimple and simply connected.

7.3.1. Corollary. *Let $\theta \in \mathbb{E}$ be an arbitrary element in the affine apartment $\mathbb{E}$, and let ρ_θ be the representation defined in Notation* 2.3.1.1*. Let $e(\theta)$ denote the rank of $(Id - \mathrm{Ad}\, \rho_\theta)$ on $Lie(K_G)$. Then,*

(7.3.1.1) $$e(\theta) = \dim_{\mathbb{R}}(K_G) - \mid S \mid - \#\{r \in R \mid (\theta, r) = \pm 1 \textit{ or } 0\}.$$

Proof. The proof is immediate from the above discussion. Note that when $\theta = \theta_\alpha$, the number $e(\theta)$ gets the explicit expression (7.2.1.1). □

Let $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$ be a set of conjugacy classes, and let $\boldsymbol{\theta}_{\boldsymbol{\tau}} = \{\theta_i\} \in \mathbb{E}^m$ be the corresponding set of points of the product of the affine apartments, with $m = |\mathcal{R}|$.

7.3.2. Theorem. *The subset* $R_o \subset R^{\boldsymbol{\tau}}(\pi, K_G)$ *of* irreducible representations *is open and nonempty and is further smooth of real dimension equal to*

$$(2g-1)\dim(K_G) + \sum_{i=1}^{m} e(\boldsymbol{\theta_\tau}). \tag{7.3.2.1}$$

Let K_G *act on* $R^{\boldsymbol{\tau}}(\pi, K_G)$ *by inner conjugation. Let* $\overline{K}_G = K_G/center$*. Then the equivalence classes of irreducible representations correspond to the quotient space* $R_o/\overline{K}_G$*; further, there is an open subset* U *of* R_o *where* $\overline{K}_G$ *acts freely. If* U *is nonempty, then the quotient* $U/\overline{K}_G$ *has the natural structure of a* real analytic manifold *of real dimension*

$$2.\dim_{\mathbb{C}}(G)(g-1) + \sum_{i=1}^{m} e(\boldsymbol{\theta_\tau}). \tag{7.3.2.2}$$

Proof. We follow the arguments of Narasimhan and Seshadri [27, Proposition 9.2] or Seshadri [36, p. 180]. Let $W = \prod W_i$, where W_i is the conjugacy class defined by $\boldsymbol{\tau}_i$. Observe that the group π is given by generators and relations as in (1.0.0.1), and the space $R^{\boldsymbol{\tau}}(\pi, K_G)$ can be identified with the inverse image of identity under the analytic map $\chi : K_G \times \cdots \times K_G \times W \to K_G$ given by $\chi(a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_m) = \prod[a_i, b_i].c_1 \cdots c_m$. As in [27] or [36], the kernel of the differential of χ at ρ is given by $Z^1(\pi, \mathrm{Ad}\,\rho)$. Also the differential is of maximal rank at ρ if and only if ρ is irreducible. Now using Proposition 7.2.1, the theorem follows as in loc.cit. □

7.3.3. Remark. It will be shown in Section 8 that the above open subset is nonempty and gets identified with the Γ-stable bundles whose automorphisms are trivial. Furthermore (see Corollary 8.1.8), the quotient $R_o/\overline{K}_G$ in fact gets the structure of a *complex analytic orbifold* (i.e., with at most finite quotient singularities) of dimension

$$\dim_{\mathbb{C}}(R_o/\overline{K}_G) = \dim_{\mathbb{C}}(G)(g-1) + \sum_{i=1}^{m} \frac{1}{2} e(\boldsymbol{\theta_\tau}). \tag{7.3.3.1}$$

8. The moduli space of parahoric torsors and the main theorem

The aim of this section is to construct the moduli space of semistable (Γ, G)-bundles on Y of local type $\boldsymbol{\tau}$ (see Definition 6.3.2) or, equivalently, by Theorem 6.3.5 the moduli space of semistable and stable parahoric torsors. We essentially follow the strategy of Balaji and Seshadri [4] and Balaji, Biswas, and Nagaraj [2]. We briefly outline a proof of [2, Theorem 5.8].

We fix a faithful representation $G \hookrightarrow GL(n)$ and consider the subscheme of a suitable "Quot"-scheme parametrizing Γ-vector bundles on the curve Y which are Γ-semistable of local type $\boldsymbol{\tau}$ and we denote this scheme by $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ (see [36] for details where this space is denoted $R^{\boldsymbol{\tau},ss}$). Equivalently, we may view the points in $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ as Γ-semistable principal $(\Gamma, GL(n))$-bundles of local type $\boldsymbol{\tau}$.

