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Moduli Spaces of Higher Spin Curves and Integrable Hierarchies

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Abstract. We prove the genus zero part of the generalized Witten conjecture, relating moduli spaces of higher spin curves to Gelfand–Dickey hierarchies. That is, we show that intersection numbers on the moduli space of stable r-spin curves assemble into a generating function which yields a solution of the semiclassical limit of the KdV_r equations. We formulate axioms for a cohomology class on this moduli space which allow one to construct a cohomological field theory of rank $r-1$ in all genera. In genus zero it produces a Frobenius manifold which is isomorphic to the Frobenius manifold structure on the base of the versal deformation of the singularity A_{r-1}. We prove analogs of the puncture, dilaton, and topological recursion relations by drawing an analogy with the construction of Gromov–Witten invariants and quantum cohomology.

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0. Introduction

The moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points is a fascinating object. Mumford [31] introduced tautological cohomology classes associated to the universal curve $\mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$. Witten [36] conjectured and Kontsevich [23] proved that certain intersection numbers of tautological cohomology classes on $\overline{\mathcal{M}}_{g,n}$ have a generating function which satisfies the equations of the Korteweg–de Vries hierarchy (more precisely, that it is a τ-function of the KdV hierarchy satisfying some additional equations). This remarkable result provided an unexpected link between the algebraic geometry of these moduli spaces and integrable systems.

The spaces $\overline{\mathcal{M}}_{g,n}$ can be generalized in two ways. The first way is by choosing a smooth projective variety V and considering the moduli space $\overline{\mathcal{M}}_{g,n}(V)$ of stable

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maps into V from genus-g, n-pointed, stable curves. When V is a point, $\overline{\mathcal{M}}_{g,n}(V)$ reduces to $\overline{\mathcal{M}}_{g,n}$.

The second way to generalize $\overline{\mathcal{M}}_{g,n}$ is by considering the moduli space $\overline{\mathcal{M}}^{1/r}_{g,n}$ of higher spin curves introduced in [17, 18]. Roughly speaking, a higher spin curve, or r-spin curve, is an algebraic curve with an rth root of its (suitably twisted) canonical bundle. Forgetting the r-spin structure reduces $\overline{\mathcal{M}}^{1/r}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$. It is natural to ask if Kontsevich's theorem admits a generalization to either of these two cases.

The case of $\overline{\mathcal{M}}_{g,n}(V)$ remains mysterious. It gives the Gromov–Witten invariants of V and their so-called gravitational descendants, which assemble into a generating function whose exponential is an analog of a τ-function. In the case where V is a point, one recovers the τ-function of the KdV hierarchy by Kontsevich's theorem. More generally, there is a conjecture of Eguchi, Hori, and Xiong [8] and of S. Katz which essentially states that this generating function is a highest weight vector for a particular representation of the Virasoro algebra. Presumably, there is some analog of an integrable system which gives rise to this Virasoro algebra action, should the conjecture hold.

On the other hand, the KdV hierarchy is just the first in a series of integrable hierarchies KdV$_r$, where $r = 2, 3, \ldots$, called the generalized KdV, or Gelfand–Dickey hierarchies. In the case of $r = 2$, this is the usual KdV hierarchy. Each of these hierarchies has a formal solution, corresponding to the unique τ-function which satisfies an additional equation known as the string (or puncture) equation. In [34, 35], Witten formulated a generalization of his original conjecture, suggesting that for each $r \geqslant 2$, there should exist moduli spaces and cohomology classes on them whose intersection numbers assemble into this τ-function of the KdV$_r$ hierarchy. The corresponding moduli spaces of higher spin curves have recently been constructed in [17, 18]. In this paper we present a precise mathematical formulation of the generalized Witten conjecture and prove it in several special cases including, in particular, the case of genus zero.

Motivated by analogy with the construction of Gromov–Witten invariants from the moduli space of stable maps, we introduce axioms which must be satisfied by a cohomology class $c^{1/r}$ (called the *virtual class*) on the moduli space of r-spin curves $\overline{\mathcal{M}}^{1/r}_{g,n}$ in order to obtain a cohomological field theory (CohFT) of rank $(r-1)$ in the sense of Kontsevich and Manin [24]. This virtual class on $\overline{\mathcal{M}}^{1/r}_{g,n}$ is an analog of the Gromov–Witten classes of a variety V (i.e. the pullbacks via the evaluation maps of elements in $H^\bullet(V)$). We realize this virtual class in genus zero as the top Chern class of a tautological bundle over $\overline{\mathcal{M}}^{1/r}_{0,n}$ associated to the r-spin structure. This yields a Frobenius manifold structure [6, 16, 27] on the state space of the CohFT which is isomorphic to the Frobenius manifold associated to the versal deformation of the A_{r-1} singularity [6]. This is an indication of the existence of a kind of "mirror symmetry" between the moduli space of r-spin curves and singularities. According to Manin [28] "isomorphisms of Frobenius manifolds of different classes remain the most direct expression of various mirror phenomena".

Proving the generalized Witten conjecture for all genera would provide further evidence of this relationship.

As in the case of Gromov–Witten invariants, one can construct a potential function from the integrals of the class $c^{1/r}$ on different components of $\overline{\mathcal{M}}_{g,n}^{1/r}$ to form the *small phase* space of the theory. The *large phase space* is constructed by introducing the tautological classes ψ associated to canonical sections of the universal curve $\mathcal{C}_{g,n}^{1/r} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r}$, and can be regarded as a parameter space for a family of CohFTs. A *very large phase space* (see [9, 21, 29]), parametrizing an even larger space of CohFTs, is obtained by considering classes λ, associated to the Hodge bundles, and classes μ, associated to the universal spin structure.

We show that the corresponding potential function satisfies analogs of the puncture and dilaton equations and also a new differential equation obtained from a universal relation involving the class μ_1. These relations hold in all genera. Topological recursion relations are also obtained from presentations of these classes in terms of boundary classes in low genera.

Finally, using the new relation involving μ_1, we show that the genus zero part of the large phase space potential $\Phi_0(\mathbf{t})$ is completely determined by the geometry, and this potential agrees with the generalized Witten conjecture in genus zero.

Some of our constructions were foreshadowed by Witten, who formulated his conjecture even before the relevant moduli spaces and cohomology classes had been constructed, just as he had done in the case of the topological sigma model and quantum cohomology. We prove that his conjecture has a precise algebro-geometric foundation, just as in the case of Gromov–Witten theory. Witten also outlined a formal argument to justify his conjecture in genus zero. Our work shows that the formulas that he ultimately obtained for the large phase space potential function in genus zero are indeed correct, provided that the geometric objects involved are suitably interpreted. This is nontrivial even in genus zero because the underlying moduli spaces are not schemes, but stacks. We proceed further to prove relations between various tautological classes associated to the r-spin structures and to derive differential equations for the potential function associated to them.

Notice also that one can introduce moduli spaces $\overline{\mathcal{M}}_{g,n}^{1/r}(V)$ of stable r-spin maps into a variety V, where one combines the data of both the stable maps and the r-spin structures. The analogous construction on these spaces yields a Frobenius manifold which combines Gromov–Witten invariants (and quantum cohomology) with the KdV_r hierarchies. Work in this direction is in progress [20].

In the first section of this paper, we review the moduli space $\overline{\mathcal{M}}_{g,n}^{1/r}$ of genus g, stable r-spin curves, which was introduced in [17, 18]. We also discuss the stratification of the boundary of $\overline{\mathcal{M}}_{g,n}^{1/r}$. The boundary strata fall into two distinct categories—the so-called Neveu–Schwarz and Ramond types.

In the second section we introduce canonical morphisms, tautological bundles, tautological cohomology classes, and cohomology classes associated to the boundary strata of $\overline{\mathcal{M}}_{g,n}^{1/r}$, and we derive a new relation involving the μ_1 class.

In the third section, we define a cohomological field theory (CohFT) in the sense of Kontsevich and Manin, its small phase space potential function, and the associativity (WDVV) equation. We then review the construction of Gromov–Witten invariants for the moduli space of stable maps and define the large and very large phase spaces in the Gromov–Witten theory. Motivated by this example, we explain how one may construct a CohFT and the various potential functions from analogous intersection numbers on $\overline{\mathcal{M}}_{g,n}^{1/r}$, assuming that the virtual class $c^{1/r}$ exists.

In the fourth section we state axioms which $c^{1/r}$ must satisfy in order to obtain a CohFT. We show that these axioms give a complete CohFT with a flat identity, and we construct the class $c^{1/r}$ in genus zero, as well as in the case $r = 2$.

In the fifth section, we obtain analogs of the string and dilaton equations for this r-spin CohFT, and we find a new equation based on the relation involving the μ_1 class. We also prove the analog of topological recursion relations in genus zero.

In the sixth section, we use the new relation for the class μ_1 to completely determine the genus-zero part of the large phase space potential.

Finally, in the seventh section, we give a precise formulation of the generalized Witten conjecture and prove that the genus-zero, large phase space potential of the r-spin CohFT yields a solution to the semiclassical limit of the KdV$_r$ hierarchy, thereby proving the Witten conjecture in genus zero. We conclude with our own W-algebra conjecture, a KdV$_r$-analog of a refinement of the Virasoro conjecture [8].

1. The Moduli Space of r-Spin Curves

In this section, we review the definition and some of the basic properties of the moduli space $\overline{\mathcal{M}}_{g,n}^{1/r}$ of genus g, n-pointed, stable r-spin curves.

1.1. AN OVERVIEW OF $\overline{\mathcal{M}}_{g,n}^{1/r}$

As the definition of $\overline{\mathcal{M}}_{g,n}^{1/r}$ is rather involved, we motivate it by starting with an intuitive approach to r-spin curves and their moduli space.

A smooth r-spin curve is essentially just a curve with an rth root of the canonical bundle ω_X (suitably twisted). In other words, it is a pair $(X, \mathcal{L})$ where X is a smooth curve and $\mathcal{L}$ is a line bundle on X such that $\mathcal{L}^{\otimes r}$ is isomorphic to the canonical bundle ω_X. Given a collection of integers $\mathbf{m} = (m_1, \ldots, m_n)$, an n-pointed smooth r-spin curve of type $\mathbf{m}$ is a smooth n-pointed curve $(X, p_1, \ldots, p_n)$ with a line bundle $\mathcal{L}$ on X, such that $\mathcal{L}^{\otimes r}$ is isomorphic to $\omega_X(-\sum_{i=1}^n m_i p_i)$. For degree reasons such a bundle exists only if $2g - 2 - \sum m_i$ is divisible by r. When this condition is met, there are r^{2g} choices of $\mathcal{L}$ on X.

If we want to compactify the space of smooth r-spin curves by allowing the curve X to degenerate to a stable curve, the above definition of an r-spin structure is insufficient. In particular, there is often no line bundle $\mathcal{L}$ such that $\mathcal{L}^{\otimes r}$ is isomorphic to $\omega_X(-\sum m_i p_i)$, even when the degree condition is satisfied. One possible solution—replacing line bundles by arbitrary rank-one, torsion-free sheaves—permits too many potential candidates. The correct structure required in this case amounts essentially to an explicit choice of isomorphism (or homomorphism when $\mathcal{L}$ is not locally free) $b: \mathcal{L}^{\otimes r} \longrightarrow \omega_X(-\sum m_i p_i)$, with some additional technical restrictions described in Definitions 1.2 and 1.3.

There are two very different types of behavior of this torsion-free sheaf $\mathcal{L}$ near a node $q \in X$. When it is still locally free, the sheaf $\mathcal{L}$ is said to be Ramond at the node q. If the sheaf $\mathcal{L}$ is not locally free at q, it is called Neveu–Schwarz.

In the Ramond case, the homomorphism b is still an isomorphism (near the node q), but in the Neveu–Schwarz case it cannot be an isomorphism. The local structure of the sheaf $\mathcal{L}$ near a Neveu–Schwarz node can be described as follows.

Near the node q, the curve X has two coordinates x and y, such that $xy = 0$; and the sheaf ω_X (or $\omega_X(-\sum m_i p_i)$) is locally generated by $dx/x = -dy/y$. Near q the sheaf $\mathcal{L}$ is generated by two elements ℓ_+ and ℓ_- supported on the x and y branches respectively (that is, $x\ell_- = y\ell_+ = 0$). The two generators may be chosen so that the homomorphism $b: \mathcal{L}^{\otimes r} \longrightarrow \omega_X(-\sum m_i p_i)$ takes $\ell_+^{\otimes r}$ to $x^{m_+ + 1}(dx/x) = x^{m_+} dx$, and so that b takes $\ell_-^{\otimes r}$ to $y^{m_- + 1}(dy/y) = y^{m_-} dy$, where $(m_+ + 1) + (m_- + 1) = r$ is the order of vanishing of b at the node q.

One more difficulty arises when r is not prime – in this case the moduli of stable curves with r-spin structure, as described above, is not smooth. The remedy is to include all d-spin structures for every d dividing r, satisfying some natural compatibility conditions. This is described in Definition 1.3.

We now give the definition of r-spin curves.

1.2. HIGHER SPIN CURVES

DEFINITION 1.1. A prestable curve is a reduced, complete, algebraic curve with at worst nodes as singularities.

DEFINITION 1.2. Let $(X, p_1, \ldots, p_n)$ be a prestable, n-pointed, algebraic curve, $\mathcal{K}$ be a rank-one, torsion-free sheaf on X, and $\mathbf{m} = (m_1, \ldots, m_n)$ be a collection of integers. A dth root of $\mathcal{K}$ of type $\mathbf{m}$ is a pair $(\mathcal{E}, b)$, where $\mathcal{E}$ is a rank-one, torsion-free sheaf, and b is an $\mathcal{O}_X$-module homomorphism $b: \mathcal{E}^{\otimes d} \longrightarrow \mathcal{K} \otimes \mathcal{O}_X(-\sum m_i p_i)$ with the following properties:

- $d \cdot \deg \mathcal{E} = \deg \mathcal{K} - \sum m_i$
- b is an isomorphism on the locus of X where $\mathcal{E}$ is locally free
- for every point $p \in X$ where $\mathcal{E}$ is not free, the length of the cokernel of b at p is $d - 1$.

The condition on the cokernel amounts essentially to the condition that the order of vanishing of b at a node should be d. For any dth root $(\mathcal{E}, b)$ of type $\mathbf{m}$, and for any $\mathbf{m}'$ congruent to $\mathbf{m}$ (mod d), we can construct a unique dth root $(\mathcal{E}', b')$ of type $\mathbf{m}'$ simply by taking $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(1/d \sum (m_i - m_i')p_i)$. Consequently, the moduli of curves with dth roots of a bundle $\mathcal{K}$ of type $\mathbf{m}$ is canonically isomorphic to the moduli of curves with dth roots of type $\mathbf{m}'$. Therefore, unless otherwise stated, we will always assume the type $\mathbf{m}$ of a dth root has the property that $0 \leqslant m_i < d$ for all i. Unfortunately, the moduli space of curves with dth roots of a fixed sheaf $\mathcal{K}$ is not smooth when d is not prime, and so we must consider not just roots of a bundle, but rather coherent nets of roots [17]. This additional structure suffices to make the moduli space of curves with a coherent net of roots smooth.

DEFINITION 1.3. Let $\mathcal{K}$ be a rank-one, torsion-free sheaf on a prestable n-pointed curve $(X, p_1, \ldots, p_n)$. A coherent net of rth roots of $\mathcal{K}$ of type $\mathbf{m} = (m_1, \ldots, m_n)$ is a pair $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ of a set of sheaves and a set of homomorphisms as follows. The set of sheaves consists of a rank-one, torsion-free sheaf $\mathcal{E}_d$ on X for every divisor d of r; and the set of homomorphisms consists of an $\mathcal{O}_X$-module homomorphism $c_{d,d'}\colon \mathcal{E}_d^{\otimes d/d'} \longrightarrow \mathcal{E}_{d'}$ for every pair of divisors d', d of r, such that d' divides d. These sheaves and homomorphisms must satisfy the following conditions:

- $\mathcal{E}_1 = \mathcal{K}$ and $c_{1,1} = \mathbf{1}$.
- For each divisor d of r and each divisor d' of d, the homomorphism $c_{d,d'}$ makes $(\mathcal{E}_d, c_{d,d'})$ into a d/d'th root of $\mathcal{E}_{d'}$ of type $\mathbf{m}'$, where $\mathbf{m}' = (m'_1, \ldots, m'_n)$ is the reduction of $\mathbf{m}$ modulo d/d' (i.e. $0 \leqslant m'_i < d/d'$ and $m_i \equiv m'_i \pmod{d)/d'}$).
- The homomorphisms $\{c_{d,d'}\}$ are compatible. That is, the diagram

$$\begin{tikzcd}
(\mathcal{E}_d^{\otimes d/d'})^{\otimes d'/d''} \arrow[r, "(c_{d,d'})^{\otimes d'/d''}"] \arrow[dr, "c_{d,d''}"'] & \mathcal{E}_{d'}^{\otimes d'/d''} \arrow[d, "c_{d',d''}"] \\
& \mathcal{E}_{d''}
\end{tikzcd}$$

  commutes for every $d''|d'|d|r$.

If r is prime, then a coherent net of rth roots is simply an rth root of $\mathcal{K}$. Even when d is not prime, if the root $\mathcal{E}_d$ is locally free, then for every divisor d' of d, the sheaf $\mathcal{E}_{d'}$ is uniquely determined, up to an automorphism of $\mathcal{E}_{d'}$. In particular, if $\mathbf{m}'$ satisfies the conditions $\mathbf{m}' \equiv \mathbf{m} \pmod{d'}$ and $0 \leqslant m'_i < d'$, then the sheaf $\mathcal{E}_{d'}$ is isomorphic to $\mathcal{E}_d^{\otimes d/d'} \otimes \mathcal{O}\big(1/d' \sum (m_i - m'_i) p_i\big)$.

DEFINITION 1.4. An n-pointed, r-spin curve of type $\mathbf{m} = (m_1, \ldots, m_n)$ is an n-pointed, prestable curve $(X, p_1, \ldots, p_n)$ with a coherent net of rth roots of ω_X of type $\mathbf{m}$, where ω_X is the (canonical) dualizing sheaf of X. An r-spin curve is called smooth if X is smooth, and it is called stable if X is stable.

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EXAMPLE 1.5. Smooth 2-spin curves of type $\mathbf{0} := (0, 0, \ldots, 0)$ correspond to classical spin curves (a curve with a theta characteristic) $(X, \mathcal{E}_2)$, with an explicit isomorphism $\mathcal{E}_2^{\otimes 2} \xrightarrow{\sim} \omega$.

DEFINITION 1.6. An isomorphism of r-spin curves

$$(X, p_1, \ldots, p_n, (\{\mathcal{E}_d\}, \{c_{d,d'}\})) \xrightarrow{\sim} (X', p'_1, \ldots, p'_n, (\{\mathcal{E}'_d\}, \{c'_{d,d'}\}))$$

of the same type $\mathbf{m}$ is an isomorphism of pointed curves

$$\tau: (X, p_1, \ldots, p_n) \xrightarrow{\sim} (X', p'_1, \ldots, p'_n)$$

and a set of sheaf isomorphisms $\{\beta_d: \tau^*\mathcal{E}'_d \xrightarrow{\sim} \mathcal{E}_d\}$, with β_1 being the canonical isomorphism $\tau^*\omega_{X'}(-\sum_i m_i p'_i) \xrightarrow{\sim} \omega_X(-\sum m_i p_i)$, and such that the homomorphisms β_d are compatible with all the maps $c_{d,d'}$ and $\tau^* c'_{d,d'}$.

Every r-spin structure on a smooth curve X is determined, up to isomorphism, by a choice of a line bundle $\mathcal{E}_r$, such that $\mathcal{E}_r^{\otimes r} \cong \omega_X(-\sum m_i p_i)$. In particular, if X has no automorphisms, then the set of isomorphism classes of r-spin structures (if non-empty) of type $\mathbf{m}$ on X is a principal homogeneous space for the group of r-torsion points of the Jacobian of X. Thus there are r^{2g} such isomorphism classes.

EXAMPLE 1.7. If $g = 1$ and $\mathbf{m} = \mathbf{0}$, then ω_X is isomorphic to $\mathcal{O}_X$, and a smooth r-spin curve is just an elliptic curve X with a line bundle $\mathcal{E}_r$ corresponding to an r-torsion point of X, together with an explicit isomorphism $\mathcal{E}_r^{\otimes r} \xrightarrow{\sim} \mathcal{O}_X$. In particular, the stack of stable, one-pointed r-spin curves of genus one and type $\mathbf{0}$ forms a gerbe over the disjoint union of modular curves $\coprod_{d|r} X_1(d)$.

DEFINITION 1.8. The stack of connected, stable, n-pointed, r-spin curves of genus g and type $\mathbf{m} = (m_1, \ldots, m_n)$ is denoted by $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. The disjoint union $\coprod_{\mathbf{m} 0 \leqslant m_i < r} \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is denoted by $\overline{\mathcal{M}}_{g,n}^{1/r}$.

Remark 1.9. As mentioned above, no information is lost by restricting $\mathbf{m}$ to the range $0 \leqslant m_i \leqslant r - 1$, since when $\mathbf{m} \equiv \mathbf{m}' \pmod r$ every r-spin curve of type $\mathbf{m}$ naturally gives an r-spin curve of type $\mathbf{m}'$ simply by

$$\mathcal{E}_d \mapsto \mathcal{E}_d \otimes \mathcal{O}\left(\sum \frac{m_i - m'_i}{d} p_i\right).$$

Thus $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is canonically isomorphic to $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}'}$.

1.3. BASIC PROPERTIES OF THE MODULI SPACE

In [17] it was shown that $\overline{\mathcal{M}}_{g,n}^{1/r}$ is a smooth Deligne–Mumford stack, finite over $\overline{\mathcal{M}}_{g,n}$, with a projective, coarse moduli space. For $g > 1$ the spaces $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ are irreducible if $\gcd(r, m_1, \ldots, m_n)$ is odd, and they are the disjoint union of two irreducible components if $\gcd(r, m_1, \ldots, m_n)$ is even. When $r = 2$ (and in fact, for all even r) this is due to the well-known fact that even and odd theta characteristics on a curve cannot be deformed into one another [30]. These two components will be denoted $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{even}}$ and $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{odd}}$ respectively.

When the genus g is zero the moduli space $\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}$ is either empty (if r does not divide $2 + \sum m_i$), or is canonically isomorphic to $\overline{\mathcal{M}}_{0,n}$. Note, however, that this isomorphism is not an isomorphism of stacks, since the automorphisms of elements of $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ vary differently from the way that automorphisms of the underlying curves vary. We will discuss this further in Section 1.6. In any case, $\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}$ is always irreducible.

When the genus g is one, the space $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m}}$ is the disjoint union of d irreducible components, where d is the number of divisors of $\gcd(r, m_1, \ldots, m_n)$. We will denote the irreducible (and connected) component indexed by a divisor e of $\gcd(r, m_1, \ldots, m_n)$ by $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m},(e)}$. When $\mathbf{m}$ is zero, as mentioned in Example 1.7, the locus of smooth r-spin curves in this component consists of n-pointed, elliptic curves with a torsion point of exact order e.

