# MODULI SPACES OF RANK 2 ACM BUNDLES ON PRIME FANO THREEFOLDS

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Abstract. Given a non-hyperelliptic prime Fano threefold $X$, we prove the existence of all rank 2 ACM vector bundles on $X$ by deformation of semistable sheaves.

## 1. Introduction

A vector bundle $F$ on a smooth polarized variety $(X, H_X)$ has no intermediate cohomology if: $\mathrm{H}^k(X, F \otimes \mathscr{O}_X(t\,H_X)) = 0$ for all $t \in \mathbb{Z}$ and $0 < k < \dim(X)$. These bundles are also called *arithmetically Cohen-Macaulay (ACM)*, because they correspond to maximal Cohen-Macaulay modules over the associated graded ring. It is known that an ACM bundle must be a direct sum of line bundles if $X = \mathbb{P}^n$ (see [Hor64]), or a direct sum of line bundles and (twisted) spinor bundles if $X$ is a smooth quadric hypersurface in $\mathbb{P}^n$ ([Knö87], [Ott89]). On the other hand, there exists a complete classification of varieties admitting, up to twist, a finite number of isomorphism classes of indecomposable ACM bundle see [EH88], [BGS87]. Only five cases exist besides rational normal curves, projective spaces and quadrics.

For varieties which are not in this list, the problem of classifying ACM bundles has been taken up only in some special cases. For instance, on general hypersurfaces of dimension at least 3, a full classification of ACM bundles of rank 2 is available, see [MKRR07a], [MKRR07b], [CM04], [CM00], [Mad00]. For dimension 2 and rank 2, a partial classification can be found in [Fae08], [CF06].

The case of smooth Fano threefolds $X$ with Picard number 1 has also been extensively studied. In this case one has $\mathrm{Pic}(X) \cong \langle H_X \rangle$, with $H_X$ ample, and $K_X = -i_X\, H_X$, where the *index* $i_X$ satisfies $1 \leq i_X \leq 4$. Recall that $i_X = 4$ implies $X \cong \mathbb{P}^3$ and $i_X = 3$ implies that $X$ is isomorphic to a smooth quadric. Thus, the class of ACM bundles is completely understood in these two cases.

In contrast to this, the cases $i_X = 2, 1$ are highly nontrivial. First of all, there are several deformation classes of these varieties, see [Isk77], [Isk78], [IP99]. A different approach to the classification of these varieties was proposed by Mukai, see for instance [Muk88], [Muk89], [Muk95].

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In second place, it is still unclear how to characterize the invariants of ACM bundles: in fact the investigation has been deeply carried out only in the case of rank 2. For $i_X = 2$, the classification was completed in [AC00]. For $i_X = 1$, a result of Madonna (see [Mad02]) implies that if a rank 2 ACM bundle $F$ is defined on $X$, then its second Chern class $c_2$ must take values in $\{1, \dots, g+3\}$ if $c_1(F) = -1$, or in $\{2, 4\}$ if $c_1(F) = 0$. Partial existence results are given in [Mad00], [AF06].

In third place, the set of ACM rank 2 bundles can have positive dimension. A natural point of view is to study them in terms of the moduli space $\mathsf{M}_X(2, c_1, c_2)$ of (Gieseker)-semistable rank 2 sheaves $F$ with $c_1(F) = c_1$, $c_2(F) = c_2$, $c_3(F) = 0$. For $i_X = 2$, such moduli space has been mostly studied when $X$ is a smooth cubic threefold, see for instance [MT01], [IM00a], [Dru00], see also [Bea02] for a survey.

If the index $i_X$ equals 1, the threefold $X$ is said to be *prime*, and one defines the *genus* of $X$ as $g = 1 + H_X^3/2$. In this case, some of the relevant moduli spaces $\mathsf{M}_X(2, 1, c_2)$ are studied in [IM00b] (for $g = 3$), [IM04], [IM07c], [BF07] (for $g = 7$), [IM07a], [IM00a] (for $g = 8$), [IR05] (for $g = 9$), [AF06] (for $g = 12$).

