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MODULI OF VECTOR BUNDLES ON CURVES WITH PARABOLIC STRUCTURES

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Let H be the upper half plane and Γ a discrete subgroup of Aut$H$. Suppose that $H \bmod \Gamma$ is of *finite measure*. This work stems from the question whether there is an algebraic interpretation for the moduli of unitary representations of Γ similar to the case when $H \bmod \Gamma$ is *compact* (cf. [3], [4], [5]). We show that this is indeed the case via the moduli of vector bundles on the compactification of $H \bmod \Gamma$, provided with some additional structures which we propose to call *parabolic structures*. The idea of parabolic structures is inspired from A. Weil's work [6, §2, Chapter I, p. 56].

Let X be a *smooth*, *irreducible*, *projective curve* defined, say, over an algebraically closed field k. By *vector bundles* on X we understand algebraic vector bundles.

DEFINITION 1. Let V be a vector bundle on X and $Q \in X$. Then a *quasi-parabolic structure of V at Q* is giving a flag on the fibre V_Q of V at Q, i.e., giving linear subspaces F^iV_Q of V_Q,

$$V_Q = F^1V_Q \supset F^2V_Q \supset \cdots \supset F^rV_Q; \quad \dim F^iV_Q = l_i; \quad l_1 > l_2 > \cdots > l_r.$$

We call $l = (l_1, \ldots, l_r)$ the *type* (or flag type) of the quasi-parabolic structure. Let $k_1 = l_1 - l_2, k_2 = l_2 - l_3, \ldots, k_{r-1} = l_{r-1} - l_r, k_r = l_r$; then k_i are called the *multiplicities* of the quasi-parabolic structure.

DEFINITION 2. Let V be a vector bundle on X and $Q \in X$. Then a *parabolic structure of V at Q* is giving

(i) a quasi-parabolic structure of V at Q; say $l = (l_1, \ldots, l_r)$ is its type and $\{k_i\}$ its multiplicities, and

(ii) constants $\alpha = (\alpha_1, \ldots, \alpha_n)$ called the *weights* of the parabolic structure such that $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n < 1$ and there are r distinct elements among α, say $\alpha' = (\alpha'_1, \ldots, \alpha'_r)$, $0 \leqslant \alpha'_1 < \alpha'_2 < \cdots < \alpha'_r < 1$, such that α'_1 occurs k_1 times, α'_2 occurs k_2 times, $\ldots$, α'_r occurs k_r times among α. We call *α'_i the weight of F^iV_Q*. Note that $l_1 = n = rkV$.

Let V, W be vector bundles on X with *quasi-parabolic* structures at Q. An isomorphism f: $V \longrightarrow W$ of vector bundles is said to be a *quasi-parabloic isomorphism* if the types of V, W at Q are the same and $f_Q(F^iV_Q) = F^iW_Q$ (f_Q:

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isomorphism induced by f on the fibers of V, W at Q). Suppose, moreover, we are given *parabolic structures* of V, W at Q consistent with the given quasi-parabolic structures; we say that f is a *parabolic isomorphism* if f is a quasi-parabolic isomorphism and weight of F^iV_Q = weight of F^iW_Q.

DEFINITION 3. Let V be as in Definition 2. Then the *parabolic degree* of V is defined by

$$\text{par deg } V = \deg V + \sum_{i=1}^{n} \alpha_i \quad (\deg V = \text{degree } V).$$

Also we write par $\mu(V)$ for the expression

$$\text{par } \mu(V) = \mu(V) + \left(\sum \alpha_i\right) \Big/ rkV; \quad \mu(V) = (\deg V)/rkV.$$

We give similar definitions when we are given parabolic structures at a finite number of points of X.

DEFINITION 4. Let W, V be vector bundles on X with parabolic structures at $Q \in X$. We say that W is a *parabolic subbundle* of V if

(i) W is a subbundle of V in the usual sense;

(ii) given i_0, $F^{i_0}W \subset F^jV$ for some j. Let j_0 be such that $F^{i_0}W \subset F^{j_0}V$ and $F^{i_0}W \not\subset F^{j_0+1}V$; then weight of $F^{j_0}V$ = weight of $F^{i_0}W$.

We define similarly the notion of a *parabolic quotient bundle* of V. Note that given an ordinary subbundle W of V (resp. quotient bundle), there exists a canonical structure of a parabolic subbundle (resp. quotient bundle) on W. Following Mumford (cf. [1]), we introduce

DEFINITION 5. Let V be a vector bundle on X with parabolic structures at a finite number of points of X. We say that V is *parabolic stable* (resp. *semi-stable*) if $\forall$ proper parabolic subbundle W of V, we have par $\mu(W) <$ par $\mu(V)$ (resp. $\leqslant$).

PROPOSITION 1. *Let V be a vector bundle on X with parabolic structures at a finite number of points of X. Suppose that V is parabolic semistable. Then $\exists$ a filtration of V by parabolic subbundles V_i, $V = V_1 \supset V_2 \supset \cdots$, such that*

(i) par $\mu(V_i)$ = par $\mu(V)$ *and V_i is parabolic semistable,*

(ii) V_i/V_{i+1} *(with the canonical parabolic structure) is parabolic stable, and*

(iii) gr $V = \bigoplus V_i/V_{i+1}$ *is well determined, i.e.,* gr V *(up to parabolic isomorphism) is independent of the filtration $\{V_i\}$ of V with properties* (i) *and* (ii).

Let $VB(d, \alpha)$ denote the category of parabolic semistable vector bundles V on X with a parabolic structure at a *single point* $Q \in X$ (we assume this for simplicity of notation) of fixed weight $\alpha = (\alpha_1, \ldots, \alpha_n)$ and fixed ordinary degree d. Let $\sim$ denote the equivalence relation in $VB(d, \alpha)$, $V_1 \sim V_2$ if gr V_1 = gr V_2. Let $M(d, \alpha)$ be the set of equivalence classes under this equivalence relation.

THEOREM 1. *Suppose that g = genus of $X \geqslant 2$. Then there is a natural structure of a normal projective variety on $M(d, \alpha)$ of dimension $n^2(g-1)+\delta$ where δ is the dimension of the variety of flags in an n-dimensional vector space of type given by the type of the underlying quasi-parabolic structure. Further $M(d, \alpha)$ is smooth at the points V where V is parabolic stable.*

Suppose now that the base field $k = \mathbf{C}$ and $X - Q = H \bmod \Gamma$ where H is the upper half plane and Γ is a discrete subgroup of $\operatorname{Aut} H$. Fix a parabolic fixed point Q', $Q' \in \bar{H}$ of Γ ($\bar{H}$ being the usual $H \cup$ certain boundary points). Let Γ_0 be the isotropy subgroup of Γ at Q'. Fix a generator γ_0 of Γ_0 ($\Gamma_0 \approx \mathbf{Z}$). Let $R(\alpha)$ denote the equivalence classes of unitary representations χ of Γ such that $\chi(\gamma_0)$ is conjugate to the diagonal matrix with entries $(e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_n})$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $0 \leqslant \alpha_1 \leqslant \alpha_2 \cdots \leqslant \alpha_n < 1$.

THEOREM 2. *Suppose that $g \geqslant 2$. Then there is a canonical identification of $R(\alpha)$ with the underlying topological space of $M(d, \alpha)$ with $d = -\Sigma_{i=1}^n \alpha_i$ (or equivalently* $\operatorname{par} \deg V = 0$, $V \in M(d, \alpha)$).

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