We then define the scheme $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}$ as the *space of* Γ*-equivariant reductions of structure group* of the bundles in $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ which consists of those (Γ, G)-bundles which are of local type $\boldsymbol{\tau}$. It is standard to show that $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}$ has the *local universal property* for families of Γ-semistable (Γ, G)-bundles of local type.

We now use the results in [36] which show that there is an action of a certain reductive group $\mathcal{H}$ on $Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$, and the good quotient $M^{\boldsymbol{\tau}}_{Y}(\Gamma, n) := Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}//\mathcal{H}$ exists and gives a coarse moduli scheme for the functor of equivalence classes of Γ-semistable principal $(\Gamma, GL(n))$-bundles on Y of local type $\boldsymbol{\tau}$.

The map $Q^{\boldsymbol{\tau}}_{(\Gamma,G)} \to Q^{\boldsymbol{\tau}}_{(\Gamma,GL(n))}$ obtained by taking an extension of structure groups via the inclusion $G \hookrightarrow GL(n)$ is shown to be *affine*, and the action of $\mathcal{H}$ lifts to $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}$ to give a good quotient $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}//\mathcal{H}$, which *we denote by* $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ (see [4] and [2]).

When G is semisimple and simply connected, we show in this paper that the points of the scheme $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ parametrize *isomorphism classes of* (Γ, G)*-bundles of local type* $\boldsymbol{\tau}$ *which are unitary* (*Definition* 8.0.4 *below*). Using this, we show that $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ is normal and projective, and we compute its dimension.

8.0.1. Remark. We note that the arguments of [2] are not sufficient for showing the last statement (i.e., the projectivity and dimension computation) since the local type of the bundles was not fixed in [2]. A key step in the arguments is the connectedness of the moduli space which fails if the local type is not fixed.

8.0.2. Remark. Note that, strictly speaking, we do not need the group G to be semisimple and simply connected, but we need only the reductivity of G to be able to talk of the space $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$.

8.0.3. Definition. A *unitary* (π, G)-bundle on $\mathbb{H}$ is defined to be the trivial G-bundle $\mathbb{H} \times G$ on $\mathbb{H}$ with the π-structure given by $\gamma(z, g) = (z, \rho(\gamma).g)$, with ρ an element of $R^{\boldsymbol{\tau}}(\pi, K_G)$.

Let V be a *unitary* (π, G)-bundle defined by $\rho : \pi \to K_{G}$. Let $r : \mathbb{H} \to Y$ be as in (2.2.1.1). Let $E(\rho) := r_*^{\pi_o}(V)$; then $E(\rho)$ is a (Γ, G)-bundle defined by the *twisted action* given by (2.2.4.1).

We observe that the *local type* $\boldsymbol{\tau}_i$ *of the bundle* $E(\rho)$ *at* y_i in the sense of Definition 2.2.6 is equivalently given by the *conjugacy class of* $\rho(C_i)$ in G. Thus if $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$, then we have

(8.0.3.1) $$\rho \text{ is of type } \boldsymbol{\tau} = \{\boldsymbol{\tau}_i\} \iff E(\rho) \text{ is of local type } \boldsymbol{\tau}.$$

8.0.4. Definition. A (Γ, G)-bundle E is called *unitary* if $E \simeq E(\rho)$ for a homomorphism $\rho : \pi \to K_G$.

8.1. Properness of the moduli of (Γ, G)-bundles. Let $H = G/Z(G)$, the associated adjoint group. Let $\mathfrak{h} = Lie(H)$. Consider the adjoint representation $\rho : H \to GL(\mathfrak{h})$. It is clear that ρ is faithful representation.

Fix the representation $\rho : H \hookrightarrow GL(n)$ (where $n = \dim\ \mathfrak{h}$) and a maximal compact K_H of H such that $K_H \hookrightarrow U(n)$. Consider the subscheme $M_Y^{\boldsymbol{\tau}}(\Gamma, n)^s \subset M_Y^{\boldsymbol{\tau}}(\Gamma, n)$ of stable $(\Gamma, GL(n))$-bundles.

