Moduli Spaces of Dynamical Systems on $\mathbb{P}^n$

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Abstract

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This thesis studies the space of morphisms on $\mathbb{P}^n$ defined by polynomials of degree d and its quotient by the conjugation action of $\mathrm{PGL}(n+1)$, which should be thought of as coordinate change. First, we construct the quotient using geometric invariant theory, proving that it is a geometric quotient and that the stabilizer group in $\mathrm{PGL}(n+1)$ of each morphism is finite and bounded in terms of n and d. We then show that when $n = 1$, the quotient space is rational over a field of any characteristic.

We then study semistable reduction in this space. For every complete curve C in the semistable completion of the quotient space, we can find curves upstairs mapping down to it; this leads to an abstract complete curve D with a projective vector bundle parametrizing maps on the curve. The bundle is trivial iff there exists a complete curve D in the semistable space upstairs mapping down to C; we show that for every n and d we can find a C for which no such D exists. Finally, in the case where D does exist, we show that, whenever it lies in the stable space, the map from D to C is ramified only over points with unusually large stabilizer, which for a fixed rational C will bound the degree of the map from D to C.

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Chapter 1

Introduction and Notation

A rational map from $\mathbb{P}^n$ to itself is determined by an $(n+1)$-tuple of polynomials in $n+1$ variables, all homogeneous of the same degree d. If this map is a morphism, it will be finite of degree d^n. In the sequel, we will refer to a rational map defined by such polynomials as a degree d map on $\mathbb{P}^n$ by abuse of terminology. The space of degree d maps on $\mathbb{P}^n$ is projective, with homogeneous coordinates coming from monomials of degree d. There are $\binom{n+d}{d}$ such monomials, so that this space has dimension $\binom{n+d}{d}(n+1)-1$. We write N_d^n for the dimension of this space, or N when d and n are clear; in the sequel, we will invariably refer to this space as $\mathbb{P}^N$.

The case of interest is morphisms on $\mathbb{P}^n$. In the sequel, we refer to the polynomials defining the map as $q_0, q_1, \dots, q_n$. Then a map $(q_0 : \dots : q_n)$ is a morphism if and only if the q_i's share no common geometric root. The q_i's only share a common root if $(q_0 : \dots : q_n)$ lies on a hypersurface of $\mathbb{P}^N$, which we call the resultant subvariety and which is defined over $\mathbb{Z}$; we denote its complement by Hom_d^n.

The space $\mathbb{P}^N$ of rational maps comes equipped with an action of $\mathrm{PGL}(n+1)$ by conjugation. The conjugation action $A \cdot \varphi = A\varphi A^{-1}$, fixes the resultant, which gives an action of $\mathrm{PGL}(n+1)$ on Hom_d^n. We study the quotient of this action, which we denote M_d^n, or M_d when $n = 1$. We will show that this quotient is geometric in the sense of geometric invariant theory [10], and compute the largest stable and semistable loci $\mathrm{Hom}_d^{n,s}$ and $\mathrm{Hom}_d^{n,ss}$, which satisfy $\mathrm{Hom}_d^n \subset \mathrm{Hom}_d^{n,s} \subset \mathrm{Hom}_d^{n,ss} \subset \mathbb{P}^N$. Therefore our first task will be to recap the results of geometric invariant theory, which we will do in section 2.

Knowing that the quotient M_d^n is well-behaved is often necessary to answer questions about the geometry of families of dynamical systems. In [12], Petsche, Szpiro, and Tepper prove that M_d^n exists as a geometric quotient in order to show that isotriviality is equivalent to potential good reduction for morphisms of $\mathbb{P}^n$ over function fields, generalizing previous results in the one-dimensional case. In [3], DeMarco uses the explicit description of the space M_2 in order to study iterations of quadratic maps on $\mathbb{P}^1$, and one can expect similar results in higher dimension given a better understanding of the structure of M_d^n.

By now the theory of morphisms on $\mathbb{P}^1$ is the standard example in dynamical systems. For a survey of the arithmetic theory, see [16]; also see a recent paper by Manes [7] about moduli of morphisms on $\mathbb{P}^1$ with a marked point of period n, which functions as a dynamical level structure. In the complex case, see an overview by Milnor [9], and the work of DeMarco [2] [3] about compactifications of the space M_d that respect the iteration map. Despite this, the higher-dimensional theory remains understudied. The only prior result in the direction of moduli of morphisms on $\mathbb{P}^n$ is the proof in [12] that,

Theorem 1.0.1. *Every* $\varphi \in \mathrm{Hom}_d^n$ *is stable.*

Remark 1.0.2. This is equivalent to the statement that M_d^n exists as a geometric quotient.

Unfortunately, the proof does not lend itself well to finding the stable and semistable spaces for the action of $\mathrm{PGL}(n+1)$ on $\mathbb{P}^N$. We construct two alternative proofs of the fact that the quotient M_d^n is geometric, first by explicitly describing the stable and semistable loci, and second by finding a uniform bound for the size of the stabilizer group in $\mathrm{PGL}(n+1)$. The former we will do in section 3.1, using the Hilbert-Mumford criterion for stability and semistability. We will see that the complements of both $\mathrm{Hom}_d^{n,s}$ and $\mathrm{Hom}_d^{n,ss}$ are equal to a finite union of linear subvarieties and their $\mathrm{PGL}(n+1)$-conjugates; this contrasts with the $n = 1$ case, when the complement is the $\mathrm{PGL}(2)$-orbit of only one linear subvariety. In section 3.2 we will provide a second alternative proof by proving:

Theorem 1.0.3. *The stabilizer of every point in* $\mathrm{Hom}_d^n, d > 1$*, is a finite group of order bounded in terms of* n *and* d*.*

Many of the above results are a natural generalization of the study of morphisms on $\mathbb{P}^1$ in [15], which refers to the space of morphisms as Rat_d and its quotient as M_d, and which proves that $\mathrm{M}_2 \cong_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{A}^2$ using the theories of fixed points and multipliers.

Specializing to the case where $n = 1$, we will prove in section 3.3 that,

Theorem 1.0.4. M_d *is rational over any field of definition.*

The proof is based on showing that M_d is birational to a vector bundle over the space $\mathrm{M}_{0,d+1}$ of $d+1$ unmarked points on $\mathbb{P}^1$, which is known to be rational.

Unfortunately, we do not see any easy generalization of rationality to M_d^n. The obstruction is that the space of unmarked points on $\mathbb{P}^n$ is not known to be rational. Clearly Hom_d^n is rational, so M_d^n is unirational, which for some applications, such as the density of points defined over a number field K, is enough. However, in order to investigate the structure of M_d^n we need more than that. We do not expect a result along the lines of that in [15], that $\mathrm{M}_2 \cong \mathbb{A}^2$, but we do expect rationality of M_d^n.

Afterward, we investigate semistable reduction in $\mathrm{Hom}_d^{n,ss}$. It is known from geometric invariant theory that,

Theorem 1.0.5. *If C is a complete curve with $K(C)$ its function field, and if $\varphi_{K(C)}$ is a semistable rational map on $\mathbb{P}^n_{K(C)}$, then there exists a curve D mapping finite-to-one onto C with a $\mathbb{P}^n$-bundle $\mathbf{P}(\mathcal{E})$ on D with a self-map Φ such that,*

1. *The restriction of Φ to the fiber of each $x \in D$, φ_x, is a semistable rational self-map.*

2. *Φ is a semistable map over $K(D)$, and is equivalent to $\varphi_{K(D)}$ under coordinate change.*

This comes from the following general result:

Theorem 1.0.6. *Let R be a discrete valuation ring with fraction field K, and let $\varphi_K \in \mathrm{M}_d^{n,ss} \times \operatorname{Spec} K$. Then for some finite extension K' of K, with R' the integral* *closure of R in K', φ_K has an integral model over R' with semistable reduction modulo the maximal ideal. In other words, for $\varphi_K \in \mathrm{Hom}_d^{n,ss} \times \mathrm{Spec}\, K$, we can find some $A \in \mathrm{PGL}(n+1, \overline{K})$ such that $A\varphi_K A^{-1}$ has semistable reduction.*

Although this is a standard proof from geometric invariant theory, we include a proof for completeness in section 4.1, as well as a more explicit coordinate-wise proof for $n = 1$ in section 4.2 in order to find a bound for the degree of the field extension.

We investigate a stronger statement than semistable reduction, claiming that for every C we obtain a complete D in $\mathrm{Hom}_d^{n,ss}$ mapping finite-to-one onto C; in the formulation of Theorem 1.0.5, this is equivalent to the existence of a trivial bundle class. Such a result is false, and we provide a counterexample for every n and d, which in the case $n = 1$ is the curve $x^d + c$, and for higher n is one of the obvious generalizations, namely $(x_0^d + cx_1^d : x_1^d : \ldots : x_n^d)$. We show this in sections 4.3 and 4.4.

In principle, we would expect almost all curves C to admit a complete D upstairs. This is because the unstable locus in $\mathbb{P}^N$ has very high codimension, and therefore the preimage of C in $\mathbb{P}^N$ should not intersect it, or should intersect it in a high-codimension locus. However, it turns out that in $\mathrm{M}_2^{ss} \cong \mathbb{P}^2$, almost all lines would not admit a complete D upstairs. To do this, we prove the following results, valid for a general geometric invariant theory setting:

Proposition 1.0.7. *Let X be a projective variety over an algebraically closed field with an action by a geometrically reductive linear algebraic group G. Using the terminology of geometric invariant theory, let D be a complete curve in the stable space X^s whose quotient by G is a complete curve C; say the map from D to C has degree*

m. Suppose the stabilizer is generically finite, of size h, and either D or C is normal. Then there exists a finite subgroup $S_D \subseteq G$, of order equal to mh, such that for all $x \in D$ and $g \in G$, $gx \in D$ iff $g \in S_D$.

Corollary 1.0.8. *With the same notation and conditions as in Proposition 4.1.5, the map from D to C is ramified precisely at points $x \in D$ where the stabilizer group is larger than h, and intersects S_D in a larger subgroup than in the generic case.*

These results constrain the possibilities of D too much, and using intersection numbers, we can derive a contradiction for most lines in M_2^{ss}. This only works because we know the isomorphism class of M_2^{ss}, so it cannot extend so easily to higher n and d.

Chapter 2

A Review of Geometric Invariant Theory

When a geometrically reductive linear algebraic group G has a linear action on a projectivized vector space $\mathbb{P}(V)$, we have,

Definition 2.0.9. A point $x \in V$ is called semistable (resp. stable) if any of the following equivalent conditions hold:

1. There exists a G-invariant homogeneous section s such that $s(x) \neq 0$ (resp. same condition, and the action of G on x is closed).
2. The closure of $G \cdot x$ does not contain 0 (resp. $G \cdot x$ is closed).
3. Every one-parameter subgroup T acts on x with both nonnegative and nonpositive weights (resp. negative and positive weights).

Remark 2.0.10. The last condition in the definition is equivalent to having nonpositive (resp. negative) weights. This is because if we can find a subgroup acting with only negative weights, then we can take its inverse and obtain only positive weights.

Observe that for every nonzero scalar k, x is stable (resp. semistable) if and only if kx is. So the same definitions of stability and semistability hold for points of $\mathbb{P}(V)$. The definitions also descend to every G-invariant projective variety $X \subseteq \mathbb{P}(V)$; in fact, in [10] they are defined for X in terms of a G-equivariant line bundle L. When L is ample, as in the case of Hom_d^n and its projective closure, this reduces to the above definition.

The importance of stability is captured in the following prior results:

Proposition 2.0.11. *The space of all stable points, X^s, and the space of all semistable points, X^{ss}, are both open and G-invariant.*

Theorem 2.0.12. *There exists a quotient $Y = X^{ss}//G$, called a good categorical quotient, with a natural map $\pi : X \to Y$, satisfying the following properties:*

1. *π is a G-equivariant map, where G acts on Y trivially.*

2. *Every G-equivariant map $X \to Z$, where G acts on Z trivially, factors through π.*

3. *π is an open submersion.*

4. *$\pi(x_1) = \pi(x_2)$ iff the closures of $G \cdot x_1$ and $G \cdot x_2$ intersect.*

5. *For every open $U \subseteq Y$, $\mathcal{O}_U = \mathcal{O}(\pi^{-1}(U))^G$.*

In addition, Y is proper.

Theorem 2.0.13. *There exists a quotient $Z = X^s//G$, called a good geometric quotient, with a natural map $\pi : X \to Z$ satisfying all enumerated conditions of a good categorial quotient, as well as the following:*

1. $\pi(x_1) = \pi(x_2)$ *iff* $G \cdot x_1 = G \cdot x_2$.

2. Z *is naturally an open subset of* $X^{ss}//G$.

Theorem 2.0.14. *On* X^s*, the dimension of the stabilizer group* $\mathrm{Stab}_G(x)$ *is constant.*

Returning to our case of self-maps of $\mathbb{P}^n$, we write the stable and semistable spaces for the conjugation action as $\mathrm{Hom}_d^{n,s}$ and $\mathrm{Hom}_d^{n,ss}$. This involves a fair amount of abuse of notation, since those two spaces are open subvarieties of $\mathbb{P}^N$ and as we will show properly contain Hom_d^n, which consists only of regular maps.

We use the Hilbert-Mumford criterion, which is the last condition in Definition 2.0.9. In more explicit terms, the criterion for semistability (resp. stability) states that for every one-parameter subgroup $T \leq \mathrm{SL}(n+1)$, the action of T on φ can be diagonalized with eigenvalues t^{a_I} and at least one a_I is nonpositive (resp. negative). Trivially, we may replace T with T^{-1}, so that semistability requires both a nonpositive weight and a nonnegative weight, and stability requires both a negative weight and a positive weight.

Remark 2.0.15. The best way to interpret the Hilbert-Mumford criterion is as follows: if there exist a positive weight and a negative weight, then $T \cdot x \subset V$ looks like a hyperbola, so that it is closed. If there exist positive weights and a zero weight, then $T \cdot x$ looks like a punctured line, with an extra closure that is not zero. And if there exist only positive weights, then $T \cdot x$ will go to zero in the limit.

If $n = 1$, we have a relatively simple description, due to Silverman [15]:

Theorem 2.0.16. *$\varphi \in \mathbb{P}^N$ is unstable (resp. not stable) iff it is equivalent under coordinate change to a map $(a_0x^d + \ldots + a_dy^d)/(b_0x^d + \ldots + b_dy^d)$, such that:*

1. *$a_i = 0$ for all $i \leq (d-1)/2$ (resp. $<$).*

2. *$b_i = 0$ for all $i \leq (d+1)/2$ (resp. $<$).*

The description for $n = 1$ can be thought of as giving a dynamical criterion for stability and semistability. A point $\varphi \in \mathbb{P}^N$ is unstable if there exists a point $x \in \mathbb{P}^1$ where φ has a bad point of degree more than $(d+1)/2$, or φ has a bad point of degree more than $(d-1)/2$ where it in addition has a fixed point. Following Rahul Pandharipande's unpublished reinterpretation of [15], we define "bad point" as a vertical component of the graph $\Gamma_\varphi \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, and "fixed point" as a fixed point of the unique non-vertical component of Γ_φ. When $n = 1, d = 2$, this condition reduces to having a fixed point at a bad point, or alternatively a repeated bad point.