Throughout this paper we will denote the forgetful morphism by $p: \overline{\mathcal{M}}_{g,n}^{1/r} \longrightarrow \overline{\mathcal{M}}_{g,n}$, and the universal curve by $\pi: \mathcal{C}_{g,n}^{1/r} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r}$. As in the case of the moduli space of stable curves, the universal curve possesses canonical sections $\sigma_i: \overline{\mathcal{M}}_{g,n}^{1/r} \longrightarrow \mathcal{C}_{g,n}^{1/r}$ for $i = 1, \ldots, n$. Unlike the case of stable curves, however, the universal curve $\mathcal{C}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is not obtained by considering $(n+1)$-pointed r-spin curves. The curve $\mathcal{C}_{g,n}^{1/r,\mathbf{m}}$ is birationally equivalent to $\overline{\mathcal{M}}_{g,n+1}^{1/r,(m_1,m_2,\ldots,m_n,0)}$, but they are not isomorphic.

There is one other canonical morphism associated to these spaces; namely, when d divides r, the morphism

$$[r/d]: \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/d,\mathbf{m}'} \text{ and } [r/d]: \overline{\mathcal{M}}_{g,n}^{1/r} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/d},$$

which forgets all of the roots and homomorphisms in the net of rth roots except those associated to divisors of d. Here $\mathbf{m}'$ is congruent to $\mathbf{m}$ (mod d) and $0 \leqslant m_i' < d$ for all $i \in \{1, \ldots, n\}$. In the case that the underlying curve is smooth, this is equivalent to replacing the line bundle $\mathcal{E}_r$ by its r/d-th tensor power (and then taking the tensor product with $\mathcal{O}(1/d \sum (m_i - m_i')p_i)$).

The two components $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},even}$ and $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},odd}$ that arise in the case that $\gcd(r, m_1, \ldots, m_n)$ is even are just the preimages of the spaces of even and odd theta-characteristics in $\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0}}$ under the map $[r/2]: \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0}}$.

1.4. BOUNDARY BEHAVIOR

1.4.1. *Neveu–Schwarz and Ramond Nodes*

At any node q of a prestable curve X, there are two types of local behavior of an rth root $(\mathcal{E}_r, b_r)$ of $\omega_X(-\sum m_i p_i)$. It is either locally free at q, in which case the homomorphism b_r is an isomorphism near q, or it is torsion-free, but not locally

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free at q. In the locally free case we will say that the root $\mathcal{E}_r$ is Ramond at q, and in the non-locally free case it will be called Neveu–Schwarz.

If the rth root sheaf $\mathcal{E}_r$ of an r-spin structure on X is Ramond at every node of X, then the whole net of roots is completely determined (up to isomorphism) by the root $(\mathcal{E}_r, c_{r,1})$ as follows:

$$\mathcal{E}_d = \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}\left(\frac{1}{d}\sum(m_i - m_i')p_i\right),$$

and $c_{d,1} = c_{r,1} \otimes I$, where I is the identity homomorphism

$$\mathcal{O}\left(\frac{1}{d}\sum(m_i - m_i')p_i\right) \longrightarrow \mathcal{O}\left(\frac{1}{d}\sum(m_i - m_i')p_i\right).$$

Remark 1.10. In the Neveu–Schwarz case, the r-spin structure maps are more complicated than those in the Ramond case, but the combinatorial structure of Neveu–Schwarz nodes is simpler than the Ramond nodes. In particular, the cohomology classes defined by boundary strata with Neveu–Schwarz nodes factor in a nice combinatorial way. Moreover, the hope of constructing a cohomological field theory from $\overline{\mathcal{M}}_{g,n}^{1/r}$ is based on the expectation that one can construct a canonical cohomology class which vanishes on the strata where the r-spin structure has an rth root sheaf which is Ramond at some node. (This would follow from Axiom 4 in Section 4.1.)

1.4.2. *Local Structure at Neveu–Schwarz Nodes*

Recall from Section 1.1 (see also [17, 18, 34]) that near a Neveu–Schwarz node q, an rth root $(\mathcal{E}_r, b_r)$ of $\omega_X(-\sum m_i p_i)$ is uniquely determined by an rth root $(\tilde{\mathcal{E}}_r, \tilde{b}_r)$ of the bundle $\omega_{\tilde{X}}(-\sum m_i p_i - m^+ q^+ - m^- q^-)$ on the normalization $\nu: \tilde{X} \longrightarrow X$ of X at the node q. Here q^+ and q^- are the inverse images of q under ν, and m^+ and m^- are non-negative integers* which sum to $r-2$. If x and y are local parameters of X near the node q satisfying the equation $xy = 0$, then the sheaf $\mathcal{E}$ is generated locally by the sections $(x^{m_+}dx)^{1/r}$ and $(y^{m_-}dy)^{1/r}$; and $\tilde{\mathcal{E}}_r$ is generated by $(x^{m_+}dx)^{1/r}$ on the x branch of $\tilde{X}$, and it is generated by $(y^{m_-}dy)^{1/r}$ on the y branch of $\tilde{X}$. The points q^+ and q^- are given by $\{x = 0\}$ and $\{y = 0\}$, respectively, on $\tilde{X}$. The sheaf $\tilde{\mathcal{E}}_r$ is simply $\nu^*\mathcal{E}_r$ modulo torsion; and $\nu_*\tilde{\mathcal{E}}_r$ is $\mathcal{E}_r$, with b_r induced from $\tilde{b}_r$ by adjointness. We will call the integers m^+ and m^- the order of the r-spin structure at the node, along the x or y branch, respectively. The order m^+ and m^- of the r-spin structure along the x or y branch of a node is not to be confused with the order of vanishing of the structure maps. Indeed, if the r-th root bundle $\mathcal{E}_r$ is Neveu–Schwarz at a node, the order of vanishing of the map $c_{1,r}$ at that node is exactly $(m^+ + 1) + (m^- + 1) = r$.

In the case that $m^+ + 1$ and $m^- + 1$ are relatively prime, one can show (see [17]) that $\tilde{\mathcal{E}}$ and $\tilde{b}$ uniquely determine the entire net. However, if $\gcd(m^+ + 1,$

*Note that the integers u and v of the papers [17] and [18] are $m^+ + 1$ and $m^- + 1$, respectively.

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$m^- + 1) = d$, then although $\mathcal{E}_r$ still completely determines the Neveu–Schwarz roots, d divides r, and the dth root $(\mathcal{E}_d, c_{d,1})$ of the net is locally free (Ramond), as are all roots $(\mathcal{E}_{d'}, c_{d',1})$ for every d' dividing d. In particular, although generators $(x^{m_+}dx)^{1/r}$ and $(y^{m_-}dy)^{1/r}$ in $\mathcal{E}_r$ determine $(x^{m_+}dx)^{1/d}$ and $(y^{m_-}dy)^{1/d}$, we must identify $(dx/x)^{1/d}$ with $(-dy/y)^{1/d}$. However, this identification is only determined by $(\mathcal{E}_r, c_{r,1})$ up to a non-canonical choice of a dth root of unity. If the normalization $\tilde{X}$ at q has two connected components, then the dth root $(\mathcal{E}_d, c_{d,1})$ is determined up to (non-canonical) isomorphism by $(\mathcal{E}_r, b_r)$, but if $\tilde{X}$ is connected, then $(\mathcal{E}_d, c_{d,1})$ is not determined by $(\mathcal{E}_r, b_r)$, since an additional choice of a dth root of unity is required to construct $\mathcal{E}_d$ from $\mathcal{E}_r^{\otimes r/d}$ (see Section 1.7).

1.4.3. *More Detailed Study of the Ramond Case*

Let $(\mathcal{E}, b)$ be an rth root of $\omega_X(-\sum m_i p_i)$ which is Ramond at a node q of X. The restriction of $\mathcal{E}$ to q gives an exact sequence $0 \longrightarrow \mathfrak{m}_q \otimes \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_q \longrightarrow 0$, where $\mathfrak{m}_q$ is the maximal ideal of the point q. The sheaf $\mathcal{E}' := \mathcal{E} \otimes \mathfrak{m}_q$ is a rank-one, torsion-free sheaf of degree $(2g - 2 - \sum m_i)/r - 1$ on X, and pulling $\mathcal{E}'$ back to the normalization $\nu\colon \tilde{X} \longrightarrow X$ of X at q gives, modulo torsion, a rank-one, torsion-free sheaf $\mathcal{E}'' := \nu^*\mathcal{E}'/\text{torsion}$, such that $\nu_*\mathcal{E}''$ is equal to $\mathcal{E}'$.

If x and y are local coordinates on $\tilde{X}$ near q^+ and q^- respectively, then $\mathcal{E}''$ is locally generated by $x(dx/x)^{1/r}$ (respectively $y(dy/y)^{1/r}$). Therefore, the homomorphism

$$b''\colon \mathcal{E}''^{\otimes r} \longrightarrow \nu^*\omega_X\left(-\sum m_i p_i\right) = \omega_{\tilde{X}}\left(-\sum m_i p_i + q^+ + q^-\right),$$

induced by $b\colon \mathcal{E}^{\otimes r} \longrightarrow \omega_{\tilde{X}}(-\sum m_i p_i)$, factors through

$$\omega_{\tilde{X}}\left(-\sum m_i p_i - (r-1)q^+ - (r-1)q^-\right) \longrightarrow \omega_{\tilde{X}}\left(-\sum m_i p_i + q^+ + q^-\right).$$

Thus the rth root $(\mathcal{E}_r, c_{r,1})$ can be Ramond at the node if and only if $m^+ = r - 1$ and $m^- = r - 1$ satisfy the degree conditions

$$\deg_{X^{(j)}} \omega_{\tilde{X}} - \sum m_i - m^+ - m^- \equiv 0 \pmod{r}$$

on every connected component $X^{(j)}$ of $\tilde{X}$. In the case that $(\mathcal{E}_r, c_{r,1})$ is Ramond, we will define the order of the r-spin structure at the node to be $m^+ = m^- = r - 1$ along both branches of the underlying curve.

Similarly,

$$(\nu^*\mathcal{E})^{\otimes r} \xrightarrow{\nu^* b} \nu^*\omega_X\left(-\sum m_i p_i\right) = \omega_{\tilde{X}}\left(-\sum m_i p_i + q^+ + q^-\right)$$

corresponds to the choice $m^+ = m^- = -1$. In this special case there is a residue map that canonically identifies $\mathcal{E}_r|_{q^+}$ and $\mathcal{E}_r|_{q^-}$ with $\mathbb{C}$.

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PROPOSITION 1.11. *If* $(\mathcal{E}_r, b)$ *is an* rth *root of* $\omega_X(-\sum m_i p_i)$ *with* $m_i = -1$ *for some i, then there is an isomorphism*

$$R_{p_i}: \mathcal{E}_r|_{p_i} \xrightarrow{\sim} \mathbb{C} \tag{1}$$

which is canonical up to a choice of an rth *root of unity.*

An immediate consequence is the following corollary:

COROLLARY 1.12. *If* σ_i *is the* ith *section of the universal curve* $\pi: \mathcal{C}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ *with* $\mathbf{m} = (m_1, \ldots, m_n)$ *and* $m_i = -1$ *for some i, then the pullback* $\sigma_i^*(\mathcal{E}_r)$ *of the universal* rth *root* $\mathcal{E}_r$ *is an r-torsion line bundle, i.e., its* rth *tensor power is isomorphic to* $\mathcal{O}_{\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}}$.

Proof of the Proposition. Let z be a local parameter on X near p, so that the sheaf $\mathcal{E}_r$ is locally generated by an element $(dz/z)^{1/r}$ which is well defined up to an rth root of unity. Define the map $R_p: \mathcal{E}_r|_p \xrightarrow{\sim} \mathbb{C}$ by $R_p((a_0 + a_1 z + \ldots)(dz/z)^{1/r}) := a_0$. It is easy to check that this definition is independent of local parameter, and hence defines a canonical isomorphism. □

1.5. GRAPHS

Much of the information about the structure of the boundary of $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ can be encoded in terms of decorated graphs.

Recall that the (dual) graph of an n-pointed prestable curve $(X, p_1, \ldots, p_n)$ consists of the following elements:

- *Vertices*, corresponding to the irreducible components of X: a vertex v is labeled with a non-negative integer $g(v)$, the (geometric) genus of the component;
- *Edges*, corresponding to the nodes of the curve: an edge connects two vertices (possibly even the same vertex, in which case the edge is called a *loop*) if and only if the corresponding node lies on the associated irreducible components;
- *Tails*, corresponding to the marked points $p_i \in X$, $i = 1, \ldots, n$: a tail labeled by the integer i is attached at the vertex associated to the component of X that contains p_i.

DEFINITION 1.13. A *half-edge* of a graph Γ is either a tail or one of the two ends of a 'real' edge of Γ. We denote by $V(\Gamma)$ the set of vertices of Γ and by $n(v)$ the number of half-edges of Γ at the vertex v.

The following definition describes a class of graphs that are dual graphs of stable pointed curves.

DEFINITION 1.14. Let Γ be a graph. The number $g(\Gamma) = \dim H^1(\Gamma) + \sum_{v \in V(\Gamma)} g(v)$ is called the genus of a graph Γ.

A graph Γ (not necessarily connected) is called stable if $2g(v) - 2 + n(v) > 0$ for every $v \in V(\Gamma)$ (in particular, it satisfies $2g(\Gamma) - 2 + n > 0$, where n is the number of tails of Γ).

To describe strata of the moduli space of r-spin curves, we decorate the graphs with additional data coming from the r-spin structure. In particular, the type $\mathbf{m} = (m_1, \ldots, m_n)$ gives a marking to each of the tails.

DEFINITION 1.15. Fix an integer $r \geqslant 2$. A decorated stable graph is a stable graph with a marking of each half-edge by a non-negative integer $m < r$, such that for each edge e the marks m^+ and m^- of the two half-edges of e satisfy

$$m^+ + m^- \equiv r - 2 \pmod{r}.$$

As mentioned in Section 1.4, decorated stable graphs with n tails and genus g correspond to boundary strata in $\overline{\mathcal{M}}^{1/r}_{g,n}$.

DEFINITION 1.16. Given a stable r-spin curve $\mathfrak{X}$ of type $\mathbf{m} = (m_1, \ldots, m_n)$, the decorated dual graph of $\mathfrak{X}$ is the dual graph Γ of the underlying curve X, with the following additional markings. The ith tail is marked by m_i, and each half-edge associated to a node of X is marked by the order (m^+ or m^-) of the r-spin structure along the branch of the node associated to that half-edge.

DEFINITION 1.17. Let Γ be a connected stable graph (or a decorated stable graph) with n tails and of genus g . We denote by $\overline{\mathcal{M}}_\Gamma$ (or by $\overline{\mathcal{M}}^{1/r}_\Gamma$) the closure in $\overline{\mathcal{M}}_{g,n}$ (or in $\overline{\mathcal{M}}^{1/r}_{g,n}$) of the moduli space of stable curves (or r-spin curves) whose dual graph is Γ. If $\Gamma = \coprod_{i \in I} \Gamma_i$ is the disjoint union of connected subgraphs Γ_i then we denote by $\overline{\mathcal{M}}_\Gamma$ the product $\prod_{i \in I} \overline{\mathcal{M}}_{\Gamma_i}$, and similarly $\overline{\mathcal{M}}^{1/r}_\Gamma = \prod_{i \in I} \overline{\mathcal{M}}^{1/r}_{\Gamma_i}$.

1.6. AUTOMORPHISMS OF r-SPIN CURVES

As mentioned in Section 1.3, even in the genus zero case, where there is a unique r-spin structure of a given type $\mathbf{m} = (m_1, \ldots, m_n)$ for each genus-zero curve (provided $\mathbf{m}$ satisfies the degree requirement $\sum m_i \equiv 2 \pmod{r}$), the automorphisms of the r-spin structure ensure that $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$ is not isomorphic, as a stack, to $\overline{\mathcal{M}}_{0,n}$. The automorphisms of r-spin structures will play an important role later in this paper, particularly in the determination of the degrees of morphisms and the properties of various cohomology classes under restriction and pullback. Consequently, we need to understand the group of automorphisms of an r-spin curve.

First, we introduce some notation. Let $\mathfrak{X} = (X, p_1, \ldots, p_n, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))$ be an r-spin curve, and let Γ be its decorated dual graph. Let V be the set of vertices of Γ and E_{nl} be the set of edges which do not start and end at the same vertex (i.e., non-loops). Furthermore, for each $v \in V$, denote by X_v the irreducible component of X associated to vertex v; and denote by F_v the set of all half-edges attached to v in Γ. For each f in F_v let p_f be the point of X_v associated to f, and let m_f

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be the marking of f. Finally, for each vertex v, if $i_v\colon X_v \longrightarrow X$ denotes the obvious inclusion, then the r-spin structure $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ pulls back to a collection $(\{i_v^*\mathcal{E}_d\}, \{i_v^* c^v_{d,d'}\})$ of sheaves and morphisms on X_v. However, these sheaves are not necessarily torsion free. Taking the quotient of each sheaf $i_v^*\mathcal{E}_d$ by its torsion submodule gives a rank-one, torsion-free sheaf, which we denote by $\mathcal{E}^v_d$. It is easy to see that the homomorphisms $\{c^v_{d,d'}\}$, induced on the $\{\mathcal{E}^v_d\}$ from the homomorphisms $i_v^*\mathcal{E}_d$, make $\mathfrak{X}_v = (X_v, p_{f_1^v}, \dots, p_{f_{k_v}^v}, (\{\mathcal{E}^v_d\}, \{c^v_{d,d'}\}))$ into an r-spin curve. We will call $\mathfrak{X}_v$ the restriction of $\mathfrak{X}$ to the curve X_v. Any $e \in E_{nl}$ consists of two half-edges f_e^+ and f_e^-; and we denote by d_e the integer $d_e := \gcd(m_{f_e^+} + 1, m_{f_e^-} + 1) = \gcd(m_{f_e^+} + 1, r)$.

PROPOSITION 1.18. *If the underlying pointed curves $(X_v, p_{f_1^v}, \dots, p_{f_{k_v}^v})$ of the r-spin curves $\mathfrak{X}_v$ have no non-trivial automorphisms (this is true for a generic curve with $g + n > 2$), then*

(1) *for each $v \in V$ the automorphism group* $\mathrm{Aut}(\mathfrak{X}_v)$ *of $\mathfrak{X}_v$ is isomorphic to μ_r, the group of* rth *roots of unity; and*
(2) *given any orientation of the edges of the dual graph Γ of $\mathfrak{X}$, the automorphism group* $\mathrm{Aut}(\mathfrak{X})$ *of $\mathfrak{X}$ fits into the following exact sequence*

$$1 \longrightarrow \mathrm{Aut}(\mathfrak{X}) \longrightarrow \prod_{v \in V} \mathrm{Aut}(\mathfrak{X}_v) \xrightarrow{\partial} \prod_{e \in E_{nl}} \mu_{r/d_e}. \tag{2}$$

Here the map ∂ is defined as follows. The orientation of each edge e determines the choice of half-edges f_e^+ and f_e^- and of corresponding vertices v_e^+ and v_e^- (the unique vertex of Γ attached to f_e^+ or f_e^-, respectively). The map ∂ maps the element $\prod \zeta_v \in \prod_{v \in V} \mu_r$ to the element $\prod (\zeta_{v_e^+}^{d_e} \zeta_{v_e^-}^{-d_e})$. Note that, although the map ∂ depends upon the given orientation of the edges of Γ, the kernel of ∂ is independent of orientation.

Proof. If $(X_v, p_{f_1^v}, \dots, p_{f_e^v})$ has no automorphisms, then each term $\mathcal{E}^v_d$ in the r-spin structure $(\{\mathcal{E}^v_d\}, \{c^v_{d,d'}\})$ is locally free (since X_v is smooth) of rank one and has automorphism group $\mathbb{C}^* = H^0(X_v, \mathcal{O}^*_{X_v})$, which acts on $\mathcal{E}_d$ by multiplication. However, an automorphism of the r-spin structure must also be compatible with the structure maps $\{c_{d,d'}\}$. In particular, compatibility with the isomorphism $c_{r,1}\colon (\mathcal{E}^v_r)^{\otimes r} \xrightarrow{\sim} \omega_{X_v}(-\sum_{f \in F_v} m_f p_f)$ shows that any automorphism σ_r of $\mathcal{E}_r$ must satisfy $(\sigma_r)^r = 1$. Moreover, compatibility with $c_{d,d'}$ shows that $\sigma_{d'} = (\sigma_d)^{d/d'}$ for every d' dividing d and d dividing r. Thus σ_r corresponds to some $\zeta \in \mu_r$, and for every d dividing r, the automorphism σ_d is just $\zeta^{r/d}$. This proves the first part of the proposition.

For the second part, it is easy to see that any automorphism of the whole r-spin curve $\mathfrak{X}$ induces, by restriction, an automorphism of $\mathfrak{X}_v$ for each $\mathfrak{X}_v$, and the map $\mathrm{Aut}(\mathfrak{X}) \longrightarrow \prod_{v \in V} \mathrm{Aut}(\mathfrak{X}_v)$ is injective.

Moreover, for any edge $e \in E_{nl}$ corresponding to half edges f^+ attached to vertex v^+ and f^- attached to vertex v^-, an automorphism σ of $\mathfrak{X}$ will induce automorphisms $\sigma_{v^+} = \zeta_+ \in \mu_r$ and $\sigma_{v^-} = \zeta_- \in \mu_r$; and these automorphisms must agree whenever $\mathcal{E}_r$ is

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Ramond (locally free) at the node p_e corresponding to edge e. Similarly, if $\mathcal{E}_d$ is Ramond at p_e, then $(\zeta_+)^{r/d}$ must equal $(\zeta_-)^{r/d}$. However, if the sheaf $\mathcal{E}_d$ is Neveu–Schwarz at p_e, then σ_{v^+} and σ_{v^-} act on distinct vector spaces (the sheaf $\mathcal{E}_d^{v^+}|_{p_{f_e^+}}$ is not the same as $\mathcal{E}_d^{v^-}|_{p_{f_e^-}}$), and so $\mathcal{E}_d$ imposes no compatibility condition on σ_{v^+} and σ_{v^-}.

Since $\mathcal{E}_d$ is Ramond at the node p_e precisely when d divides both $m_{f^+}+1$ and $m_{f^-}+1$, we have that the condition imposed by compatibility for σ_{v^+} and σ_{v^-} is precisely the equality $(\sigma_{v^+})^{r/d} = (\sigma_{v^-})^{r/d}$ for

$$d = d_e := \gcd(m_{f^+}+1, m_{f^-}+1) = \gcd(m_{f^+}+1, r).$$

It is clear that any $\prod \sigma_v \in \prod \mathrm{Aut}(\mathfrak{X}_v)$ which meets this compatibility condition at each edge $e \in E_{nl}$ defines a global automorphism $\sigma \in \mathrm{Aut}(\mathfrak{X})$. $\square$

Of course, since we only care about the kernel of ∂, the right-most term in exact sequence (2) might as well include all the edges, including loops. This is because for any loop e, the map ∂ will always map every element of $\prod_{v\in V} \mathrm{Aut}(\mathfrak{X}_v)$ to $1 \in \mu_{r/d_e}$, since the vertices v^+ and v^- (and hence also σ_{v^+} and σ_{v^-}) are the same.

For r-spin curves of genus 1 and 2 with a generic involution (i.e. in $\overline{\mathcal{M}}_{1,1}^{1/r}$ and $\overline{\mathcal{M}}_{2,0}^{1/r}$), we have the following description of the group of automorphisms. Note that when $g = n = 1$, degree requirements force m_1 to be zero. Also when $g = 2$ and $n = 0$, degree requirements force r to be 2.