8.1.1. Lemma. *Let* $\phi : M_Y^{\boldsymbol{\tau}}(\Gamma, H) \longrightarrow M_Y^{\boldsymbol{\tau}}(\Gamma, n)$ *be the morphism induced by the representation* ρ *and the map of Quot schemes. Let* $M_Y^{\boldsymbol{\tau}}(\Gamma, H)^o := \phi^{-1}(M_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ *be the inverse image of the stable points. Then* $M_Y^{\boldsymbol{\tau}}(\Gamma, H)^o$ (*when nonempty*) *is open and consists of unitary* (Γ, H)*-bundles which are* Γ*-stable as well.*

Proof. We claim that a principal (Γ, H) bundle E is unitary if and only if the associated $(\Gamma, GL(\mathfrak{h}))$-bundle $E(\mathfrak{h})$ is so. If E is unitary, then obviously so is $E(\mathfrak{h})$.

We now show the converse. Let $A(\mathfrak{h})$ denote the stabilizer of the $GL(\mathfrak{h})$-action on the tensor space $\mathfrak{h}^* \otimes \mathfrak{h}^* \otimes \mathfrak{h}$ at the point $[\ ,\ ]$, i.e., the Lie bracket. Since we have assumed that H is of adjoint type, it implies that $A(\mathfrak{h}) = Aut(\mathfrak{h})$.

Now assume that $E(\mathfrak{h})$ comes from a unitary representation of π. Then we take the Lie bracket morphism $E(\mathfrak{h}) \otimes E(\mathfrak{h}) \to E(\mathfrak{h})$. Both $E(\mathfrak{h}) \otimes E(\mathfrak{h})$ and $E(\mathfrak{h})$ come from unitary representations of π and, by *local constancy* ([24, Proposition 1.2]), morphisms of such bundles are induced by morphisms of π-modules. It now follows that $E(\mathfrak{h})$ gets a reduction of structure group to the group $A(\mathfrak{h}) = Aut(\mathfrak{h})$.

Since H is a connected adjoint group, firstly, $\mathrm{Ad}\,(H) = H$ and secondly, it gets identified with the group of inner automorphisms. Thus we have a short exact sequence

$$1 \to H \to A(\mathfrak{h}) \to F \to 1,$$

where elements of $F \simeq A(\mathfrak{h})/H$ are the *outer automorphisms*. Again we have a similar exact sequence of compact groups

$$1 \to K_H \to K_{A(\mathfrak{h})} \to F \to 1.$$

The bundle E is therefore such that $E(A(\mathfrak{h}))$ is a unitary bundle and comes from a representation $\bar{\chi} : \pi \to K_{A(\mathfrak{h})}$. Furthermore, the extended bundle $E(A(\mathfrak{h}))(F)$ is trivial since it comes with a section (giving E). By composing the representation $\bar{\chi}$ with the map $K_{A(\mathfrak{h})} \to F$, we see that the triviality of $E(A(\mathfrak{h}))(F)$ forces the composite to be the trivial homomorphism, implying that $\bar{\chi}$ factors via $\chi : \pi \to K_H$ to give the bundle E (cf. Atiyah and Bott [1, Lemma 10.12]).

Now using the main theorem of [36] we see that points of $M_Y^{\boldsymbol{\tau}}(\Gamma, n)^s$, being stable bundles, are all unitary. Hence by the claim above the bundles in the inverse image $\phi^{-1}(M_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ are also unitary.

It follows easily from Remark 6.3.3 (cf. [31, Remark 2.2]) that a (Γ, H)-bundle is Γ-stable if and only if the associated Lie algebra bundle $E(\mathfrak{h})$ has no Γ-invariant parabolic subalgebra bundles of degree ≥ 0. It is now easy to see that a (Γ, H)-bundle is Γ-stable if the associated Lie algebra bundle is a Γ-stable vector bundle since $E(\mathfrak{h})$ has no Γ-subbundles of degree ≥ 0 and, in particular, no Γ-invariant parabolic subalgebra bundles of degree ≥ 0. □

8.1.2. Proposition. *Assume that H is simple of adjoint type. Let ρ be the adjoint representation of H. Then the inverse image of $M_Y^{\boldsymbol{\tau}}(\Gamma, n)^s$ by the induced morphism ϕ is nonempty.*

Proof. Recall that the Fuchsian group π can be identified with the group generated by $2g + m$ elements A_i, B_i, C_i, modulo relations given by (1.0.0.1) and (1.0.0.2).