The conditions for higher n are not as geometric. However, if we interpret fixed points liberally enough, there are still strong parallels with the $n = 1$ case. One can show that the unstable space for $n = 2$ and $d = 2$ consists of two irreducible components, which roughly generalize the $n = 1, d = 2$ condition of having a fixed point at a bad point; in this case, one needs to define a limit of the value of $\varphi(x)$ as x approaches the bad point, though this limit can be defined purely in terms of degrees of polynomials, without needing to resort to a specific metric on the base field.

Chapter 3

The Spaces Hom_d^n and M_d^n

3.1 The Construction of M_d^n

The space Hom_d^n of degree-d morphisms on $\mathbb{P}^n$ arises as the subset of $\mathbb{P}^N = \{(q_0 : q_1 : \ldots : q_n)\}$, defined by the condition that the q_i's share no common root. In order to give this space an algebraic structure, we investigate its complement. We will show the following result, proven by Macaulay [6] and reinterpreted here in modern language (see also [4] for a more complete treatment):

Theorem 3.1.1. *The maps on $\mathbb{P}^n$ of degree d such that the q_i's share a nonzero root form a closed, irreducible subvariety of $\mathbb{P}^N$ of codimension 1, which is defined over $\mathbb{Z}$.*

Proof. Consider the variety $V = \mathbb{P}^n \times \mathbb{P}^N$. We think of V as representing a set of polynomials $(q_0 : q_1 : \ldots : q_n)$ acting on the point $(x_0 : x_1 : \ldots : x_n)$. Consider the resultant subvariety $U \subset V$ defined by the condition that $q_i(\mathbf{x}) = 0$ for all i. This variety clearly has codimension at most $n+1$. If we denote the variables defining $\mathbb{P}^N$ as $a^i_{j_0^i j_1^i \ldots j_n^i}$ with $j_0^i + \ldots + j_n^i = d$, representing the $x_0^{j_0^i} \ldots x_n^{j_n^i}$ monomial of q_i, then we see that U is defined by equations that are bihomogeneous of degree 1 in the a^i_J's and

d in the x_i's.

We claim that U is irreducible. The claim follows from a generalization of the fact that a primitive polynomial is irreducible over a domain whenever it is irreducible over its fraction field. More precisely, let R be a domain with fraction field K, and let I be an ideal of $R[y_1, \ldots, y_m]$ that is not contained in any prime of R. We have a natural map f from $\operatorname{Spec} K[y_1, \ldots, y_m]$ to $\operatorname{Spec} R[y_1, \ldots, y_m]$. If $V(I)$ is reducible over R, say $V(I) = V_1 \cup V_2$ with V_i nonempty, then either $V(I)$ is reducible over K, or one $f^{-1}(V_i)$, say $f^{-1}(V_1)$, is empty. In the latter case, $I(V_1)$ may not contain nonconstant polynomials, so it contains at least one prime constant. This contradicts the assumption that I is not contained in any prime of R; hence, $V(I)$ is reducible over K.

With the above generalization, suppose that U is reducible. Then it is also reducible as a subvariety of $\mathbb{A}^{n+1} \times \mathbb{A}^{N+1}$. Further, by letting $R = \mathbb{Z}[x_0, \ldots, x_n]$ and K be its fraction field, we see that either U is contained in a prime of R, or U is reducible in $\mathbb{A}_K^{N+1}$. The former case is impossible since U is not contained in any prime of $\mathbb{Z}$ or any relevant prime ideal of the ring of polynomials over $\mathbb{Z}$, and the latter is impossible since it is defined by linear equations in the a_J^i's. Either way this is a contradiction, so U is irreducible and the claim is proven.

Finally, the maps on $\mathbb{P}^n$ of degree d whose polynomials have a common nonzero root arise as the projection of U onto the second factor of $\mathbb{P}^n \times \mathbb{P}^N$. It is irreducible because the projection map is surjective. It is closed because the map is proper. It has codimension at most 1 because almost all polynomials in U share just one root, so that the dimension of U and its image are equal. It has exact codimension 1 because some maps, for instance $q_i = x_i^d$, are morphisms. And it is defined over $\mathbb{Z}$ because every construction we have made in this proof is defined over $\mathbb{Z}$. $\square$

We call the image of U the resultant subvariety of $\mathbb{P}^N$; we call its generating polynomial the Macaulay resultant and denote it by Res_d^n. Macaulay proved the theorem by constructing the resultant explicitly for homogeneous polynomials p_i of arbitrary degrees, and showing that it has integer coefficients and is irreducible. His explicit construction shows that if the polynomials $p_0, p_1, \ldots, p_n$ are homogeneous of degrees $d_0, d_1, \ldots, d_n$, then the resultant is $(n+1)$-homogeneous in the coefficients of each polynomial p_i of degree $\prod_{j \neq i} d_j$. In our case, all the degrees are equal to d, so that the resultant is $(n+1)$-homogeneous in the coefficients of each q_i of degree d^n. In particular, the resultant subvariety is a hypersurface of degree $(n+1)d^n$.

Theorem 3.1.1 shows that the space of morphisms is the complement of the resultant subvariety, and is therefore affine and of dimension N; we will refer to it as Hom_d^n, and to the resultant subvariety as Res_d^n by abuse of notation. Silverman [15], who only considers the case $n = 1$, denotes the space of morphisms by Rat_d, and Petsche, Szpiro, and Tepper [12] denote the space of morphisms by End_d^n.

The action of $\mathrm{PGL}(n+1)$ on $\mathbb{P}^n$ leads to a conjugation action on Hom_d^n, wherein $A \in \mathrm{PGL}(n+1)$ acts on a rational map φ by sending it to $A\varphi A^{-1}$. The property of being ill-defined at a point is stable under both the left action mapping φ to $A\varphi$ and the right action mapping φ to φA^{-1}; hence, the conjugation action is well-defined on Hom_d^n. The space of endomorphisms of $\mathbb{P}^n$ defined by degree-d polynomials may be regarded as the quotient of Hom_d^n by the conjugation action.

A priori, we only know that over an algebraically closed field, the quotient exists as a set. In order to give it algebraic structure, we need to pass to the stable or semistable space in geometric invariant theory. Fortunately, we have the following result:

Theorem 3.1.2. *Every $\varphi \in \mathrm{Hom}_d^n$ is stable.*

Proof. We use the Hilbert-Mumford criterion. To do that, we pull back the action of $\mathrm{PGL}(n+1)$ on $\mathbb{P}^N$ to the action of $\mathrm{SL}(n+1)$ on $\mathbb{A}^{N+1}$, and consider one-parameter subgroups of $\mathrm{SL}(n+1)$. Recall that a point lies in the stable space $\mathrm{Hom}_d^{n,s}$ (respectively, the semistable space $\mathrm{Hom}_d^{n,ss}$) iff for every such subgroup, its action on the point can be diagonalized with diagonal elements t^{a_I}, and at least one a_I is negative (resp. non-positive).

Note that the action of $A \in \mathrm{SL}(n+1)$ on $\varphi \in \mathbb{A}^{N+1}$ is conjugate to the action of BAB^{-1} on $B\varphi B^{-1}$. In particular, it will have the same eigenvalues, so the action of a one-parameter subgroup $T = \mathbb{G}_m$ will have the same a_I's. Therefore, we may conjugate T to be diagonal, which will be enough to give us criteria for stability and semistability up to conjugation. So from now on, we assume T is the diagonal subgroup whose ith diagonal entry is $t^{a_i}, a_i \in \mathbb{Z}$. Here we label the rows and columns from 0 to n, in parallel with the label for the q_i's. We have $a_0 + \ldots + a_n = 0$. We may also assume that $a_0 \geq a_1 \geq \ldots \geq a_n$, after conjugation if necessary, and that the a_i's are coprime.

The action of T on $\mathbb{A}^{N+1}$ is already diagonal. We denote the $\mathbf{x}^{\mathbf{d}}$ coefficient of q_i

by $c_{\mathbf{d}}(i)$; then T multiplies $c_{\mathbf{d}}(i)$ by $t^{a_i}t^{-(a_0d_0+\ldots+a_nd_n)}$. A point φ is not stable (resp. unstable) if for some choice of T, all the $c_{\mathbf{d}}(i)$'s for which $a_0d_0+\ldots+a_nd_n > a_i$ (resp. $a_0d_0+\ldots+a_nd_n \geq a_i$) are zero. Let us observe that this means that, for $d > 1$, every x_0^d coefficient has to be zero, as we will have $da_0 > a_0 \geq a_i$ for every i. This means that φ has no x_0^d term, so that the q_i's have a nontrivial zero at $(1 : 0 : \ldots : 0)$, and $\varphi \notin \mathrm{Hom}_d^n$. The property of not being a morphism is preserved under conjugation, proving the theorem. $\square$

Since Hom_d^n is stable, it has a natural geometric quotient induced by the $\mathrm{PGL}(n+1)$ action on $\mathbb{P}^N$, which we denote by M_d^n; as Hom_d^n is affine, M_d^n is affine, with structure sheaf $\mathcal{O}_{\mathrm{Hom}_d^n}^{\mathrm{SL}(n+1)}$. We may also write $\mathrm{M}_d^{n,s}$ for the quotient of the stable space and $\mathrm{M}_d^{n,ss}$ for the quotient of the semistable space. The latter quotient is only categorical, rather than geometric, but will be proper over $\operatorname{Spec}\mathbb{Z}$ (all spaces in question, as well as $\mathrm{SL}(n+1)$, are defined over $\mathbb{Z}$; hence, so are the quotients), so it can be written as $\operatorname{Proj}\mathcal{O}_{\mathbb{P}^N}^{\mathrm{SL}(n+1)}$.

In the $n = 1$ case, T depends only on a_0, which may be taken to be 1. This gives us only one criterion for stability (resp. semi-stability), which means that the not-stable (resp. unstable) space is irreducible (in fact, it will be a linear subvariety and its orbit under $\mathrm{PGL}(2)$-conjugation). The only T has $a_0 = 1, a_1 = -1$, so $a_0d_0 + a_1d_1 = d_0 - d_1 = 2d_0 - d$. When d is even, $2d_0 - d$ is always even, so the conditions $a_0d_0 + a_1d_1 > a_i$ and $a_0d_0 + a_1d_1 \geq a_i$ coincide, and the stable and semistable spaces are the same. This is the proof given in [15] for Theorem 2.0.16.

When $n > 1$, T depends on multiple variables, and we can find many infinite families of coprime a_i's that sum to 0 and are in decreasing order. However, the not-stable (resp. unstable) space will still be a union of finitely many linear subvarieties and their orbits under conjugation by $\mathrm{PGL}(n+1)$, where the number of linear subvarieties generally grows with d and n. This is because there are only 2^{N+1} linear spaces defined by conditions of the form $c_{\mathbf{d}}(i) = 0$ for a collection J of $(\mathbf{d}, i)$ pairs. For each such space, either there exists a T such that $(\mathbf{d}, i) \in J$ if and only if $a_0 d_0 + \ldots + a_n d_n > a_i$ (resp. $a_0 d_0 + \ldots + a_n d_n \geq a_i$), or there doesn't. Of course, a given J may correspond to infinitely many T, which will in general have ratios $a_0 : \ldots : a_n$ that are close in the archimedean metric.

We omit the calculation of the linear subvarieties that occur as the not-stable (resp. unstable) space for each d and n, as well as the number of such varieties. We will just note that there are far fewer than 2^{N+1} such varieties: for a start, we have already seen that $((d, 0, \ldots, 0), i) \in J$ for all i. One more constraint that follows trivially from the definition of the a_i's is that if $(\mathbf{d}, i) \in J$, then so is $(\mathbf{d}, j)$ for $j > i$. Put another way, not being stable (resp. instability) imposes more conditions on q_j than on q_i for $j > i$. It may also be shown that for each T the number of conditions is roughly between one half and e^{-1} times N; we omit the proof, as this result will not be relevant in the remainder of this thesis.

Unlike in the case of $n = 1$, we have:

Proposition 3.1.3. *For all $d, n > 1$, we have* $\mathrm{Hom}_d^{n,s} \subsetneq \mathrm{Hom}_d^{n,ss}$.

Proof. First, observe that if we set $a_0 = 1$, $a_n = -1$, and $a_i = 0$ for $i \neq 0, n$, we obtain $a_0 d_0 + \ldots + a_n d_n = d_0 - d_n$, which may take any value between $-d$ and d

inclusive. Hence, the conditions $a_0d_0+\ldots+a_nd_n > a_i$ and $a_0d_0+\ldots+a_nd_n \geq a_i$ will not coincide.

Now, suppose that φ is a point that is not stable, with $c_{\mathbf{d}}(i) = 0$ if and only if $d_0 - d_n > a_i$ with a_i as above. If φ is unstable, then we can find some T such that if $a_0d_0+\ldots+a_nd_n \geq a_0$ then $d_0-d_n > 1$, and if $a_0d_0+\ldots+a_nd_n \geq a_i$ for $i \neq 0, n$, then $d_0-d_n > 0$. If for that T we have $a_1 \geq 0$, then looking at the $x_0x_1^{d-1}$ monomial, we get $a_0d_0+\ldots+a_nd_n = a_0+(d-1)a_1 \geq a_0$ but $d_0-d_n = 1$, a contradiction. If $a_1 < 0$, then we must have $a_i < 0$ for all $i > 0$, so $a_0+a_n > 0$. For $d = 2k+1$, we consider the $x_0^{k+1}x_n^k$ monomial, for which $a_0d_0+\ldots+a_nd_n = k(a_0+a_n)+a_0 > a_0$ but $d_0-d_n = 1$; for $d = 2k$, we consider the $x_0^kx_n^k$ monomial, for which $a_0d_0+\ldots+a_nd_n = k(a_0+a_n) > 0 > a_1$ but $d_0-d_n = 0$. Either way, we have a contradiction, so φ is semistable but not stable. $\square$

We will conclude this section with the following strict containment:

Proposition 3.1.4. $\mathrm{Hom}_d^n \subsetneq \mathrm{Hom}_d^{n,s}$.

Proof. Observe that the linear subvarieties defined above are invariant under conjugation by every upper triangular matrix, at least when we ensure $a_0 \geq a_1 \geq \ldots \geq a_n$. Hence, the codimension of the not-stable space is equal to the codimension of the largest linear subvariety, minus $n(n+1)/2$. It suffices to show this codimension is more than 1, or, in other words, that every linear subvariety has codimension at least $n(n+1)/2+2$. We will consider two cases.

Case 1. $a_1 \geq 0$. When $d_0 > 0$, the $x_0^{d_0}x_1^{d_1}$ monomial has $a_0d_0+a_1d_1 > a_1$, so it is zero for all q_i's except q_0; when $d_0 > 1$ it is also zero for q_0, since $a_0d_0+a_1d_1 \geq 2a_0$.