PROPOSITION 1.19.

(1) *If the underlying curve (X, p) of a smooth r-spin curve $\mathfrak{X} \in \overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0},(j)}$ of index j has no automorphisms other than the elliptic involution, then the automorphism group of $\mathfrak{X}$ is*

$$\mathrm{Aut}\mathfrak{X} = \begin{cases} \mu_r \times \mathbb{Z}/2, & \text{if } j = 1 \text{ or } j = 2, \\ \mu_r, & \text{if } j > 2. \end{cases}$$

(2) *If the underlying curve X of a smooth 2-spin curve $\mathfrak{X} \in \mathcal{M}_{2,0}^{1/2}$ has no automorphisms other than the hyperelliptic involution, then the automorphism group of $\mathfrak{X}$ is* $\mathrm{Aut}\mathfrak{X} = \mu_r \times \mathbb{Z}/2$.

Proof. In the case of $g = n = 1$, smooth r-spin curves of type 0 and index j correspond to the torsion points of X of exact order j. It is well known that the involution $i: X \longrightarrow X$ acts without fixed points on the points of exact order j, unless j is 1 or 2, in which case the involution fixes all 2-torsion points (including the identity, corresponding to the trivial bundle $\mathcal{O}_X = \omega_X$).

It is easy to check (e.g. by explicitly writing out the coordinates) that for sheaves $\mathcal{E}_r$ of index 2 (or 1), corresponding to 2-torsion, there is a canonical choice of isomorphism τ: $i^*\mathcal{E}_r \longrightarrow \mathcal{E}_r$ such that

- $i^*\tau \circ \tau$ is trivial,

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- any other isomorphism $i^*\mathcal{E}_r \longrightarrow \mathcal{E}_r$ differs from τ by an element $\zeta \in \mu_r$, and
- τ commutes with all elements of μ_r.

Thus $\mathrm{Aut}\mathfrak{X}$ has order $2r$ and is Abelian; and if r is even, then every automorphism has order dividing r. Thus the proposition follows in genus 1.

The proof in genus 2 is similar, but simpler, since every 2-spin structure is fixed by the involution. $\square$

1.7. GLUING

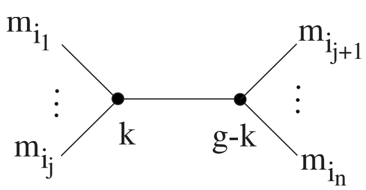
In the case of moduli spaces of ordinary stable curves, if a graph Γ is obtained from another graph $\tilde{\Gamma}$ (not necessarily connected) by gluing together two tails of $\tilde{\Gamma}$ (thus producing a new edge), there is a natural gluing morphism

$$\rho\colon \overline{\mathcal{M}}_{\tilde{\Gamma}} \longrightarrow \overline{\mathcal{M}}_{\Gamma} \hookrightarrow \overline{\mathcal{M}}_{g,n}. \tag{3}$$

It corresponds to gluing together the punctures on a curve $X \in \overline{\mathcal{M}}_{\tilde{\Gamma}}$ associated to the two tails, and thus the curve $\rho(X) \in \overline{\mathcal{M}}_{\Gamma}$ will have an additional node and two fewer punctures than X. A similar gluing operation sometimes exists for r-spin curves, but even then we often need to include extra data.

1.7.1. *Two Irreducible Components*

Consider the case of an r-spin curve with a single Neveu–Schwarz node. Assume the normalization of the underlying curve X at the node has two connected components: X^+ of genus k, and X^- of genus $g-k$. The decorated dual graph of the r-spin curve looks like this



The rth root bundle $\mathcal{E}_r$ factors as $\mathcal{E}_r = \mathcal{E}_r^+ \oplus \mathcal{E}_r^-$, with $\mathcal{E}_r^+$ an rth root of $\omega_{X^+}(-\sum_{l=1}^{j} m_{i_l} p_{i_l} - m^+ q^+)$ on X^+ and $\mathcal{E}^-$ an rth root of $\omega_{X^-}(-\sum_{l=j+1}^{n} m_{i_l} p_{i_l} - m^- q^-)$ on X^-. Here m^+ and m^- can easily be calculated (for degree reasons) as the unique non-negative integers summing to $r-2$ such that the rth roots of the corresponding (twisted) canonical bundles exist; namely,

$$m^+ \equiv 2k - 2 - \sum_{l=1}^{j} m_{i_l} \pmod r \tag{4}$$

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and

$$m^- \equiv 2(g-k) - 2 - \sum_{l=j+1}^{n} m_{i_l} \pmod r. \tag{5}$$

Of course, since the degree of the original bundle was divisible by r, each of the two relations implies the other. The Neveu–Schwarz case occurs exactly when the solutions m^+ and m^- to the congruences (4) and (5) lie in $\{0, 1, \ldots, r-2\}$ (whereas the Ramond case occurs when the solution is $m^+ \equiv r-1$, $m^- \equiv r-1$) [18]. If $\gcd(m^+ + 1, m^- + 1)$ is one, then these data completely determine the r-spin structure on X, and in a canonical way, so that in this case we have a well-defined gluing morphism

$$\rho\colon \overline{\mathcal{M}}^{1/r}_{\Gamma_1 \sqcup \Gamma_2} = \overline{\mathcal{M}}^{1/r}_{\Gamma_1} \times \overline{\mathcal{M}}^{1/r}_{\Gamma_2} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}. \tag{6}$$

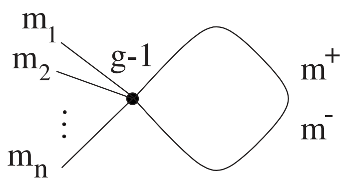
If, however, $d := \gcd(m^+ + 1, m^- + 1)$ is greater than one, there is no canonical morphism of stacks, as there is in (6). To construct a gluing morphism would require an isomorphism $\phi\colon \mathcal{E}^+_d|_{q^+} \xrightarrow{\sim} \mathcal{E}^-_d|_{q^-}$ that makes $\mathcal{E}_d$ Ramond at the node. In particular, ϕ must be compatible with the isomorphism $(\mathcal{E}^+_d)^{\otimes d} \xrightarrow{\sim} \omega(-\sum m_i p_i)$ and $(\mathcal{E}^-_d)^{\otimes d} \xrightarrow{\sim} \omega(-\sum m_i p_i)$. We call such an isomorphism ϕ a gluing datum.

DEFINITION 1.20. Given a prestable $(n+2)$-pointed curve $(X, p_1, \ldots, p_{n+2})$ (not necessarily connected) and a dth root $(\mathcal{E}, b)$ of $\omega_X(-\sum m_i p_i)$, such that $m_{n+1} = m_{n+2} = -1$, denote by $\overline{X}$ the curve obtained from X by identifying the point p_{n+1} with p_{n+2}. A gluing datum for $(\mathcal{E}, b)$ is an isomorphism $\phi\colon \mathcal{E}|_{p_{n+1}} \xrightarrow[b^{-1}]{\sim} \mathcal{E}|_{p_{n+1}}$ which is compatible with the d-th root maps $\mathcal{E}|^{\otimes d}_{p_{n+1}} \xrightarrow{b} \omega_{\overline{X}}(-\sum_{i=1}^{n} m_i p_i) \longrightarrow \mathcal{E}|^{\otimes d}_{p_{n+2}}$.

Returning to the case of a curve X with a single node and two irreducible components, since the normalized curve has two connected components, the root $(\mathcal{E}_d, c_{d,1})$ is determined *up to non-canonical isomorphism* by $(\mathcal{E}_r, c_{r,1})$. Still, choosing one gluing datum $\phi\colon \mathcal{E}^+_d|_{q^+} \xrightarrow{\sim} \mathcal{E}^-_d|_{q^-}$ does not give a morphism of stacks because an automorphism of the r-spin structure on X^+ or on X^- changes the gluing datum ϕ, and thus induces a different (but isomorphic) r-spin structure on the curve X.

1.7.2. *Irreducible Curve with One Node*

In the case of an r-spin curve X with a single component and one node we have the dual graph



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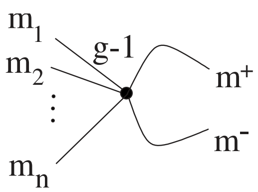
This determines an r-spin structure on the normalized curve $\tilde{X}$, with the dual graph



If $d = \gcd(m^+ + 1, m^- + 1) = \gcd(m^+ + 1, r) = 1$, then all roots $\mathcal{E}_d$ in the r-spin structure are of the Neveu–Schwarz type and we can define the gluing morphism

$$\rho\colon \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma_{irr}} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \tag{7}$$

in the obvious way.

However, if $d = \gcd(m^+ + 1, r)$ is greater than one, then, as mentioned above, an additional gluing datum is required to construct $(\mathcal{E}_d, c_{d,1})$ from $(\mathcal{E}_r^{\otimes r/d}, c_{r,1})$. In this case, set $u^+ = (m^+ + 1/d)$ and $u^- = (m^- + 1/d)$ and define

$$\mathcal{E}'_d := \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}(u^+p^+ + u^-p^-),$$

where p^+ and p^- are the inverse images under normalization of the node. This shows that $\mathcal{E}'_d$ is a dth root of $\omega_{\tilde{X}}(-\sum m_i p_i + p^+ + p^-)$; that is, m^+ and m^- are both replaced with -1. To construct $\mathcal{E}_d$ from $\mathcal{E}'_d$ we need to choose an isomorphism $\phi\colon \mathcal{E}'_d|_{p^+} \longrightarrow \mathcal{E}'_d|_{p^-}$ compatible with the isomorphisms $\mathcal{E}'^{\otimes d}_d|_{p^+} \xrightarrow{\sim} \omega_{\tilde{X}}(p^+ + p^-) \xrightarrow{\sim} \mathcal{E}'^{\otimes d}_d|_{p^-}$, and there are exactly d such isomorphisms.

Unlike in the case of the tree, an automorphism of the r-spin structure on the normalized curve induces the same automorphism on both sides of the gluing datum ϕ, and thus it preserves ϕ. Consequently, we expect $d = \gcd(m^+ + 1, r)$ gluing morphisms $\rho_\phi\colon \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma_{irr}} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$, indexed by the set of different choices of ϕ. However, in order to construct such a morphism, one needs to be able to define the gluing data in families. That is, if $\pi\colon \mathcal{C}^{1/r}_{\tilde{\Gamma}_{irr}} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}}$ is the universal curve over $\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}}$, and if D^+ and D^- are the loci in of the two sections σ^+ and σ^- of π to be glued, then we need to define an isomorphism $\phi\colon \mathcal{E}'_d|_{D^+} = \sigma^{+*}(\mathcal{E}'_d) \xrightarrow{\sim} \sigma^{-*}(\mathcal{E}'_d) = \mathcal{E}'_d|_{D^-}$. Such an isomorphism may not exist because $\mathcal{E}'_d|_{D^+}$ and $\mathcal{E}'_d|_{D^-}$ may differ by an r-torsion line bundle on $\overline{\mathcal{M}}^{1/r,(\mathbf{m},-1,-1)}_{g-1,n+2}$ But if $\mathfrak{g}$ is the set of isomorphisms $\mathcal{E}'_d|_{D^+} \xrightarrow{\sim} \mathcal{E}'_d|_{D^-}$ of the sheaf $\mathcal{E}'_d = \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}(u^+D^+ + u^-D^-)$, we have a morphism $\rho_{\Gamma_{irr}}\colon \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}_{irr}} \times \mathfrak{g} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma_{irr}} \hookrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$.

2. Tautological Cohomology Classes

Unless otherwise stated, all cohomology groups in the paper are considered with coefficients in $\mathbb{Q}$.

2.1. DEFINITIONS

There are many natural cohomology classes in $H^\bullet(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}, \mathbb{Q})$; these include the classes induced by pullback from $\overline{\mathcal{M}}_{g,n}$, as well as classes induced by replacing the canonical (relative dualizing) sheaf ω_π with $\mathcal{E}_r$ in the usual constructions of tautological cohomology classes on $\overline{\mathcal{M}}_{g,n}$.

In particular, we have the ith Chern classes λ_i of the Hodge bundle $\pi_*\omega_\pi$. However, $\pi_*\mathcal{E}_r$ is not especially well behaved. Instead, we prefer to use the K-theoretic pushforward $\pi_!\mathcal{E}_r$ (also called $R\pi_*\mathcal{E}$).* Recall that for any coherent sheaf $\mathcal{F}$ on the universal curve $\pi: \mathcal{C}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$, the element $\pi_!\mathcal{F}$ of $K_0(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}})$ is the difference $\pi_*\mathcal{F} - R^1\pi_*\mathcal{F}$ (π has relative dimension 1). Here $R^1\pi_*\mathcal{F}$ is the sheaf whose fiber over a point p of the base $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is the vector space $H^1(\pi^{-1}(p), \mathcal{F}|_{\pi^{-1}(p)})$. Serre duality shows that $H^1(X, \omega)$ is canonically isomorphic to $\mathbb{C}$, and so $R^1\pi_*\omega_\pi$ is a trivial line bundle. Hence $\pi_*\omega_\pi = \pi_!\omega_\pi + \mathcal{O}$ in $K_0(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}})$. Therefore, we have an equality of Chern polynomials $c_t\pi_*\omega_\pi = c_t\pi_!\omega_\pi = 1 + \lambda_1 t + \lambda_2 t^2 + \dots$. Tautological classes v_i are defined as components of the Chern character of the Hodge bundle

$$ch_t\pi_*\omega_\pi = 1 + ch_t\pi_!\omega_\pi = g + v_1 t + v_2 t^3 + v_3 t^5 + \dots. \tag{8}$$

(The even components of $ch_t\pi_*\omega_\pi$ vanish by Mumford's theorem [31].) Similarly, we define classes μ_i as components of the Chern character of $\pi_!\mathcal{E}_r$:

$$ch_t\pi_!\mathcal{E}_r = D + \mu_1 t + \mu_2 t^2 + \dots. \tag{9}$$

Here $-D$ is the Euler characteristic $\chi(\mathcal{E}_r|_{\mathcal{C}_{\bar{s}}})$ of $\mathcal{E}_r$ on any geometric fiber $\mathcal{C}_{\bar{s}}$ of π, and by Riemann–Roch

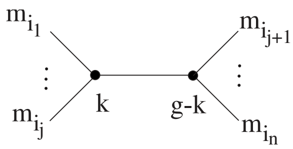
$$D = \frac{1}{r}\left((r-2)(g-1) + \sum_i m_i\right). \tag{10}$$

Serre duality shows that for any $\mathcal{F}$ we have $\pi_!(\mathcal{H}om(\mathcal{F}, \omega_\pi)) = \pi_!\mathcal{F}$, so $\pi_!\mathcal{O} = \pi_!\omega_\pi$. More importantly, for purposes of comparison with Witten's calculations of [34], in the special case that $R^1\pi_*\mathcal{H}om(\mathcal{E}_r, \omega_\pi) = 0$ (or equivalently, $\pi_*\mathcal{E}_r = 0$), the bundle $\mathcal{V} := \pi_*\mathcal{H}om(\mathcal{E}_r, \omega_\pi)$ of [34] corresponds to $\pi_!\mathcal{E}_r$.

In addition to the Hodge-like classes, there are those induced by the canonical sections σ_i of $\pi: \mathcal{C}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. These are classes $\psi_i := c_1(\sigma_i^*(\omega_\pi))$ and $\tilde{\psi}_i := c_1(\sigma_i^*(\mathcal{E}_r))$ (and also class $\psi_i^{(d)}$ for each divisor d of r). When working in $Pic\,\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$, we will abuse notation and use ψ_i to indicate the line bundle $\sigma_i^*(\omega_\pi)$, and $\tilde{\psi}_i$ the line bundle $\sigma_i^*(\mathcal{E}_r)$. Finally, there are the boundary classes. In particular, if $A \sqcup B$ is a partition of $\{1, \dots, n\}$ into two subsets, we denote by $\alpha_{k;A}$ the class

*The notation $\pi_!$ is from algebraic topology and is not to be confused with the sheaf-theoretic *direct image with compact supports*, which we never use in this paper.

of the divisor associated to r-spin curves with the dual graph Γ of the form



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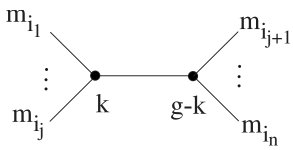
with $\{i_1, \ldots, i_j\} = A$ and $\{i_{j+1}, \ldots, i_n\} = B$. Of course there is an obvious equality: $\alpha_{k;A} = \alpha_{g-k;B}$.

Since the graph Γ is a tree, there is a unique choice of m^+ and m^-, given the original type $\mathbf{m}$ and the partition $A \sqcup B$.

If g is greater than 1, and if r and all of the m_i are even, then the moduli space has two components, $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{even}}$ and $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{odd}}$. If $2 \leqslant k \leqslant g-2$, then $\alpha_{k;A}$ is the sum of four divisors—two on each irreducible component of the moduli space. In particular, there are two divisors in $Pic\,\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{even}}$ with dual graph



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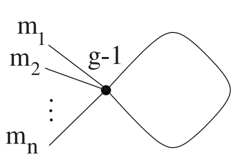
The first is the locus where both vertices of the graph (irreducible components of the underlying curve) are endowed with an even r-spin structure; and the second is where both vertices are endowed with an odd r-spin structure. Similarly, in $Pic\,\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},\mathrm{odd}}$, the two divisors correspond to the two ways of endowing the vertices with r-spin structures of differing parities.

In the case of $k = 0$ and $g > 1$, the divisor $\alpha_{0;k}$ is the sum of only two divisors, corresponding to the parity of the r-spin structure on the other vertex (of genus g). If $k = 1$ then $\alpha_{1;A}$ is the sum of (potentially many) divisors corresponding to the choices of index for the r-spin structure on the vertex of genus 1, (as well as the choices of index or parity for the remaining vertex).

Finally, denote by $\tilde{\delta}_{irr,m^+}$ the divisor associated to the graph



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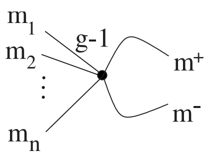
with the r-spin structure inducing



on the normalization; and denote the divisor corresponding to the Ramond root by $\tilde{\delta}_{\mathrm{irr},r-1}$.⋆ The divisor $\tilde{\delta}_{\mathrm{irr},m^+}$ is not necessarily irreducible, since different choices of gluing will induce distinct (and disjoint) divisors, all in $\tilde{\delta}_{\mathrm{irr},m^+}$. Again, there is an obvious equality $\tilde{\delta}_{\mathrm{irr},m} = \tilde{\delta}_{\mathrm{irr},r-2-m}$.

2.2. BASIC PROPERTIES OF THE TAUTOLOGICAL CLASSES

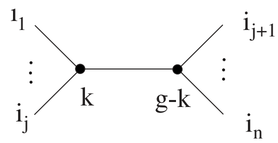
The following Proposition describes relations between various elements in the Picard group of $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$. It is a straightforward generalization of the corresponding result for the case $\mathbf{m} = \mathbf{0}$ proved in [19].

PROPOSITION 2.1.

- *The forgetful map* $p\colon \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ *induces an injection*

$$p^*\colon Pic\,\overline{\mathcal{M}}_{g,n} \otimes \mathbb{Q} \longrightarrow Pic\,\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \otimes \mathbb{Q}.$$

- *Let* $\delta_{k;A}$ *denote the pullback to* $Pic\,\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ *of the class in* $Pic\,\overline{\mathcal{M}}_{g,n}$ *associated to the union of all strata in* $\overline{\mathcal{M}}_{g,n}$ *with the dual graph*



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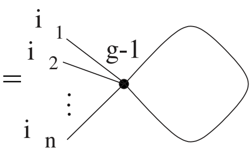
with $A = \{i_1, \ldots, i_j\}$. *The pullback* $\delta_{k;A}$ *is related to* $\alpha_{k;A}$ *as follows:*

$$\delta_{k;A} = \frac{r}{\gcd(m^+ + 1, r)}\alpha_{k;A},$$

where m^+ *is determined by* k, $\mathbf{m}$, *and* A, *as in Section 1.7.1.*
- *Let* δ_{irr} *be the pullback to* $Pic\,\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ *of the divisor of all curves in* $\overline{\mathcal{M}}_{g,n}$ *with dual graph*

$\Gamma_{\mathrm{irr}} =$



.

⋆Beware that the index m of $\tilde{\delta}_{\mathrm{irr},m}$ differs from the index j of γ in [19] by one.

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The pullback δ_{irr} can be expressed in terms of the $\tilde{\delta}_{\mathrm{irr},m}$ as follows:

$$\delta_{\mathrm{irr}} = \sum_{r/2-1 \leqslant m<r} \frac{r}{\gcd(r, m+1)} \tilde{\delta}_{\mathrm{irr},m}.$$

The fact that, for an rth root $(\mathcal{E}, b)$ of $\omega(-\sum m_i p_i)$, the map b is almost an isomorphism means that $\tilde{\psi}_i$ and ψ_i are closely related.

PROPOSITION 2.2. *The line bundles $\sigma_i^*\omega$ and $\sigma_i^*(\mathcal{E}_d)$ on the stack $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ are related by*

$$r\sigma_i^*(\mathcal{E}_r) \cong (m_i+1)\sigma_i^*(\omega) \quad \textit{and} \quad d\sigma_i^*(\mathcal{E}_d) \cong (m_i'+1)\sigma_i^*(\omega),$$

where m_i' is the smallest non-negative integer congruent to m_i (mod d). Therefore, in $\mathrm{Pic}\, \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \otimes \mathbb{Q}$ we have

$$\tilde{\psi}_i = \frac{m_i+1}{r}\psi_i \quad \textit{and} \quad \tilde{\psi}_i^{(d)} = \left(\frac{m_i'+1}{m_i+1}\right)\frac{r}{d}\tilde{\psi}_i.$$

Before proving the proposition, we recall the following well-known fact.

LEMMA 2.3. *Let $\pi\colon \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the universal n-pointed curve, and σ_i the ith tautological section of π. If D_i is the divisor of $\mathcal{C}_{g,n}$ associated to σ_i, and if $\omega = \omega_\pi$ is the canonical (relative dualizing) sheaf, then*

$$\sigma_i^*(\mathcal{O}(-D_j)) \cong \begin{cases} \sigma_i^*(\omega|_{D_i}) = \psi_i, & \text{if } i=j,\\ \mathcal{O}, & \text{if } i \neq j. \end{cases}$$

Proof. When $i \neq j$ the bundle $\sigma_i^*(\mathcal{O}(-D_j))$ is trivial because the sections are disjoint. In the case $i=j$ the result follows from the fact that taking residues gives an isomorphism between $\omega_\pi(D_i)|_{D_i}$ and $\mathcal{O}_{D_i}$. □

Proof of Proposition 2.2. The map $c_{d,1}\colon \mathcal{E}_d^{\otimes d} \longrightarrow \omega(-\sum m_j' p_j)$ pulls back, via σ_i^*, to give $\sigma_i^* c_{d,1}\colon (\sigma_i^*\mathcal{E})^{\otimes r} \longrightarrow \sigma_i^*\omega\left(-\sum m_j' p_j\right)$. Since $\mathrm{im}(\sigma_i)$ is disjoint from the nodes of X, $\sigma_i^* c_{d,1}$ is an isomorphism, even on the boundary strata where $\mathcal{E}$ fails to be locally free.