So to prove that the inverse image $\phi^{-1}(\mathsf{Bun}_Y^{\boldsymbol{\tau}}(\Gamma, n)^s)$ is nonempty, we need to exhibit a representation $\chi : \pi \to K_H$ such that the composition

$$\rho \circ \chi : \pi \to U(n) \text{ is } \textit{irreducible}. \tag{8.1.2.1}$$

Choose elements $h_1, \ldots, h_m \in K_H$ so that they are elements of order n_i, where $i = 1, \ldots, m$ (these correspond to fixing the *local type* $\boldsymbol{\tau}$ of our bundles).

It is a well-known fact that every element of a compact connected real semisimple Lie group is a commutator. Another well-known fact is that there exists a dense subgroup $\langle \alpha, \beta \rangle$ of K_H generated by two general elements $\{\alpha, \beta\}$ (see for example [38, Lemma 3.1]). Recall that the genus $g \geq 2$, and define the representation $\chi : \pi \to K_H$ as

$$\chi(A_1) = \alpha,\ \chi(B_1) = \beta,\ \chi(A_2) = \beta,\ \chi(B_2) = \alpha, \tag{8.1.2.2}$$

$$\begin{gathered}\chi(A_i) = a_i,\ \chi(B_i) = b_i,\ for\ i = 3, \cdots, g,\\ \chi(C_j) = h_j,\ and\ j = 1, \cdots, m.\end{gathered} \tag{8.1.2.3}$$

It is clear that χ gives a representation of the group π. Since H is *simple*, ρ is irreducible, and the image of χ contains a dense subgroup, the composition $\rho \circ \chi$ gives an irreducible representation of π in the unitary group $U(n)$.

Therefore, it gives a *stable* Γ-linearized vector bundle, which comes as the extension of structure group of a H-bundle. This completes the proof of the proposition. □

8.1.3. Corollary. *There is a nonempty Zariski open subscheme $M_Y^{\boldsymbol{\tau}}(\Gamma, H)^o$ of $M_Y^{\boldsymbol{\tau}}(\Gamma, H)$ consisting of unitary bundles of local type $\boldsymbol{\tau}$ which are also Γ-stable.*

Proof. Corollary 8.1.3 follows from Lemma 8.1.1 and Proposition 8.1.2. We observe that since H is semisimple of adjoint type, it can be written as a direct product $\prod H_i$ of simple groups of adjoint type. Now a (Γ, H)-bundle (resp. unitary) is the same as a product of (Γ, H_i)-bundles (resp. unitary). Likewise by [31, Proposition 7.1], a Γ-stable (Γ, H)-bundle is the same as a product of Γ-stable (Γ, H_i)-bundles. For each factor H_i, Lemma 8.1.1 and Proposition 8.1.2 applies and the result follows. □

We now return to G which is as before a semisimple, simply connected algebraic group.

8.1.4. Proposition. *The subscheme of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ consisting of stable unitary bundles of local type $\boldsymbol{\tau}$ is* non-empty *and contains a Zariski open subset.*

Proof. Let $\eta : M_Y^{\boldsymbol{\tau}}(\Gamma, G) \to M_Y^{\boldsymbol{\tau}}(\Gamma, H)$ be the morphism induced by the quotient map $G \to H$. Let $M_Y^{\boldsymbol{\tau}}(\Gamma, H)^o$ be as in Corollary 8.1.3. We claim that the required Zariski open subset of $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is

$$M_Y^{\boldsymbol{\tau}}(\Gamma, G)^o := \eta^{-1}(M_Y^{\boldsymbol{\tau}}(\Gamma, H)^o). \tag{8.1.4.1}$$

Let E be a (Γ, G)-bundle in $\eta^{-1}(M_Y^{\boldsymbol{\tau}}(\Gamma, H)^o)$. By Corollary 8.1.3, the H-bundle $E(H)$ comes from a unitary representation $\rho : \pi \to K_H$.

Recall that, by the structure of π described above, there is a central extension

$$1 \to Z_{\tilde{\pi}} \to \tilde{\pi} \to \pi \to 1, \tag{8.1.4.2}$$

where $\tilde{\pi}$ is generated by $A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_m$ together with a central element J satisfying the extra relation

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_1 \cdots C_m = J. \tag{8.1.4.3}$$

It is easy (as in [27]) by adding an extra lasso around a dummy point (other than the parabolic points) to choose a lift of ρ to a representation $\tilde{\rho} : \tilde{\pi} \to K_G$ so that the associated (Γ, G)-bundle $E(\tilde{\rho})$ also maps to $E(H)$. Thus, both E and $E(\tilde{\rho})$ give $E(H)$ under the quotient map $G \to H$. Therefore, by twisting by a central character of $\tilde{\pi}$, we get a representation $\tilde{\pi} \to K_G$ which gives the (Γ, G)-bundle E (cf. [31, p. 148]).