The goal of our paper is to provide the classification of rank 2 ACM bundles $F$ on a smooth prime Fano threefold $X$, i.e. in the case $i_X = 1$. Note that we can assume $c_1(F) \in \{0, 1\}$. Combining our existence theorems (namely Theorem 3.6 and Theorem 4.10) with the results of Madonna and others mentioned above, we obtain the following classification.

**Theorem.** *Let $X$ be a smooth prime Fano threefold, with $-K_X$ very ample. Then there exists an ACM vector bundle $F$ of rank 2*

*i) with $c_1(F) = 1$ if and only if $c_2(F) = 1$ or $\lceil \frac{g+2}{2} \rceil \leq c_2(F) \leq g + 3$,*
*ii) with $c_1(F) = 0$ if and only if $c_2(F) = 2, 4$,*

*where in case* (i) *existence holds under the further assumption that $X$ contains a line $L$ with normal bundle $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$.*

Note that the assumption that $-K_X$ is very ample (the threefold $X$ is thus called *non-hyperelliptic*) excludes two families of prime Fano threefolds, one with $g = 2$, the other with $g = 3$. These two cases will be discussed in a forthcoming paper.

On the other hand, the assumption that $X$ contains a line $L$ with normal bundle $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$ (in this case we will say that $X$ is *ordinary*) is always verified if $g \geq 9$ unless $X$ is the Mukai-Umemura threefold of genus 12. However this condition is verified by a general prime Fano threefold of any genus, see section 2.3 for more details.

The paper is organized as follows. In the following section we give some preliminary notions. Section 3 is devoted to the proof of part (i) of the main theorem. Section 4 concerns the case (ii) of the main theorem. We end the paper describing some properties of the moduli space $\mathsf{M}_X(2, 0, 4)$.

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## 2. Preliminaries

Given a smooth complex projective $n$-dimensional polarized variety $(X, H_X)$ and a sheaf $F$ on $X$, we write $F(t)$ for $F \otimes \mathscr{O}_X(tH_X)$. Given a subscheme $Z$ of $X$, we write $F_Z$ for $F \otimes \mathscr{O}_Z$ and we denote by $\mathcal{I}_{Z,X}$ the ideal sheaf of $Z$ in $X$, and by $N_{Z,X}$ its normal sheaf. We will frequently drop the second subscript.

Given a pair of sheaves $(F, E)$ on $X$, we will write $\mathrm{ext}^k_X(F, E)$ for the dimension of the Čech cohomology group $\mathrm{Ext}^k_X(F, E)$, and similarly $\mathrm{h}^k(X, F) = \dim \mathrm{H}^k(X, F)$. The Euler characteristic of $(F, E)$ is defined as $\chi(F, E) = \sum_k (-1)^k \mathrm{ext}^k_X(F, E)$ and $\chi(F)$ is defined as $\chi(\mathscr{O}_X, F)$. We denote by $p(F, t)$ the Hilbert polynomial $\chi(F(t))$ of the sheaf $F$. The degree $\deg(L)$ of a divisor class $L$ is defined as the degree of $L \cdot H_X^{n-1}$.

### 2.1. ACM sheaves.

Let $(X, H_X)$ be a $n$-dimensional polarized variety, and assume $H_X$ very ample, so we have $X \subset \mathbb{P}^m$. We denote by $I_X$ the ideal of $X$ in $\mathbb{P}^m$, and by $R(X)$ the coordinate ring of $X$. Given a sheaf $F$ on $X$, we define the following $R(X)$-modules:

$$\mathrm{H}^k_*(X, F) = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^k(X, F(t)), \qquad \text{for each } k = 0, \dots, n.$$

The variety $X$ is said to be arithmetically Cohen-Macaulay (ACM) if $R(X)$ is a Cohen-Macaulay ring. This is equivalent to $\mathrm{H}^1_*(\mathbb{P}^m, \mathcal{I}_{X,\mathbb{P}^m}) = 0$ and $\mathrm{H}^k_*(\mathbb{P}^m, \mathscr{O}_X) = 0$ for $0 < k < n$. A sheaf $F$ on a variety $X$ is called locally Cohen-Macaulay if for any point $x \in X$ we have $\mathrm{depth}(F_x) = \dim(\mathscr{O}_x)$.