This gives us a total codimension of $n^2 + (n-1)$, which is larger than $n(n+1)/2+1$ for all $n \geq 2$. When $n = 1$ this case is impossible because we need to have $a_0 + a_1 = 0$.

Case 2. $a_1 < 0$. We have $a_0 = -(a_1 + \ldots + a_n) > -a_i$ for all i; therefore, the $x_0^{d-1}x_i$ monomial is zero in every q_j except q_0; the x_0^d monomial is always zero. This gives us a codimension of $n^2 + n + 1$, which is large enough for all n. □

Remark 3.1.5. The larger spaces $\mathrm{Hom}_d^{n,s}$ and $\mathrm{Hom}_d^{n,ss}$ are more interesting in the study of moduli spaces more than in that of dynamical systems. The problem is that we cannot always iterate rational maps which are not morphisms, even if they are stable: the image may not be dense, and may eventually map to a locus on which the map is ill-defined. A map of the form $(q : 0 : 0 : \ldots : 0)$ with $q(1, 0, \ldots, 0) = 0$ will be impossible to iterate. For general q, it will also be stable for large d, because we will have $a_0d_0 + \ldots + a_nd_n > a_0$ for many different $\mathbf{d}$'s no matter how we choose the a_i's, even after conjugation. When $n = 1$, it suffices to have $d \geq 4$, because then φ is unstable only if it is of the form $(p : q)$ with p and q sharing a common root of multiplicity at least $(d-1)/2$, and we may pick a map $(q : 0)$ with q having distinct roots. For one approach for giving a completion of Hom_d^n in a way that permits iteration at the boundary, see [2].

3.2 Stabilizer Groups

Each morphism in Hom_d^n, and more generally each rational map, has a well-defined stabilizer group in $\mathrm{PGL}(n+1)$. This group remains well-defined up to conjugation after descent to M_d^n, or more generally $\mathrm{M}_d^{n,ss}$. This stabilizer will be finite, at least on

$\mathrm{M}_d^{n,s}$, from standard facts from geometric invariant theory. We will study the possible subgroups of $\mathrm{PGL}(n+1)$ that may occur as stabilizers of morphisms. We gain very little by assuming Theorem 3.1.2, so we might as well not assume it *a priori*; this will provide an alternative proof for it.

Note that the resultant is a $\mathrm{PGL}(n+1)$-invariant section of a $\mathrm{PGL}(n+1)$-linearizable divisor on $\mathbb{P}^N$ that is nonzero on Hom_d^n. Therefore, on Hom_d^n stability is equivalent to having closed fibers, which is equivalent to having a stabilizer group of the lowest possible dimension (see chapter 1 of [10]). Hence, to provide a second proof of Theorem 3.1.2, it suffices to show that the stabilizer of every $\varphi \in \mathrm{Hom}_d^n$ is finite. This was done in [12]. We will prove a stronger result:

Theorem 3.2.1. *The stabilizer of every point in* $\mathrm{Hom}_d^n, d > 1$, *is a finite group of order bounded in terms of* n *and* d.

Proof. Note that if $A \in \mathrm{Stab}(\varphi)$, then $BAB^{-1} \in \mathrm{Stab}(B\varphi B^{-1})$. Therefore, when considering individual stabilizing matrices, we may assume they are in Jordan canonical form. We use the following result:

Lemma 3.2.2. *If* $A \in \mathrm{Stab}(\varphi)$, *and* φ *is not purely inseparable, then* A *is diagonalizable.*

Proof. In characteristic zero, this is trivial given Theorem 3.1.2. However, it is not trivial in characteristic p; the proof works for every characteristic, so we lose nothing from not using Theorem 3.1.2.

We will assume that A is not diagonalizable and derive a contradiction. It suffices to assume that A is a Jordan matrix whose largest Jordan block is of size $r > 1$. After conjugation and scaling, we may assume that the first Jordan block is also the largest, and has eigenvalue 1. We will label the rows and columns from 0 to n, in parallel with the labels for the q_i's. We will also write $\varphi = (q_0 : q_1 : \ldots : q_n)$, $k_i = a_{ii}$ for the eigenvalue in the ith position, and r_i for the size of the Jordan block containing a_{ii}. We have $r_0 = r, k_0 = 1, r_i \leq r$.

Note that the inverse of the first Jordan block is the matrix with zeroes below the main diagonal and $a_{ij} = (-1)^{i-j}$ on or above it. Therefore, each vector $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is transformed to:

$$\mathbf{x}' = (x_0 - x_1 + \ldots \pm x_{r-1}, x_1 - x_2 + \ldots \mp x_{r-1}, \ldots, x_{r-1}, \ldots, \frac{1}{k_n} x_n)$$

We write $q_i'(\mathbf{x}) = q_i(\mathbf{x}')$. Similarly, A transforms $\varphi = (q_0, \ldots, q_n)$ to:

$$\varphi' = (q_0' + q_1', q_1' + q_2', \ldots, q_{r-1}', \ldots, k_n q_n')$$

Since A stabilizes φ, we need φ' to be a scalar multiple of φ.

For each $\mathbf{d} \in \mathbb{Z}^{n+1}$, we denote the $\mathbf{x}^{\mathbf{d}}$ coefficient of q_i (respectively q_i') by $c_{\mathbf{d}}(i)$ (resp. $c_{\mathbf{d}}'(i)$). We suppress trailing zeroes for simplicity, so that c_d denotes the x_0^d coefficient. We are looking for the largest i such that $c_d(i) \neq 0$; such an i exists, or else $(1 : 0 : \ldots : 0)$ is a common root of all the q_i's. As the only x_0^d term in $\mathbf{x'^d}$ comes from $x_0'^d$, we have $c_d'(j) = c_d(j)$ for all j. Now in φ', the ith term is either q_i' or $q_i' + k_i q_{i+1}'$, so that its x_0^d coefficient is $k_i c_d(i)$. This implies that the scaling factor is k_i, i.e. $\varphi' = k_i \varphi$.

Now, assume that i is not at the beginning of its Jordan block, that is that

$a_{i-1,i} = 1$. Then $k_{i-1} = k_i$, and the fact that $\varphi' = k_i\varphi$ implies that:

$$k_{i-1}c'_d(i-1) + c'_d(i) = k_i c_d(i-1)$$

This reduces to $c_d(i) = 0$, a contradiction. Therefore, i is at the beginning of its Jordan block.

Let us now consider the $x_0^{d-1}x_1$ coefficients, and assume throughout that all indices are in the same Jordan block as i. We have $c'_{d-1,1}(j) = c_{d-1,1}(j) - dc_d(j)$. For $j > i$, this reduces to $c'_{d-1,1}(j) = c_{d-1,1}(j)$. Conversely, the corresponding term to $c_{d-1,1}$ in $\varphi' = k_i\varphi$ will be $k_i c'_{d-1,1}(j) + c'_{d-1,1}(j+1) = k_i c_{d-1,1}(j)$. When $j > i$, this implies that $c'_{d-1,1}(j+1) = 0$, so that $c_{d-1,1}(j) = 0$ for $j > i+1$; conversely, for $i+1$, we obtain $k_i c'_{d-1,1}(i) + c'_{d-1,1}(i+1) = k_i c_{d-1,1}(i)$, which reduces to $c_{d-1,1}(i+1) = k_i d c_d(i) \neq 0$. This shows that $i+1$ is the largest index with a nonzero $x_0^{d-1}x_1$ coefficient, at least in the Jordan block containing i.

We may apply induction on $s(\mathbf{d}) = d_1 + 2d_2 + \ldots + (r-1)d_{r-1}$, and find that in the Jordan block containing i, the largest index with a nonzero $\mathbf{x}^{\mathbf{d}}$ coefficient is $i + s(\mathbf{d})$. Note that the Jordan block has $r_i \leq r$ elements, but the number of monomial indices attached to the first Jordan block is $(r-1)d+1$, which is strictly greater than r when $d, r > 1$. This is a contradiction: the last element of the Jordan block has $k_i c'_{\mathbf{d}} = k_i c_{\mathbf{d}}$ for all $\mathbf{d}$, i.e. $c_{\mathbf{d}}(i + r_i - 1)' = c_{\mathbf{d}}(i + r_i - 1)$, but that last equality is only true when $s(\mathbf{d}) \leq r_i$, which is not the case for all $\mathbf{d}$. Since we are assuming $d > 1$, we must have $r = 1$, and we are done.

The careful reader may note that the proof that $i + s$ is the largest index with a nonzero $\mathbf{x}^{\mathbf{d}}$ coefficient for $s(\mathbf{d}) = s$ makes an assumption about the characteristic we are working in. In characteristic zero, $d \neq 0$ and there is no problem. In characteristic p, we need to treat separately the case when $p < d$. Then for example we may have $p \mid d$, so that $c'_{d-1,1}(j) = c_{d-1,1}(j)$ for all j, and $c_{d-1,1}(i+1)$ may be zero. Note that the number of monomial indices containing x_0^{d-2} attached to the first Jordan block is $2(r-1)+1$, which is strictly greater than r when $r > 1$; when $p \nmid d(d-1)$, we may restrict ourselves to such monomials, and the proof proceeds as in characteristic zero.

When $p \mid d-1$, we may restrict ourselves to monomials containing x_0^{d-1}, and proceed with the proof. We will only encounter an obstruction if $r_i = r$ and only at the end of the Jordan block, where the existence of a nonzero $x_0^{d-1}x_{r-1}$ monomial does not guarantee that of $x_0^{d-2}x_1x_{r-1}$. However, the action of A on q_{i+r-1} takes it to $k_iq'_{i+r-1}$, and we must have $c'_{\mathbf{d}}(i+r-1) = c_{\mathbf{d}}(i+r-1)$ for all $\mathbf{d}$. If we write $d-1 = p^lm, m \nmid p$, then we see that $x_0^{d-1}x_{r-1}$ is transformed to $k_i(x_0 - x_1 + \ldots \pm x_{r-1})^{d-1}x_{r-1} = k_i(x_0^{p^l} - \ldots \pm x_{r-1}^{p^l})^m x_{r-1}$ which shows that the $x_0^{p^l(m-1)}x_1^{p^l}$ monomial does not satisfy $c'_{\mathbf{d}}(i+r-1) = c_{\mathbf{d}}(i+r-1)$. This yields a contradiction.

Finally, when $p \mid d$, we may write $d = p^lm$ with $p \nmid m$. When $m > 1$, we apply exactly the same proof as in characteristic zero, except that we write m instead of d and $m_j = d_j/p^l$ instead of d_j; then we define $s(\mathbf{d}) = m_1 + \ldots + (r-1)m_{r-1}$, and in the Jordan block containing i, the largest index with a nonzero $\mathbf{x}^{\mathbf{d}}$ coefficient is $i + s(\mathbf{d})$. As $m > 1$, we have $(r-1)m + 1 > r$ for $r > 1$, and we have the same contradiction as in the characteristic zero case. Note that when $m = 1$, we may derive the same contradiction from any nonzero monomial not of the form x_j^d, which must exist if φ

is not purely inseparable. Hence, if φ has a non-diagonalizable stabilizer then it is purely inseparable and we are done. $\square$

With the above lemma, we know that any abelian subgroup of $\mathrm{Stab}(\varphi) \in \mathrm{GL}(n+1)$ will be simultaneously diagonalizable, unless φ is purely inseparable. We will prove the following uniform bound on the size of abelian stablizing subgroups:

Lemma 3.2.3. *Every diagonal subgroup stabilizing $\varphi \in \mathrm{Hom}_d^n$ is of size at most d^{n+1}.*

Proof. A diagonal matrix A with diagonal entries $(a_0, a_1, \dots, a_n)$ acts on each q_i by multiplying $c_{\mathbf{d}}(i)$ by $a_i / \left(a_0^{d_0} \dots a_n^{d_n}\right)$. Our case of interest will be the x_j^d coefficients. Each has to be nonzero for at least one i, which induces the equation $a_i = a_j^d$. Note that we may set the scaling factor k to be 1, since the scalar matrix $k^{1/(1-d)}$ multiplies every coefficient by k.

Now, we have at least $n+1$ different relations $a_i = a_j^d$. We may drop relations until each j has just one i such that such a relation holds; dropping relations will increase the size of the group, so by bounding the size of the larger group, we will bound the size of any automorphism group.

We obtain a function $j \mapsto i$. If the function is bijective, we may write it as a product of disjoint cycles, and conjugate to get the cycles to be $(0\ 1\ \dots\ s_1-1)\dots(n-s_k+1\ \dots\ n)$, where here r_i denotes the length of the ith cycle, and has nothing to do with the definition in Lemma 3.2.2. Then $a_0^{d^{r_1}} = a_0$ and a_0 is a root of unity of order dividing $d^{r_1}-1$, the choice of which uniquely determines $a_i, 0 \le i \le r_1 - 1$. We have similar results for $a_{r_1}, \dots, a_{n-r_k+1}$; since $\sum r_i = n+1$, this bounds the size of the group by d^{n+1}.

In general, of course, the function $j \mapsto i$ may not be bijective, so we can only write it as a product of precycles, whose cycles are disjoint. Here a precycle means a cycle and zero or more tails. The above discussion applies to the cycles. For the tails, suppose without loss of generality that $(0\ 1\ \ldots\ r)$ is a tail where r and no element before it is part of a cycle; then the choice of a_r determines a choice of d possibilities for a_{r-1} and in general d^s for a_{r-s} subject to the obvious compatibility condition. This clearly respects the bound of d^{n+1}: if m is the total number of elements in cycles, then we have at most m^{n+1} possibilities for the cycles, each of which gives us exactly $(d-m)^{n+1}$ possibilities for the tails. $\square$

The bound d^{n+1} works for abelian stabilizing subgroups in the purely inseparable case as well. We may view a purely inseparable φ as the action of raising every coefficient to the dth power followed by the matrix B. Then $A\varphi A^{-1} = \varphi$ if and only if $ABA_d^{-1} = B$, where A_d is the image of the matrix A under the homomorphism of raising every entry to the dth power; we need to show that the group of such A, which we will write as $\mathrm{Stab}(B)$, is finite. Since A and A_d are conjugate, all eigenvalues of A are in $\mathbb{F}_d$.

We may conjugate an abelian stabilizing subgroup G to obtain a block diagonal group with each block upper triangular and with its (i,j) entry depending only on $j-i$. We may also fix one element, C to be in Jordan canonical form, in which case we will have $C_d = C$ and thus $BC = CB$. Then B is in block form; labeling the blocks by r, s and the rth block of C by C_r, we see that the B_{rs} is nonzero if and only if the blocks r and s are of the same size and equal for every element of $\mathrm{Stab}(B)$, and in any case B_{rs} commutes with $C_r = C_s$, so it is upper triangular with its (i, j) entry depending only on $j - i$. In particular, it commutes with every $A_r = A_s$, so that B commutes with G. Hence for all $A \in G$, we have $AB = BA$ and $ABA_d^{-1} = B$, so that $A = A^d$ and A has entries in $\mathbb{F}_d$. Furthermore, for each block in G of size r, we have r positive possibilities for $j - i$, inducing d^r possible blocks, and d^{n+1} possible matrices in G.

