

## MODULI OF VECTOR BUNDLES ON CURVES WITH PARABOLIC STRUCTURES

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Let  $H$  be the upper half plane and  $\Gamma$  a discrete subgroup of  $\text{Aut}H$ . Suppose that  $H \bmod \Gamma$  is of *finite measure*. This work stems from the question whether there is an algebraic interpretation for the moduli of unitary representations of  $\Gamma$  similar to the case when  $H \bmod \Gamma$  is *compact* (cf. [3], [4], [5]). We show that this is indeed the case via the moduli of vector bundles on the compactification of  $H \bmod \Gamma$ , provided with some additional structures which we propose to call *parabolic structures*. The idea of parabolic structures is inspired from A. Weil's work [6, §2, Chapter I, p. 56].

Let  $X$  be a *smooth, irreducible, projective curve* defined, say, over an algebraically closed field  $k$ . By *vector bundles* on  $X$  we understand algebraic vector bundles.

DEFINITION 1. Let  $V$  be a vector bundle on  $X$  and  $Q \in X$ . Then a *quasi-parabolic structure* of  $V$  at  $Q$  is giving a flag on the fibre  $V_Q$  of  $V$  at  $Q$ , i.e., giving linear subspaces  $F^i V_Q$  of  $V_Q$ ,

$$V_Q = F^1 V_Q \supset F^2 V_Q \supset \cdots \supset F^r V_Q; \quad \dim F^i V_Q = l_i; \quad l_1 > l_2 > \cdots > l_r.$$

We call  $l = (l_1, \dots, l_r)$  the *type* (or flag type) of the quasi-parabolic structure. Let  $k_1 = l_1 - l_2, k_2 = l_2 - l_3, \dots, k_{r-1} = l_{r-1} - l_r, k_r = l_r$ ; then  $k_i$  are called the *multiplicities* of the quasi-parabolic structure.

DEFINITION 2. Let  $V$  be a vector bundle on  $X$  and  $Q \in X$ . Then a *parabolic structure* of  $V$  at  $Q$  is giving

(i) a quasi-parabolic structure of  $V$  at  $Q$ ; say  $l = (l_1, \dots, l_r)$  is its type and  $\{k_i\}$  its multiplicities, and

(ii) constants  $\alpha = (\alpha_1, \dots, \alpha_n)$  called the *weights* of the parabolic structure such that  $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n < 1$  and there are  $r$  distinct elements among  $\alpha$ , say  $\alpha' = (\alpha'_1, \dots, \alpha'_r), 0 \leq \alpha'_1 < \alpha'_2 < \cdots < \alpha'_r < 1$ , such that  $\alpha'_1$  occurs  $k_1$  times,  $\alpha'_2$  occurs  $k_2$  times,  $\dots$ ,  $\alpha'_r$  occurs  $k_r$  times among  $\alpha$ . We call  $\alpha'_i$  the *weight* of  $F^i V_Q$ . Note that  $l_1 = n = rk$ .

Let  $V, W$  be vector bundles on  $X$  with *quasi-parabolic* structures at  $Q$ . An isomorphism  $f: V \rightarrow W$  of vector bundles is said to be a *quasi-parabolic isomorphism* if the types of  $V, W$  at  $Q$  are the same and  $f_Q(F^i V_Q) = F^i W_Q$  ( $f_Q$ :

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isomorphism induced by  $f$  on the fibers of  $V, W$  at  $Q$ ). Suppose, moreover, we are given *parabolic structures* of  $V, W$  at  $Q$  consistent with the given quasi-parabolic structures; we say that  $f$  is a *parabolic isomorphism* if  $f$  is a quasi-parabolic isomorphism and  $\text{weight of } F^i V_Q = \text{weight of } F^i W_Q$ .

DEFINITION 3. Let  $V$  be as in Definition 2. Then the *parabolic degree* of  $V$  is defined by

$$\text{par deg } V = \text{deg } V + \sum_{i=1}^n \alpha_i \quad (\text{deg } V = \text{degree } V).$$

Also we write  $\text{par } \mu(V)$  for the expression

$$\text{par } \mu(V) = \mu(V) + \left( \sum \alpha_i \right) / \text{rk } V; \quad \mu(V) = (\text{deg } V) / \text{rk } V.$$

We give similar definitions when we are given parabolic structures at a finite number of points of  $X$ .