Consequently, we have

$$\sigma_i^*(c_{d,1})\colon \sigma_i^*(\mathcal{E}_d^{\otimes d}) \xrightarrow{\sim} \sigma_i^*\left(\omega \otimes \mathcal{O}\left(-\sum_j m_j' D_j\right)\right) = \psi_i + m_i'\psi_i = (m_i'+1)\psi_i.$$

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2.3. NON-TRIVIAL RELATIONS INVOLVING THE CLASS μ_1.

PROPOSITION 2.4. *Define the boundary divisors* ε *and* δ *by*

$$\varepsilon = \sum_{k,A} \frac{(m^+ + 1)(m^- + 1)}{u_{k;A}} \alpha_{k;A} + \sum_{\frac{r}{2}-1 \leqslant m < r-1} \frac{(m+1)(r-m-1)}{v_m} \tilde{\delta}_{\mathrm{irr},m}$$

and $\delta = \sum \delta_{k;A} = \sum_{\frac{r}{2}-1 \leqslant m < r} \frac{r}{v_m} \tilde{\delta}_{irr,m} + \sum_{k,A} \frac{r}{u_{k;A}} \alpha_{k;A}$, *where* $v_m = \gcd(m+1, r)$, $u_{k;A} = \gcd(m^+ + 1, r)$, *and* m^+ *is determined by* k, A, *and* $\mathbf{m}$ *via relation (4).*
Then the following relation holds in $\mathrm{Pic}\,\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$:

$$r\varepsilon = (2r^2 - 12r + 12)\lambda_1 - 2r^2\mu_1 + (r-1)\delta + \sum_{1 \leqslant i \leqslant n} m_i(r - 2 - m_i)\psi_i.$$

The proof of the Proposition is almost identical to its counterpart in [19, Theorem 4.3.3] except that $\mathcal{E}_r$ is not an rth root of ω, but rather of $\omega(-\sum m_i p_i)$. The only extra information necessary to prove the proposition is the content of the following two lemmas and the fact that the divisor we have called ε is the product $< \mathcal{E}_r, \tilde{\mathfrak{E}} >$ in [19].

LEMMA 2.5. *Let* $\pi: \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ *be the universal curve and*

$$\langle , \rangle : \mathit{Pic}\,\mathcal{C}_{g,n} \times \mathit{Pic}\,\mathcal{C}_{g,n} \longrightarrow \mathit{Pic}\,\overline{\mathcal{M}}_{g,n}$$

be Deligne's bilinear product defined by

$$\langle \mathcal{L}, \mathcal{M} \rangle := \det(\pi_!(\mathcal{L} \otimes \mathcal{M}) - \pi_!\mathcal{L} - \pi_!\mathcal{M} + \pi_!\mathcal{O}).$$

If D *is the image of a section* $\sigma: \overline{\mathcal{M}}_{g,n} \longrightarrow \mathcal{C}_{g,n}$, *then for any line bundle* $\mathcal{L}$ *on* $\mathcal{C}_{g,n}$ *the product* $\langle \mathcal{L}, \mathcal{O}_{\mathcal{C}}(D) \rangle$ *is equal to the restriction of* $\mathcal{L}$ *to* D; *that is, it is just* $\sigma^*\mathcal{L}$.

This is proved in [4, Prop. 6.1.3], but it also follows from the fact that when the base is a smooth curve B, then $\deg \langle \mathcal{L}, \mathcal{M} \rangle$ is exactly the usual intersection number $(\mathcal{L}.\mathcal{M})$. Since line bundles on $\overline{\mathcal{M}}_{g,n}$ are completely determined by their degree on smooth curves in $\overline{\mathcal{M}}_{g,n}$ [2], and since in the case of a smooth, one-parameter base the degrees agree, the lemma is true in general.

LEMMA 2.6. *We have*

$$\langle \mathcal{O}(D_i), \mathcal{O}(D_j) \rangle = \begin{cases} -\sigma_i^*\omega_\pi = -\psi_i, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

This follows immediately from Lemmas 2.3 and 2.5.

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COROLLARY 2.7. *If $g = 0$ then classes λ_1 and $\tilde{\delta}_{irr,m}$ vanish (for all m), and*

$$\begin{aligned} 2r^2\mu_1 &= (r-1)\delta + \sum_i m_i(r-2-m_i)\psi_i - r\varepsilon \\ &= \sum_{A\subseteq[n]} \frac{r}{u_{0;A}}[r-1-(m^+ + 1)(m^- + 1)]\alpha_{0;A} + \\ &\quad + \sum_{1\leqslant i\leqslant n} m_i(r-2-m_i)\psi_i. \end{aligned}$$

3. Cohomological Field Theory

In this section we begin with a review of the notion of a cohomological field theory (CohFT) in the sense of Kontsevich and Manin [24]. This is an object which formalizes the expected factorization properties of the theory of topological gravity coupled to topological matter. The Gromov–Witten invariants associated to a smooth, projective variety V correspond to the physical situation where the matter sector arises from the topological sigma model [24, 32]. The analogous intersection numbers associated to the moduli space of r-spin curves have their physical origins in a different choice of the matter sector. Our goal in this section is to give a precise formulation of these notions in terms of the moduli spaces described above.

3.1. AXIOMS OF CohFT

DEFINITION 3.1. A (*complete*) *cohomological field theory* (*CohFT*) *of rank d* (denoted by $(\mathcal{H}, \eta, \Lambda)$ or just $(\mathcal{H}, \eta)$) is a d-dimensional vector space $\mathcal{H}$ with a metric η and a collection $\Lambda := \{\Lambda_{g,n}\}$ of n-linear $H^\bullet(\overline{\mathcal{M}}_{g,n})$-valued forms on $\mathcal{H}$

$$\Lambda_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{H}^{*\otimes n} = \mathrm{Hom}(\mathcal{H}^{\otimes n}, H^\bullet(\overline{\mathcal{M}}_{g,n})) \tag{11}$$

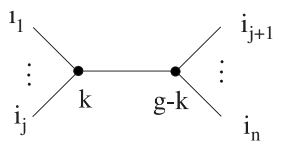
defined for stable pairs (g, n) and satisfying the following axioms **C1–C3** (where $\{e_0, \ldots, e_{d-1}\}$ is a fixed basis of $\mathcal{H}$, $\eta^{\mu\nu}$ is the inverse of the matrix of the metric η in this basis, and the summation convention has been used).

C1. The element $\Lambda_{g,n}$ is invariant under the action of the symmetric group S_n.

C2. Let

$$\rho_{\text{tree}}: \overline{\mathcal{M}}_{\Gamma_1\sqcup\Gamma_2} = \overline{\mathcal{M}}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2} \longrightarrow \overline{\mathcal{M}}_{\Gamma_{\text{tree}}} \hookrightarrow \overline{\mathcal{M}}_{g,n} \tag{12}$$

be the gluing morphism (3) corresponding to the stable graph



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and the two graphs Γ_1 and Γ_2 obtained by cutting the edge of Γ_{tree}. Then the forms $\Lambda_{g,n}$ satisfy the composition property

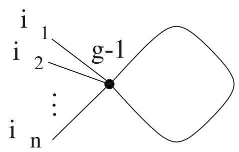
$$\begin{aligned}&\rho_{\text{tree}}^{*}\Lambda_{g,n}(\gamma_1, \gamma_2 \ldots, \gamma_n)\\ &\quad = \Lambda_{k,j}(\gamma_{i_1}, \ldots, \gamma_{i_j}, e_\mu)\eta^{\mu\nu} \otimes \Lambda_{g-k,n-j}(e_\nu, \gamma_{i_{j+1}}, \ldots, \gamma_{i_n})\end{aligned} \tag{13}$$

for all $\gamma_i \in \mathcal{H}$.

C3. Let

$$\rho_{\text{loop}}\colon \overline{\mathcal{M}}_{\tilde{\Gamma}} = \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{\Gamma_{\text{loop}}} \hookrightarrow \overline{\mathcal{M}}_{g,n} \tag{14}$$

be the gluing morphism (3) corresponding to the stable graph



(15)

and the graph $\tilde{\Gamma}$ obtained by cutting the loop of Γ_{loop}. Then

$$\rho_{\text{loop}}^{*}\,\Lambda_{g,n}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \Lambda_{g-1,n+2}\,(\gamma_1, \gamma_2, \ldots, \gamma_n, e_\mu, e_\nu)\,\eta^{\mu\nu}. \tag{16}$$

The pair $(\mathcal{H}, \eta)$ is called the state space of the CohFT.

An element $e_0 \in \mathcal{H}$ is called a flat identity of the CohFT if, in addition, the following equations hold.

C4a. For all γ_i in $\mathcal{H}$ we have

$$\Lambda_{g,n+1}(\gamma_1, \ldots, \gamma_n, e_0) = \pi^{*}\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n), \tag{17}$$

where $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal curve on $\overline{\mathcal{M}}_{g,n}$ and

C4b.

$$\int_{\overline{\mathcal{M}}_{0,3}} \Lambda_{0,3}(\gamma_1, \gamma_2, e_0) = \eta(\gamma_1, \gamma_2). \tag{18}$$

A CohFT with flat identity is denoted by $(\mathcal{H}, \eta, \Lambda, e_0)$. A *genus* $\tilde{g}$ *CohFT* on the state space $(\mathcal{H}, \eta)$ is the collection of forms $\{\Lambda_{g,n}\}_{g \leqslant \tilde{g}}$ that satisfy only those of the Equations (13), (16), (17), and (18), where $g \leqslant \tilde{g}$.

Remarks 3.2. (1) In general, the state space $\mathcal{H}$ of CohFT is $\mathbb{Z}_2$-graded, but here, for simplicity, we are assuming that $\mathcal{H}$ contains only even elements, since this is the only case that will arise in this paper.

(2) The definition of a CohFT given above has an equivalent dual description in terms of homology. Consider the maps $H_\bullet(\overline{\mathcal{M}}_{g,n}) \to T^n\mathcal{H}^*$ given by $[c] \longmapsto \int_{[c]} \Lambda_{g,n}$. These maps are called the (n-point) correlators of the CohFT. A structure of a (complete) CohFT on $(\mathcal{H}, \eta)$ is equivalent to the requirement that these

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correlators endow $(\mathcal{H}, \eta)$ with the structure of an algebra over the modular operad $\{H_\bullet(\overline{\mathcal{M}}_{g,n})\}$ in the sense of Getzler and Kapranov [14].

(3) Clearly, the definition of a cohomological field theory extends from $\mathbb{C}$ to more general ground rings $\mathcal{K}$.

Let Γ be a stable graph, then there is a canonical composition map ρ_Γ

$$\rho_\Gamma\colon \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \tag{19}$$

where $V(\Gamma)$ denotes the set of vertices of Γ. Since the map ρ_Γ can be constructed from gluing morphisms (12) and (14), the forms $\Lambda_{g,n}$ satisfy a restriction property

$$\rho_\Gamma^*\Lambda_{g,n} = \rho_\Gamma^{-1}\Big(\bigotimes_{v\in V(\Gamma)} \Lambda_{g(v),n(v)} \Big) \tag{20}$$

where $\rho_\Gamma^{-1}: \bigotimes_{v\in V(\Gamma)} T^{n(v)}\mathcal{H}^* \to T^n\mathcal{H}^*$ contracts the factors $T^n\mathcal{H}^*$ by means of the inverse of the metric η and successive application of Equations (13) and (16). There is a parameter which can be introduced into the definition of a CohFT. This parameter can be regarded as a coupling constant in the theory.

LEMMA 3.3. *Let $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ be a CohFT with flat identity e_0 and let λ be a nonzero parameter. If we define $\Lambda = \{\Lambda_{g,n}\}$, where $\Lambda_{g,n} := \lambda^{2g-2}\widetilde{\Lambda}_{g,n}$ and $\eta := \lambda^{-2}\widetilde{\eta}$, then $(\mathcal{H}, \eta, \Lambda, e_0)$ is a CohFT with flat identity.*

The proof is obvious.

DEFINITION 3.4. The *small phase space potential function* of the CohFT $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda})$ is a formal series $\Phi \in \mathbb{C}[[\mathcal{H}]]$ given by

$$\Phi(\mathbf{x}) := \sum_{g=0}^{\infty} \Phi_g(\mathbf{x}), \tag{21}$$

where $\Phi_g(\mathbf{x}) := \sum_n \frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}} \left\langle \widetilde{\Lambda}_{g,n}\,,\, \mathbf{x}^{\otimes n} \right\rangle$. Here $\langle \cdots \rangle$ denotes evaluation, the sum over n is understood to be over the stable range, and $\mathbf{x} = \sum_\alpha x^\alpha e_\alpha$, where $\{e_\alpha\}$ is a basis of $\mathcal{H}$.

Remark 3.5. The small phase space potential function of the CohFT $(\mathcal{H}, \eta, \Lambda, e_0)$ associated to $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ as in Lemma 3.3 may be regarded as an element in $\lambda^{-2}\mathbb{C}[[\mathcal{H}, \lambda^2]]$.

All of the information of a genus zero CohFT is encoded in this potential.

THEOREM 3.6 ([24, 27]). *An element Φ_0 in $\mathbb{C}[[\mathcal{H}]]$ is the potential of a rank d, genus zero CohFT $(\mathcal{H}, \eta)$ if and only if it contains only terms which are of cubic and higher order in the coordinates $x^0, \ldots, x^{d-1}$ (corresponding to a basis $\{e_0, \ldots e_{d-1}\}$ of $\mathcal{H}$) and it satisfies the* associativity, or WDVV (*Witten–Dijkgraaf–Verlinde*[2])

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equation $\partial_a\partial_b\partial_e\Phi_0\,\eta^{ef}\,\partial_f\partial_c\partial_d\Phi_0 = \partial_b\partial_c\partial_e\Phi_0\,\eta^{ef}\,\partial_f\partial_a\partial_d\Phi_0$, *where* η^{ab} *is the inverse matrix of the matrix of* η *in the basis* $\{e_a\}$, ∂_a *is derivative with respect to* x^a, *and the summation convention has been used.*

Conversely, a genus zero CohFT structure on $(\mathcal{H}, \eta)$ *is uniquely determined by its potential* Φ_0, *which must satisfy the WDVV equation.*

A genus zero CohFT with flat identity is essentially equivalent to endowing the state space $(\mathcal{H}, \eta)$ with the structure of a formal Frobenius manifold [6, 16, 27]. The theorem follows from the work of Keel [22], who proved that $H^\bullet(\overline{\mathcal{M}}_{0,n})$ is generated by boundary classes and that all relations between boundary divisors arise from lifting the basic codimension one relation on $\overline{\mathcal{M}}_{0,4}$.

3.2. GROMOV–WITTEN INVARIANTS AND THEIR POTENTIALS

Our construction of CohFTs from the moduli space of stable r-spin curves is guided by analogy with the moduli space of stable maps and Gromov–Witten invariants. Let us briefly review this construction. Let V be a smooth projective variety, $\mathcal{H} = H^\bullet(V, \mathbb{C})$, and η the Poincaré pairing. Let $\overline{\mathcal{M}}_{g,n}(V)$ be the moduli stack of stable maps into V of genus g with n marked points. The *Gromov–Witten invariants of* V are multilinear maps $\mathcal{H}^{\otimes n} \to \mathbb{C}$ given by

$$\langle\, \tau_0(\gamma_1) \cdots \tau_0(\gamma_n) \,\rangle_g = \lambda^{2g-2} \int_{[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}} \mathrm{ev}_1^*\,\gamma_1 \cup \cdots \cup \mathrm{ev}_n^*\,\gamma_n, \tag{22}$$

where $[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli stack $\overline{\mathcal{M}}_{g,n}(V)$ and λ is a formal parameter. The corresponding *small phase space potential* $\Phi(\mathbf{x})$ is defined by (21) where the genus g part is given by

$$\Phi_g(\mathbf{x}) = \sum_n \lambda^{2g-2} \int_{[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}} \mathrm{ev}_1^*\,\mathbf{x} \cup \cdots \cup \mathrm{ev}_n^*\,\mathbf{x},$$

$\mathbf{x} = \sum_a x^a e_a$, and $\{ e_0, \ldots, e_r \}$ is a basis for $\mathcal{H}$ such that e_0 is the identity element. If V is a convex variety, then $\overline{\mathcal{M}}_{0,n}(V)$ is a smooth stack and its virtual fundamental class coincides with its topological fundamental class. In this situation, [11] shows that $(\mathcal{H}, \eta)$ forms a genus zero CohFT with potential Φ_0. This result can be generalized to higher genera and to more general varieties, as well.

Remark 3.7. In the usual definition of Gromov–Witten invariants, there is no factor of λ^{2g-2} in the definition of the correlators but this factor is inserted into the potential function by hand. We have chosen our conventions so that this factor appears instead in the correlator but is not explicitly inserted into the potential function.

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The *gravitational descendants* are defined by twisting the Gromov–Witten classes with the tautological ψ classes as follows:

$$\langle \tau_{a_1}(\gamma_1)\cdots\tau_{a_n}(\gamma_n)\rangle_g := \lambda^{2g-2}\int_{[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{vir}}} \mathrm{ev}_1^*\gamma_1 \cup \psi_1^{a_1} \cup \cdots \cup \mathrm{ev}_n^*\gamma_n \cup \psi_n^{a_n} \tag{23}$$

for all $a_1, \ldots, a_n = 0, 1, 2, \ldots$ and $\gamma_1, \ldots, \gamma_n$ in $H^\bullet(V)$. This gives rise to the *large phase space potential* $\Phi(\mathbf{t}) \in \lambda^{-2}\mathbb{C}[[\mathbf{t}, \lambda^2]]$ where $\mathbf{t} = (\mathbf{t}_0, \mathbf{t}_1, \ldots)$ and $\mathbf{t}_n = (t_n^0, \ldots, t_n^r)$, which is defined by

$$\Phi(\mathbf{t}) := \sum_g \Phi_g(\mathbf{t}) \tag{24}$$

and

$$\Phi_g(\mathbf{t}) := \sum \left\langle \tau_{a_1}(e_{\alpha_1}) \cdots \tau_{a_n}(e_{\alpha_n}) \right\rangle_g t_{a_1}^{\alpha_1} \cdots t_{a_n}^{\alpha_n} \frac{1}{n!}. \tag{31}$$

Setting $t_n^\alpha = 0$ for $n \geqslant 1$ and $x^\alpha = t_0^\alpha$ reduces the large phase space potential $\Phi(\mathbf{t})$ to the small phase space potential $\Phi(\mathbf{x})$.

When V is a point, Kontsevich's theorem gives that $Z(\mathbf{t}) := \exp(\Phi(\mathbf{t}))$ is a τ-function of the KdV hierarchy. In addition, Kontsevich showed that $Z(\mathbf{t})$ is a highest weight vector for the Virasoro Lie algebra, a condition which allows one to completely solve for these intersection numbers. The existence of a similar Virasoro algebra action has been conjectured by Eguchi, Hori, and Xiong [8] in the case where V is not a point. Evidence for this conjecture is mounting [10, 15]. A *very large phase space* has recently been introduced in [21, 29] for the case where V is a point and for more general varieties in [9] by including variables corresponding to the Hodge classes v_i as well. These additional variables parametrize an even larger family of CohFTs than just the large phase space coordinates [21]. We will shortly introduce, in addition, variables associated to the r-spin structure (see (52)).

3.3. r-SPIN CohFT

We perform an analogous construction of a very large phase space where the role of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(V)$ is played by the moduli space of stable r-spin curves $\overline{\mathcal{M}}_{g,n}^{1/r}$. Unlike the moduli space of general stable maps, the moduli space of stable r-spin curves is a smooth stack. Intersection theory is therefore simpler in this case than for the case of stable maps. However, the difficulty lies instead in the construction of the analogs of the Gromov–Witten classes.

In the next section, we will introduce axioms which a collection of cohomology classes (called a virtual class) $c_{g,n}^{1/r}(\mathbf{m})$ in $H^\bullet(\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}})$ must satisfy in order to insure that the following result holds.

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THEOREM 3.8. *Let* $(\mathcal{H}^{(r)}, \widetilde{\eta})$ *be a vector space of dimension* $r-1$ *with a basis* $\{e_0, \ldots, e_{r-2}\}$ *and metric* $\widetilde{\eta}$ *given by*

$$\widetilde{\eta}(e_\mu, e_\nu) := \widetilde{\eta}_{\mu\nu} := \frac{1}{r}\,\delta_{\mu+\nu,r-2}. \tag{26}$$

Let $c^{1/r}_{g,n}(\mathbf{m})$ *be a virtual class in* $H^\bullet(\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n})$ *satisfying Axioms 1 through 5 from the next section, and let* $p : \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ *be the map which forgets the r-spin structure. Let* $\widetilde{\Lambda} := \{\widetilde{\Lambda}_{g,n}\}$ *be defined by*

$$\widetilde{\Lambda}^{(\mathbf{s},\mathbf{u})}_{g,n}(e_{m_1}, \ldots, e_{m_n}) := p_*\left(c^{1/r}_{g,n}(\mathbf{m})\,\exp(\mathbf{s}\cdot\boldsymbol{\mu} + \mathbf{u}\cdot\boldsymbol{\nu})\right), \tag{27}$$

where these forms have values in the ring $\mathbb{C}[[\mathbf{s}, \mathbf{u}]]$, *then* $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}^{(\mathbf{s},\mathbf{u})}, e_0)$ *is a CohFT satisfying Axiom C4a. Furthermore, if*

$$\widetilde{\Lambda}_{g,n} := \widetilde{\Lambda}^{(\mathbf{0},\mathbf{0})}_{g,n}, \tag{28}$$

then $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ *is a CohFT with flat identity. The latter will be called* the r-spin CohFT. *Restricting the* r*-spin CohFT to genus zero shows that* $(\mathcal{H}^{(r)}, \eta)$ *is endowed with the structure of a Frobenius manifold.*

This theorem is proved in Section 4.2.