Note that we may have additional stabilizing matrices in $\mathrm{PGL}(n+1)$. These occur when there exists an automorphism of the set $\{0, 1, \ldots, n\}$ that does not leave the diagonal vector $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{A}^{n+1}$ fixed, but does fix $\mathbf{a} = (a_0 : \ldots : a_n) \in \mathbb{P}^n$. Since the automorphism has to fix $a_0 a_1 \ldots a_n$, we see that it must send each a_i to ζa_i where ζ is a root of unity of order at most $n + 1$; hence there are at most $n + 1$ possibilities for such an automorphism, modulo automorphisms that fix $\mathbf{a} \in \mathbb{A}^{n+1}$ and are hence simultaneously block-diagonalizable with A.

We will rely on one final bound, due to G. A. Miller [8]:

Proposition 3.2.4. *The size of a finite group is bounded in terms of the size of its largest abelian subgroup.*

Proof. It suffices to show this for p-groups. For each n, we let $k(n)$ be the minimal exponent of the largest abelian subgroup of any p-group of exponent n. Furthermore, for each $l \leq n$, we let $k(n, l)$ be the minimal exponent subject to the restriction that $Z = Z(G)$ have exponent l, so that $k(n) = \min\{k(n, l)\}$. It is enough to show that $\lim_{n\to\infty} k(n) = \infty$.

It is trivial to show that $k(2) = 2$. In general, for a p-group of exponent n and center of exponent l, let g be such that $g \notin Z$, $g^p \in Z$, and $gZ \in Z(G/Z)$. Unless G is abelian, in which case the result is trivial, we may take g to be a preimage of a nontrivial element in the socle of G/Z, i.e. a nontrivial element of $Z(G/Z)$ of order p. For every $h \in G$, $hgh^{-1} = gz$ for some $z \in Z$; we obtain a group homomorphism $h \mapsto z$ from G to Z. The homomorphism has kernel K of exponent at least $n - l$ and center containing $\langle Z, g \rangle$. Any abelian subgroup of K will be an abelian subgroup of G, so that we obtain $k(n, l) \geq k(n - l, l + 1)$. It easily follows that $k(n) \geq 2\sqrt{n}$. $\square$

The bound in the above proposition is very weak. It is known that for odd p we have $k(n) \leq (n + 4)/3$ and for $p = 2$ we have $k(n) \leq 2(n + 3)/5$ [1], but little more. However, we can show,

Proposition 3.2.5. *The bound in Theorem 3.2.1 is sub-exponential in* d^{n+1}.

Proof. The fact that $k(n) \geq 2\sqrt{n}$ is equivalent to the fact that for a fixed $r = k(n)$, $n \leq r(r + 1)/2$. Now if G is a finite group, and for every p dividing the order of G, the largest abelian p-subgroup of G has size p^{r_p}, then $|G| \leq \prod p^{r_p(r_p+1)/2}$. Trivially, we have $r_p \leq \log_2 p^{r_p} \leq \log_2 |G|$, and trivially we have $\prod p^{r_p} \leq |G|$; therefore, $|G|$ is bounded by $(\prod p^{r_p})^{(1+\log \prod p^{r_p})/2}$.

Now, let us return to the notation of the main theorem, where n is the ambient dimension rather than the exponent of a group. We list all the primes dividing $(n+1)$, d, and $d^i - 1$ for $1 \leq i \leq n+1$, together with the maximal multiplicities with which they could divide the order of an abelian stabilizer that is bounded by $(n+1)d^{n+1}$. It would be enough to list the primes and multiplicities dividing $(n+1)d^{n+1}\prod_{ij\leq n+1}(d^i-1)^j$. Now we have,

$$(n+1)d^{n+1}\prod_{ij\leq n+1}(d^i-1)^j \leq (n+1)d^{n+1}d^{(n+1)(1+\frac{1}{2}+\ldots+\frac{1}{n+1})} \leq (n+1)d^{(n+1)(2+\log n)}$$

Using the bound for a general G above, we can bound the stabilizer by,

$$\left((n+1)d^{(n+1)(2+\log n)}\right)^{\frac{1+\log\left((n+1)d^{(n+1)(2+\log n)}\right)}{2}}$$

The logarithm of this expression grows roughly as $\log^2\left((n+1)d^{(n+1)(2+\log n)}\right)$, which grows more slowly than d^{n+1}. Hence the bound is sub-exponential. □

Note that there is no way to make the bound polynomial in d without improving the bound in Proposition 3.2.4 to $k(n) \geq n/r$ for a fixed r.

Note also that proposition 3.2.4 does not show *a priori* that the group has to be finite, only that if it is finite then its size is bounded. We may use Theorem 3.1.2 and finish. However, with little additional effort, we may prove finiteness directly, providing an alternative proof that all morphisms are stable. The fact that finite implies uniformly bounded means that it is enough to show that every finitely generated stabilizing subgroup is finite. More precisely:

Proposition 3.2.6. *Every finitely generated subgroup of* $\mathrm{PGL}(n)$ *contained in finitely many finite-order conjugacy classes is finite.*

Proof. Let R be the $\mathbb{Z}$-algebra generated by the finitely many coefficients of the generators. Then the group is contained in $\mathrm{PGL}(n,R)$, and we may project it into the finite group $\mathrm{PGL}(n, R/\mathfrak{m})$ where $\mathfrak{m}$ is a maximal ideal in R; we will show the map can be chosen to be injective. In fact, each non-unipotent conjugacy class i contains two different eigenvalues, a_{i_1}, a_{i_2}; therefore, if we choose $\mathfrak{m}$ not to contain $a_{i_1} - a_{i_2}$, which we can since there are only finitely many such elements, then the map will have unipotent kernel. In characteristic 0, the only finite-order unipotent matrix is the identity, so the map is injective and we are done.

In characteristic p, we obtain a finite-index and hence finitely generated unipotent group. We may conjugate it by some matrix P to be upper triangular; then matrix multiplication is equivalent to addition of the $(r, r+1)$ entry for any r, and the finite generation implies that the set of all $(r, r+1)$ entries lies in a finitely generated $\mathbb{Z}/p\mathbb{Z}$-vector space, which is finite. For the matrices with all $(r, r+k)$ entries for all $k \leq l$, matrix multiplication corresponds to addition of $(r, r+l+1)$ entries, and we may add those entries to our vector space, which will remain finite. We may now construct $\mathfrak{m}$ to avoid the finite vector space and the determinant of P, as well as the eigenvalue differences described above. The map will then be injective. □

Note that in the proof of proposition we make no assumption on the base ring. Of course, the argument in the proposition applies to $\mathrm{GL}(n+1)$, and shows that the answer to Burnside's problem, which asks whether a finitely generated group of bounded exponent is necessarily finite, is yes when restricted to subgroups with faithful finite-dimensional representations over some field. □

For each stabilizer group $G \in \mathrm{PGL}(n+1)$, there is a closed subscheme $\mathrm{Fix}(G) \in \mathrm{Hom}_d^n$ consisting of all φ with stabilizer group containing G. Theorem 3.2.1 states that every G with nonempty $\mathrm{Fix}(G)$ is finite and of bounded order. Furthermore, each nontrivial stabilizing matrix is, up to conjugation, one of the d^{n+1} possibilities for each of the $(n+1)^{n+1}$ functions on the set $\{0, 1, \ldots, n\}$. We may strengthen this result as follows:

Corollary 3.2.7. *There are only finitely many G with nonempty* $\mathrm{Fix}(G)$ *up to conjugation. In particular, on an open dense set of* Hom_d^n*, which descends to* M_d^n*, the stabilizer group is trivial.*

Remark 3.2.8. The statement that there are only finitely many such G up to conjugation is stronger than the statement that there are only finitely many G up to isomorphism, which follows trivially from the bound on the size of G.

Proof. Since the size of G is bounded, it suffices to show that each stabilizing subgroup has finitely many projective $n+1$-dimensional representations up to conjugacy. This is always true when the representation is completely reducible, which will be true if the ambient characteristic p does not divide $|G|$. But when $\mathrm{Fix}(G)$ is not purely inseparable, every element will be diagonalizable, so it will have order not divisible by p, so that G has order not divisible by p. In the purely inseparable case, we have $\mathrm{PGL}(n+1)$ acting on itself stably and with finite stabilizers, so that each orbit is of dimension $(n+1)^2 - 1$ and thus consists of all of $\mathrm{PGL}(n+1)$. In other words, every purely inseparable map is, up to conjugation, $(x_0^d : \ldots : x_n^d)$, so that its stabilizer group is conjugate to $\mathrm{PGL}(n+1, \mathbb{F}_d)$.

It remains to be shown that the complement of $\bigcup_{G \supset I} \mathrm{Fix}(G)$ is dense; its openness follows from the fact that the condition $A\varphi A^{-1} = \varphi$ is closed. It suffices to show that each $\mathrm{Fix}(G)$ is a proper subset of Hom_d^n. We lose nothing if we ignore purely separable maps. From the proof of Lemma 3.2.3, each of the finitely many elements that may occur in G, a diagonal matrix with ith entry a_i, multiplies $c_{\mathbf{d}}(i)$ by $a_i/\mathbf{a}^{\mathbf{d}}$, and hence induces the relation $c_{\mathbf{d}}(i) = 0$ outside a set of $(\mathbf{d}, i)$'s for which $a_i/\mathbf{a}^{\mathbf{d}}$ is constant. If $a_i/\mathbf{a}^{\mathbf{d}}$ is constant for all $(\mathbf{d}, i)$, then we have $a_i = k\mathbf{a}^{\mathbf{d}}$; choosing a constant $\mathbf{d}$, we see that a_i is constant, so A is a scalar matrix. Hence no non-trivial A fixes all of Hom_d^n. □

Note that when $n = 1$, [15] has an explicit bound on the size of $\mathrm{Stab}(\varphi)$ of $n_1!n_2!n_3!$, where the n_i's are indices for which there exist periodic points for φ of exact order n_i. The technique we use improves on that bound. Following the proof of Lemma 3.2.3, we have three possibilities for the map $j \mapsto i$ up to conjugation: $(1,2) \mapsto (1,2)$, $(1,2) \mapsto (2,1)$, and $(1,2) \mapsto (1,1)$. In the first case, $a_0 = \zeta_{d-1}^i$ and $a_1 = \zeta_{d-1}^j$, where we use ζ_i to denote an ith root of unity; modulo multiplying both a_0 and a_1 by some ζ_{d-1}, we obtain a cyclic group of order $d-1$. In the second case, we have $a_0 = \zeta_{d^2-1}^i, a_1 = a_0^d$, and modulo multiplying both by $\zeta_{d^2-1}^{d+1}$, we obtain a cyclic group of order $d+1$. In the third case, $a_0 = \zeta_{d-1}$ and $a_1^d = a_0$, and modulo multiplying both by ζ_{d-1}, we obtain a cyclic group of order d.

Thus every diagonalizable abelian subgroup A of $\mathrm{Stab}(\varphi)$ will be cyclic of size dividing $d-1$, d, or $d+1$. Furthermore, the only non-diagonalizable element commuting with A can be the matrix M corresponding to the automorphism permuting x_0 and x_1; we have $M^{-1} = M$ and $MAM = A$ in $\mathrm{PGL}(2)$ if and only if $a_1/a_0 = a_0/a_1$, or, equivalently, $a_i = \pm 1$ for $i = 0, 1$. In other words, the only possible non-diagonalizable abelian subgroup A is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Now, the only finite subgroups of PGL(2) are, up to conjugation, cyclic, dihedral, tetrahedral, octahedral, or icosahedral [14]. The last three groups are of order at most 60; only the first two are infinite families. Since the largest abelian subgroup of the dihedral group of order $2k$ is of order k, we see that for large d, the order of $\operatorname{Stab}(\varphi)$ is bounded by $2(d+1)$.

We conclude this section with a remark that $\mathrm{M}_d^n(k)$, consisting of all k-rational points in $\mathrm{M}_d^n(\overline{k})$, is not the same as the quotient $\mathrm{Hom}_d^n(k)/\operatorname{PGL}(n+1,k)$. The latter parametrizes morphisms of $\mathbb{P}_k^n$ up to conjugation defined over k, the former up to conjugation defined over $\overline{k}$. There exist maps defined over k which are conjugate over $\overline{k}$ but not over k itself. For examples, see [15] and §§4.7-4.10 of [16].

3.3 Rationality of M_d

In this section, we show that when $n = 1$, the variety $\mathrm{M}_d = \mathrm{M}_d^1$ is rational. This partly generalizes Silverman's result in [15] that $\mathrm{M}_2 \equiv \mathbb{A}^2$ over $\mathbb{Z}$. We do so by parametrizing fixed points of φ. The fixed point set of φ, $\operatorname{Fix}(\varphi)$, is the intersection of two curves in $\mathbb{P}^1 \times \mathbb{P}^1$, the graph Γ_φ and the diagonal embedding Δ. As Δ is irreducible and not contained in Γ_φ for $d > 1$, this is a proper intersection of divisors of type $(1,1)$ and $(d,1)$, so it has $d+1$ points, counting multiplicity. We have:

Theorem 3.3.1. *M_d is birational to the total space of a rank-d vector bundle on $\mathrm{M}_{0,d+1}$, the space of unmarked $d+1$ points on $\mathbb{P}^1$. Since $\mathrm{M}_{0,d+1}$ is rational, it follows that M_d is rational.*

Proof. We explicitly write $\varphi(x : y) = (p : q)$ where $p(x, y) = a_d x^d + \ldots + a_0 y^d$ and $q(x, y) = b_d x^d + \ldots + b_0 y^d$. The fixed points of φ are those for which $(p : q) = (x : y)$, which are the roots of the homogeneous degree-$(d + 1)$ polynomial $py - qx$. The polynomial $py - qx$ induces a map from Rat_d to $(\mathbb{P}^1)^{d+1}/S_{d+1}$ where S_{d+1} acts by permutation of the factors. We will call this map Fix. We use the following lemma:

Lemma 3.3.2. *The map* Fix *is surjective, and has rational fibers.*

Proof. A point $(x : y)$ is fixed if and only if we have $py = qx$, i.e. $a_d x^d y + \ldots + a_0 y^{d+1} = b_d x^{d+1} + \ldots + b_0 x y^d$. This is a homogeneous linear condition in the coefficients of φ, and we have $d + 1$ such conditions compared with $2d + 2$ variables. Once we show surjectivity, from elementary linear algebra, we have a solution space of linear dimension $d + 1$, or projective dimension d; it is a linear subvariety of $\mathbb{P}^{2d+1}$, so it is rational.