We observe that this representation $\tilde{\pi} \to K_G$ in fact descends to a representation $\pi \to K_G$. This follows from the fact that the local type of E at the dummy point is trivial.

From this we can now conclude that all bundles in $M^{\boldsymbol{\tau}}_Y(\Gamma, G)^o$ are unitary (cf. [1, Lemma 10.12]). Furthermore, since $G \to H$ is surjective, it is not hard to see that a (Γ, G)-bundle is Γ-stable if and only if the associated (Γ, H)-bundle is so (cf. [31, Proposition 7.1]). It follows that all points of $M^{\boldsymbol{\tau}}_Y(\Gamma, G)^o$ are also Γ-stable (Γ, G)-bundles, completing the proof of the proposition. □

We now have a canonical continuous map

$$\psi : R^{\boldsymbol{\tau}}(\pi, K_G) \to M^{\boldsymbol{\tau}}_Y(\Gamma, G), \tag{8.1.4.4}$$

which sends ρ to the class of $E(\rho)$. This map is obtained following [35, p. 334]. First we consider the space $R^{\boldsymbol{\tau}}(\pi, G)$ of all homomorphisms $\pi \to G$ of local type $\boldsymbol{\tau}$. Let $R^{\boldsymbol{\tau}}(\pi, G)^{ss}$ be the subset of $R^{\boldsymbol{\tau}}(\pi, G)$ consisting of points ρ such that $E(\rho)$ is Γ-semistable. One can easily construct an analytic family of (π, G)-bundles on $\mathbb{H} \times R^{\boldsymbol{\tau}}(\pi, G)$; the subgroup π_o acts freely on $\mathbb{H}$ and this family is easily seen to come down to an analytic family of (Γ, G)-bundles on Y parametrized by $R^{\boldsymbol{\tau}}(\pi, G)^{ss}$.

Further, since a (Γ, G)-bundle is Γ-semistable if and only if the associated Γ-vector bundle is so (see for example the proof of [2, Proposition 3.2]), it follows that the subset $R^{\boldsymbol{\tau}}(\pi, G)^{ss}$ is nonempty and open in $R^{\boldsymbol{\tau}}(\pi, G)$ and contains the space $R^{\boldsymbol{\tau}}(\pi, K_G)$ of all unitary representations.

By the local universal property of $Q^{\boldsymbol{\tau}}_{(\Gamma,G)}$, given a $\rho \in R^{\boldsymbol{\tau}}(\pi, G)^{ss}$, we get an analytic neighborhood U of ρ together with an analytic map $U \to (Q^{\boldsymbol{\tau}}_{(\Gamma,G)})^{ss}$. These maps glue to give an analytic morphism $\psi : R^{\boldsymbol{\tau}}(\pi, G)^{ss} \to M^{\boldsymbol{\tau}}_Y(\Gamma, G)$. Restricting this map to $R^{\boldsymbol{\tau}}(\pi, K_G)$ gives the continuous map ψ. The image of ψ consists of (Γ, G)-bundles which are *unitary*.

The following irreducibility result is an immediate consequence of Theorem 5.3.1, [21, Theorem 2], and [21, Proposition 1].

8.1.5. Proposition. *The moduli stack* $\mathsf{Bun}^{\boldsymbol{\tau}}_Y(\Gamma, G)$ *of* (Γ, G)*-bundles on* Y *of local type* $\boldsymbol{\tau}$ *is irreducible and smooth when the group* G *is semisimple and simply connected.*

8.1.6. Remark. We now indicate a different proof of the connectedness from the picture of Hecke correspondences shown in (8.2.1.2). By Drinfeld and Simpson [12], the moduli stack $\mathsf{Bun}_X(G)$ is irreducible because G is semisimple and simply connected. Further, the morphism $\mathsf{Bun}(\mathcal{G}_{\mathcal{I},X}) \to \mathsf{Bun}_X(G)$ is surjective and has fiber G/B, B being the Borel subgroup. Hence, $\mathsf{Bun}(\mathcal{G}_{\mathcal{I},X})$ is connected. Now observe that the map $\mathsf{Bun}(\mathcal{G}_{\mathcal{I},X}) \to \mathsf{Bun}(\mathcal{G}_{\Omega,X})$ given by (8.2.1.2) is also surjective since it comes from the inclusion $\mathcal{I} \subset \mathcal{P}_\Omega(K)$. Hence $\mathsf{Bun}(\mathcal{G}_{\Omega,X})$ is connected. The irreducibility follows from the formal smoothness of the functor of torsors (see [21, Proposition 1]); the obstruction to smoothness vanishes since we work on curves.