**Definition 2.1.** A sheaf $F$ on a ACM variety $X$ is called *ACM* (*arithmetically Cohen-Macaulay*) if $F$ is locally Cohen-Macaulay and it has no intermediate cohomology, i.e.:

$$\mathrm{H}^k_*(X, F) = 0 \quad \text{for any } 0 < k < 0\ . \tag{2.1}$$

By [CH04, Proposition 2.1], there is a one-to-one correspondence between ACM sheaves on $X$ and graded maximal Cohen-Macaulay modules on $R(X)$, given by $F \mapsto \mathrm{H}^0_*(X, F)$.

If $X$ is smooth and $F$ is locally free, then $F$ is ACM if and only if the cohomological condition (2.1) holds. Moreover if $X$ is smooth, any ACM sheaf is locally free.

### 2.2. Stability and moduli spaces.

Let us now recall a few well-known facts about semistable sheaves on projective varieties. We refer to the book [HL97] for a more detailed account of these notions.

Let $(X, H_X)$ be a smooth complex projective $n$-dimensional polarized variety. We recall that a torsionfree coherent sheaf $F$ on $X$ is (Gieseker) *semistable* if for any coherent subsheaf $E$, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $p(E, t)/\mathrm{rk}(E) \leq p(F, t)/\mathrm{rk}(F)$ for $t \gg 0$. The sheaf $F$ is called *stable* if the inequality above is always strict.

The *slope* of a sheaf $F$ of positive rank is defined as $\mu(F) = \deg(c_1(F))/\mathrm{rk}(F)$, where $c_1(F)$ is the first Chern class of $F$. We say that $F$ is *normalized* if $-1 < \mu(F) \leq 0$. We recall that a torsionfree coherent sheaf $F$ is *$\mu$-semistable* if for any coherent subsheaf $E$, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $\mu(E) < \mu(F)$. The sheaf $F$ is called *$\mu$-stable* if the above inequality is always strict. The two notions are related by the following implications:

$$\mu\text{-stable} \quad \Rightarrow \quad \text{stable} \quad \Rightarrow \quad \text{semistable} \quad \Rightarrow \quad \mu\text{-semistable}$$

Recall that by Maruyama's theorem, see [Mar80], if $\dim(X) = n \geq 2$ and $F$ is a $\mu$-semistable sheaf of rank $r < n$, then its restriction to a general hypersurface of $X$ is still $\mu$-semistable.

We will make use of different notions of moduli spaces. Let us introduce here some notation. We denote by $\mathsf{M}_X(r, c_1, \ldots, c_n)$ the moduli space of $S$-equivalence classes of rank $r$ torsionfree semistable sheaves on $X$ with Chern classes $c_1, \ldots, c_n$, where $c_k$ lies in $\mathrm{H}^{k,k}(X)$. The Chern class $c_k$ will be denoted by an integer as soon as $\mathrm{H}^{k,k}(X)$ has dimension 1. We will drop the last values of the classes $c_k$ when they are zero.

We denote by $\mathsf{M}^{\mu}_X(r, c_1, \ldots, c_r)$ the moduli space of $S$-equivalence classes of $\mu$-semistable sheaves with fixed rank and Chern classes. Given a filtration of a sheaf $F$, we denote by $\mathrm{gr}(F)$ the associated graded object.

The moduli space of simple sheaves with fixed rank and Chern classes will be denoted by $\mathsf{Spl}_X(r, c_1, \ldots, c_n)$. For an account on this moduli space we refer to [AK80], [KO89], [LO87], see also [Muk84].

We denote by $\mathscr{H}^g_d(X)$ the union of components of the Hilbert scheme of closed $Z$ subschemes of $X$ with Hilbert polynomial $p(\mathscr{O}_Z, t) = dt + 1 - g$, containing integral curves of degree $d$ and arithmetical genus $g$.

2.3. **Prime Fano threefolds.** Let now $X$ be a smooth projective variety of dimension 3. Recall that $X$ is called *Fano* if its anticanonical divisor class $-K_X$ is ample. A Fano threefold is called *non-hyperelliptic* if $-K_X$ is very ample.