We can also show that the fibers will not be contained in the resultant locus. We fix a set of fixed points and write r for the polynomial having those fixed points as roots. We need to show r is of the form $py - qx$ for some p and q sharing no common root. By conjugating, we may assume neither $(0 : 1)$ nor $(1 : 0)$ is a root of r, so that it has a nonzero x^{d+1} coefficient, which we may take to be 1, and a nonzero y^{d+1} coefficient. Now we let $q = -x^d$ so that $r + qx$ is divisible by y, yielding $p = (r + qx)/y$. Now $r + qx$ has a nonzero y^{d+1} coefficient, so p has a nonzero y^d coefficient; therefore, p does not have $(0 : 1)$ as a root, so it shares no root with y. $\square$

Now, Fix descends to a rational map $\mathrm{Fix}' : \mathrm{M}_d \to (\mathbb{P}^1)^{d+1}/S_{d+1}\,\mathrm{PGL}(2)$ where $\mathrm{PGL}(2)$ acts diagonally; we are restricting to the open set of M_d whose fixed points are in the stable space of the action of PGL(2) on $(\mathbb{P}^1)^{d+1}/S_{d+1}$. With this restriction, the image is $\mathrm{M}_{0,d+1}$, so it suffices to show the general fiber of Fix′ is rational. Lemma 3.3.2 says that the fiber of Fix is rational, so it suffices to show that the automorphism group of the general point in $(\mathbb{P}^1)^{d+1}/S_{d+1}$ is small enough that the quotient of the fiber by it is still rational. Using Noether's problem [11] [13], we will show a stabilizer of size 4 or 6 is small enough.

Lemma 3.3.3. *Let $d > 1$. The automorphism group of a general configuration of $d+1$ unmarked points in $\mathbb{P}^1$ is trivial, unless $d = 2$, in which case it is S_3, or $d = 3$, in which case it is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. We will use inhomogeneous coordinates. For $d = 2$, we can conjugate the three points to be $0, 1, \infty$; the set is then stabilized by every permutation in S_3, so it has size 6. For $d > 3$, we will show that the stabilizer is generically trivial, and on the way show that for $d = 3$ the stabilizer is generically of order 4, consisting of all elements in S_4 of cycle type $(2, 2)$. This will be enough to prove the theorem.

First, note that if a $(d+1)$-cycle stabilizes the set of points, then by conjugation we may assume it sends 0 to 1, 1 to λ, μ to ∞, and ∞ to 0. The cycle, regarded as an element of PGL(2), is of the form $(ax + b)/(cx + e)$; then $b/e = 1$, $a = 0$, $(a + b)/(c + e) = \lambda$, and $c\mu + e = 0$. These equations together imply that $\lambda = e/(c+e) = e/(e-\frac{e}{\mu}) = \mu/(\mu-1)$. For a generic choice of μ, λ, this can never happen, so no $(d+1)$-cycle is in the stabilizer. This remains true for $d = 3$, in which case we are forced to have $\lambda = \mu$, since generically $\lambda \neq \lambda/(\lambda - 1)$.

Observe that if an automorphism of cycle type $(c_1, \ldots, c_k)$ stabilizes the set, then each subset corresponding to the ith cycle is stabilized by a c_i-cycle. Therefore, the above discussion shows that no cycle of length 4 or more stabilizes a generic set. We have reduced to the case when all cycles are of size 1, 2, or 3. Now, if we have a stabilizing automorphism which includes a 3-cycle, we may conjugate the 3-cycle to be $(0\ 1\ \infty)$, forcing it to act on $\mathbb{P}^1$ as $1/(1-x)$. Generically, if λ is a fourth point, none of the points in the set (including λ) will be $1/(1-\lambda)$. We are left with cycles of size 1 or 2. If we have a stabilizing automorphism with two 2-cycles, then up to conjugation we may assume the element acts on four points as $(0\ \infty)(1\ \lambda)$, so that it maps x to λ/x. If $d = 3$ then this will stabilize the set regardless of what λ is. If $d > 3$ then we have an additional point μ, and generically λ/μ will not be in our set.

We are left with automorphisms that act as single 2-cycles, fixing $d-1$ points. For $d \geq 4$, they will fix 3 points and therefore act trivially. For $d = 3$, we may assume by conjugation that the element acts as $(0\ 1)$ and fixes ∞; this forces it to be the automorphism $1-x$, which generically does not fix λ. This leaves us with automorphisms consisting only of 1-cycles, i.e. the identity. $\square$

We will return to Noether's problem now. Let us work over a fixed field k. Recall [11] that if $K = k(x_1, \ldots, x_m)$ is a purely transcendental field, and G is a finite group of size 2, 3, 4, or 6 permuting the x_i's, then K^G is purely transcendental as well. In particular, if R is the graded k-algebra $k[x_1, \ldots, x_m]$, and G acts on it by permutation of the x_i's, then $\operatorname{Proj} R^G$ is rational. We will show this to be the case when R is the fiber of Fix in the $d = 2$ and $d = 3$ cases, by finding an orbit $y_1, \ldots, y_m$ generating R over k.

When $d = 2$, we have a 2-dimensional fiber. Explicitly, we have six homogeneous variables $a_i, b_i, 0 \leq i \leq 2$, on which the automorphism group PGL(2) acts linearly. The fiber we are interested in consists of maps fixing the points $0, 1, \infty$, corresponding to the linear conditions $a_0 = 0$, $a_0 + a_1 + a_2 = b_0 + b_1 + b_2$, $b_2 = 0$, respectively. The values of a_2, a_1, b_0 uniquely determine that of b_1, so we may write the fiber as $\operatorname{Proj} k[a_2, a_1, b_0]$. The group S_3 acts linearly and faithfully on the k-vector space spanned by a_2, a_1, b_0. Let us consider the action of the automorphism $(0\ \infty) = 1/x$:

$$\varphi(x) = \frac{a_2x^2 + a_1x}{b_1x + b_0}$$

$$\frac{1}{\varphi(1/x)} = \frac{b_0x^2 + b_1x}{a_1x + a_2}$$

$$a_2 \mapsto b_0$$

$$a_1 \mapsto b_1 = a_2 + a_1 - b_0$$

$$b_0 \mapsto a_2$$

Observe that this automorphism fixes $a_2 + b_0$. Let us also consider the action of the automorphism $(0\ 1) = 1 - x$:

$$\begin{aligned} 1 - \varphi(1-x) &= 1 - \frac{a_2(1-x)^2 + a_1(1-x)}{b_1(1-x) + b_0} \\ &= \frac{-a_2(1-x)^2 + (b_1 - a_1)(1-x) + b_0}{b_1(1-x) + b_0} \end{aligned}$$

$$a_2 \mapsto -a_2$$

$$a_1 \mapsto 2a_2 + a_1 - b_1 = a_2 + b_0$$

$$b_0 \mapsto b_0 + b_1 = a_2 + a_1$$

This automorphism does not stabilize $a_2 + b_0$; hence, $a_2 + b_0$ has stabilizer of order 2, and orbit of size 3. By repeating the maps $1 - x$ and $\frac{1}{x}$, we can compute the orbit as $\{a_2 + b_0, a_1, a_2 + a_1 - b_0\}$. This generates R as long as $\operatorname{char} k \neq 2$. When $\operatorname{char} k = 2$, the automorphism $1 - x$ fixes a_2, whose orbit is then $\{a_2, b_0, a_2 + a_1\}$. In either case, we can construct the action of S_3 as an action of generators, reducing the quotient to Noether's problem.

When $d = 3$, we similarly obtain a 3-dimensional fiber, fixing the points $0, 1, \lambda, \infty$. We obtain the linear conditions:

$$a_0 = 0$$

$$b_3 = 0$$

$$a_3 + a_2 + a_1 = b_2 + b_1 + b_0$$

$$\lambda^2 a_3 + \lambda a_2 + a_1 = \lambda^2 b_2 + \lambda b_1 + b_0$$

We may write R as $k[a_3, a_2, b_1, b_0]$. We look at the automorphism $(0\ \infty)(1\ \lambda) = \lambda/x$:

$$\varphi(x) = \frac{a_3 x^3 + a_2 x^2 + a_1 x}{b_2 x^2 + b_1 x + b_0}$$

$$\frac{\lambda}{\varphi(\lambda/x)} = \frac{\lambda}{\frac{a_3\lambda^3 + a_2 x\lambda^2 + a_1 x^2 \lambda}{b_2 x\lambda^2 + b_1 x^2 \lambda + b_0 x^3}} = \frac{b_0 x^3 + b_1 \lambda x^2 + b_2 \lambda^2 x}{a_1 x^2 + a_2 \lambda x + a_3 \lambda^2}$$

$$a_3 \mapsto b_0$$

$$a_2 \mapsto \lambda b_1$$

$$b_1 \mapsto \lambda a_2$$

$$b_0 \mapsto \lambda^2 a_3$$

We may scale down by a factor of λ to obtain $(\lambda^{-1}b_0, b_1, a_2, \lambda a_3)$, which is equivalent to picking the representative function $\sqrt{\lambda}/(\sqrt{\lambda^{-1}}x)$. Let us also consider the action of the automorphism $(0\ \lambda)(1\ \infty) = (x-\lambda)/(x-1)$:

$$\varphi\left(\frac{x-\lambda}{x-1}\right) = \frac{a_3(x-\lambda)^3 + a_2(x-\lambda)^2(x-1) + a_1(x-\lambda)(x-1)^2}{b_2(x-\lambda)^2(x-1) + b_1(x-\lambda)(x-1)^2 + b_0(x-1)^3}$$

We obtain:

$$\frac{a_3(x-\lambda)^3 + (a_2-\lambda b_2)(x-\lambda)^2(x-1) + (a_1-\lambda b_1)(x-\lambda)(x-1)^2 - \lambda b_0(x-1)^3}{a_3(x-\lambda)^3 + (a_2-b_2)(x-\lambda)^2(x-1) + (a_1-b_1)(x-\lambda)(x-1)^2 - b_0(x-1)^3}$$

$$a_3 \mapsto a_3 + a_2 + a_1 - \lambda(b_2+b_1+b_0)$$

We will show the orbit of a_3 generates R. But first, note that $a_3+a_2+a_1 = b_2+b_1+b_0$ implies that $a_1 = b_2+b_1+b_0-a_2-a_3$, and then $\lambda^2 a_3+\lambda a_2+a_1 = \lambda^2 b_2+\lambda b_1+b_0$ implies that $(\lambda^2-1)a_3+(\lambda-1)a_2 = (\lambda^2-1)b_2+(\lambda-1)b_1$, that is, $b_2 = a_3+(a_2-b_1)/(\lambda+1)$.

We have $(x-\lambda)/(x-1)$ mapping a_3 to $a_3+a_2+a_1-\lambda(b_2+b_1+b_0)$. Now we have:

$$a_3+a_2+a_1-\lambda(b_2+b_1+b_0) = (1-\lambda)(b_2+b_1+b_0) = (1-\lambda)(a_3+b_0+(a_2+\lambda b_1)/(\lambda+1))$$

If we then apply the map λ/x, we obtain $(1-\lambda)(\lambda^{-1}b_0+\lambda b_3+(b_1+\lambda a_2)/(\lambda+1))$. The orbit is, up to scaling, $\{a_3, b_0, a_3+b_0+(a_2+\lambda b_1)/(\lambda+1), \lambda^{-1}b_0+\lambda b_3+(b_1+\lambda a_2)/(\lambda+1)\}$, which generates R. Again, we apply Noether's problem and obtain a rational quotient, as desired. $\square$

Unfortunately, this proof does not seem to generalize to M_d^n. Although Lemma 3.3.3 is true for all $n, d > 1$, there are two significant obstructions. First, the dimension of the target space of the map Fix will be $n(1 + d + \ldots + d^n)$, which is larger than N_d^n unless n and d are very small. This means that the map will not be surjective, though the fibers are still rational whenever they are nonempty. And second, even for small n and d the base space for the vector bundle is not $\mathrm{M}_{0,d+1}$, which is relatively tame, but rather the space of $1 + d + \ldots + d^n$ points on $\mathbb{P}^n$, a much more complex object. All we can say at this stage is that M_d^n is unirational, which follows trivially from the fact that it is covered by Hom_d^n.

Chapter 4

Semistable Reduction

4.1 Introduction and the Proof of Semistable Reduction

The semistable reduction theorem states the following, answering in the affirmative a conjecture for $\mathbb{P}^1$ in [17]:

Theorem 4.1.1. *If C is a complete curve with $K(C)$ its function field, and if $\varphi_{K(C)}$ is a semistable rational map on $\mathbb{P}^n_{K(C)}$, then there exists a curve D mapping finite-to-one onto C with a $\mathbb{P}^n$-bundle $\mathbf{P}(\mathcal{E})$ on D with a rational map $\Phi : \mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{E})$ such that,*

1. *The restriction of Φ to the fiber of each $x \in D$, φ_x, is a semistable rational self-map.*
2. *Φ is a semistable map over $K(D)$, and is equivalent to $\varphi_{K(D)}$ under coordinate change.*

This can be seen by using an alternative formulation. Semistable reduction can be thought of as extending a rational map defined over a field K to a rational map defined over a discrete valuation ring R whose fraction field is K, in a way that is not too degenerate. The reason a discrete valuation ring suffices is that once we know we can extend to a discrete valuation ring, we can extend to some larger integral domain.

We thus obtain the following equivalent formulation of semistable reduction, stated in full generality for any geometrically reductive G acting on a complete variety X:

Theorem 4.1.2. *Let R be a discrete valuation ring with fraction field K, and let $x_K \in X_K^s$. Then for some finite extension K' of K, with R' the integral closure of R in K', x_K has an integral model over R' with semistable reduction modulo the maximal ideal. In other words, we can find some $A \in G(\overline{K})$ such that $A \cdot x_K$ has semistable reduction. If $x_K \in X_K^{ss}$, then the same result is true, except that $x_{R'}$ could be an integral model for some $x'_{K'}$ mapping to the same point of $X^{ss}//G$ such that $x'_{K'} \notin G \cdot x_K$.*

Proof. Let C be the Zariski closure of x_K in $X_R^{ss}//G$, and reduce it modulo the maximal ideal to obtain x_k, where k is the residue field of R. Observe that C is a one-dimensional subscheme of $X_{\bar{k}}^{ss}//G$ which is isomorphic to $\operatorname{Spec} R$, and is as a result connected. Since G is connected, the preimage $\pi^{-1}(C)$ is also connected: when x_K is stable it follows from the fact that $\pi^{-1}(C)$ is the Zariski closure of $G \cdot x_K$ in X^{ss}, and even when it is not, $\pi^{-1}(C)$ is the union of connected orbits whose closures intersect. Since further $\pi^{-1}(C)$ surjects onto C, we can find an integral one-dimensional subscheme mapping surjectively to C. This subscheme necessarily maps finite-to-one onto C by dimension counting, so it is isomorphic to some finite extension ring R', giving us K' as its fraction field. $\square$

This leads to the natural question of which vector bundle classes can occur for each C, and more generally for each choice of n and d. One interesting subquestion is whether, for every C, we can choose the bundle to be trivial. Equivalently, it asks whether for each C we can find a proper $D \subseteq \mathrm{Hom}_d^{n,ss}$ that maps finite-to-one onto C. For most curves upstairs, the answer should be positive, by simple dimension counting: as demonstrated in [15] and [5], the complement of $\mathrm{Hom}_d^{n,ss}$ has high codimension, equal to about half of N. However, it turns out that the answer is sometimes negative, and in fact, for every n and d we can find a C with only nontrivial bundle classes. More precisely:

Theorem 4.1.3. *For every n and d, there exists a curve with no trivial bundle class satisfying semistable reduction.*

Remark 4.1.4. An equivalent formulation for Theorem 4.1.3 is that for every n and d we can find a curve $C \subseteq \mathrm{M}_d^{n,ss}$ such that there does not exist a curve $D \subseteq \mathrm{Hom}_d^{n,ss}$ mapping onto C under π.