Since we work over char 0, the connectedness of the moduli space of (Γ, G)-bundles of local type $\boldsymbol{\tau}$ could also be carried out following [31] and [1, Proposition 4.2].

We have a morphism $f : \mathsf{Bun}^{\boldsymbol{\tau}}_{Y}(\Gamma, G)^{ss} \to M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$, namely, the canonical quotient map obtained by the categorical quotient property of the moduli space $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$. The map f is surjective on points; therefore by Proposition 8.1.5, this implies that $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ is *irreducible*.

8.1.7. Theorem. *The map* $\psi : R^{\boldsymbol{\tau}}(\pi, K_G) \to M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ *obtained in* (8.1.4.4) *is surjective and hence* $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ *is compact. Further, the variety* $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ *gets a structure of a normal projective variety.*

Proof. By the Proposition 8.1.4, the subset $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)^o$ is nonempty and consists entirely of unitary bundles. Thus it is a subset of the image $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ in $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$, i.e., the image $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ contains a Zariski open subset of $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$. Since $R^{\boldsymbol{\tau}}(\pi, K_G)$ is compact, the image $\psi(R^{\boldsymbol{\tau}}(\pi, K_G))$ is therefore the whole of $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$, because $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ is *irreducible*.

This proves that $M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ is topologically compact and hence by GAGA it is a projective variety. The normality follows from the smoothness of the stack $\mathsf{Bun}^{\boldsymbol{\tau}}_{Y}(\Gamma, G)^{ss}$, again by Proposition 8.1.5. □

8.1.8. Corollary. (1) *Let* $g(X) \geq 2$. *Then the map* $\psi : R^{\boldsymbol{\tau}}(\pi, K_G) \to M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ *defined above descends to a map*

$$\psi^* : R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G \to M^{\boldsymbol{\tau}}_{Y}(\Gamma, G), \tag{8.1.8.1}$$

which gives a homeomorphism of topological spaces. Further, the subset $R_o/\overline{K}_G$ *of equivalence classes of irreducible unitary representations maps bijectively onto the subset of stable* (Γ, G)*-bundles.*

(2) *Let* $g(X) < 2$. *When* $X = \mathbb{P}^1$ *and* $|\mathcal{R}| \geq 3$ *or when* X *is an elliptic curve and* $\mathcal{R} \neq \emptyset$*, the map* ψ^* *in* (8.1.8.1) *is a homeomorphism provided there exists an irreducible representation* $\rho : \pi_1(X - \mathcal{R}) \to K_G$ *with preassigned conjugacy classes of images of lassos around the points of* $\mathcal{R}$*.*

Proof. The surjectivity of the map $\psi^* : R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G \to M^{\boldsymbol{\tau}}_{Y}(\Gamma, G)$ follows from surjectivity statement in Theorem 8.1.7.

For the injectivity of ψ^*, suppose that $\psi(\rho_1) = \psi(\rho_2)$, i.e., we have an isomorphism $E_{\rho_1} \simeq E_{\rho_2}$ of the unitary bundles defined by the ρ_i. Now we follow Ramanathan [31, Proposition 6.2] and work with our Γ instead of $\pi_1(X - x_o)$. The proof simply goes through, and this implies that the ρ_i are in the same orbit of $\overline{K}_G$. One could also argue as in Lemma 8.1.1 to get the injectivity statement.