COROLLARY 3.9. *Let* $(\mathcal{H}, \eta, \Lambda, e_0)$ *be constructed from* $(\mathcal{H}, \widetilde{\eta}, \widetilde{\Lambda}, e_0)$ *above by setting*

$$\eta(e_\mu, e_\nu) := \eta_{\mu\nu} := \frac{1}{r\lambda^2}\,\delta_{\mu+\nu,r-2}, \tag{29}$$

$$\Lambda^{(\mathbf{s},\mathbf{u})}_{g,n}(e_{m_1}, \ldots, e_{m_n}) := \lambda^{2g-2}\,\widetilde{\Lambda}^{(\mathbf{s},\mathbf{u})}_{g,n}(e_{m_1}, \ldots, e_{m_n}), \tag{30}$$

and

$$\Lambda_{g,n} := \Lambda^{(\mathbf{0},\mathbf{0})}_{g,n}, \tag{31}$$

then $(\mathcal{H}, \eta, \Lambda^{(\mathbf{s},\mathbf{u})}, e_0)$ *is a CohFT satisfying Axiom C4a and* $(\mathcal{H}, \eta, \Lambda, e_0)$ *is a CohFT with flat identity.*

Proof. This is a direct consequence of Lemma 3.3 and the previous theorem. □

The classes $c^{1/r}_{g,n}(\mathbf{m})$ are analogs of the Gromov–Witten classes in this theory. The analogs of the gravitational descendants (23) are given by

$$\langle \tau_{a_1}(e_{m_1}) \cdots \tau_{a_n}(e_{m_n}) \rangle_g := \lambda^{2g-2} \int_{\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}\, c^{1/r}_{g,n}(\mathbf{m}), \tag{32}$$

and the large phase space potential function is defined by Equations (24) and (25). The small phase space potential function is defined by restricting (32) to correlators with $a_i = 0$. *We will see that the case of* $\lambda = 1/\sqrt{r}$ *corresponds to the generalized*

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Witten conjecture. This corresponds to the metric

$$\eta(e_{m_1}, e_{m_2}) = \delta_{m_1+m_2, r-2} \tag{33}$$

and the forms

$$\Lambda_{g,n}(e_{m_1}, \ldots, e_{m_n}) := r^{1-g} p_* c^{1/r}_{g,n}(\mathbf{m}). \tag{34}$$

Remark 3.10. Strictly speaking, the state space of this r-spin CohFT (34) should be, instead, $(\hat{\mathcal{H}}^{(r)}, \hat{\eta})$, where $\hat{\mathcal{H}}^{(r)}$ is an r-dimensional vector space, with basis $\{e_0, \ldots, e_{r-1}\}$ and a metric given by

$$\hat{\eta}(e_a, e_b) = \begin{cases} 1 & \text{if } a+b \equiv (r-2) \mod r \\ 0 & \text{otherwise.} \end{cases}$$

However, it follows from the axioms for $c^{1/r}$ in Section 4.1 that the obvious orthogonal decomposition $\hat{\mathcal{H}}^{(r)} = \mathcal{H}^{(r)} \oplus \mathcal{H}'$, where $\mathcal{H}'$ is the trivial one-dimensional CohFT with basis $\{e_{r-1}\}$, is a direct sum of CohFTs. For this reason, we can (and will) restrict ourselves to the state space $(\mathcal{H}^{(r)}, \eta)$.

4. Virtual Classes

To endow the pair $(\mathcal{H}^{(r)}, \eta)$ from Theorem 3.8 with the structure of a CohFT by Equation (34), we must define cohomology classes $c^{1/r}_{g,n}(\mathbf{m})$. We will call this collection of classes an *r-spin virtual class.* It should satisfy the axioms described below. *Throughout this section, we will restrict ourselves to the case where the coupling constant λ is $1/\sqrt{r}$, unless otherwise stated.* This is done purely for convenience as analogous results hold for general λ as well.

4.1. AXIOMS FOR THE VIRTUAL CLASS

DEFINITION 4.1. An r-spin virtual class is an assignment of a cohomology class

$$c^{1/r}_{\Gamma} \in H^{2D}(\overline{\mathcal{M}}^{1/r}_{\Gamma}, \mathbb{Q}) \tag{35}$$

to every genus g, stable, decorated graph Γ with n-tails. Here, if the tails of Γ are marked with the n-tuple $\mathbf{m} = (m_1, \ldots, m_n)$, then the dimension D is

$$D = \frac{1}{r}\left((r-2)(g-\alpha) + \sum_{i=1}^{n} m_i\right), \tag{36}$$

and α is the number of connected components of Γ. In the special case where Γ has one vertex and no edges, we denote $c^{1/r}_{\Gamma}$ by $c^{1/r}_{g,n}(\mathbf{m})$. These classes must satisfy the axioms below.

Axiom 1a (Connected Graphs): Let Γ be a connected, genus g, stable, decorated graph with n tails. Let $E(\Gamma)$ denote the set of edges of Γ. For each edge e of Γ, let

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$l_e := \gcd(m_e^+ + 1, r)$, where m_e^+ is an integer decorating a half-edge of e. The classes $c_\Gamma^{1/r}$ and $c_{g,n}^{1/r}(\mathbf{m})$ are related by

$$c_\Gamma^{1/r} = \left(\prod_{e\in E(\Gamma)} \frac{r}{l_e}\right)\tilde{i}^*\, c_{g,n}^{1/r}(\mathbf{m}) \in H^{2D}(\overline{\mathcal{M}}_\Gamma^{1/r}), \tag{37}$$

where $\tilde{i}\colon \overline{\mathcal{M}}_\Gamma^{1/r} \hookrightarrow \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is the canonical inclusion map.

Axiom 1b (Disconnected Graphs): Let Γ be a stable, decorated graph which is the disjoint union of connected graphs $\Gamma^{(d)}$, then the classes $c_\Gamma^{1/r}$ and $c_{\Gamma^{(d)}}^{1/r}$ are related by

$$c_\Gamma^{1/r} = \bigotimes_d c_{\Gamma^{(d)}}^{1/r} \in H^\bullet(\overline{\mathcal{M}}_\Gamma^{1/r}).$$

Axiom 2 (Convexity): Consider the universal r-spin structure $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ on the universal curve $\pi\colon \mathcal{C}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. For each irreducible (and connected) component of $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ (denoted here by $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},(d)}$ for some index d), if $\pi_*\mathcal{E}_r = 0$ on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},(d)}$, then $c_{g,n}^{1/r}(\mathbf{m})$ restricted to $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m},(d)}$ is $c_D(-R^1\pi_*\mathcal{E}_r)$, the top Chern class of the bundle with fiber $H^1(X,\mathcal{E}_r)^*$ at $[(X, p_1, \ldots, p_n, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))] \in \overline{\mathcal{M}}_{g,n}^{1/r}$.

Axiom 3 (Cutting edges): Given any genus g decorated stable graph Γ with n tails marked with $\mathbf{m}$, we have a diagram

$$\begin{tikzcd}
 & \overline{\mathcal{M}}_{\tilde\Gamma} \times_{\overline{\mathcal{M}}_\Gamma} \overline{\mathcal{M}}_\Gamma^{1/r} \arrow[r, "\tilde\mu"] \arrow[dl, "p_1"'] \arrow[dd, "p"] & \overline{\mathcal{M}}_\Gamma^{1/r} \arrow[r, hook, "\tilde{i}"] \arrow[dd, "p"] & \overline{\mathcal{M}}_{g,n}^{1/r} \arrow[dd, "p"] \\
\overline{\mathcal{M}}_{\tilde\Gamma}^{1/r} \arrow[dr, "p_2"'] & & & \\
 & \overline{\mathcal{M}}_{\tilde\Gamma} \arrow[r, "\mu"] & \overline{\mathcal{M}}_\Gamma \arrow[r, hook, "i"] & \overline{\mathcal{M}}_{g,n}.
\end{tikzcd} \tag{38}$$

where $\overline{\mathcal{M}}_{\tilde\Gamma}$ is the stack of stable curves with graph $\tilde\Gamma$, the graph obtained by cutting all edges of Γ, and $\overline{\mathcal{M}}_{\tilde\Gamma}^{1/r}$ is the stack of stable r-spin curves with graph $\tilde\Gamma$ (still marked with $m^\pm$ on each half edge). p_1 is the following morphism: The fiber product consists of triples of an r-spin curve $(X/T, \{\mathcal{E}_d, c_{d,d'}\})$, a stable curve $\tilde{X}/T$, and a morphism $\nu\colon \tilde{X} \longrightarrow X$, making $\tilde{X}$ into the normalization of X. Also, the dual graphs of X and $\tilde{X}$ are Γ and $\tilde\Gamma$, respectively. The associated r-spin curve in $\overline{\mathcal{M}}_{\tilde\Gamma}^{1/r}$ is simply $(\tilde{X}/T, \nu^*\{\mathcal{E}_d, c_{d,d'}\})$. We require that $p_{1*}\tilde\mu^* c_\Gamma^{1/r} = r^{|E(\Gamma)|} c_{\tilde\Gamma}^{1/r}$, where $E(\Gamma)$ is the set of edges of Γ that are cut in $\tilde\Gamma$.

Axiom 4 (Vanishing): If Γ contains a tail marked with $m_i = r-1$, then $c_\Gamma^{1/r} = 0$.

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Axiom 5 (Forgetting tails): Let $\widehat{\Gamma}$ be a stable graph whose ith tail is marked by $m_i = 0$, Γ be the stable graph obtained by removing the ith tail, and $\pi\colon \overline{\mathcal{M}}^{1/r}_{\hat{\Gamma}} \longrightarrow \overline{\mathcal{M}}^{1/r}_{\Gamma}$ be the forgetful morphism. The classes $c^{1/r}_{\hat{\Gamma}}$ and $\pi^* c^{1/r}_{\Gamma}$ are related by $c^{1/r}_{\hat{\Gamma}} = \pi^* c^{1/r}_{\Gamma}$.

Remarks 4.2.

(1) The factor of $\prod_e (r/l_e)$ in Axiom 1 arises from the fact that the right hand square in Equation (38) is not quite Cartesian. Rather, because of ramification of p over $\overline{\mathcal{M}}_\Gamma$, we have that for any cohomology class c on $\overline{\mathcal{M}}^{1/r}_{g,n}$,

$$i^* p_* c = \left(\prod_e \frac{r}{l_e}\right) p_* \tilde{i}^* c. \tag{39}$$

(2) Notice that in Axiom 3, unlike the case of a tree, if Γ contains a loop and if the r-spin structure is Ramond at the corresponding node, the dimensions D_Γ and $D_{\tilde{\Gamma}}$ of the virtual classes $c^{1/r}_\Gamma$ and $c^{1/r}_{\tilde{\Gamma}}$ are different. Thus Axiom 3 actually requires the vanishing of both $p_{1*}\tilde{\mu}^* c^{1/r}_\Gamma$ and $c^{1/r}_{\tilde{\Gamma}}$ in this case. Of course, for the Ramond case, Axiom 4 already requires the vanishing of $c^{1/r}_{\tilde{\Gamma}}$, since the cut half-edges are both marked with $r-1$. Thus for any graph (tree or otherwise), in the Ramond case Axiom 3 amounts essentially to the requirement that $p_{1*}\tilde{\mu}^* c^{1/r}_\Gamma$ vanish.

(3) Although the vanishing of H^0 or π_* is often called concavity, the Serre dual $\mathcal{H}om(\mathcal{E}_r, \omega)$, corresponding to Witten's sheaf $\mathcal{V}$ in [34], is convex (H^1 vanishes) exactly when $\mathcal{E}_r$ is concave. Moreover, $\pi_! \mathcal{V} = \pi_! \mathcal{E}_r$. Therefore, we use the term convex to describe the case when $\pi_* \mathcal{E}_r = 0$.

(4) One might think that the class $c_D(\pi_! \mathcal{E}_r)$ would be a good candidate for a virtual class, since it coincides with $c^{1/r}$ in the convex case. However, this is not the case (see Section 4.4).

(5) Witten has described [34, Section 1.3] an analytic construction of a class that he calls the 'top Chern class,' but it is not clear that this class satisfies the above axioms. Witten's index-like construction is reminiscent of the analytic construction of a virtual fundamental class of the moduli space of stable maps in the theory of Gromov–Witten invariants. Ideally, one should be able to construct $c^{1/r}$ by methods similar to those used in algebraic constructions of the fundamental class.

(6) Although, as explained in Remark 1.9, the restriction $0 \leqslant m_i < r$ does not change the moduli space $\overline{\mathcal{M}}^{1/r}_{g,n}$, it does give a different choice of $c^{1/r}$. Indeed, replacing m_i by $m_i + r$ changes the dimension of $c^{1/r}$ by 1 and corresponds (up to a multiplicative constant) to the first descendant of the classes associated to m_i. Thus on a given moduli space $\overline{\mathcal{M}}^{1/r}_{g,n}$ there are potentially several (but still only finitely many, for dimensional reasons) choices of $c^{1/r}$ and the corresponding *CohFT*. However, without the restriction $0 \leqslant m_i < r$, the corresponding metric η is not necessarily invertible, and several other unusual considerations also arise.

These issues will be treated in a forthcoming paper [20]. In the remainder of this paper, we will assume that $0 \leqslant m_i < r$ except where explicitly stated.

4.2. VERIFICATION OF THE CohFT AXIOMS FOR $(\mathcal{H}^{(r)}, \eta)$

In this section we give a proof of Theorem 3.8, first for the case where $\mathbf{s} = \mathbf{u} = \mathbf{0}$, and then in general.

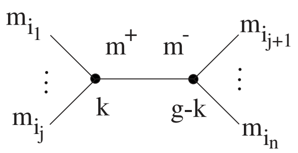
4.2.1. *The Case* $\mathbf{s} = \mathbf{u} = \mathbf{0}$

Let $c^{1/r}_{g,n}$ be a cohomology class on $\overline{\mathcal{M}}^{1/r}_{g,n}$ satisfying the axioms of Section 4.1. We will show that the collection of classes $\{\Lambda_{g,n}\}$ given by (40) satisfies the CohFT axioms C1–C4 with state space $(\mathcal{H}^{(r)}, \eta)$. Axiom C1 clearly holds by the definition of $\{\Lambda_{g,n}\}$.

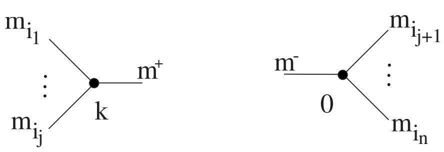
Let $\rho = \rho_{tree}$ be the gluing morphism (12). Condition (13) of Axiom C2 is equivalent to

$$\begin{aligned}
&\rho^* p_* c^{1/r}_{g,n}(\mathbf{m}) r^{1-g} \\
&\quad = \sum_{a,b=0}^{r-2} r^{1-k} p_* c^{1/r}_{k,j+1}(m_{i_1}, \dots, m_{i_j}, a) \otimes r^{1-(g-k)} p_* c^{1/r}_{g-k,n-j+1}(b, m_{i_{j+1}}, \dots, m_{i_n}) \eta^{ab}
\end{aligned} \tag{40}$$

for all $0 \leqslant m_i \leqslant r - 1$ and $\mathbf{m} = (m_1, \dots, m_n)$. Consider the decorated stable graph



and denote the graph obtained from cutting its edge by $\tilde{\Gamma} = \Gamma_1 \sqcup \Gamma_2$ where



Since the spaces $\overline{\mathcal{M}}^{1/r}_{\Gamma_i}$ are non-empty for $0 \leqslant m^{\pm} \leqslant r - 1$ only when $m^{\pm}$ are determined by the conditions (4) and (5), the sum in the right-hand side of Equation (40) has only one non-vanishing term. By the definition of the metric η (33), Axiom C2 reduces to the following:

$$\rho^* p_* c^{1/r}_{g,n}(\mathbf{m}) r^{1-g} = \begin{cases} (p \times p)_*(c^{1/r}_{\Gamma_1} r^{1-k} \otimes c^{1/r}_{\Gamma_2} r^{1-(g-k)}), & \text{if } 0 \leqslant m^+ \leqslant r-2 \\ 0 & \text{if } m^+ = r-1, \end{cases} \tag{41}$$

with $m^- = r - 2 - m^+$. In other words, we must show that

$$\rho^* p_* c^{1/r}_{g,n}(\mathbf{m}) = r(p_* c^{1/r}_{\Gamma_1} \otimes p_* c^{1/r}_{\Gamma_2}) = r \cdot p_{2*}(c^{1/r}_{\tilde{\Gamma}}), \tag{42}$$

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where we have the diagram

$$\begin{tikzcd}
& \overline{\mathcal{M}}_{\tilde{\Gamma}} \times_{\overline{\mathcal{M}}_{\Gamma}} \overline{\mathcal{M}}_{\Gamma}^{1/r} \arrow[dl, "p_1"'] \arrow[dd, "p"] \arrow[r, "\tilde{\chi}"] & \overline{\mathcal{M}}_{\Gamma}^{1/r} \arrow[r, "\tilde{i}"] \arrow[dd, "p"] & \overline{\mathcal{M}}_{g,n}^{1/r} \arrow[dd, "p"] \\
\overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r} \arrow[dr, "p_2"'] & & & \\
& \overline{\mathcal{M}}_{\tilde{\Gamma}} \arrow[r, "\chi"'] & \overline{\mathcal{M}}_{\Gamma} \arrow[r, "i"'] & \overline{\mathcal{M}}_{g,n},
\end{tikzcd}$$

and the map ρ is $i \circ \chi$. However, if we let l be $\gcd(m^+ + 1, r)$, then

$$\begin{aligned}
\rho^* p_* c_{g,n}^{1/r}(\mathbf{m}) &= \chi^* i^* p_* c_{g,n}^{1/r} = \chi^* p_* \tilde{i}^* (r/l) c_{g,n}^{1/r} \\
(\text{by the def. of } c_{\Gamma}^{1/r}) &= \chi^* p_* c_{\Gamma}^{1/r} = p_* \tilde{\chi}^* c_{\Gamma}^{1/r} = p_{2*} p_{1*} \tilde{\chi}^* c_{\Gamma}^{1/r} \\
(\text{by Axiom 3}) &= (p_{2*} c_{\tilde{\Gamma}}^{1/r}) r.
\end{aligned}$$

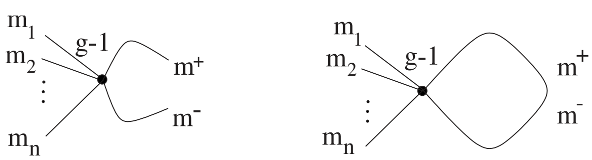
This gives (40); therefore, Axiom C2 is verified. The statement (16) of Axiom C3 is equivalent by (34) and (33) to

$$\rho^* p_* c_{g,n}^{1/r}(\mathbf{m}) = r \sum_{m^+=0}^{r-2} p_* \, c_{g-1,n+2}^{1/r}(m_1, \ldots, m_n, m^+, m^-), \tag{43}$$

where $\rho = \rho_{\text{loop}}$ is the gluing morphism (14) and

$$m^- = \begin{cases} r - 2 - m^+, & \text{if } 0 \leqslant m^+ \leqslant r - 2, \\ r - 1, & \text{if } m^+ = r - 1. \end{cases} \tag{44}$$

Let



(45)

be decorated stable graphs. Let $\tilde{\Gamma}$ and $\Gamma = \Gamma_{\text{loop}}$ be the corresponding underlying

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(undecorated) graphs, respectively. We have the commuting diagram

$$\begin{tikzcd}
 & \bigsqcup\limits_{(r-2)/2\leq m^+<r} F_{\Gamma,m^+} \arrow[r, "\tilde{\chi}"] \arrow[dl, "p_1"'] \arrow[dd, "p"] & \bigsqcup\limits_{(r-2)/2\leq m^+<r} \overline{\mathcal{M}}^{1/r}_{\Gamma,m^+} \arrow[r, "\tilde{i}"] \arrow[dd, "p"] & \overline{\mathcal{M}}^{1/r}_{g,n} \arrow[dd, "p"] \\
\bigsqcup\limits_{(r-2)/2\leq m^+<r} \overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}} \arrow[dr, "p_2"'] & & & \\
 & \overline{\mathcal{M}}_{\tilde{\Gamma}} \arrow[r, "\chi"'] & \overline{\mathcal{M}}_{\Gamma} \arrow[r, "i"'] & \overline{\mathcal{M}}_{g,n}.
\end{tikzcd}$$

where $F_{\Gamma,m^+} := \overline{\mathcal{M}}_{\tilde{\Gamma}} \times_{\overline{\mathcal{M}}_{\Gamma}} \overline{\mathcal{M}}^{1/r}_{\Gamma,m^+}$. Therefore, if l_e is defined to be $\gcd(m^+ + 1, r)$, then

$$\begin{aligned}
\rho^* p_* c^{1/r}_{g,n} &= \chi^* p_* \sum_{m^+} (\tilde{i}^* c_{g,n}) r/l_e = \chi^* p_* \sum_{m^+} c^{1/r}_{\Gamma,m^+} \\
&= p_* \tilde{\chi}^* \sum_{m^+} c^{1/r}_{\Gamma,m^+} = p_{2*} p_{1*} \tilde{\chi}^* \sum_{m^+} c^{1/r}_{\Gamma,m^+} = r \cdot p_{2*} \sum_{m^+} c^{1/r}_{\tilde{\Gamma},m^+}.
\end{aligned}$$

This proves axiom C3. To prove Axiom C4a, consider the Cartesian square

$$\begin{tikzcd}
\overline{\mathcal{M}}^{1/r,(m_1,\ldots,m_n,0)}_{g,n+1} \arrow[r, "\tilde{\pi}"] \arrow[d, "p"] & \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n} \arrow[d, "p"] \\
\overline{\mathcal{M}}_{g,n+1} \arrow[r, "\pi"] & \overline{\mathcal{M}}_{g,n}.
\end{tikzcd}$$

By (11) and Axiom 5 (forgetting tails) we have the required Equation (17)

$$\begin{aligned}
\pi^* \Lambda_{g,n}(e_{m_1}, \ldots, e_{m_n}) &= r^{1-g}\, \pi^* p_* c^{1/r}_{g,n}(\mathbf{m}) = r^{1-g}\, p_* \tilde{\pi}^* c^{1/r}_{g,n}(\mathbf{m}) \\
&= r^{1-g}\, c^{1/r}_{g,n+1}(m_1, \ldots, m_n, 0) = \Lambda_{g,n+1}(e_{m_1}, \ldots, e_{m_n}, e_0).
\end{aligned}$$

Finally, a direct calculation yields Axiom 4b (see Proposition 6.1).

4.2.2. *The General Case*

The proofs of Axioms C1 and C4a remain the same. We only need to prove Axioms C2 and C3. Before doing so, we will need a lemma on regular imbeddings and base change.

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LEMMA 4.3. *Let i be a regular imbedding, and let*

$$\begin{tikzcd} Y' \arrow[r, "\tilde{\imath}"'] \arrow[d, "\pi'"'] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

be a Cartesian square with π and π' proper and flat, and $\mathcal{E}$ a coherent sheaf on X', flat over X. If X and Y both carry an ample invertible sheaf, then the Chern character commutes with base change, that is $i^* ch\pi_!\mathcal{E} = ch\pi'_!\tilde{\imath}^*\mathcal{E}$.

Proof. First, we claim that because i is a regular imbedding it has finite Tor dimension; that is, there is an integer N such that for every coherent $\mathcal{O}_X$-module $\mathcal{F}$, the $\mathcal{O}_Y$-modules $\mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})$ vanish for $j > N$. This can be seen as follows. We may assume that X is Spec$A$ and Y is Spec$A/(x)$ for some regular element x in a ring A. This gives the free resolution $0 \longrightarrow A \xrightarrow{(x)} A \longrightarrow A/(x) \longrightarrow 0$ of $\mathcal{O}_Y$, and shows i has finite Tor dimension.

Since π is flat, the sheaves $\mathcal{O}_{X'}$ and $\mathcal{O}_Y$ are Tor independent over X; that is, $Tor_j^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) = 0$ for all $j > 0$.

Let Li^* and $L\tilde{\imath}^*$ be the left derived functors of i^* and $\tilde{\imath}^*$, respectively. Proposition 5.13 of [33] states that if i has finite Tor dimension, and if $\mathcal{O}_{X'}$ and $\mathcal{O}_Y$ are Tor-independent over X, then $Li^*\pi_! = \pi'_! L\tilde{\imath}^*$.

However, since $\mathcal{E}$ is flat over X, we have $L\tilde{\imath}^*\mathcal{E} := \sum_{j=0}^{\infty}(-1)^j Tor_j^X(\mathcal{E}, \mathcal{O}_Y) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \tilde{\imath}^*\mathcal{E}_j$; and since Li^* commutes with the Chern character, the lemma holds. $\square$

Now we prove that Axioms C2 and C3 hold.