Although most curves in $\mathrm{Hom}_d^{n,ss}$ can be completed, it does not imply we can find a nontrivial bundle on an open dense set of the Chow variety of $\mathrm{M}_d^{n,ss}$. In fact, as we will see in section 4.5, there exist components of the Chow variety of $\mathrm{M}_d^{n,ss}$ where, at least generically, a nontrivial bundle is required.

Our study of bundle classes will now split into two cases. In the case of curves satisfying semistable reduction with a trivial bundle, the reformulation of Remark 4.1.4, in its positive form, means that we can study D directly as a curve in $\mathbb{P}^N$. We can bound the degree of the map from D to C in terms of the stabilizer groups that occur on D. Let us restate Propositions 1.0.7 and 1.0.8:

Proposition 4.1.5. *Let X be a projective variety over an algebraically closed field with an action by a geometrically reductive linear algebraic group G. Using the terminology of geometric invariant theory, let D be a complete curve in the stable space X^s whose quotient by G is a complete curve C; say the map from D to C has degree m. Suppose the stabilizer is generically finite, of size h, and either D or C is normal. Then there exists a finite subgroup $S_D \subseteq G$, of order equal to mh, such that for all $x \in D$ and $g \in G$, $gx \in D$ iff $g \in S_D$.*

Corollary 4.1.6. *With the same notation and conditions as in Proposition 4.1.5, the map from D to C is ramified precisely at points $x \in D$ where the stabilizer group is larger than h, and intersects S_D in a larger subgroup than in the generic case.*

If the genus of C is 0, then the only way that the map from D to C could have high degree is if it ramifies over many points; therefore, Corollary 4.1.6 forces the degree to be small, at least as long as C is contained in the stable locus.

In the case of curves that only satisfy semistable reduction with a nontrivial bundle, we do not have a description purely in terms of coordinates. Instead, we will study which bundle classes can be attached to every curve C. The question of which bundles occur is an invariant of C; therefore, it is essentially an invariant that we can use to study the scheme $\mathrm{Hom}(C, \mathrm{M}_d^{n,ss})$. In the sequel, we will study the scheme using the bundle class set and height invariants.

For the study of which nontrivial bundle classes can occur, first observe that fixing a D for which a bundle exists, we can apply the reformulation of Theorem 4.1.2 to obtain a unique extension of φ locally. This can be done at every point, so it is true globally, so we have,

Proposition 4.1.7. *Using the notation of Theorem 4.1.1, the bundle class* $\mathbf{P}(\mathcal{E})$ *depends only on* D *and its trivialization* U_i*,* $U_i \hookrightarrow \mathrm{Hom}_d^{n,ss}$*.*

Note that the bundle class does not necessarily depend only on D, regarded as an abstract curve with a map to C. The reason is that a point of D may not be stable, which means it may correspond to one of several different orbits, whose closures intersect. However, there are only finitely many orbits corresponding to each point, so the bundle class depends on D up to a finite amount; if C happens to be contained in the stable locus, then it depends only on D.

Thus we can study which bundle classes occur for a given C. We will content ourselves with rational curves, for which there is a relatively easy description of all projective bundles. Recall that every vector bundle over $\mathbb{P}^1$ splits as a direct sum of line bundles, and that the bundle $\bigoplus_i \mathcal{O}(m_i)$ is projectively equivalent to $\bigoplus_i \mathcal{O}(l+m_i)$ for all $l \in \mathbb{Z}$. In other words, a $\mathbb{P}^n$-bundle over $\mathbb{P}^1$ can be written as $\mathcal{O} \oplus \mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_n)$; if the m_i's are in non-decreasing order, then the expression uniquely determines the bundle's class. We will show that,

Proposition 4.1.8. *There exists a curve* C *for which multiple non-isomorphic bundle classes can occur. In fact, suppose* C *is isomorphic to* $\mathbb{P}^1$*, and there exists some* $U \subseteq \mathrm{Hom}_d^{n,ss}$ *mapping finite-to-one into* C *such that* U *is a projective curve minus a point. Then there are always infinitely many possible classes: if the class of* U *is* *thought of as splitting as* $\mathbf{P}(\mathcal{E}) = \mathcal{O} \oplus \mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_n)$*, where* $m_i \in \mathbb{N}$*, then for every integer* l *the class* $\mathcal{O} \oplus \mathcal{O}(lm_1) \oplus \ldots \oplus \mathcal{O}(lm_n)$ *also occurs.*

Proposition 4.1.8 frustrated our initial attempt to obtain an easy classification of bundles based on curves. However, it raises multiple interesting questions instead. First, the construction uses a rational D mapping finite-to-one onto C, and going to higher m involves raising the degree of the map $D \to C$. It may turn out that bounding the degree bounds the bundle class; we conjecture that if we fix the degree of the map then we obtain only finitely many bundle classes. Furthermore, in analogy with the consequences of Corollary 4.1.6, we should conversely be able to bound the degree of the map in terms of C and the bundle class, at least for rational C.

Second, it is nontrivial to find the minimal m_i's for which a bundle splitting as $\mathcal{O} \oplus \mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_n)$ would satisfy semistable reduction; the case of $n = 1$ could be stated particularly simply, as the question would be about the minimal m for which $\mathcal{O} \oplus \mathcal{O}(m)$ occurs.

4.2 A Coordinate-Based Proof of Semistable Reduction

In this section, we will reprove Theorem 4.1.2 specifically for rational maps on $\mathbb{P}^1$ using the explicit coordinate form. This will let us bound the degree of the finite extension we need to take.

Fix the following notation in this section: K is a local field with valuation ring R, uniformizer t, and residue field $R/(t) = k$. φ_K is a rational map of degree d on $\mathbb{P}^1_K$;

then φ_K has a minimal model over R, which in the sequel we will identify with φ_K. We can then speak of the reduction of φ_K modulo t, φ_k. We have:

Theorem 4.2.1. *Fix $n = 1$. Then under the conjugation action of $\mathrm{PGL}(2, \overline{K})$ on $\mathrm{Rat}_d^{ss}(\overline{K})$, there exists a model for φ_K, over $\overline{K}$, which remains semistable after reduction modulo the maximal ideal.*

Proof. Assume φ_k is unstable. In other words, writing φ_k as $f_k(x, y)/g_k(x, y)$, there exists a root $\alpha x - \beta y$ appearing in both f_k and g_k to multiplicity at least $\frac{d}{2} + 1$, or appearing to multiplicity $\frac{d}{2}$ with $\varphi_k(\alpha x - \beta y) = \alpha x - \beta y$. We call this root the bad point of φ_k; such a root is unique if it exists, because if two bad points exist, then $f_k = g_k = 0$, which is impossible.

The map φ_k is unstable iff upstairs we can find a sufficient number of roots of f_K and g_K that reduce to the same root mod t. We may assume after conjugating downstairs by some matrix in $\mathrm{PGL}(2, \overline{k})$ that this shared root is y, i.e. that there are roots to the required multiplicity, all of the form $t^a px - y$ with $a \in \mathbb{Q}^+$ and $p \in R^\times$. The condition for instability can now be phrased without the word "or": φ_K has unstable reduction iff f_K has at least $\frac{d}{2}$ roots of the form $t^a px - y$ with $a > 0$ and g_K has at least $\frac{d}{2} + 1$ such roots.

Let us now eliminate the bad point y by conjugating φ_K by a diagonal matrix, with diagonal entries $(t^c, 1)$, with $c \in \mathbb{Q}$. This will replace every instance of x with $t^{-c}x$ as well as multiply f_K by t^c. The model for φ_K we get may not be integral. To get an integral form, we need to replace every root of the form $t^a px - y$ for which $a < 0$ with $x - t^{-a}p^{-1}y$, multiplying the polynomial it appears in (f or g) by t^a;

we may need to multiply both f and g by some power of t to get an integral form with nonzero reduction. Observe that the process of scaling involves no choice–it is determined purely by which point of Rat_d^{ss} we choose as a model.

Now, there is a minimal power t^c conjugating by which removes just the right number of powers of t from x to ensure that, after reduction mod t, y is no longer a bad point. All we need to do now is ensure that this does not create new bad points.

If we conjugate by t^c where c is very large, then we instead make x bad: we will get too many roots of the form $x - t^a py$ with $a > 0$. But if c is the minimal number for which y is no longer a bad point, then this will not happen. If we conjugate by $t^{c-\epsilon}$ for any $\epsilon > 0$, then by definition y occurs to multiplicity at least $\frac{d}{2}$ in f_k and $\frac{d}{2} + 1$ in g_k. None of these roots will actually be conjugated to x downstairs: they'll either remain at y or go to $x - sy$ with $s \in k^\times$. Now if x is a bad point then it occurs in f_k to multiplicity $\frac{d}{2} + 1$ and in g_k to multiplicity $\frac{d}{2} + 1$, which is impossible unless $f_k = g_k = 0$.

We will now show that we can make sure φ_k is not unstable because of any other point. If $x = sy, s \in k^\times$ is a bad point, then it means that the corresponding roots of f_K and g_K agree beyond the first term of their power series expansions. In that case, we conjugate the bad point to be $y = 0$ using triangular matrices with 1s on the diagonal, and repeat the above process of conjugating by t^c; we pick c based on the new roots, not the old roots.

The above-described process is bound to terminate. The reason is that conjugating by t^c multiplies or divides a power series by some power of t, but does not affect agreement to higher order. The other process, of moving the bad point, does not affect agreement at all – it only turns the first term of the power series, on which the roots agree, to 0; effectively, it moves the problem to the next term. After finitely many steps, we will necessarily remove a common root of f_k and g_k, or else the power series are equal and φ_K is unstable. Those removed common roots will automatically go to $x = 0$ after conjugation by t^c, and this process will not collapse different roots mod t to the same root anywhere else. Thus we will eventually obtain some φ_k which is semistable. $\square$

Remark 4.2.2. The proof invokes power series, as if $K = k((t))$. Although this is the motivating case of interest, this is equally true for p-adic fields. The power series are invoked as a way of writing down elements, without properties unique to power series rings such as that $k \subset K$.

Remark 4.2.3. This should generalize to maps over $\mathbb{P}^n$ with $n > 1$. While we do not have a linear factorization of the polynomials for higher n, the steps could potentially be done by looking at the coefficients of the polynomials. The second step could easily be replaced by conjugation by a unipotent matrix. The first step, involving t^c, seems harder to generalize. The correct generalization of bad points seems to be bad flags, which would require a good notion of what it means for φ to be unstable with respect to multiple flags.

The bound for the degree of the extension K'/K comes from bounding the denominator of c. Since c depends only on the valuations of the roots of f and g, the minimal t that makes the valuation of enough roots nonnegative has denominator bounded by $1/d$. We need to repeat this process at most r times, where r is the maximal power of t such that φ_K has unstable reduction over $R/(t^r)$. There are at most dr terms, so we get:

Corollary 4.2.4. *Suppose φ_K has semistable reduction modulo t^r. Then it has an integral model over K', a field extension of K of degree at most r^{dr}, with semistable reduction modulo t.*

Remark 4.2.5. We can do better in cases where roots of f and g not only coincide modulo high powers of t, but also are highly t-divisible, because then we can conjugate by one t^c and get rid of a high power of t at once. The worst case as far as $[K' : K]$ goes seems to be when the roots that coincide are defined over fields of high degree over K, and then the roots resulting after the first conjugation are again defined over fields of high degree over K, and so on.

4.3 Examples of Nontrivial Bundles

The space $\mathrm{Rat}_2 = \mathrm{Hom}_2^1$ and its quotient M_2 have been analyzed with more success than the larger spaces, yielding the following prior structure result [9] [15]:

Theorem 4.3.1. $\mathrm{M}_2 = \mathbb{A}^2$*;* $\mathrm{M}_2^s = \mathrm{M}_2^{ss} = \mathbb{P}^2$*. The first two elementary symmetric polynomials in the multipliers of the fixed points realize both isomorphisms.*

Recall that within $\mathbb{P}^N = \mathbb{P}^5$, a map $(a_0x^2 + a_1xy + a_2y^2)/(b_0x^2 + b_1xy + b_2y^2)$ is unstable iff it is in the closure of the PGL(2)-orbit of the subvariety $a_0 = b_0 = b_1 = 0$. In other words, it is unstable iff there the map is degenerate and has a double bad point, or a fixed point at a bad point.

Definition 4.3.2. A map on $\mathbb{P}^1$ is a polynomial iff there exists a totally invariant fixed point. Taking such a point to infinity turns the map into a polynomial in the ordinary sense. In Rat_d, or generally in $\mathbb{P}^N = \mathbb{P}^{2d+1}$, a map is polynomial iff it is in the closure of the PGL(2)-orbit of the subvariety defined by zeros in all coefficients in the denominator except the y^d-coefficient.

Remark 4.3.3. A totally invariant fixed point is not necessarily a totally fixed point. A totally invariant fixed point is one that is totally ramified. A totally fixed point is the root of the fixed point polynomial when it is unique, i.e. when the polynomial is a power of a linear term. In fact by an easy computation, a map has a totally invariant, totally fixed point x iff it is degenerate linear with a multiplicity-$d-1$ bad point at x, in which case it is necessarily unstable.

The polynomial maps define a curve in M_2^{ss} (in fact a line in $\mathbb{P}^2$); we will show,

Proposition 4.3.4. *The polynomial curve in* M_2^{ss} *only satisfies semistable reduction with nontrivial bundles.*

Proof. First, note that in $\mathbb{P}^5$, the polynomial maps are those that can be conjugated to the form $(a_0x^2 + a_1xy + a_2y^2)/b_2y^2$, in which case the totally invariant fixed point is $\infty = (1 : 0)$. We will call the polynomial map locus X. If $a_0 = 0$ then the map is unstable; we will show that every curve in X contains a map for which $a_0 = 0$. Clearly, the set of all maps with a given totally invariant fixed point is isomorphic to $\mathbb{P}^3$, and the unstable locus within it is isomorphic to $\mathbb{P}^2$ as a linear subvariety, so for there to be any hope of a trivial bundle, a curve in X cannot lie entirely over one totally invariant point.

Now, the fixed point equation for a map of the form f/g is $fy - gx$; the homogeneous roots of this equation are the fixed points, with the correct multiplicities. For our purposes, when the totally invariant point is ∞, the fixed point equation is $a_0x^2y + (a_1 - b_2)xy^2 + a_2y^3$. We get that $a_0 = 0$ iff the totally invariant point is a repeated root of the fixed point equation.