We say that $X$ is *prime* if the Picard group is generated by the canonical divisor class $K_X$. If $X$ is a prime Fano threefold we have $\mathrm{Pic}(X) \cong \mathbb{Z} \cong \langle H_X \rangle$, where $H_X = -K_X$ is ample. These varieties are classified up to deformation, see for instance [IP99, Chapter IV]. The number of deformation classes is 10. It is well-known that a smooth prime Fano threefold is ACM.

One defines the *genus* of a prime Fano threefold $X$ as the integer $g$ such that $\deg(X) = H_X^3 = 2\,g - 2$. It is known that the genus of a non-hyperelliptic prime Fano threefolds takes values in $\{3, \ldots, 10, 12\}$, while there are two families (one for $g = 2$, another for $g = 3$) that consist of hyperelliptic threefolds. A hyperelliptic prime Fano threefold of genus 3 is a flat specialization of a quartic hypersurface in $\mathbb{P}^4$, see [Man93] and references therein.

It is well known that any prime Fano threefold $X$ contains lines and conics. The Hilbert scheme $\mathscr{H}^0_1(X)$ of lines contained in $X$ is a projective curve. It is well known that the surface swept out by the lines of a prime Fano threefold $X$ is linearly equivalent to the divisor $rH_X$, for some $r \geq 2$, see e.g. [IP99, Table at page 76]. Moreover every line meets finitely many other lines, see [Isk78, Theorem 3.4, iii] and [Man93].

A prime Fano threefold $X$ is said to be *exotic* if the Hilbert scheme $\mathscr{H}^0_1(X)$ has a nonreduced component. By [Isk78, Lemma 3.2], this is equivalent to the fact that for any line $L \subset X$ of this component, the normal bundle $N_L$ splits as $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-2)$. It is well-known that a general prime anticanonical threefold $X$ is not exotic. On the other hand, for $g \geq 9$, the results of [GLN06] and [Pro90] imply that $X$ is nonexotic unless $g = 12$ and $X$ is the Mukai-Umemura threefold, see [MU83]. We say that a prime Fano threefold $X$ is *ordinary* if the Hilbert scheme $\mathscr{H}_1^0(X)$ has a reduced component. This is equivalent to the fact that there exists a line $L \subset X$ whose normal bundle $N_L$ splits as $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$.

If $X$ is a non-hyperelliptic prime Fano threefold, the Hilbert scheme $\mathscr{H}_2^0(X)$ of conics contained in $X$ is a projective surface, and a general conic $C$ in $X$ has trivial normal bundle, see [Isk78, Proposition 4.3 and Theorem 4.4]. Moreover, the threefold $X$ is covered by conics. Moreover if $X$ is a general prime Fano threefold, then $\mathscr{H}_2^0(X)$ is a smooth surface, see [IM07b] for a survey.

Recall also that a smooth projective surface $S$ is a *K3 surface* if it has trivial canonical bundle and irregularity zero. A general hyperplane section $S$ of a prime Fano threefold $X$ is a K3 surface, polarized by the restriction $H_S$ of $H_X$ to $S$. If $X$ has genus $g$, then $S$ has (sectional) genus $g$, and degree $H_S^2 = 2\,g - 2$. If $X$ is non-hyperelliptic, by Moishezon's theorem [Moĭ67], we have $\mathrm{Pic}(S) \cong \mathbb{Z} = \langle H_S \rangle$.

Note that a general hyperplane section of a hyperelliptic prime Fano threefold is still a K3 surface of Picard number 1 if $g = 2$. This is no longer true in the other hyperelliptic case, i.e. for $g = 3$. Indeed, let $X$ be a double cover of a smooth quadric in $\mathbb{P}^4$ ramified along a general octic surface. Then the general hyperplane section is a K3 surface of Picard number 2.