First consider the Cartesian square

$$\begin{tikzcd} \mathcal{C}_\Gamma \arrow[r, "\hat{\imath}"] \arrow[d, "\pi"'] & \mathcal{C} \arrow[d, "\pi"] \\ \overline{\mathcal{M}}_\Gamma^{1/r} \arrow[r, "\tilde{\imath}"'] & \overline{\mathcal{M}}_{g,n}^{1/r}, \end{tikzcd}$$

where π is the universal curve. Let $\mathcal{E}_r$ be the rth root from the universal r-spin structure on $\mathcal{C}$. The morphism π is projective, and $\overline{\mathcal{M}}_{g,n}^{1/r}$ carries an ample invertible line bundle [17, 3.1.1], so Lemma 4.3 gives $\mathrm{ch}_i(\pi_!\hat{\imath}^*\mathcal{E}_r) = \mathrm{ch}_i(\tilde{\imath}^*\pi_!\mathcal{E}_r)$. Note that $\hat{\imath}^*\mathcal{E}_r$ is the rth root from the universal r-spin structure $\mathcal{C}_\Gamma$.

Let F be the fiber product

$$F := \overline{\mathcal{M}}_{\tilde{\Gamma}} \times_{\overline{\mathcal{M}}_\Gamma} \overline{\mathcal{M}}_\Gamma^{1/r}. \tag{46}$$

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We have

$$
\begin{tikzcd}
\mathcal{C}_{\tilde{\Gamma}} \arrow[dd] & \hat{\mathcal{C}}_F \arrow[l] \arrow[d, "\theta"'] & \\
 & \mathcal{C}_F \arrow[r] \arrow[d, "j"'] & \mathcal{C}_\Gamma \arrow[d] \\
\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}} & F \arrow[l, "p_1"'] \arrow[r] & \overline{\mathcal{M}}^{1/r}_{\Gamma},
\end{tikzcd}
$$

where $\mathcal{C}_{\tilde{\Gamma}}$ is defined to be the universal curve over $\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}}$, and $\hat{\mathcal{C}}_F$ and $\mathcal{C}_F$ are the fibered products $F \times_{\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}}} \mathcal{C}_{\tilde{\Gamma}}$ and $F \times_{\overline{\mathcal{M}}^{1/r}_{\Gamma}} \mathcal{C}_\Gamma$, respectively. The morphism θ is just normalization, and in fact, if $\hat{\mathcal{C}}_\Gamma \xrightarrow{\theta_\Gamma} \mathcal{C}_\Gamma$ is the normalization of $\mathcal{C}_\Gamma$, then we have the following fibered diagram (all rectangles are Cartesian).⋆

$$
\begin{tikzcd}
\mathcal{C}_{\tilde{\Gamma}} \arrow[dd, "\pi"'] & \hat{\mathcal{C}}_F \arrow[l, "\hat{p}_1"] \arrow[r, "\hat{\chi}"'] \arrow[d, "\theta"'] & \hat{\mathcal{C}}_\Gamma \arrow[d, "\theta_\Gamma"'] \\
 & \mathcal{C}_F \arrow[r] \arrow[d, "\pi"'] & \mathcal{C}_\Gamma \arrow[d, "\pi"'] \\
\overline{\mathcal{M}}^{1/r}_{\tilde{\Gamma}} & F \arrow[l, "p_1"'] \arrow[r, "\tilde{\chi}"] & \overline{\mathcal{M}}^{1/r}_{\Gamma}
\end{tikzcd}
$$

Moreover, if $\mathcal{E}_{\tilde{\Gamma}}, \mathcal{E}_\Gamma, \hat{\mathcal{E}}_\Gamma, \mathcal{E}_F$, and $\hat{\mathcal{E}}_F$ are the r-th roots of the universal r-spin structures on $\mathcal{C}_{\tilde{\Gamma}}, \mathcal{C}_\Gamma, \hat{\mathcal{C}}_\Gamma, \mathcal{C}_F$, and $\hat{\mathcal{C}}_F$, respectively, we have $\hat{\mathcal{E}}_F = \hat{p}_1^*\mathcal{E}_{\tilde{\Gamma}}$. Also, since θ is finite, $(\pi \circ \theta)_!\hat{\mathcal{E}}_F = \pi_!\mathcal{E}_F$. Since $\tilde{\chi}$ is flat, we have $\tilde{\chi}^*\pi_!\mathcal{E}_\Gamma = \pi_!\tilde{\chi}^*\mathcal{E}_\Gamma = \pi_!\mathcal{E}_F = (\pi \circ \theta)_!\hat{\mathcal{E}}_F$. Also, since p_1 is flat $p_1^*\pi_!\mathcal{E}_{\tilde{\Gamma}} = (\pi \circ \theta)_! p_1^*\mathcal{E}_{\tilde{\Gamma}} = (\pi \circ \theta)_!\hat{\mathcal{E}}_F$. Furthermore, $p_1^*\pi_! = (\pi \circ \theta)_! p_1^*$. If $\tilde{i}\colon \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is the inclusion map, then the above implies that

$$
\begin{aligned}
&\tilde{\chi}^*\,\tilde{i}^*\mathrm{ch}_i(\pi_!\,\mathcal{E}) = \tilde{\chi}^*\,\mathrm{ch}_i(\pi_!\,\mathcal{E}_\Gamma) = \mathrm{ch}_i(\pi_!\,\hat{\chi}^*\,\mathcal{E}_\Gamma) = \\
&\mathrm{ch}_i((\pi \circ \theta)_!\,\hat{p}_1^*\,\mathcal{E}_{\tilde{\Gamma}}) = \mathrm{ch}_i(p_1^*\,\pi_!\,\mathcal{E}_{\tilde{\Gamma}}) = p_1^*\,\mathrm{ch}_i(\pi_!\,\mathcal{E}_{\tilde{\Gamma}}).
\end{aligned}
\tag{47}
$$

The previous equation, the fact that ν_i on $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{g,n}$ is the lift of ν_i on $\overline{\mathcal{M}}_{g,n}$, and the projection formula yield the desired result.

This finishes the proof of Theorem 3.8.

⋆Elsewhere in the paper the maps $\tilde{\chi}$ and θ were called $\tilde{\mu}$ and ν, respectively, but we have renamed them here in order to avoid confusion with the cohomology classes μ_i and ν_i.

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4.3. THE GENUS-ZERO CASE

As we explained in Remark 4.2.3, we call the r-spin structure $(\{\mathcal{E}_d\}, \{c_{d,d'}\})$ on the universal curve $\mathcal{C}_\Gamma^{1/r}$ convex if $\pi_*\mathcal{E}_r$ is identically zero. This occurs, for example, when $g = 0$, as is shown in the following proposition.

PROPOSITION 4.4. *Let X be a prestable curve of genus zero with n punctures and markings $(m_1, \ldots, m_n)$, such that $-1 \leqslant m_i \leqslant r-1$ for all i and $m_i \geqslant 0$ for all i except at most one. Then, if $(\mathcal{E}, b)$ is an r-th root of $\omega_X(-\sum m_i p_i)$, we have $H^0(X, \mathcal{E}) = 0$.*

Proof. The degree of $\mathcal{E}$ is an integer and is equal to $-(2+\sum m_i)/r$. Thus $\sum m_i \geqslant r-2$, and the degree of $\mathcal{E}$ is strictly negative. Therefore, when X is irreducible, $\mathcal{E}$ has no global sections. When X is not irreducible, but $\mathcal{E}$ is locally free (Ramond) at each node, the same argument holds. If $\mathcal{E}$ is Neveu–Schwarz at some nodes, then normalization $\nu\colon \tilde{X} \longrightarrow X$ at the nodes of X where $\mathcal{E}$ is not locally free gives $\mathcal{E} = \nu_*\mathcal{F}$, where $\mathcal{F}$ is locally free on $\tilde{X}$. Restricting $\mathcal{F}$ to $\tilde{X}$ we obtain an r-th root of $\omega_{\tilde{X}}(-\sum \tilde{m}_i\tilde{p}_i)$, where the points $\tilde{p}_i$ are either marked points or inverse images of nodes, and thus the collection $\tilde{m}_i$ still meets the hypotheses of the proposition, but now $\mathcal{F}$ is locally free on each component, and hence has no global sections. Since ν is finite, $H^0(X, \mathcal{E}) = H^0(\tilde{X}, \mathcal{F}) = 0$. $\square$

The previous proposition shows that if a class $c^{1/r}$ on $\overline{\mathcal{M}}_{0,n}^{1/r}$ satisfying Axioms 1–5 exists, then by Axiom 2 it must be the top Chern class of the bundle with fiber $H^1(X, \mathcal{E}_r)^*$ at $[(X, p_1, \ldots, p_n, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))] \in \overline{\mathcal{M}}_{0,n}^{1/r}$. In this case, it does indeed satisfy the required properties.

THEOREM 4.5. *Define cohomology classes on $\overline{\mathcal{M}}_{0,n}^{1/r}$ by*

$$c_{0,n}^{1/r}(\mathbf{m}) = c_D(\pi_!\mathcal{E}_r) = (-1)^D c_D(R^1\pi_*\mathcal{E}_r), \tag{48}$$

where

$$D = \frac{1}{r}\left(2 - r + \sum_{i=1}^{n} m_i\right)$$

and $\mathcal{E}_r$ is the rth root sheaf of the universal r-spin structure. Then the collection of classes $c_\Gamma^{1/r}$ defined by (37) for decorated stable graphs of genus zero satisfies Axioms 1–5.

Proof. It is clear from the construction of the classes $c_\Gamma^{1/r}$ that they satisfy Axiom 2 (convexity).

Axiom 1 follows from the fact that since $\mathcal{E}_{r,\Gamma} = \mathcal{E}_{r,\Gamma_1} \oplus \mathcal{E}_{r,\Gamma_2}$, the top-dimensional Chern class $c_{D(\Gamma)} = c_{top}$ of $\pi_!\mathcal{E}_{r,\Gamma}$, is simply the product of the top-dimensional classes of $\pi_!\mathcal{E}_{r,\Gamma_1}$ and $\pi_!\mathcal{E}_{r,\Gamma_2}$.

Now we will show that Axiom 3 holds. If $\mathcal{E}_r$ is the r-th root from the universal r-spin structure on $\mathcal{C}_{g,n}^{1/r} \to \overline{\mathcal{M}}_{g,n}^{1/r}$, then since $g = 0$, $\mathcal{E}_r$ is convex. Repeating the argument in the proof of Axiom 3 in Section 4.2.2 with the Chern character replaced by the top Chern class yields

$$c_{\Gamma}^{1/r} = \left(r^{|E|} / \prod_{e \in E} l_e \right) c_D(-R^1 \pi_* \mathcal{E}_r),$$

where E denotes the edge set of Γ. Also, we can compute the degree of $p_1 \colon F \longrightarrow \overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r}$ from the diagram

$$\begin{tikzcd}
& F \arrow[r, "\tilde{\chi}"] \arrow[dl, "p_1"'] \arrow[dd, "\tilde{p}"] & \overline{\mathcal{M}}_{\Gamma}^{1/r} \arrow[dd, "p_{\Gamma}"] \\
\overline{\mathcal{M}}_{\tilde{\Gamma}}^{1/r} \arrow[dr, "p_2"'] & & \\
& \overline{\mathcal{M}}_{\tilde{\Gamma}} \arrow[r] & \overline{\mathcal{M}}_{\Gamma}.
\end{tikzcd}$$

The morphism $\overline{\mathcal{M}}_{\tilde{\Gamma}} \longrightarrow \overline{\mathcal{M}}_{\Gamma}$ has degree 1, since Γ is a tree, and therefore $\tilde{\chi}$ also has degree 1. The morphism p_{Γ} has degree $\prod_e l_e r^{-|V(\Gamma)|}$, as can be seen from the fact that the coarse moduli space map induced by p_{Γ} has degree 1, and there are $r^{|V(\Gamma)|}/\prod_e l_e$ automorphisms of a generic r-spin structure. Thus the map $\tilde{p}$ also has degree $\prod_e l_e r^{-|V(\Gamma)|}$, the map p_2 has degree $r^{-|V(\Gamma)|}$, so p_1 has degree exactly $\prod_e l_e$. Now we can compute $p_{1*}\tilde{\chi}^* c_{\Gamma}^{1/r}$ using Equation (47) (replacing ch_i with c_D) to obtain

$$\begin{aligned}
p_{1*}(r^{|E|}/\prod_e l_e)\tilde{\chi}^* c_D(-R^1\pi_*\mathcal{E}_{\Gamma}) &= (r^{|E|}/\prod_e l_e) p_{1*} c_D(p_1^*(-R^1\pi_*\mathcal{E}_{\tilde{\Gamma}})) \\
&= (r^{|E|}/\prod_e l_e) p_{1*} p_1^* c_D(-R^1\pi_*\mathcal{E}_{\tilde{\Gamma}}) = r^{|E|} c_D(-R^1\pi_*\mathcal{E}_{\tilde{\Gamma}}) = r^{|E|} c_{\tilde{\Gamma}}^{1/r}
\end{aligned}$$

as desired. All that remains to check is Axiom 4 (vanishing). Let p be a point corresponding to a tail marked by $m = r - 1$. Taking the tensor product of $\mathcal{E}_r$ with the exact sequence $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(p) \longrightarrow \mathcal{O}(p)|_p \longrightarrow 0$ gives the exact sequence $0 \longrightarrow \mathcal{E}_r \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}'|_p \longrightarrow 0$, where $\mathcal{E}' = \mathcal{E}_r \otimes \mathcal{O}(p)$. Thus $\mathcal{E}'$ corresponds to a root with p_i marked by $m' = -1$. Since $R^1\pi_*(\mathcal{E}'|_p) = 0$, and since the residue isomorphism (1) $R_p \colon \pi_*(\mathcal{E}'|_p) \xrightarrow{\sim} \mathcal{O}$ shows that $\pi_*(\mathcal{E}'|_p)$ is a trivial bundle, we have $\pi_!(\mathcal{E}') = \mathcal{O} + \pi_!(\mathcal{E}_r)$. By Proposition 4.4 the sheaves $\mathcal{E}_r$ and $\mathcal{E}'$ are both convex; thus $\pi_!\mathcal{E}_r = -R^1\pi_*\mathcal{E}_r$ and $\pi_!\mathcal{E}' = -R^1\pi_*\mathcal{E}'$ are both locally free and have the same Chern classes in all dimensions. However, the vector bundle $\pi_!\mathcal{E}'$ has dimension $D' = D - 1$, and so $c^{1/r} = c_D(\pi_!\mathcal{E}_r) = 0$. This gives Axiom 4. □

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4.4. THE CASE $\overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0}}$

In this section we will calculate the virtual class on $\overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0}}$ from the axioms. Let Γ_{m^+} be the decorated graph as in (45) with the underlying graph $\Gamma = \Gamma_{loop}$ with one tail, one node of genus zero, and one loop whose one half-edge is marked with m^+ and the other with m^- given by (44).

The stack $\overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0}}$ is a disjoint union $\coprod_{d|r} \overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0},(d)}$, where the component indexed by d has a generic geometric point corresponding to a smooth r-spin curve $(X, \mathcal{E}_r, c_{r,1})$ with $\mathcal{E}_r^{\otimes d}$ isomorphic to $\mathcal{O}_X$. That is, $\mathcal{E}_r$ is a d-torsion point of the Jacobian of X. Since $\mathcal{E}_r$ has global sections if and only if $d = 1$, the case of $d > 1$ is convex. Since the dimension (36) of the virtual class $c_{1,1}^{1/r,\mathbf{0}}$ is 0, we have that $c_{1,1}^{1/r,\mathbf{0},(d)} = 1$ for $d > 1$. Moreover, consider the graph Γ_{r-1}. By Axiom 3, $p_{1*}\tilde{\mu}^* c_{\Gamma_{r-1}} = c_{\widetilde{\Gamma}_{r-1}}$, where $\widetilde{\Gamma}_{r-1}$ is Γ_{r-1} with the loop cut. Since both the half-edges of the cut loop will be labeled by $r-1$, by Axiom 4 the corresponding class must vanish. Therefore, the (Ramond) case of Γ_{r-1} with a trivial gluing (i.e., $\mathcal{E}_r = \mathcal{O}_X$) yields

$$c_{\Gamma_{r-1}}^{1/r,\mathbf{0},(1)} = i^* c_{1,1}^{1/r,\mathbf{0},(1)} = -(r-1), \tag{49}$$

since all the remaining Ramond components have $i^* c_{1,1}^{1/r,\mathbf{0},(d)} = +1$, and there are $r-1$ of them. Since $D = 0$, this means $c_{1,1}^{1/r,\mathbf{0},(1)}$ is also equal to $-(r-1)$. Notice that this differs from the top Chern class of the bundle $c_D(\pi_! \mathcal{E}_r) = c_0(\pi_! \mathcal{E}_r) = 1$.

Now, the map $p^{(1)}: \overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0},(1)} \to \overline{\mathcal{M}}_{1,1}$ has degree $1/r$, and $p^{(d)}: \overline{\mathcal{M}}_{1,1}^{1/r,\mathbf{0},(d)} \to \overline{\mathcal{M}}_{1,1}$ has degree $d^2/r \prod_{p|d}(1-1/p^2)$. The latter is $1/r$ times the number of points of order precisely d on the Jacobian of the

$$\begin{aligned} p_* c_{1,1}^{1/r} &= 1/r \sum_{d>1} d^2 \prod_{p|d} (1-1/p^2) - (r-1)/r \\ &= \frac{r^2-1}{r} - \frac{r-1}{r} = r-1. \end{aligned}$$

Therefore,

$$\langle \tau_{1,0} \rangle_1 = \int_{\overline{\mathcal{M}}_{1,1}^{1/r,0}} \psi_1 c_{1,1}^{1/r,0} = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 p_* c_{1,1}^{1/r,0} = (r-1) \int_{\overline{\mathcal{M}}_{1,1}} \psi_1,$$

and we conclude that

$$\langle \tau_{1,0} \rangle_1 = \frac{r-1}{24}. \tag{50}$$

Equation (50) is consistent with the prediction from CohFT stated in Equation (61).

4.5. THE CASE $r = 2$

In this section we will show that in the case of theta-characteristics (i.e. when $r = 2$) there exists a unique virtual class $c^{1/2}$ satisfying the axioms of Section 4.1.

THEOREM 4.6. *The collection of cohomology classes* $c_{g,n}^{1/2}(\mathbf{m}) \in H^\bullet(\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{m}})$ *satisfies Axioms 1–5 of Section 4.1 if and only if* $c_{g,n}^{1/2}(\mathbf{m}) = 0$ *for* $\mathbf{m} \neq \mathbf{0}$, *and for* $\mathbf{m} = \mathbf{0}$ *the class* $c_{g,n}^{1/2}(\mathbf{0})$ *belongs to* $H^0(\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0}})$ *and is given by*

$$c_{g,n}^{1/2}(\mathbf{0}) = \begin{cases} 1, & \text{on } \overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0},\text{even}}, \\ -1, & \text{on } \overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0},\text{odd}}. \end{cases} \tag{51}$$

Proof. Let us show first that the conditions of the theorem are necessary. If $\mathbf{m} \neq \mathbf{0}$, then by Axiom 4, $c_{g,n}^{1/2}(\mathbf{m})$ must vanish; therefore, we can assume that $\mathbf{m} = \mathbf{0}$. In this case the dimension D (given in Equation (42)) of the class $c_{g,n}^{1/2}$ is equal to zero, and since $\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0}}$ has two connected components ($\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0},\text{even}}$ and $\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0},\text{odd}}$) it will be sufficient to find $c_\Gamma^{1/2}$ for two graphs Γ_0 and Γ_1, such that the intersections $\overline{\mathcal{M}}_{\Gamma_0}^{1/2} \cap \overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0},\text{even}}$ and $\overline{\mathcal{M}}_{\Gamma_1}^{1/2} \cap \overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0},\text{odd}}$ are non-empty.

Let Γ_0 be the graph with one genus-zero vertex, n tails, and g Neveu–Schwarz (i.e., all half-edges are decorated with zeroes) loops. In this case, $\overline{\mathcal{M}}_{\tilde{\Gamma}_0}^{1/2} = \overline{\mathcal{M}}_{\tilde{\Gamma}_0} \times_{\overline{\mathcal{M}}_{\Gamma_0}} \overline{\mathcal{M}}_{\Gamma_0}^{1/2}$ in (38), so by Axiom 3 (cutting edges) the class $c_{\Gamma_0}^{1/2}$ pulls back to $c_{\tilde{\Gamma}_0}^{1/2}$, where $\tilde{\Gamma}_0$ is the graph with one vertex of genus zero and $n+2g$ tails. Since the genus is zero, the universal square root $\mathcal{E}$ of ω on the universal curve over $\overline{\mathcal{M}}_{0,n+2g}^{1/2,\mathbf{0}}$ is convex by Proposition 4.4. Therefore, if E is the set of edges of Γ_0, we have

$$c_{g,n}^{1/2,\text{even}}(\mathbf{0}) = \frac{1}{2^{|E|}} c_{\Gamma_0}^{1/2,\text{even}}(\mathbf{0}) = c_{\tilde{\Gamma}_0}^{1/2} = c_0(-R^1\pi_*\mathcal{E}) = 1,$$

where the first equality follows from Axiom 1, the second from Axiom 3, and the third from Axiom 2. To find $c^{1/2,\text{odd}}$ consider the graph Γ_1 with a single vertex of genus one, n tails, and $g-1$ Neveu–Schwarz loops. Axiom 3 again shows that $c_{\Gamma_1}^{1/2}$ pulls back to $c_{\tilde{\Gamma}_1}^{1/2}$, where $\tilde{\Gamma}_1$ has a single vertex of genus one and $2g-2+n$ tails. Since $\mathbf{m} = \mathbf{0}$, Axiom 5 (forgetting tails) shows that $c^{1/2,\text{odd}}$ is a pullback from $\overline{\mathcal{M}}_{1,1}^{1/2,\mathbf{0},\text{odd}} = \overline{\mathcal{M}}_{1,1}^{1/2,(1)}$, and $c_{1,1}^{1/2,(1)} = -1$ by Equation (49).

Now let us show that the classes $c_{g,n}^{1/2}(\mathbf{m})$ defined above for $r = 2$ indeed satisfy Axioms 1–5.

Axiom 2 (convexity) holds when $\mathbf{m} = \mathbf{0}$, since in this case the class has dimension 0, and if $\mathcal{E}$ is convex (and, therefore, even) $R^1\pi_*\mathcal{E} = 0$ and $c^{1/2} = c_{\text{top}} = 1$ as required.

If $\mathbf{m} \neq \mathbf{0}$ then $\mathcal{E}$ is not convex on the universal curve over $\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{m}}$ for any $g > 0$. In particular, consider the degenerate curve of genus g which has two irreducible components E and C joined at a single node, the component E of genus zero, containing all n marked points, and the component C of genus g. For degree reasons the node must be Neveu–Schwarz, and so $\mathcal{E}$ corresponds to $\mathcal{E}_E \oplus \mathcal{E}_C$, for $\mathcal{E}_E$ a square root of type $\mathbf{m}$ of $\omega_E = \mathcal{O}_E(-2 - \sum m_i)$, and for $\mathcal{E}_C$ a theta-characteristic on C. In general, $\mathcal{E}_C$ has non-zero global sections. Thus $\mathcal{E}$ also has non-zero global sections, and the universal square root is not convex.