There exists a map from X to $\mathbb{P}^1 \times \mathbb{P}^2$, mapping φ to its totally invariant point in $\mathbb{P}^1$, and to the two elementary symmetric polynomials in the two other fixed points in $\mathbb{P}^2$. Write $(x : y)$ for the image in $\mathbb{P}^1$ and $(a : b : c)$ for the image in $\mathbb{P}^2$. Now $(x : y)$ is a repeated root if $ax^2 + bxy + cy^2 = 0$. The equation defines an ample divisor, so every curve in $\mathbb{P}^1 \times \mathbb{P}^2$ will meet it. Finally, a curve in X maps either to a single point in $\mathbb{P}^1 \times \mathbb{P}^2$, in which case it must contain points with $a_0 = 0$ as above, or to a curve, in which case it intersects the divisor $ax^2 + bxy + cy^2 = 0$. In both cases, the curve contains unstable points. Thus there is no global semistable curve D in $\mathrm{Hom}_d^{n,ss}$ mapping down to C. □

Note that in the above proof, maps conjugate to x^2 have two totally invariant points, so a priori the map from X to $\mathbb{P}^1 \times \mathbb{P}^2$ is not well-defined at them. However, for any curve D in X, there is a well-defined completion of this map, whose value at x^2 on the $\mathbb{P}^1$ factor is one of the two totally invariant points. Thus this complication does not invalidate the above proof.

Let us now compute the vector bundle classes that do occur for the polynomial curve. We work with the description $x^2 + c$, which yields an affine curve that maps one-to-one into C, missing only the point at infinity, which is conjugate to $\frac{x^2-x}{0}$. To hit the point at infinity, we choose the alternative parametrization $cx^2 - cx + 1$, which, when $c = \infty$, corresponds to the unique (up to conjugation) semistable degenerate constant map. For any c, this map is conjugate to $x^2 - cx + c$ and thence $x^2 + c/2 - c^2/4$, using the transition function $[c, -1/2; 0, 1]$. Thus the bundle splits as $\mathcal{O} \oplus \mathcal{O}(1)$.

This bundle depends on the choice of D. In fact, if we choose another parametrization for D, for example $c^2x^2 - c^2x + 1$, then the transition function $[c^2, -1/2; 0, 1]$, which leads to the bundle $\mathcal{O} \oplus \mathcal{O}(2)$. This is not equivalent to $\mathcal{O} \oplus \mathcal{O}(1)$. This then leads to the question of which classes of bundles can occur over each C. In the example we have just done, the answer is every nontrivial class: for every positive integer m, we can use $c^m x^2 - c^m x + 1$ as a parametrization, leading to $\mathcal{O} \oplus \mathcal{O}(m)$, which exhausts all nontrivial projective bundle classes.

Recall the result of Proposition 4.1.8:

Proposition 4.3.5. *Suppose C is isomorphic to $\mathbb{P}^1$, and there exists $U \subseteq \mathrm{Hom}_d^{n,ss}$ mapping finite-to-one into C such that U is a projective curve minus a point. Then there are always infinitely many possible classes: if the class of U is thought of as splitting as $\mathbf{P}(\mathcal{E}) = \mathcal{O} \oplus \mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_n)$, where $m_i \in \mathbb{N}$, then for every integer l the class $\mathcal{O} \oplus \mathcal{O}(lm_1) \oplus \ldots \oplus \mathcal{O}(lm_n)$ also occurs.*

Proof. Imitating the analysis of the polynomial curve above, we can parametrize C by one variable, say c, and choose coordinates such that the sole bad point in the closure of U corresponds to $c = \infty$. Now, we can by assumption find a piece U' above the infinite point with a transition function determining the vector bundle $\mathcal{O} \oplus \mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_n)$. Now let V be the composition of U' with the map $c \mapsto c^l$.

Then U and V determine a vector bundle satisfying semistable reduction, of class $\mathcal{O} \oplus \mathcal{O}(lm_1) \oplus \ldots \oplus \mathcal{O}(lm_n)$, as required. $\square$

The example in Theorem 4.3.4, of polynomial maps, is equivalent to a multiplier condition. When $d = 2$, a map is polynomial iff it has a fixed point whose multiplier is zero; see the description in the first chapter of [16]. One can imitate the proof that semistable reduction does not hold for a more general curve, defined by the condition that there exists a fixed point of multiplier $t \neq 1$. In that case, the condition $b_1 = 0$ is replaced by $b_1 = ta_0$, and the point is a repeated root of the fixed point equation iff $a_0 = b_1$, in which case we clearly have $a_0 = b_1 = 0$ and the point is unstable.

When the multiplier is 1, the fixed point in question is automatically a repeated root, with $b_1 = a_0$. The condition that the point is the only fixed point corresponds to $b_2 = a_1$, which by itself does not imply that the map fails to be a morphism, let alone that it is unstable.

Instead, the condition that gives us $b_1 = a_0 = 0$ is the condition that the fixed point is totally invariant. Specifically, the fixed point's two preimages are itself and one more point; when the fixed point is ∞, the extra point is $-b_2/b_1$. Now we can map X to $\mathbb{P}^1 \times \mathbb{P}^1$ where the first coordinate is the fixed point and the second is its preimage. This map is well-defined on all of X because only one point can be a double root of a cubic. Now the diagonal is ample in $\mathbb{P}^1 \times \mathbb{P}^1$, so the only way a curve D can avoid it is by mapping to a single point; but in that case, D lies in a fixed variety isomorphic to $\mathbb{P}^3$ where the unstable locus is $\mathbb{P}^2$, so it will intersect the unstable locus.

The fact that any condition of the form "there exists a fixed point of multiplier t" induces a curve for which semistable reduction requires a nontrivial bundle means that there is no hope of enlarging the semistable space in a way that ensures we always have a trivial bundle. We really do need to think of semistable reduction as encompassing nontrivial bundle classes as well as trivial ones.

Specifically: it is trivial to show that the closure of the polynomial locus in Rat_2 includes all the unstable points (fix ∞ to be the totally invariant point and let a_0 go to zero). At least some of those unstable points will also arise as closures of other multiplier-t conditions. However, different multiplier-t conditions limit to different points in $\mathrm{M}_2^{ss} \setminus \mathrm{M}_2$.

4.4 The General Case

So far we have talked about nontrivial classes in M_2. But we have a stronger result, restating Theorem 4.1.3:

Theorem 4.4.1. *For all n and d, over any base field, there exists a curve with no trivial bundle class satisfying semistable reduction.*

Proof. In all cases, we will focus on **polynomial maps**, which we will define to be maps that are $\mathrm{PGL}(n+1)$-conjugate to maps for which the last polynomial q_n has zero coefficients in every monomial except possibly x_n^d.

Lemma 4.4.2. *The set of polynomial maps, defined above, is closed in $\overline{\mathrm{Hom}_d^n} = \mathbb{P}^N$.*

Proof. Clearly, the set of polynomial maps with respect to a particular hyperplane – for example, $x_n = 0$ – is closed. Now, for each hyperplane $a_0x_0 + \ldots + a_nx_n = 0$, we can check by conjugation to see that the condition that the map is polynomial corresponds to the condition that $a_0q_0 + \ldots + a_nq_n = c(a_0x_0 + \ldots + a_nx_n)^d$, where c may be zero. As $\mathbb{P}^n$ is proper, it suffices to show that the condition "φ is polynomial with respect to $a_0x_0 + \ldots + a_nx_n = 0$" is closed in $(\mathbb{P}^n)^* \times \mathbb{P}^N$.

Now, we may construct a rational function f from $(\mathbb{P}^n)^* \times \mathbb{P}^N$ to $\mathrm{Sym}^d(\mathbb{P}^n) \times \mathrm{Sym}^d(\mathbb{P}^n)$ by $((a_0x_0 + \ldots + a_nx_n), \varphi) \mapsto ((a_0x_0 + \ldots + a_nx_n)^d, a_0q_0 + \ldots + a_nq_n)$. The map φ is polynomial with respect to $a_0x_0 + \ldots + a_nx_n = 0$ iff f is ill-defined at $((a_0x_0 + \ldots + a_nx_n), \varphi)$ or $f((a_0x_0 + \ldots + a_nx_n), \varphi) \in \Delta$, the diagonal subvariety. The ill-defined locus of f is closed, and the preimage of Δ is closed in the well-defined locus. $\square$

In fact, the condition of φ being polynomial with respect to any number of distinct hyperplanes in general position – in other words, the condition that φ is conjugate to a map for which $q_i = c_i x_i^d$ for all $i > 0$ (or $i > 1$, etc.) – is more or less closed as well. It is not closed, but a sufficiently good condition is closed. Namely:

Lemma 4.4.3. *For each $1 \leq i \leq n$, consider the* $\mathrm{PGL}(n+1)$*-orbit of the space of maps in which, for each $j \geq i$, q_j has zero coefficients in every monomial containing any term x_k with $k < j$. This orbit is closed in $\mathbb{P}^N$.*

Proof. Observe that the above-defined space of maps consists of maps that are polynomial with respect to $x_n = 0$, such that the induced map on the totally invariant hyperplane $x_n = 0$ is polynomial with respect to $x_{n-1} = 0$, and so on until we reach the induced map on the totally invariant subspace $x_{i+1} = \ldots = x_n = 0$.

Now we use descending induction. Lemma 4.4.2 is the base case, when $i = n$. Now suppose it is true down to i. Then for $i-1$, the condition of having no nonzero x_k term in q_{i-1} with $k < i-1$ is equivalent to the condition that the induced map on the totally invariant subspace $x_i = x_{i+1} = \ldots = x_n = 0$ is polynomial; this condition is closed in the space of all maps that are polynomial down to x_i, which we assume closed by the induction hypothesis. □

Definition 4.4.4. We call maps of the form in the above lemma **polynomial with respect to** B, where B is the Borel subgroup preserving the ordered basis of conditions. In the case above, B is the upper triangular matrices.

We need one final result to make computations easier:

Lemma 4.4.5. *Let X be a curve of polynomial maps, all with respect to a Borel subgroup B, and let φ be a semistable map in $\overline{\mathrm{PGL}(n+1) \cdot X}$. Then $\varphi \in \overline{B \cdot X}$.*

Proof. Let C be the closure of the image of X in $\mathrm{M}_d^{n,ss}$. By semistable reduction, there exists some affine curve $Y \ni \varphi$ mapping finite-to-one to C, i.e. dominantly. We need to find some open $Z \subseteq Y$ containing φ and some $f : Z \to \mathrm{PGL}(n+1)$ such that $f(\varphi)$ is the identity matrix, and $Z' = \{(f(z) \cdot z)\}$ consists of maps which are polynomial with respect to B. Such a map necessarily exists: we have a map h from Y to the flag variety of $\mathbb{P}^n$ sending each y to the subgroup with respect to which it is polynomial (possibly involving some choice if generically y is polynomial with respect to more than one flag), which then lifts to G, possibly after deleting finitely many points. Generically, a point of X maps to a point of C that is in the image of Z;

therefore, picking the correct points in X, we get that $\varphi \in \overline{B \cdot X}$. $\square$

With the above lemmas, let us now prove the theorem with $n = 1$, which is slightly easier than the higher-n case, where the more complicated Lemma 4.4.3 is needed. We will use the family $x^d + c$, where $c \in \mathbb{A}^1$. In projective notation, this is $\frac{a_0 x^d + a_d y^d}{b_d y^d}$, which is a one-dimensional family modulo conjugation. We have,

Lemma 4.4.6. *Let V be the closure of the* PGL(2)*-orbit of the family $\frac{a_0 x^d + a_d y^d}{b_d y^d}$ in $\mathbb{P}^N$. Then:*

1. *In characteristic* 0 *or $p \nmid d$, every $\varphi \in V$ is actually in the* PGL(2)*-orbit of the family, or else it is a degenerate linear map, conjugate to $\frac{a_{d-1} x y^{d-1} + a_d y^d}{b_d y^d}$.*
2. *In characteristic $p \mid d$, with $p^m \mid\mid d$ and $p^m \neq d$, every $\varphi \in V$ is in the* PGL(2)*-orbit of the family or is a degenerate map conjugate to $\frac{a_{d-p^m} x y^{d-p^m} + a_d y^d}{b_d y^d}$.*
3. *In characteristic p with $d = p^m$, set V to be the closure of the orbit of the family $\frac{a_0 x^d + a_{d-1} x y^{d-1}}{b_d y^d}$; then every $\varphi \in V$ is actually in the orbit of the family, or else it is a degenerate linear map, conjugate to $\frac{a_{d-1} x y^{d-1} + a_d y^d}{b_d y^d}$, and furthermore $a_{d-1} = b_d$.*

Proof. Observe that the first two cases are really the same: case 2 is reduced to case 1 viewed as a degree-$\frac{d}{p^m}$ map in $(x^{p^m} : y^{p^m})$. So it suffices to prove case 1 to prove 2; we will start with the family $\frac{a_0 x^d + a_d y^d}{b_d y^d}$ and see what algebraic equations its orbit satisfies. As polynomials are closed in $\overline{\mathrm{Rat}_d}$, every point in the closure of the orbit is a polynomial. We may further assume it is polynomial with respect to $y = 0$; therefore, by Lemma 4.4.5, it suffices to look at the action of upper triangular matrices. Further, the condition of being within the family $\frac{a_0x^d+a_dy^d}{b_dy^d}$ is stabilized by diagonal matrices; therefore, it suffices to look at the action of matrices of the form $[1,t;0,1]$.

Now, the conjugation action of $[1,t;0,1]$ fixes b_dy^d and maps $a_0x^d + a_dy^d$ to $a_0(x-ty)^d + (a_d+tb_d)y^d$. Clearly, there is no hope of obtaining any condition on b_d or a_d. Now, the conditions on the terms $a_0, \ldots, a_{d-1}$ are that for some t, they fit into the pattern $a_0(x^d - dtx^{d-1}y + \ldots \pm dt^{d-1}xy^{d-1})$, i.e. $a_i = (-t)^i\binom{d}{i}a_0$. To remove the dependence on t, note that when $i+j=k+l$, we have $\binom{d}{i}\binom{d}{j}a_ia_j = \binom{d}{k}\binom{d}{l}a_ka_l$, as long as $i,j,k,l<d$.

Let us now look at what those conditions imply. Setting $j=i, k=i-1, l=i+1$, we get conditions of the form $\binom{d}{i}^2a_i^2 = \binom{d}{i-1}\binom{d}{i+1}a_{i-1}a_{i+1}$, whenever $i+1<d$. If $a_0 \neq 0$, then the value of a_1 uniquely determines the value of a_2 by the condition with $i=1$; the value of a_2 uniquely determines a_3 by the condition with $i=2$; and so on, until we uniquely determine a_{d-1}. In this case, choosing $t = -\frac{a_1}{da_0}$ will conjugate this map back to the family $\frac{a_0x^d+a_dy^d}{b_dy^d}$. If $a_0=0$, then the equation with $i=1$ will imply that $a_1=0$; then the equation with $i=2$ will imply that $a_2=0$; and so on, until we set $a_{d-2}=0$. We cannot ensure $a_{d-1}=0$ because a_{d-1} always appears in those equations multiplied by a different a_i, instead of squared. Hence we could get a degenerate-linear map.