Since $R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G$ is compact and $M_Y^{\boldsymbol{\tau}}(\Gamma, G)$ is Hausdorff in the usual topology, the map ψ^* is a homeomorphism. The fact that irreducible representations give stable bundles and vice versa follows exactly as in [31]. The second part when the genus is 0 or 1 follows from Remark 5.3.3 and Theorem 7.3.2. □

Let $\mathcal{G}_{\Omega,X}$ be a parahoric Bruhat–Tits group scheme associated to a collection of facets $\Omega = \{\Omega_i\}$. Choose $\boldsymbol{\tau} = \{\tau_i\}$ and $\boldsymbol{\theta_\tau} \in (Y(T) \otimes \mathbb{Q})^m$, so that $\mathcal{G}_{\Omega,X} \simeq \mathcal{G}_{\boldsymbol{\theta_\tau},X}$. Recall that Theorem 6.3.5 identifies stable (resp. semistable) families of parahoric $\mathcal{G}_{\Omega,X}$-torsors with stable (resp. semistable) (Γ, G)-bundles of local type $\boldsymbol{\tau}$ on the ramified cover Y.

8.1.9. Definition. Say two parahoric $\mathcal{G}_{\Omega,X}$-torsors $(E, \boldsymbol{\theta})$ and $(F, \boldsymbol{\theta})$ on X are S-equivalent if the corresponding (Γ, G)-bundles on Y are S-equivalent.

8.1.10. Remark. Recall that notion of S-equivalence of principal bundles in [32]. It is routine to extend this notion to (Γ, G)-bundles as well (see [2] and [40]) . The notions of *admissible reduction of structure group* is made with the additional Γ-equivariance property in [2] and [3]. This gives the analogous definitions of (Γ, G)-polystable bundles and Γ-associated graded of a (Γ, G)-semistable bundle ([40]). We omit the details.

Let

$$M(\mathcal{G}_{\boldsymbol{\theta},X}) := \left\{ \begin{array}{l} \text{the set of } S\text{-equivalence classes of} \\ \text{semistable parahoric } \mathcal{G}_{\Omega,X}\text{-torsors on } X \end{array} \right\}, \tag{8.1.10.1}$$

and let $M(\mathcal{G}_{\boldsymbol{\theta},X})^s \subset M(\mathcal{G}_{\boldsymbol{\theta},X})$ denote the subset of *stable* torsors.

By definition we have the set-theoretic identification

$$M_Y^{\boldsymbol{\tau}}(\Gamma, G) \simeq M(\mathcal{G}_{\boldsymbol{\theta_\tau},X}), \tag{8.1.10.2}$$

and by transport of structure we get the structure of a variety on $M(\mathcal{G}_{\boldsymbol{\theta},X})$. We summarize this discussion in the following theorem which is immediate from Theorem 8.1.7:

8.1.11. Theorem. *The set $M(\mathcal{G}_{\boldsymbol{\theta_\tau},X})$ gets a natural structure of an irreducible normal projective variety with $M(\mathcal{G}_{\boldsymbol{\theta_\tau},X})^s$ as an open subset. It gives a* coarse moduli space *for the substack $\mathsf{Bun}(\mathcal{G}_{\boldsymbol{\theta_\tau},X})^{ss}$ of semistable torsors in $\mathsf{Bun}(\mathcal{G}_{\boldsymbol{\theta_\tau},X})$. Furthermore, we have a homeomorphism*

$$\phi^* : R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G \to M(\mathcal{G}_{\boldsymbol{\theta_\tau},X}), \tag{8.1.11.1}$$

which identifies $R_o/\overline{K}_G$ with $M(\mathcal{G}_{\boldsymbol{\theta_\tau},X})^s$.

The next corollary follows from Theorem 7.3.2 and Theorem 8.1.11.

8.1.12. Corollary. *Let $\boldsymbol{\theta_\tau} = \{\theta_i\} \in \mathbb{E}^m$ be the corresponding point in the product of the affine apartment. Then the dimension of the moduli space*

$M(\mathcal{G}_{\boldsymbol{\theta_\tau},X})$ *is given by*

$$\dim_{\mathbb{C}}(G)(g-1)+\sum_{i=1}^{m}\frac{1}{2}e(\boldsymbol{\theta_\tau}). \tag{8.1.12.1}$$

8.2. Extension to the case when the structure group is reductive. We indicate briefly how to extend the construction of the moduli space of (Γ, H)-bundles to the case when the structure group H is a connected reductive algebraic group and identify it with the space of homomorphisms from π to K_H. However, the corresponding relationship with parahoric torsors for reductive G needs a closer analysis of Bruhat–Tits theory for reductive groups.