DEFINITION 4. Let  $W, V$  be vector bundles on  $X$  with parabolic structures at  $Q \in X$ . We say that  $W$  is a *parabolic subbundle* of  $V$  if

- (i)  $W$  is a subbundle of  $V$  in the usual sense;
- (ii) given  $i_0, F^{i_0} W \subset F^j V$  for some  $j$ . Let  $j_0$  be such that  $F^{j_0} W \subset F^{j_0} V$  and  $F^{i_0} W \not\subset F^{j_0+1} V$ ; then  $\text{weight of } F^{j_0} V = \text{weight of } F^{j_0} W$ .

We define similarly the notion of a *parabolic quotient bundle* of  $V$ . Note that given an ordinary subbundle  $W$  of  $V$  (resp. quotient bundle), there exists a canonical structure of a parabolic subbundle (resp. quotient bundle) on  $W$ .

Following Mumford (cf. [1]), we introduce

DEFINITION 5. Let  $V$  be a vector bundle on  $X$  with parabolic structures at a finite number of points of  $X$ . We say that  $V$  is *parabolic stable* (resp. *semi-stable*) if  $\forall$  proper parabolic subbundle  $W$  of  $V$ , we have  $\text{par } \mu(W) < \text{par } \mu(V)$  (resp.  $\leq$ ).

PROPOSITION 1. Let  $V$  be a vector bundle on  $X$  with parabolic structures at a finite number of points of  $X$ . Suppose that  $V$  is parabolic semistable. Then  $\exists$  a filtration of  $V$  by parabolic subbundles  $V_i, V = V_1 \supset V_2 \supset \dots$ , such that

- (i)  $\text{par } \mu(V_i) = \text{par } \mu(V)$  and  $V_i$  is parabolic semistable,
- (ii)  $V_i/V_{i+1}$  (with the canonical parabolic structure) is parabolic stable, and
- (iii)  $\text{gr } V = \bigoplus V_i/V_{i+1}$  is well determined, i.e.,  $\text{gr } V$  (up to parabolic isomorphism) is independent of the filtration  $\{V_i\}$  of  $V$  with properties (i) and (ii).

Let  $VB(d, \alpha)$  denote the category of parabolic semistable vector bundles  $V$  on  $X$  with a parabolic structure at a single point  $Q \in X$  (we assume this for simplicity of notation) of fixed weight  $\alpha = (\alpha_1, \dots, \alpha_n)$  and fixed ordinary degree  $d$ . Let  $\sim$  denote the equivalence relation in  $VB(d, \alpha), V_1 \sim V_2$  if  $\text{gr } V_1 = \text{gr } V_2$ . Let  $M(d, \alpha)$  be the set of equivalence classes under this equivalence relation.

**THEOREM 1.** *Suppose that  $g = \text{genus of } X \geq 2$ . Then there is a natural structure of a normal projective variety on  $M(d, \alpha)$  of dimension  $n^2(g-1) + \delta$  where  $\delta$  is the dimension of the variety of flags in an  $n$ -dimensional vector space of type given by the type of the underlying quasi-parabolic structure. Further  $M(d, \alpha)$  is smooth at the points  $V$  where  $V$  is parabolic stable.*

Suppose now that the base field  $k = \mathbf{C}$  and  $X - Q = H \bmod \Gamma$  where  $H$  is the upper half plane and  $\Gamma$  is a discrete subgroup of  $\text{Aut } H$ . Fix a parabolic fixed point  $Q', Q' \in \bar{H}$  of  $\Gamma$  ( $\bar{H}$  being the usual  $H \cup$  certain boundary points). Let  $\Gamma_0$  be the isotropy subgroup of  $\Gamma$  at  $Q'$ . Fix a generator  $\gamma_0$  of  $\Gamma_0$  ( $\Gamma_0 \approx \mathbf{Z}$ ). Let  $R(\alpha)$  denote the equivalence classes of unitary representations  $\chi$  of  $\Gamma$  such that  $\chi(\gamma_0)$  is conjugate to the diagonal matrix with entries  $(e^{2\pi i \alpha_1}, \dots, e^{2\pi i \alpha_n})$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n < 1$ .

**THEOREM 2.** *Suppose that  $g \geq 2$ . Then there is a canonical identification of  $R(\alpha)$  with the underlying topological space of  $M(d, \alpha)$  with  $d = -\sum_{i=1}^n \alpha_i$  (or equivalently  $\text{par deg } V = 0, V \in M(d, \alpha)$ ).*

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