In case 3, we again look at the action of matrices of the form $[1,t;0,1]$. Such matrices map $\frac{a_0x^d+a_{d-1}xy^{d-1}}{b_dy^d}$ to $\frac{a_0x^d+a_{d-1}xy^{d-1}+(-a_0t^d-a_{d-1}t+b_dt)y^d}{b_dy^d}$. Now the only way a map of the form $\frac{a_0x^d+a_{d-1}xy^{d-1}+a_dy^d}{b_dy^d}$ could degenerate is if the image of the polynomial map $t \mapsto -a_0t^d - a_{d-1}t + b_dt$ misses a_d, which could only happen if the polynomial were constant, i.e. $a_0 = 0$ and $a_{d-1} = b_d$, giving us a degenerate-linear map. $\square$

Remark 4.4.7. The importance of the lemma is that in all degenerate cases, the map is necessarily unstable, since $d-1$ (or, in case 2, $d-p^m$) is always at least as large as $d/2$.

We can now prove the theorem for $n = 1$. So if we can always find a $D \subseteq \mathrm{Hom}_d^{n,ss}$ that works globally, we can find one over a family in which every map is conjugate to $\frac{a_0x^d+a_dy^d}{b_dy^d}$, or, in characteristic p with $d = p^m$, $\frac{a_0x^d+a_{d-1}xy^{d-1}}{b_dy^d}$. It suffices to show that there exists a map with $a_0 = 0$. For this, we use the fixed point polynomial, which is well-defined on this family. If the polynomial is fixed, then all maps in the family may be simultaneously conjugated to the form $\frac{a_0x^d+a_dy^d}{b_dy^d}$ (or $\frac{a_0x^d+a_{d-1}xy^{d-1}}{b_dy^d}$), and then one map must have $a_0 = 0$. If the polynomial varies, then some map will have the point at infinity colliding with another fixed point. This will force the map to be ill-defined at infinity; recall that totally invariant points are simple roots of the fixed point polynomial, unless they are bad. This will force a_0 to be zero, again.

For higher n, the proof is similar. The lemma we need is similar to the lemma we use above, but is somewhat more complicated:

Lemma 4.4.8. *Let V be the closure of the* $\mathrm{PGL}(n+1)$*-orbit of the family* $(c_0x_0^d + bx_1^d : q_1 : \ldots : q_n)$*, where q_i is x_j-free for all $j < i$.*

1. *If the characteristic does not divide d, then every $\varphi \in V$ is actually in the* $\mathrm{PGL}(n+1)$*-orbit of the family, or else it is a degenerate map, whose only possible nonzero coefficients in q_0 are those without an x_0 term and those of the form x_0p_0 where there is no nonzero x_0-term in p_0.*

2. *If the characteristic p satisfies $p \mid d$, with $d \neq p^m \mid\mid d$ then the same statement as in case* 1 *holds as long as each q_i is in terms of $x_j^{p^m}$, but with $x_0 p_0$ replaced by $x_0^{p^m} p_0$.*

3. *If the characteristic p satisfies $d = p^m$ then, changing the family to $(c_0 x_0^d + b x_0 x_1^{d-1} : q_1 : \ldots : q_n)$, with q_i in terms of x_j^d as in case* 2*, the same statement as in case* 1 *holds.*

Proof. As in the one-dimensional case, case 2 is reducible to case 1 with d replaced with $\frac{d}{p^m}$ and x_i with x_i^m. By Lemma 4.4.5, we only need to conjugate by upper triangular matrices. Further, we only need to conjugate by just matrices of the family E, with first row $(1, t_1, \ldots, t_n)$ and other rows the same as the identity matrix. This is because we can control the diagonal elements because the condition of being in the family is diagonal matrix-invariant, and we can control the rest by projecting any curve Z of unipotent upper triangular matrices onto E.

Set $a_{\mathbf{d}}$ to be the $\mathbf{x}^{\mathbf{d}}$-coefficient in q_0. For all vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l}$ with $\mathbf{i} + \mathbf{j} = \mathbf{k} + \mathbf{l}$, we have $\binom{d}{\mathbf{i}}\binom{d}{\mathbf{j}} a_{\mathbf{i}} a_{\mathbf{j}} = \binom{d}{\mathbf{k}}\binom{d}{\mathbf{l}} a_{\mathbf{k}} a_{\mathbf{l}}$, as long as none of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, or $\mathbf{l}$ is in the span of $\mathbf{e}_i$ for $i > 0$. Note that i and $\mathbf{i}$ are two separate quantities, one an index of coordinates and one an index of monomials.

As in the one-dimensional case, we may set $\mathbf{j} = \mathbf{i}$ and $\mathbf{k} = \mathbf{i} - \mathbf{e}_0 + \mathbf{e}_i$. If $c_0 = a_{(d,0,\ldots,0)} \neq 0$, then by the same argument as before, the values of the $x_0^{d-1} x_i$-coefficients determine all the rest, and we can conjugate the map back to the desired form. And if $c_0 = 0$, then the value of every coefficient that can occur as $\mathbf{i}$ in the above construct is zero; the only coefficients that cannot are those with no x_0 component and those with a linear x_0 component.

In case 3, we restrict to matrices of the same form as in case 1, and observe that those matrices only generate extra x_i^d and $x_i x_1^{d-1}$ in q_0. The statement is vacuous if $c_0 = 0$, so assume $c_0 \neq 0$. For $i = 1$, this is identical to the one-dimensional case, so if $c_0 \neq 0$ then we can find an appropriate t_1. For higher i, if $b \neq 0$ then we can extract t_i from the $x_i x_1^{d-1}$ coefficient, which will necessarily work for the x_i^d coefficient as well, making the map conjugate to the family; if $b = 0$, then the same equations as for $i = 1$ hold for higher i, and we can again find t_i's conjugating the map to the family. $\square$

While we could also control the terms involving a linear (or p-power) x_0 coefficient in the above construction, it is not necessary for our purposes.

To finish the proof of the theorem, first note that in the closure of the family above, any map for which $c_0 = 0$ is unstable. Indeed, the one-parameter subgroup of $\mathrm{PGL}(n+1)$ with diagonal coefficients $t_0 = n, t_i = -1$ for $i > 0$, shows instability. Recall that a map is unstable with respect to such a family if $t_i > t_0 d_0 + \ldots + t_n d_n$ whenever the $x_0^{d_0} \ldots x_n^{d_n}$-coefficient of q_i is nonzero. With the above one-parameter subgroup, we have $t_0 d_0 + \ldots + t_n d_n = -d < -1$ for the only nonzero monomials in q_i with $i > 0$; in q_0, the maximal value of $t_0 d_0 + \ldots + t_n d_n$ is $t_0 + t_i(d-1) = n - (d-1) < n$.

Now we need to show only that for some map in the family, c_0 will indeed be zero. So suppose on the contrary that c_0 is never zero. Then all maps are, after conjugation, in the family $(c_0 x_0^d + b x_1^d : q_1 \ldots : q_n)$, where the linear subvariety $q_i = q_{i+1} = \ldots = q_n$ is totally invariant. Now look at the action on the line $x_2 = \ldots = x_n = 0$. Every morphism will induce a morphism on this line, so there will be three fixed points on it, counting multiplicity. We now imitate the proof in the one-dimensional case: the totally invariant fixed point on this line, $(1 : 0 : \ldots : 0)$, will collide with another fixed point, so the map will be ill-defined at it. This means that $(1 : 0 : \ldots : 0)$ is a bad point, which cannot happen unless $c_0 = 0$. □

Trivially, the above theorem for curves shows the same for higher-dimensional families in $\mathrm{M}_d^{n,ss}$. An interesting question could be to generalize semistable reduction to higher-dimensional families, for which we may get projective vector bundles as in the case of curves. Trivially, if we have two proper subvarieties of $\mathrm{M}_d^{n,ss}$, $V_1 \subseteq V_2$, and a bundle class occurs for V_2, then its restriction to V_1 occurs for V_1. In particular, if we have the trivial class over V_2 then we also have it over V_1, as well as any other subvariety of V_2. This leads to the following question: if the trivial class occurs for every proper closed subvariety of V_2, does it necessarily occur for V_2? What if we weaken the condition and only require the trivial class to occur for subvarieties that cover V_2?

4.5 The Trivial Bundle Case

For most curves $C \subseteq \mathrm{M}_d^{n,ss}$, there occurs a trivial bundle. Since the complement of $\mathrm{Hom}_d^{n,ss}$ in $\mathbb{P}^N$ has high codimension, this is true by simple dimension counting. Therefore, it is useful to analyze those curves separately, as we have more tools to work with. Specifically, we can use more machinery from geometric invariant theory. We will start by proving Proposition 1.0.7, restated below:

Proposition 4.5.1. *Let X be a projective variety over an algebraically closed field with an action by a geometrically reductive linear algebraic group G. Using the terminology of geometric invariant theory, let D be a complete curve in the stable space X^s whose quotient by G is a complete curve C; say the map from D to C has degree m. Suppose the stabilizer is generically finite, of size h, and either D or C is normal. Then there exists a finite subgroup $S_D \subseteq G$, of order equal to mh, such that for all $x \in D$ and $g \in G$, $gx \in D$ iff $g \in S_D$.*

Proof. For $x \in D$, we define $S_D(x) = \{g \in G : gx \in D\}$. This is a map of sets from an open dense subset of D to $\mathrm{Sym}^{mh}(G)$, and is regular on an open dense subset. We have:

Lemma 4.5.2. *The map from $\mathrm{Sym}^{mh}(G) \times X^s$ to $\mathrm{Sym}^{mh}(X^s) \times X^s$ defined by sending each $(\{g_1, \ldots, g_{mh}\}, x)$ to $(\{g_1 \cdot x, \ldots, g_{mh} \cdot x\}, x)$ is proper.*

Proof. By standard geometric invariant theory, the map from $G \times X^s$ to $X^s \times X^s$, $(g, x) \mapsto (g \cdot x, x)$, is proper. Thus the map from $G^{mh} \times (X^s)^{mh}$ to $(X^s)^{mh} \times (X^s)^{mh}$ defined by $(g_i, x_i) \mapsto (g_i \cdot x_i, x_i)$ is also proper, as the product of proper maps. Now closed immersions are proper, so the map remains proper if we restrict it to $G^{mh} \times X^s$ where we embed X^s into $(X^s)^{mh}$ diagonally; the image of this map is contained in $(X^s)^{mh} \times X^s$. Finally, we quotient out by the symmetric group S_k, obtaining:

$$\begin{tikzcd}
G^{mh} \times X^s \arrow[r] \arrow[d, "\pi"'] & (X^s)^{mh} \times X^s \arrow[d, "\pi"] \\
\mathrm{Sym}^{mh}(G) \times X^s \arrow[r] & \mathrm{Sym}^{mh}(X^s) \times X^s
\end{tikzcd}$$

The map on the bottom is already separated and finite-type; we will show it is universally closed. Extend it by some arbitrary scheme Y. If $V \subseteq \mathrm{Sym}^{mh}(G) \times X^s \times Y$ is closed, then so is $\pi^{-1}(V) \subseteq G^{mh} \times X^s \times Y$. The map on top is universally closed, so its image is closed in $(X^s)^{mh} \times X^s \times Y$. But the map on the right is proper, so the image of V is also closed in $\mathrm{Sym}^{mh}(X^s) \times X^s \times Y$. □

Now, the rational map $f_D(x) = S_D(x) \cdot x \in \mathrm{Sym}^{mh}(D)$ can be extended to a morphism on all of D, since both D and $\mathrm{Sym}^{mh}(D)$ are proper. This is trivial if D is normal; if it is not normal, but C is normal, then observe that the map factors through C since it is constant on orbits, and then analytically extend it through C. But now $(f_D(x), x)$ embeds into $\mathrm{Sym}^{mh}(X^s) \times X^s$ as a proper curve. The preimage in $\mathrm{Sym}^{mh}(G) \times X^s$ of this curve is also proper; for each $(f_D(x), x)$, it is a finite set of points of the form (S, x) satisfying $S \cdot x = f_D(x)$, including $(S_D(x), x)$. Projecting onto the $\mathrm{Sym}^{mh}(G)$ factor, we still get a proper set, which means it must be a finite set of points, as $\mathrm{Sym}^{mh}(G)$ is affine. One of these points will be S_D, which is then necessarily finite.

Finally, if $g, h \in S_D$ and $x \in D$ then $g \cdot h \cdot x \in g \cdot D = D$; therefore S_D is a group. □

Remark 4.5.3. The proposition essentially says that the cover $D \to C$ is necessarily Galois. The generic stabilizer is necessarily a group H, normal in S_D.

Corollary 4.5.4. *With the same notation and conditions as in Proposition 4.5.1, the map from D to C ramifies precisely at points $x \in D$ such that* $\mathrm{Stab}(x)$ *intersects*

S_D in a strictly larger group than H. Furthermore, the ramification degree is exactly $[\mathrm{Stab}(x) \cap S_D : H]$.

For high n or d, the stabilized locus of Hom_d^n is of high codimension. Furthermore, most curves in $\mathrm{Hom}_d^{n,ss}$ lie in $\mathrm{Hom}_d^{n,s}$. Therefore, generically not only is H trivial, but also there are no points on D with nontrivial stabilizer. Thus for most C and D, the map $D \to C$ must be unramified. Thus, when C is rational, generically the degree is 1.

It's based on this observation that we conjecture the bounds for the nontrivial bundle case in both directions – that is, that if we fix C and the bundle class $\mathbf{P}(\mathcal{E})$, then the degree of the map $\pi : D \to C$ is bounded.

Using the structure result on $\mathrm{M}_2^{ss} = \mathbb{P}^2$, we can prove much more:

Proposition 4.5.5. *If C is a generic line in M_2^{ss}, then it requires a nontrivial bundle.*

Proof. Generically, C is not the line consisting of the resultant locus, $\mathrm{M}_2^{ss} \setminus \mathrm{M}_2$. So it intersects this line at exactly one point. Furthermore, since the resultant Res_2 is an SL(2)-invariant section, we have $D. \mathrm{Res}_2 = m \cdot C. \mathrm{Res}_2$; we abuse notation and use Res_d^n to refer to the resultant divisor both upstairs and downstairs. Since the degree of the resultant upstairs is $(n+1)d^n = 4$ [4], we obtain $4 \cdot D.\mathcal{O}(1) = m$. In other words, $m \geq 4$.

However, using Proposition 4.5.1, we will show $m \leq 2$ generically. The generic stabilizer is trivial, and the stabilized locus is a cuspidal cubic in $\mathbb{P}^2$, on which the stabilizer is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, except at the cusp, where it is S_3. The generic line C

will intersect this cuspidal curve at three points, none of which is the cusp. Therefore, $h = 1$, and there are at most three points of ramification, with ramification degree 2. By Riemann-Hurwitz, the maximum m is 2, contradicting $m \geq 4$. $\square$

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