When $g = 0$ and $\mathbf{m} \neq 0$, the sheaf $\mathcal{E}$ is convex by Proposition 4.4. By Theorem 4.5 (and Axiom 4) the class $c_D(-R^1\pi_*\mathcal{E})$ vanishes, and so agrees with our definition of $c^{1/2}_{g,n}(\mathbf{m})$. Axiom 1 holds because of the simple observation that the parity of a root $\mathcal{E}$ over a curve with the graph $\Gamma_1 \sqcup \Gamma_2$ is equal to the sum modulo 2 of the parities of the restrictions of $\mathcal{E}$ to the components corresponding to Γ_1 and Γ_2.

To prove Axiom 3, we may assume that $\mathbf{m} = \mathbf{0}$, and it is sufficient to check the case that Γ has only one edge. We have by definition $c_\Gamma^{1/2,\mathrm{even}} = 2$ and $c_\Gamma^{1/2,\mathrm{odd}} = -2$ for Γ a tree. Let F be defined as in Equation (46). The canonical morphism $p_1 : F \longrightarrow \overline{\mathcal{M}}^{1/2}_{\tilde\Gamma}$ is actually an isomorphism if Γ is a tree, and so we get $p_{1*}\tilde\mu^* c_\Gamma^{1/2} = 2 \cdot c^{1/2}_{\tilde\Gamma}$ since the parity of $\mathcal{E}_2$ does not change when restricting to the normalization. In the case of a loop, there are two subcases. First when $m^+ = 1$, F is isomorphic to two copies of $p_1(F) \subseteq \overline{\mathcal{M}}^{1/2}_{\tilde\Gamma}$ (because of the two choices of gluing data – see Section 1.7.2). In the second case $m^+ = 0$, and F is isomorphic to $\overline{\mathcal{M}}^{1/2}_{\tilde\Gamma}$, so p_1 has degree 1. Also, $c^{1/2}_\Gamma$ is $2\tilde{i}^* c^{1/2}_{g,n}$ if $m^+ = 0$, and $\tilde{i}^* c^{1/2}_{g,n}$ if $m^+ = 1$. Thus, when $m^+ = 0$, $p_{1*}\tilde\mu^* c_\Gamma^{1/2} = 2c^{1/2}_{\tilde\Gamma}$ as desired.

When $m^+ = 1$, the two choices of gluing give different parities. Since parity is deformation invariant, this can be seen by degenerating to the special case of the curve X, whose partial normalization $\tilde X$ at one node q consists of two irreducible components joined at a single node q. One component C is of genus $g-1$ and contains the marked points $p_1, \ldots, p_n$. The other component $\tilde E$ is of genus zero and contains marked points q^+ and q^-. Degree reasons force the node to be Neveu–Schwarz, and so $\tilde{\mathcal{E}}$ is simply a direct sum $\mathcal{E}_C \oplus \mathcal{E}_{\tilde E}$. Moreover, since $\tilde{\mathcal{E}}_{\tilde E}$ is a square root of $\omega_{\tilde E} = \mathcal{O}_{\tilde E}(-2)$ of type $(0,-1,-1)$, $\mathcal{E}_{\tilde E}$ must be trivial ($\mathcal{E}_{\tilde E} \cong \mathcal{O}_{\tilde E}$). Gluing $\mathcal{E}_{\tilde E}$ via $+1$ and -1 yields the trivial bundle $\mathcal{O}_E = \mathcal{E}_E^+$ and another non-trivial bundle $\mathcal{E}_E^-$ of degree zero, respectively. Consequently, $h^0(\mathcal{E}^+) = h^0(\mathcal{E}_C \oplus \mathcal{O}) = 1 + h^0(\mathcal{E}_C \oplus \mathcal{E}_E^-)$. Since the parities of $\mathcal{E}^+$ and $\mathcal{E}^-$ are simply the parities of $h^0(\mathcal{E}^+)$ and $h^0(\mathcal{E}^-)$, respectively, $\mathcal{E}^+$ and $\mathcal{E}^-$ have different parities. The different parities under the two gluings give $p_{1*}\tilde\mu^* c_\Gamma^{1/2} = 0 = c^{1/2}_{\tilde\Gamma}$, since $\tilde\Gamma$ has a tail marked with $+1 = m^+$. This completes the proof of Axiom 3.

Axiom 5 is true because the projection $\overline{\mathcal{M}}^{1/2,(m_1,\ldots,m_n,0)}_{g,n+1} \longrightarrow \overline{\mathcal{M}}^{1/2,\mathbf{m}}_{g,n}$ respects the parity of components. Axiom 4 follows from the definition of the classes $c^{1/2}$.

This theorem together with Theorem 3.8 implies that in the $r = 2$ case we obtain a well-defined complete CohFT of rank one. It turns out to be the same as the Witten–Kontsevich rank-one CohFT of the pure topological gravity. Namely, we have the following result (cf. also [34]).

COROLLARY 4.7. *The class $\Lambda_{g,n}(e_{m_1}, \ldots, e_{m_n}) \in H^\bullet(\overline{\mathcal{M}}_{g,n})$ of the 2-spin CohFT given by (34) is equal to 1 if $m_1 = m_2 = \ldots = m_n = 0$ and 0 otherwise.*

Proof. We only need to check the case $m_1 = m_2 = \ldots = m_n = 0$. Since the class has dimension 0, Equation (34) gives

$$\Lambda_{g,n}(e_0, \ldots, e_0) = 2^{1-g} p_* c^{1/2}_{g,n}(\mathbf{0}) = 2^{1-g}\big(2^{g-1}(2^g+1) - 2^{g-1}(2^g-1)\big)/2 = 1,$$

where $2^{g-1}(2^g \pm 1)$ are the numbers of even/odd theta characteristics on a smooth curve of genus g and the last factor of $1/2$ is the local (orbifold) degree of the map p near a generic point of $\overline{\mathcal{M}}_{g,n}^{1/2,\mathbf{0}}$. □

In Section 7.2 we will see that this corollary together with Kontsevich's theorem gives the generalized Witten conjecture for $r = 2$.

5. Intersection Numbers and Recursion Relations

In this section, we use relations between boundary classes and tautological classes in order to derive recursion relations between intersection numbers on the moduli space of stable r-spin curves. Throughout this section, we assume the existence of a virtual class $c^{1/r}$ satisfying Axioms 1–5 of Section 4.1. This class was shown in Section 4 to exist in genus zero for arbitrary r, and in arbitrary genus for $r = 2$. Let $(\mathcal{H}^{(r)}, \eta, \Lambda, e_0)$ be the r-spin CohFT with the standard basis $\{e_0 \ldots, e_{r-2}\}$ described in Corollary 3.9.

We will find it convenient to introduce the notion of a very large phase space potential in this section. For all $v_1, \ldots, v_n$ in $\mathcal{H}^{(r)}$, define

$$\langle\langle\, \tau_{a_1}(v_1) \cdots \tau_{a_n}(v_n) \,\rangle\rangle_g := \langle\, \tau_{a_1}(v_1) \cdots \tau_{a_n}(v_n) \exp(\mathbf{t} \cdot \boldsymbol{\tau} + \mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}) \,\rangle_g,$$

where

$$\mathbf{t} \cdot \boldsymbol{\tau} = \sum_{\substack{0 \leqslant m \leqslant r-2 \\ a \geqslant 0}} \tau_{a,m}\, t_a^m$$

and $\tau_{a,m} = \tau_a(e_m)$. Here

$$\mathbf{u} \cdot \boldsymbol{\nu} = \sum_{i \geqslant 1} u_i\, \nu_i, \qquad \mathbf{s} \cdot \boldsymbol{\mu} = \sum_{i \geqslant 1} s_i \mu_i,$$

where the classes ν_i and μ_i are defined by (8) and (9) as components of the Chern characters of the Hodge bundle and its r-spin analog. These expressions should be understood as formal power series in variables t_a^m, u_i, v_i. The correlators are defined by

$$\langle \tau_{a_1}(e_{m_1}) \cdots \tau_{a_n}(e_m n) \rangle_g := \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}} \psi_1^{a_1} \cdots \psi_n^{a_n} c_{g,n}^{1/r}(\mathbf{m}) \exp(\mathbf{s} \cdot \boldsymbol{\mu} + \mathbf{u} \cdot \boldsymbol{\nu}),$$

where $\mathbf{m} = (m_1, \ldots, m_n)$. In particular, the *very large phase space potential* is

$$\Phi(\mathbf{t}, \mathbf{s}, \mathbf{u}) = \sum_{g=0}^{\infty} \Phi_g(\mathbf{t}, \mathbf{s}, \mathbf{u}), \tag{52}$$

where $\Phi_g(\mathbf{t}, \mathbf{s}, \mathbf{u}) = \langle\langle\ \rangle\rangle_g$. The other two potentials are restrictions of $\Phi(\mathbf{t}, \mathbf{s}, \mathbf{u})$. The large phase space potential $\Phi(\mathbf{t}) = \Phi(\mathbf{t}, \mathbf{0}, \mathbf{0})$ corresponds to setting all the $\mathbf{s}$ and $\mathbf{u}$ variables to zero. The small phase space potential is $\Phi(\mathbf{x}) := \Phi(x^0, \ldots, x^{r-2})$, where

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$x^i := t_0^i$ and all other variables are set to zero. The function $\Phi(\mathbf{x})$ is the small phase space potential of the r-spin CohFT. It will also be useful to define

$$Z(\mathbf{t},\mathbf{s},\mathbf{u}) := \exp(\Phi(\mathbf{t},\mathbf{s},\mathbf{u})). \tag{53}$$

5.1. THE EULER VECTOR FIELD

We begin with a differential equation arising from the grading. The dimensions of the moduli spaces and cohomology classes induce a grading on the potential function.

DEFINITION 5.1. The *Euler vector field* E is the differential operator

$$E = \sum_{\substack{a\geqslant 0\\ 0\leqslant m\leqslant r-2}} \left(a-1+\frac{m}{r}\right) t_a^m \frac{\partial}{\partial t_a^m} + \sum_{a\geqslant 1}\left(a s_a \frac{\partial}{\partial s_a} + (2a-1)u_a\frac{\partial}{\partial u_a}\right).$$

PROPOSITION 5.2. *The very large phase space potential* $\Phi(\mathbf{t},\mathbf{u},\mathbf{s})$ *satisfies the* grading equation

$$E\Phi = \left(1+\frac{1}{r}\right)\lambda\frac{\partial}{\partial\lambda}\Phi. \tag{54}$$

Proof. This follows from the definition of the potential, the dimensions of the cohomology classes, and the dimensions of the moduli spaces $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. It encodes the fact that intersection numbers between cohomology classes vanish if the classes do not have proper the dimension. □

This equation encodes the fact that the potential function is invariant under the rescaling $t_a^m \longmapsto \varepsilon^{a-1+\frac{m}{r}} t_a^m$, $u_a \longmapsto \varepsilon^{2a-1} u_a$, $s_a \longmapsto \varepsilon^a s_a$, and $\lambda \longmapsto \varepsilon^{-1-\frac{1}{r}}\lambda$.

Remark 5.3. This grading shows that our small phase space potential function *cannot* arise as the small phase space potential associated to the Gromov–Witten invariants of a smooth, projective variety. This is because the elements in $\mathcal{H}^{(r)}$ have fractional dimension with respect to this Euler vector field, whereas cohomology classes of a space always have integral dimension.

5.2. THE STRING EQUATION AND ITS COUSINS

We begin by proving the analog of the string equation, the dilaton equation, and a new equation arising from our identity on the μ_1 class.

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THEOREM 5.4. *Let (g, n) be a pair of nonnegative integers such that $2g-2+n>0$. The following identities are satisfied:*

$$\langle\, \tau_{0,0}\, \tau_{0,m_1}\, \tau_{0,m_2}\, \exp(\mathbf{s}\cdot\boldsymbol{\mu}+\mathbf{u}\cdot\boldsymbol{\nu})\,\rangle_0 = \eta_{m_1 m_2} \tag{55}$$

and

$$\begin{aligned}&\langle\tau_{0,0}\tau_{a_1,m_1}\cdots\tau_{a_n,m_n}\exp(\mathbf{s}\cdot\boldsymbol{\mu}+\mathbf{u}\cdot\boldsymbol{\nu})\rangle_g\\ &\quad=\sum_{i=1}^{n}\langle\tau_{a_1,m_1}\cdots\tau_{a_i-1,m_i}\cdots\tau_{a_n,m_n}\exp(\mathbf{s}\cdot\boldsymbol{\mu}+\mathbf{u}\cdot\boldsymbol{\nu})\rangle_g,\end{aligned}$$

where we assume that the terms in the sum containing $\tau_{a,m}$ with $a<0$ vanish.

These two equations are equivalent to the string (or puncture) equation

$$\mathcal{L}_{-1}Z=0, \tag{56}$$

where

$$\mathcal{L}_{-1}:=-\frac{\partial}{\partial t_0^0}+\sum_{0\leqslant m_1,m_2\leqslant r-2}\frac{1}{2}\eta_{m_1m_2}t_0^{m_1}t_0^{m_2}+\sum_{\substack{0\leqslant m\leqslant r-2\\ a\geqslant 0}}t_{a+1}^m\frac{\partial}{\partial t_a^m}.$$

Similarly, the following identities are satisfied:

$$\langle\,\tau_{1,0}\,\exp(\mathbf{s}\cdot\boldsymbol{\mu}+\mathbf{u}\cdot\boldsymbol{\nu})\,\rangle_1 = \frac{r-1}{24} \tag{57}$$

and

$$\begin{aligned}&\langle\tau_{0,0}\tau_{a_1,m_1}\cdots\tau_{a_n,m_n}\exp(\mathbf{s}\cdot\boldsymbol{\mu}+\mathbf{u}\cdot\boldsymbol{\nu})\rangle_g\\ &\quad=(2g-2+n)\langle\tau_{a_1,m_1}\cdots\tau_{a_n,m_n}\exp(\mathbf{s}\cdot\boldsymbol{\mu}+\mathbf{u}\cdot\boldsymbol{\nu})\rangle_g,\end{aligned}$$

These two equations are equivalent to the dilaton equation

$$\mathcal{D}Z=0, \tag{58}$$

where

$$\mathcal{D}=-\frac{\partial}{\partial t_1^0}+\sum_{\substack{0\leqslant m\leqslant r-2\\ a\geqslant 0}}t_a^m\frac{\partial}{\partial t_a^m}+\lambda\frac{\partial}{\partial\lambda}+\frac{r-1}{24}. \tag{59}$$

Finally, if $\mathcal{L}_0$ denotes the differential operator

$$\begin{aligned}\mathcal{L}_0:=&-\left(1+\frac{1}{r}\right)\frac{\partial}{\partial t_1^0}+\sum_{\substack{0\leqslant m\leqslant r-2\\ a\geqslant 0}}\left(a+\frac{m+1}{r}\right)t_a^m\frac{\partial}{\partial t_a^m}+\frac{r^2-1}{24r}+\\ &+\sum_{a\geqslant 1}\left((2a-1)u_a\frac{\partial}{u_a}+a\frac{\partial}{\partial s_a}\right),\end{aligned}$$

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then the following equation holds:

$$\mathcal{L}_0 Z = 0. \tag{60}$$

Furthermore, $[\mathcal{L}_0, \mathcal{L}_{-1}] = \mathcal{L}_{-1}$. *When restricted to the large phase space* $(\mathbf{u} = \mathbf{s} = \mathbf{0})$*, the operators* $\mathcal{L}_0$ *and* $\mathcal{L}_{-1}$ *become the usual generators* L_0 *and* L_1 *in the Virasoro Lie algebra.*

Proof. Recall that on the moduli space of stable curves, $\overline{\mathcal{M}}_{g,n}$, the ψ_i classes (where $i = 1, \ldots, n$) satisfy the equation $\psi_i = \pi^* \psi_i + D_{i,n+1}$, where $D_{i,n+1}$ is the image of the i-th canonical section and $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Let p be the forgetful morphism $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathcal{M}}_{g,n}$, then since the class ψ_i on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ is the pullback via p of the ψ_i class on $\overline{\mathcal{M}}_{g,n}$, one can lift the same formula to $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ to obtain $\psi_i = \tilde{\pi}^* \psi_i + D_{i,n+1}$, where this equation is now regarded as being on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$, $D_{i,n+1}$ is the pullback via p of the divisors with the same name on $\overline{\mathcal{M}}_{g,n}$, and $\tilde{\pi}$ is the forgetful morphism $\overline{\mathcal{M}}_{g,n+1}^{1/r,\mathbf{m}\sqcup 0} \to \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$. Suppose that $(g, n+1) \neq (0, 3), (1, 1)$. Using the lifting formula and canceling trivial terms, we obtain

$$\psi_j^a = \tilde{\pi}^* \psi_j^a + D_{j,n+1}\, \tilde{\pi}^*(\psi_j^{a-1}).$$

Since $\tilde{\pi}^* c^{1/r} = c^{1/r}$, we have

$$\begin{aligned}
&\langle \tau_{0,0} \tau_{a_1,m_1} \cdots \tau_{a_n,m_n} \rangle_g \\
&\quad = \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,n+1}^{1,r,\mathbf{m}}} \psi_1^{a_1} \cdots \psi_n^{a_n} c^{1/r} \\
&\quad = \sum_{1 \leqslant i \leqslant n} \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,n+1}^{1,r,\mathbf{m}}} D_{i,n+1} \psi_1^{a_1} \cdots \psi_i^{a_i - 1} \cdots \psi_n^{a_n} c^{1/r},
\end{aligned}$$

where the right hand side is understood to vanish if an exponent is negative. Integration over the fiber of $\overline{\mathcal{M}}_{g,n+1}^{1/r,\mathbf{m}\sqcup 0} \to \overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ yields the desired result. Inclusion of the additional μ and ν classes into the correlators does not change the argument, since $\tilde{\pi}^* \mu_i = \mu_i$, and similarly for ν_i.

Finally, the exceptional cases follow from dimensional considerations and the fact that on $\overline{\mathcal{M}}_{0,3}^{1/r,\mathbf{m}}$, $c^{1/r}$ is the identity element in cohomology provided that $m_1 + m_2 + m_3 = r - 2$. The dilaton equation is proved by a similar analysis, where the exceptional case can be computed by using the explicit presentation for ψ_1 on $\overline{\mathcal{M}}_{1,1}^{1/r,0}$ to obtain

$$\langle \tau_{1,0} \rangle_1 = \frac{1}{24} \eta^{m_+ m_-} \langle \tau_{0,0} \tau_{0,m_+} \tau_{0,m_-} \rangle_0 = \frac{r-1}{24}. \tag{61}$$

Finally, the equation $\mathcal{L}_0 Z = 0$ is obtained by combining the dilaton equation and the grading Equation (54). $\square$

The new relation for the μ_1 class yields new equations between correlators.

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THEOREM 5.5. *Let* $\xi : \mathcal{H}^{(r)} \to \mathcal{H}^{(r)}$ *be defined by*

$$\xi_{m_+}{}^{m_-} := \frac{m_+ m_-}{r^2}\,\delta_{m_+ + m_-, r-2},$$

relative to the standard basis, and let $\xi^{m_+,m_-} := \eta^{m_+ m}\,\xi_m{}^{m_-}$. *The following differential equation holds:*

$$\frac{\partial\Phi}{\partial s_1} = \frac{r^2-6r+6}{r}\frac{\partial\Phi}{\partial u_1} + \sum_{\substack{a\geqslant 0\\ 0\leqslant m_+,m_-\leqslant r-2}} \frac{1}{2} t_a^{m_+}\xi_{m_+}{}^{m_-}\frac{\partial\Phi}{\partial t_{a+1}^{m_-}} - \\ - \sum_{m_+,m_-}\frac{1}{2}\frac{\partial\Phi}{\partial t_0^{m_+}}\xi^{m_+,m_-}\frac{\partial\Phi}{\partial t_0^{m_-}} - \sum_{m_+,m_-}\frac{1}{4}\frac{\partial^2\Phi}{\partial t_0^{m_+}\partial t_0^{m_-}}\xi^{m_+,m_-},$$

where the summation over m_+ *and* m_- *runs over* $0,\ldots,r-2$. *This is equivalent to the following relations between the correlators:*

$$\begin{aligned}
&\langle\langle\tau_{a_1,m_1}\cdots\tau_{a_n,m_n}\mu_1\rangle\rangle_g\\
&\quad= \frac{r^2-6r+6}{r}\langle\langle\tau_{a_1,m_1}\cdots\tau_{a_n,m_n}\nu_1\rangle\rangle_g +\\
&\quad\quad+\sum_{\substack{1\leqslant i\leqslant n\\ m_i'}}\frac{1}{2}\xi_{m_i}{}^{m_i'}\langle\langle\tau_{a_1,m_1}\cdots\tau_{a_i+1,m_i'}\cdots\tau_{a_n,m_n}\rangle\rangle_g -\\
&\quad\quad-\sum_{\substack{I_+\sqcup I_-=[n]\\ g_++g_-=g\\ m_+,m_-}}\frac{1}{2}\left\langle\left\langle\left(\prod_{i\in I_+}\tau_{a_i,m_i}\right)\tau_{0,m_+}\right\rangle\right\rangle_{g_+}\xi^{m_+,m_-}\left\langle\left\langle\tau_{0,m_-}\left(\prod_{j\in I_-}\tau_{a_j,m_j}\right)\right\rangle\right\rangle_{g_-} -\\
&\quad\quad-\sum_{m_+,m_-}\frac{1}{4}\langle\langle\tau_{a_1,m_1}\cdots\tau_{a_n,m_n}\tau_{0,m_+}\tau_{0,m_-}\rangle\rangle_{g-1}\xi^{m_=,m_-},
\end{aligned}$$

where we use the notation $[n] = \{1,2,\ldots,n\}$.

The class ν_1 *which appears above is precisely* λ_1. *This class vanishes on* $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ *for* $g = 0$.

Proof. The proof follows from the facts that $p_*(c^{1/r}\exp(\mathbf{s}\boldsymbol{\mu}))$ forms a CohFT, and that ν and ψ are lifts of the analogous classes on the moduli space of stable curves, and from Proposition 2.4. □

5.3. TOPOLOGICAL RECURSION RELATIONS

Topological recursion relations are relations between correlators which arise from presentations of tautological classes in terms of boundary classes.