2.4. **Summary of basic formulas.** From now on, $X$ will be a prime smooth Fano threefold of genus $g$, polarized by $H_X$. Let $F$ be a sheaf of (generic) rank $r$ on $X$ with Chern classes $c_1, c_2, c_3$. Recall that these classes will be denoted by integers, since $\mathrm{H}^{k,k}(X)$ is generated by the class of $H_X$ (for $k = 1$), the class of a line contained in $X$ (for $k = 2$), the class of a closed point of $X$ (for $k = 3$). We will say that $F$ is an $r$-bundle if it is a vector bundle (i.e. a locally free sheaf) of rank $r$. We recall that the *discriminant* of $F$ is

$$\Delta(F) = 2\,r\,c_2 - (r-1)(2\,g-2)\,c_1^2. \tag{2.2}$$

Bogomolov's inequality, see for instance [HL97, Theorem 3.4.1], states that if $F$ is a $\mu$-semistable sheaf, then we have:

$$\Delta(F) \geq 0. \tag{2.3}$$

Applying to the sheaf $F$ the theorem of Hirzebruch-Riemann-Roch, we obtain the following formulas:

$$\chi(F) = r + \frac{11+g}{6}\,c_1 + \frac{g-1}{2}\,c_1^2 - \frac{1}{2}c_2 + \frac{g-1}{3}\,c_1^3 - \frac{1}{2}\,c_1\,c_2 + \frac{1}{2}\,c_3,$$
$$\chi(F,F) = r^2 - \frac{1}{2}\Delta(F).$$

We recall by [Muk84], see also [HL97, Part II, Chapter 6] that, given a simple sheaf $F$ of rank $r$ on a K3 surface $S$ of genus $g$, with Chern classes $c_1, c_2$, the dimension at the point $[F]$ of the moduli space $\mathsf{Spl}_S(r, c_1, c_2)$ is:

$$\Delta(F) - 2\,(r^2 - 1), \tag{2.4}$$

where $\Delta(F)$ is still defined by (2.2). If the sheaf $F$ is stable, this value also equals the dimension at the point $[F]$ of the moduli space $\mathsf{M}_S(r, c_1, c_2)$.

We recall now the well-known Hartshorne-Serre correspondence, relating vector bundles of rank 2 over $X$ with subvarieties $Z$ of $X$ of codimension 2. For more details we refer to [Har74], [Har78], [Har80, Theorem 4.1], see also [Arr07] for a survey.

**Proposition 2.2.** *Let $X$ be a smooth prime Fano threefold, and fix the integers $c_1, c_2$. Then we have a one-to-one correspondence between*

1. *the pairs $(F, s)$, where $F$ is a rank 2 vector bundle on $X$ with $c_i(F) = c_i$ and $s$ is a global section of $F$ whose zero locus has codimension 2, and*
2. *the locally complete intersection Cohen-Macaulay curves $Z \subset X$, with $\omega_Z \cong \mathscr{O}_Z(c_1 - 1)$, and of degree $c_2$.*

Recall that in the above correspondence $Z$ has arithmetical genus $p_a(Z) = 1 - \frac{d(1-c_1)}{2}$. The zero locus of a nonzero global section $s$ of a rank 2 vector bundle $F$ has codimension 2 if $F$ is globally generated and $s$ is general, or if $\mathrm{H}^0(X, F(-1)) = 0$.

**Lemma 2.3.** *Let $X$ be a smooth prime Fano threefold and let $s$ be a nonzero global section of a locally free sheaf $F$ on $X$, whose zero locus is a curve $D \subset X$. Then we have:*

$$\mathrm{H}^1_*(X, F) \cong \mathrm{H}^1_*(X, \mathcal{I}_D(1)). \tag{2.5}$$

*In particular $F$ is ACM if and only if $D$ is projectively normal.*

*Proof.* The section $s$ gives the exact sequence:

$$0 \to \mathscr{O}_X \to F \to \mathcal{I}_D(c_1(F)) \to 0, \tag{2.6}$$

and taking cohomology we obtain the required isomorphism (2.5).