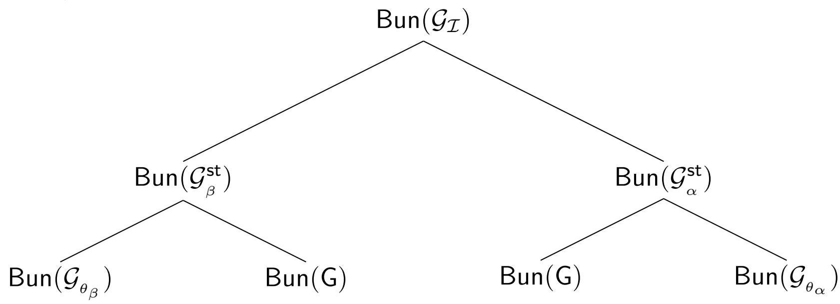
Let $S = [H, H]$ be the derived group, i.e., the *maximal connected semisimple subgroup* of H. Let Z_0 be the connected component of the center of H (which is a torus), and one knows that S and Z_0 together generate H. Let $G = Z_0 \times S$. Then in fact, $G \to H$ is a finite covering map. It is easy to see (following [31, p. 145]) that (Γ, G)-bundles gives rise to (Γ, H)-bundles and the stability and semistability of the associated (Γ, H)-bundles follows immediately from that of the (Γ, G)-bundles.

The problem of handling the reductive group G reduces to the problem of handling the semisimple group H but which is *not simply connected.* Let $\tilde{H}$ be the semisimple, simply connected algebraic group which is the covering group of H.

We are in the situation of Proposition 8.1.4. Recall the central extension (8.1.4.2). By adding a dummy point other than the parabolic point, the theory of (π, H)-bundles is recovered from that of $(\tilde{\pi}, \tilde{H})$-bundles. Notice that a homomorphism $\pi \to K_H$ has as many liftings $\tilde{\pi} \to K_{\tilde{H}}$ as the order of the center of $\tilde{H}$. It follows quite easily, following arguments as in Lemma 8.1.1, that the number of connected components of the moduli space in the nonsimply connected case is given by the order of the center of $\tilde{H}$. In fact, $Hom(\tilde{\pi}, K_{\tilde{H}})$ is a union of spaces labeled by elements of the center of $\tilde{H}$. Let $Z_0 = Ker(\tilde{H} \to H)$. Then there is an action of $H^1(X, Z_0)$ on a specific labeled subset of $Hom(\tilde{\pi}, K_{\tilde{H}})$. A component of the moduli space of representations into K_H can be obtained as a quotient of each of these by the action of $H^1(X, Z_0)$. Details of these ideas are again found in [31, p. 148], and follow the ideas of Narasimhan and Seshadri [27], where the data over a dummy point is called a *special parabolic structure.*

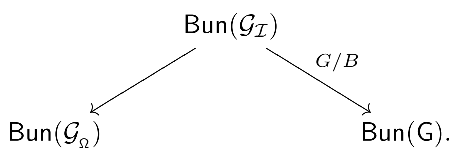
8.2.1. Hecke correspondences. Recall that for the case of linear groups, one has the classical Hecke correspondences due to Narasimhan and Ramanan [26]. In what follows, we consider parahoric subgroups $\mathcal{P}_{\Omega}(K)$ of $G(K)$ which contain a fixed Iwahori subgroup $\mathcal{I}$ (see subsection 2.1.11 for notation). Using (2.1.11.2), we get $\mathcal{I} \subset \mathcal{P}^{st}_{\alpha}(K) \subset \mathcal{P}_{\theta_\alpha}(K) \cap \mathcal{P}_{0}(K)$. These maps of parahoric groups induce morphisms of the corresponding parahoric Bruhat–Tits group schemes, $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}^{st}_{\alpha}$ and $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}_{\theta_\alpha}$ and morphisms at the level of stacks and we obtain the following generalized Hecke correspondences. The dimension formulae (see Corollary 8.1.12) get reflected accurately in the picture.

(8.2.1.1)



For instance, we have the following picture of a Hecke correspondence induced by the morphisms $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}_{\Omega}$ and $\mathcal{G}_{\mathcal{I}} \to \mathcal{G}_{0} (= G \times X)$:

(8.2.1.2)



8.2.2. Remark. It would be interesting to express these relations as morphisms between moduli spaces $M(\mathcal{G}_{\Omega,X})$; even the existence of suitable morphisms between the moduli spaces would involve choice of polarization (in the sense of GIT), which would be needed for an algebro-geometric construction of the moduli spaces of parahoric torsors.

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