THEOREM 5.6. *The following topological recursion relations hold in genus zero:*

$$\langle\langle\tau_{a_1+1,m_1}\tau_{a_2,m_2}\tau_{a_3,m_3}\rangle\rangle_0 = \sum_{m_+,m_-}\langle\langle\tau_{a_1,m_1}\tau_{0,m_+}\rangle\rangle_0\eta^{m_+m_-}\langle\langle\tau_{0,m_-}\tau_{a_2,m_2}\tau_{a_3,m_3}\rangle\rangle_0,$$

which is equivalent to the differential equation

$$\frac{\partial^3\Phi_0}{\partial t^{m_1}_{a_1+1}\partial t^{m_2}_{a_2}t^{m_3}_{a_3}} = \sum_{m_+,m_-}\frac{\partial^2\Phi_0}{\partial t^{m_1}_{a_1}\partial t^{m_+}_0}\eta^{m_+m_-}\frac{\partial^3\Phi_0}{\partial t^{m_-}_0\,\partial t^{m_2}_{a_2}\,\partial t^{m_3}_{a_3}}.$$

Proof. On $\overline{\mathcal{M}}^{1/r,\mathbf{m}}_{0,n}$, the class ψ_1 can be written in terms of boundary classes as

$$\psi_1 = \sum_{\substack{I_+\sqcup I_-=[n]\\ n-1,n\in I_+\\ 1\in I_-}}\delta_{0;I_+}.$$

This equation is obtained from lifting the analogous relation on $\overline{\mathcal{M}}_{0,n}$. The classes ψ_i can be written similarly by applying an element of the permutation group S_n. The recursion relation follows from this presentation and the restriction properties of the ψ_i to the boundary strata. □

THEOREM 5.7. *The following topological recursion relation holds in genus one:*

$$\begin{aligned}\langle\langle\tau_{a_1+1,m_1}\rangle\rangle_1 &= \frac{1}{24}\langle\langle\tau_{a_1,m_1}\tau_{0,m_+}\tau_{0,m_-}\rangle\rangle_0\eta^{m_+m_-}+\\ &\quad+\sum_{m_=,m_-}\langle\langle\tau_{a_1,m_1}\tau_{0,m_+}\rangle\rangle_0\eta^{m_+m_-}\langle\langle\tau_{0,m_-}\rangle\rangle_1.\end{aligned}$$

This is equivalent to

$$\frac{\partial\Phi_1}{\partial t^{m_1}_{a_1+1}} = \frac{1}{24}\sum_{m_+,m_-}\frac{\partial^3\Phi_0}{\partial t^{m_1}_{a_1}\,\partial t^{m_+}_0\,\partial t^{m_-}_0}\eta^{m_+m_-} + \sum_{m_+,m_-}\frac{\partial^2\Phi_0}{\partial t^{m_1}_{a_1}\partial t^{m_+}_0}\eta^{m_+m_-}\frac{\partial\Phi_1}{\partial t^{m_-}_0}.$$

The topological recursion relation for $v_1 = \lambda_1$ *from* [21]

$$\langle\langle v_1\rangle\rangle_1 = \frac{1}{24}\sum_{m_+,m_-}\eta^{m_+m_-}\langle\langle\tau_{0,m_+}\tau_{0,m_-}\rangle\rangle_0$$

can be written as

$$\frac{\partial\Phi_1}{\partial u_1} = \frac{1}{24}\sum_{m_+,m_-}\eta^{m_+m_-}\frac{\partial^2\Phi_0}{\partial t^{m_+}_0\partial t^{m_-}_0}.$$

Proof. The proof of the first topological recursion relation arises from the relation on $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m}}$

$$\psi_1 = \frac{1}{12}\delta_{irr} + \sum_{\substack{I_+\sqcup I_-=[n]\\ n-1,n\in I_+\\ 1\in I_-}} \delta_{0;I_+},$$

which is obtained from lifting the analogous relation from $\overline{\mathcal{M}}_{1,n}$. The action of S_n yields ψ_i. This, combined with the restriction properties of the ψ classes, yields the desired result. The second comes from the presentation of λ_1 on $\overline{\mathcal{M}}_{1,n}^{1/r,\mathbf{m}}$ $\lambda_1 = [\frac{1}{12}]\delta_{\mathrm{irr}}$ and the restriction properties of $\lambda_1 = \nu_1$. □

These two relations allow one to completely reduce the large phase space potential to the small phase space potential in genus zero and one. Combined with the previous equations, we can compute $\Phi(\mathbf{t}, \mathbf{s}, \mathbf{u})$ in genus zero and one when we set $s_i = u_i = 0$ for all $i \geqslant 2$.

The genus one potential satisfies an analog of the WDVV equation due to Getzler [12], which arises from relations between codimension-two boundary classes on $\overline{\mathcal{M}}_{1,4}$. Using this equation, Dubrovin and Zhang [7] showed that if the Frobenius manifold is semisimple, then the genus-one potential is determined by the Frobenius structure. Since the Frobenius manifold structure on $(\mathcal{H}^{(r)}, \eta)$ associated to $\Phi_0(\mathbf{x})$ is known to be semisimple [6], $\Phi_1(\mathbf{x})$ is determined. On the other hand, the latter must vanish due to dimensional considerations. Together with the topological recursion relations in genus zero and one, we obtain the following corollary.

COROLLARY 5.8. *Let* $\mathbf{s} = \mathbf{u} = \mathbf{0}$ *so that we are on the large phase. Let* $v^m := \sum_{m=0}^{r-2}\langle\langle \tau_{0,0}\tau_{0,l}\rangle\rangle_0 \eta^{lm}$. *Let* $\Delta(\mathbf{t})$ *denote the matrix with entries* $\partial v^m \partial t_0^l$ *where* $m, l = 0, \ldots, r-2$*, then*

$$\Phi_1(\mathbf{t}) = \frac{1}{24}\ln\det\Delta(\mathbf{t}). \tag{62}$$

Proof. Dijkgraaf and Witten [5] write down a formula (see Theorem 15 of [13] for an explicit proof) for the large phase space potential $\Phi_1(\mathbf{t})$ which in our case is $\Phi_1(\mathbf{t}) = \langle \exp(\tau_0(v))\rangle_1 + \frac{1}{24}\ln\det\Delta(\mathbf{t})$, where $v := \sum_{m=0}^{r-2} v^m e_m$. The term $\langle \exp(\tau_0(v))\rangle_1$ is equal to the small phase space potential $\Phi_1(\mathbf{x})$, evaluated at $x^m = v^m$ for all $m = 0, \ldots, r-2$, but the small phase potential $\Phi_1(\mathbf{x})$ vanishes. □

In genus 2, there exist relations among products of psi classes and boundary classes, which give rise to topological relations [3, 13]. However, unlike in genus 0 and 1, these relations do not allow the expression of the large phase space potential as a differential polynomial of the small phase space potential, since some of the terms in these relations involve a single descendant, which cannot be eliminated.*

*We would like to thank the referee for both the content and the wording of this paragraph.

6. The Genus-Zero Large Phase Space Potential

In this section, we compute the genus-zero, three- and four-point correlators and show that they completely determine the genus-zero, large phase space potential function $\Phi_0(\mathbf{t})$. We roughly follow the outline provided by Witten [34] and are able to rigorously prove the validity of his computations, now that the relevant moduli spaces and classes have been constructed.

Throughout the rest of this paper, we will consider only the large phase space potential $\Phi(\mathbf{t})$ (setting the other variables u_a and s_a of the very large phase space potential to zero). We will also fix the coupling constant λ as $\lambda = (1/\sqrt{r})$, since this is the value which is relevant to the generalized Witten conjecture.

The following proposition rigorously demonstrates the formulas from [34], but the idea of our proof is quite different, as it uses our new relation for the μ_1 class.

PROPOSITION 6.1. *The three-point and four-point correlators of the r-spin CohFT are given by the following formulas:* $\langle \tau_{0,m_1} \tau_{0,m_2} \tau_{0,m_3} \rangle_0 = \delta_{m_1+m_2+m_3,r-2}$ *and* $\langle \tau_{0,m_1} \dots \tau_{0,m_4} \rangle_0 = \frac{1}{r} \mathrm{Min}_{1 \leqslant i \leqslant 4}(m_i, r-1-m_i)$, *where* Min *is minimum value.*

Proof. $\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}$ is nonempty if and only if $(2 + \sum_i m_i)/r \in \mathbb{Z}$, where $\mathbf{m} = (m_1, \dots, m_n)$ and $m_i = 0 \dots r-1$ for all i. The genus zero correlators are given by

$$\langle \tau_{0,m_1} \dots \tau_{0,m_n} \rangle_0 = r \int_{\overline{\mathcal{M}}_{0,n}^{1/r,\mathbf{m}}} c^{1/r},$$

where $c^{1/r} = c_D(-R^1 \pi_* \mathcal{E}_r)$ is the (top) Chern class of degree

$$D = -1 + \frac{2}{r} + \frac{1}{r} \sum_i m_i.$$

The class $c^{1/r}$ vanishes unless $m_i = 0, \dots r-2$ for all i by Theorem 4.5. Furthermore, the correlator can only be nonzero if $D = n-3$.

If $n = 3$ then the dimensionality condition becomes $m_1 + m_2 + m_3 = r-2$, in which case $c^{1/r}$ is the identity. This proves the first part of the proposition.

If $n = 4$ then the dimensionality condition becomes $m_1 + \dots + m_4 = 2r-2$. If this condition is satisfied, then $c^{1/r} = \mu_1$. The correlator is

$$\langle \tau_{0,m_1} \dots \tau_{0,m_4} \rangle_0 = r \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \mu_1.$$

The right hand side can be computed using the relation for the class μ_1 in Proposition 2.4, which becomes, in genus zero,

$$\mu_1 = \sum_{1 \leqslant i \leqslant n} \frac{m_i(r-2-m_i)}{2r^2} \psi_i + \sum_{I_+ \subset [n]} \frac{r-1-(m_+ + 1)(m_- + 1)}{2r^2} \delta_{0;I_+},$$

where m_+ and m_- are uniquely determined by the divisor $\delta_{0:I_+}$. Let $\delta_{ij,kl}$ denote the divisor $\delta_{0;\{i,j\}}$ on $\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}$.

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Plugging in this formula, one obtains (after doing a little case by case analysis to write m_+ and m_- in terms of $m_1, \ldots, m_4$)

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{0,4}^{1/r,m}} \mu_1 = \sum_{i=1}^{4} \frac{m_i(r-2-m_i)}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,m}} \psi_1 + \\ + \frac{r-1-(\chi_{12,34}+1)(r-1-\chi_{12,34})}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,m}} \delta_{12,34} + \\ + \frac{r-1-(\chi_{13,24}+1)(r-1-\chi_{13,24})}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,m}} \delta_{13,24} + \\ + \frac{r-1-(\chi_{14,23}+1)(r-1-\chi_{14,23})}{2r^2} \int_{\overline{\mathcal{M}}_{0,4}^{1/r,m}} \delta_{14,23},\end{aligned}$$

where $\chi_{ij,kl} := Min(m_i + m_j, m_k + m_l)$. Since each $\delta_{ij,kl}$ is Poincaré dual to the (topological) homology class represented by a point, one has $r \int_{\overline{\mathcal{M}}_{0,4}^{1/r,\mathbf{m}}} \delta_{ij,kl} = 1$. Similarly, each class ψ_i can be represented by $\delta_{ij,kl}$ for some i, j, k, l. Therefore, one obtains

$$\begin{aligned}r \int_{\overline{\mathcal{M}}_{0,4}^{1/r,m}} \mu_1 = \sum_{i=1}^{4} \frac{m_i(r-2-m_i)}{2r^2} + \\ + \frac{r-1-(\chi_{12,34}+1)(r-1-\chi_{12,34})}{2r^2} + \\ + \frac{r-1-(\chi_{13,24}+1)(r-1-\chi_{13,24})}{2r^2} + \\ + \frac{r-1-(\chi_{14,23}+1)(r-1-\chi_{14,23})}{2r^2}.\end{aligned}$$

The right-hand side of this equation can be shown to be equal to $1/r \, \mathrm{Min}_{1 \leqslant i \leqslant 4}(m_i, r-1-m_i)$, an elementary but not obvious identity. □

PROPOSITION 6.2. [34] *The genus zero potential* $\Phi_0(\mathbf{t})$ *is completely determined by* $\langle \tau_{a_1,m_1} \tau_{a_2,m_2} \tau_{a_3,m_3} \tau_{a_4,m_4} \rangle_0$ *and the fact that* $\langle \tau_{a_1,m_1} \tau_{a_2,m_2} \tau_{a_3,m_3} \rangle_0 = \delta_{m_1+m_2+m_3,r-2}$.

Proof. The large phase space, genus zero potential $\Phi_0(\mathbf{t})$ is completely determined by its values on the small phase space by the topological recursion relations. Let $\Phi_0(\mathbf{x})$ denote the small phase space potential, which must satisfy the WDVV equation since $c^{1/r}$ yields a CohFT. Furthermore, the grading Equation (54) shows that the small phase space potential is a polynomial in the variables $\{x^0, \ldots, x^{r-2}\}$ of degree of at most $r+1$. One then performs an induction on the degree of the polynomial to show that $\Phi_0(\mathbf{x})$ is uniquely determined by the above data and the WDVV equation. The proof is straightforward. □

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7. Gelfand–Dickey Hierarchies and the Generalized Witten Conjecture

In this section we present a mathematical formulation of the generalized Witten conjecture, relating intersection theory on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ with Gelfand–Dickey integrable hierarchies and prove this conjecture in various cases.

7.1. GELFAND–DICKEY HIERARCHIES AND THEIR POTENTIALS

In order to fix notation and normalization constants for the generalized Witten conjecture, we recall the definition of the Gelfand–Dickey hierarchies KdV_r and their special solutions. A more detailed review can be found, for example, in [26, 34].

Fix an integer $r \geqslant 2$ and consider the space

$$\mathcal{D} = \left\{ D^r - \sum_{m=0}^{r-2} u_m(x) D^m \right\} \tag{63}$$

of differential operators in $D = (i/\sqrt{r})(\partial/\partial x)$ (the factor $(i/\sqrt{r})$ is added for convenience), where u_m are formal functional variables. For every operator $L \in \mathcal{D}$ there exists a unique pseudodifferential operator $L^{1/r} = D + \sum_{m>0} w_m D^{-m}$, such that $(L^{1/r})^r = L$. All coefficients w_m of $L^{1/r}$ are differential polynomials in $u_0, u_1, \ldots, u_{r-2}$.

For a pseudodifferential operator $Q = \sum_{m \geqslant -n} v_m D^{-m}$, denote by $Q_+ = \sum_{m=-n}^{0} v_m D^{-m}$ its differential part, and consider the following infinite family of differential equations on $\mathcal{D}$:

$$i \frac{\partial L}{\partial t_n^m} = \frac{k_{n,m}}{\sqrt{r}} \Big[(L^{n+\frac{m+1}{r}})_+, L \Big], \tag{64}$$

where the constants

$$k_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\ldots(nr+m+1)}$$

have been introduced for convenience. It can be shown that the corresponding flows on $\mathcal{D}$ commute, and thus the following definition makes sense.

DEFINITION 7.1. The infinite system (64) of partial differential equations with $r-1$ unknown functions $u_i(x, t_n^m)$, $i = 0, \ldots, r-2$, $m = 0, \ldots, r-1$, $n \geqslant 0$ is called the rth Gelfand–Dickey hierarchy or KdV_r.

The KdV_2 hierarchy is the usual Korteweg–de Vries hierarchy.

For $L = D^r - \sum_{m=0}^{r-2} u_m(x) D^m$, consider the functions

$$v_n = -\frac{r}{n+1} \operatorname{res}(L^{1/r})^{n+1}, \tag{65}$$

where the residue of a pseudodifferential operator is defined as the coefficient of D^{-1}. The functions v_k can be expressed in terms of u_j by a triangular system of differential

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polynomials. This means that u_j can be expressed in terms of v_n in a similar way, and we may regard $v_0, v_1, \ldots, v_{r-2}$ as a new system of coordinates for $\mathcal{D}$.

DEFINITION 7.2. A formal power series $\Psi(\mathbf{t})$, in variables t_n^m, $m = 0, \ldots, r-2$, $n \geqslant 0$, is called a potential of the KdV_r hierarchy if it satisfies the following conditions:

(1) $\Psi(\mathbf{0}) = 0$,
(2) the functions $v_m(\mathbf{t}) = \partial^2\Psi(\mathbf{t})/\partial t_0^0 \partial t_0^m$ satisfy the Equations (64) with $x = t_0^0$ and u_j related to v_m via (65),
(3) $\Psi(\mathbf{t})$ satisfies the string equation

$$\frac{\partial\Psi(\mathbf{t})}{\partial t_0^0} = \frac{1}{2}\sum_{m,n=0}^{r-2} \eta_{mn} t_0^m t_0^n + \sum_{k=0}^{\infty}\sum_{m=0}^{r-2} t_{k+1}^m \frac{\partial\Psi(\mathbf{t})}{\partial t_k^m}, \tag{66}$$

where $\eta_{mn} = \delta_{m+n,r-2}$.

It can be shown that the potential $\Psi(\mathbf{t})$ is uniquely determined by these conditions (cf. [36]).

Finally, we introduce the semiclassical limit of the hierarchy KdV_r (87) and its potential.

For a differential operator $L = D^r - \sum_{m=0}^{r-2} u_m(x) D^m \in \mathcal{D}$, denote by $\widetilde{L} = p^r - \sum_{m=0}^{r-2} u_m(x) p^m$ the polynomial in a formal variable p obtained by replacing D with p. The commutator $[L, Q]$ of differential operators will be replaced in (64) by the Poisson bracket

$$\{\widetilde{L}, \widetilde{Q}\} = \frac{\partial\widetilde{L}}{\partial p}\frac{\partial\widetilde{Q}}{\partial x} - \frac{\partial\widetilde{Q}}{\partial p}\frac{\partial\widetilde{L}}{\partial x}.$$

DEFINITION 7.3. The semiclassical limit KdV_r^s of the KdV_r hierarchy, is the system of equations

$$\frac{\partial\widetilde{L}}{\partial t_n^m} = \frac{k_{m,n}}{r}\left\{\widetilde{L^{n+\frac{m+1}{r}}}, \widetilde{L}\right\} \tag{67}$$

in unknown functions $u_0, \ldots, u_{r-2}$.

The corresponding potential function $\Psi_0(\mathbf{t})$ is defined as the unique function satisfying the string Equation (66) and the condition $\Psi_0(\mathbf{0}) = 0$, and such that the functions $u_0, \ldots, u_{r-2}$ given by (65) and

$$v_m(\mathbf{t}) = \frac{\partial^2\Psi_0(\mathbf{t})}{\partial t_0^0 \partial t_0^m} \tag{68}$$

satisfy the equations of the hierarchy (67).

7.2. THE GENERALIZED WITTEN CONJECTURE

Even before the moduli space $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ of r-spin curves was constructed, Witten [34] conjectured that these moduli spaces would exist, and that intersection numbers on them would assemble into the potential $\Psi(\mathbf{t})$ of the KdV_r hierarchy. Now we can give this conjecture the following mathematical formulation.

CONJECTURE 7.4. *There exists an r-spin virtual (cohomology) class $c^{1/r}$ on $\overline{\mathcal{M}}_{g,n}^{1/r,\mathbf{m}}$ satisfying Axioms 1—5 of Definition 4.1, such that the large phase space potential $\Phi(\mathbf{t})$ of the r-spin CohFT (34) coincides with the potential function $\Psi(\mathbf{t})$ of the KdV_r hierarchy.*

Using results from Sections 4, 5, and 6, we prove this conjecture in two special cases.

THEOREM 7.5. *Conjecture 7.4 holds for $r = 2$ and arbitrary g.*

Proof. Theorem 4.6 shows that when $r = 2$ the class given by (51) satisfies the axioms of a virtual class, and Corollary 4.7 implies that the large phase space potential of the corresponding 2-spin CohFT is equal to the generating function of tautological intersection numbers on $\overline{\mathcal{M}}_{g,n}$ (the large space potential of pure topological gravity). By Kontsevich's theorem [23], this generating function coincides with the potential function of the Korteweg–de Vries hierarchy, which is the same as the KdV_2 hierarchy. □

THEOREM 7.6. *Conjecture 7.4 holds for $g = 0$ and arbitrary r.*

Proof. In this case, the conjecture means that the genus zero part $\Phi_0(\mathbf{t})$ of the large phase space potential (24) of the r-spin CohFT (34) coincides with the potential $\Psi_0(\mathbf{t})$ of the semiclassical limit of the KdV_r hierarchy.

In genus zero the virtual class $c^{1/r}$ exists by Theorem 4.5. From Theorem 5.4 it follows that the corresponding potential function Φ_0 satisfies the string Equation (66).

Because of the uniqueness of the potential function of the KdV_r hierarchy (and its semiclassical approximation) all that remains is the proof of the following proposition. □

PROPOSITION 7.7. *The functions $u_m(\mathbf{t})$, $m = 0, \ldots, r-2$, given by (65) and*

$$v_m(\mathbf{t}) = \frac{\partial^2 \Phi_0(\mathbf{t})}{\partial t_0^0 \partial t_0^m}, \tag{69}$$

satisfy the equations of the semiclassical limit of the KdV_r hierarchy (67).

Proof. By Proposition 6.2 there is a unique formal power series $\Phi_0(\mathbf{t})$, of the proper grading, satisfying the equations of Proposition 6.1, WDVV, and the genus-zero topological recursion relations. Witten [34] shows by a straightforward computation that any such power series yields a solution of the semiclassical limit of the *KdV_r* hierarchy. □

COROLLARY 7.8. *The Frobenius manifold structure on* $(\mathcal{H}^{(r)}, \eta)$*, defined by Theorems 3.8 and 4.5, is isomorphic to the Frobenius structure on the base of the versal deformation of the* A_{r-1} *singularity.*

Proof. The proof follows from Theorem 7.6 and the fact that the potential of the Frobenius structure on the base of the versal deformation of the A_{r-1} singularity is equal to the potential Ψ_0 of the semiclassical limit of the KdV_r hierarchy (cf. [6]). □

Remarks 7.9. (1) The generalized Witten conjecture, as it is stated here, should be viewed as a refinement of Witten's original formulation of his conjecture [34], since it is not clear that his construction yields a class with the desired factorization properties.

(2) The coincidence of Frobenius structures given by Corollary 7.8 appears to be a genus zero manifestation of some mirror phenomenon [28], relating the moduli space of r-spin curves and singularities of type A_{r-1}.

(3) There is additional evidence for Conjecture 7.4 in genus one for arbitrary r. Witten [34] states that the formula (50) for the intersection numbers when $g = 1$ can be derived from the conjecture for all $r \geqslant 2$. Furthermore, when $r \leqslant 4$, it can be shown that Equation (62) holds for the genus-one part of the potential of the KdV_r hierarchy (cf. [5, 7]).

(4) The fact that $\Psi(\mathbf{t})$ (in all genera) is independent of the variables t_n^{r-1} for all $n \geqslant 0$ is consistent with Axiom 4 (Vanishing) of the virtual class.

The exponential of the KdV potential function is called a τ-function and can be defined as the unique function $Z(\mathbf{t})$ annihilated by certain differential operators L_i, $i \geqslant -1$, generating (a part of) the Virasoro Lie algebra. This gives an alternate formulation of the original Witten conjecture. Similarly, the exponential $Z(\mathbf{t})$ of the KdV_r potential is annihilated by a series of differential operators which forms a so-called W_r^+-algebra [1, 25] (part of which forms a subalgebra isomorphic to (half of) the Virasoro algebra). Thus we obtain an alternate formulation of the generalized Witten conjecture.

CONJECTURE 7.10 (W-algebra conjecture). *There exist a collection of differential operators forming a* W_r^+ *algebra (in which the generators* $\{L_n\}_{n \geqslant -1}$ *of the Virasoro algebra form a subset) which annihilates and completely determines* $Z(\mathbf{t})$.

This conjecture can be regarded as the KdV_r analog of a refinement of the Virasoro conjecture [8]. When $r = 2$, this conjecture reduces to the usual Virasoro highest weight condition.

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