Let now $X \subset \mathbb{P}^m$ and $D \subset \mathbb{P}^\ell$ with $\ell \geq 2$, $m \geq 4$. By definition, the curve $D$ is projectively normal if and only if $\mathrm{H}^1_*(\mathbb{P}^\ell, \mathcal{I}_{D,\mathbb{P}^\ell}) = 0$. Clearly, this is equivalent to $\mathrm{H}^1_*(\mathbb{P}^m, \mathcal{I}_{D,\mathbb{P}^m}) = 0$. We have the exact sequence:

$$0 \to \mathcal{I}_{X,\mathbb{P}^m} \to \mathcal{I}_{C,\mathbb{P}^m} \to \mathcal{I}_{C,X} \to 0.$$

Taking cohomology, we get that $D$ is projectively normal if and only if $\mathrm{H}^1_*(X, \mathcal{I}_D(1)) = 0$, which is equivalent to $F$ being ACM by (2.5) and Serre duality. $\square$

2.5. **ACM bundles of rank 2.** In this section, we recall Madonna's result in the case of bundles of rank 2 on a smooth prime Fano threefold.

**Proposition 2.4** ([Mad02])**.** *Let $X$ be a smooth prime Fano threefold of genus $g$, and let $F$ be a normalized ACM 2-bundle on $X$. Then the Chern classes $c_1$ and $c_2$ of $F$ satisfy the following restrictions:*

$$c_1 = 0 \Rightarrow c_2 \in \{2, 4\},$$
$$c_1 = -1 \Rightarrow c_2 \in \{1, \dots, g+3\}.$$

**Remark 2.5.** Let $F$, $c_1$, $c_2$ be as above, and $t_0$ be the smallest integer $t$ such that $\mathrm{H}^0(X, F(t)) \neq 0$. In [Mad02] the author computes the following values of $t_0$:

a) if $(c_1, c_2) = (-1, 1)$, then $t_0 = 0$,
b) if $(c_1, c_2) = (0, 2)$, then $t_0 = 0$,
c) if $(c_1, c_2) = (-1, c_2)$, with $2 \leq c_2 \leq g + 2$, then $t_0 = 1$,
d) if $(c_1, c_2) = (0, 4)$, then $t_0 = 1$,
e) if $(c_1, c_2) = (-1, g + 3)$, then $t_0 = 2$.

We observe that $F$ is not semistable in cases (a) and (b), and strictly $\mu$-semistable in case (b). In fact, these two cases correspond respectively to lines and conics contained in $X$, and the existence of $F$ is well-known. On the other hand, in the remaining cases, if $F$ exists then it is a $\mu$-stable sheaf.

The following lemma (see [BF07, Lemma 3.1]) gives a small (sharp) restriction on the values in Madonna's list.

**Lemma 2.6.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g$ and set:*

$$m_g = \left\lceil \frac{g+2}{2} \right\rceil. \tag{2.7}$$

*Then the moduli space $\mathsf{M}_X(2, 1, d)$ is empty for $d < m_g$. In particular, we get the further restriction $c_2 \geq m_g$ in case* (c).

**Remark 2.7.** The assumption non-hyperelliptic cannot be dropped. Indeed if $X$ is a hyperelliptic Fano threefold of genus 3, then the moduli space $\mathsf{M}_X(2, 1, 2)$ is not empty. Indeed let $Q \in \mathbb{P}^4$ be a smooth quadric and $\pi : X \to Q$ be a double cover ramified along a general octic surface. Let $\mathscr{S}$ be the spinor bundle on $Q$ and set $F = \pi^*(\mathscr{S})$. Then $F$ is a stable vector bundle on $X$ lying in $\mathsf{M}_X(2, 1, 2)$. Notice that the restriction $F_S$ to any hyperplane section $S \subset X$ is strictly semistable.

## 3. Bundles with odd first Chern class

Throughout the paper, we denote by $X$ a smooth non-hyperelliptic prime Fano threefold of genus $g$. In this section we will prove the existence of the semistable bundles appearing in the (restricted) Madonna's list, whose first Chern class is odd. We will study first the case of minimal $c_2$ and then we proceed recursively. We conclude this section studying the curves arising as the zero locus of a global section of a rank 2 ACM bundle.

### 3.1. **Moduli of ACM 2-bundles with minimal $c_2$.**

In this section we study the moduli space with odd $c_1$ and minimal $c_2$. Let $F$ be a sheaf of rank 2. If the sheaf $F$ is locally free, we have $F \cong F^*(c_1(F))$. Notice that, if $c_1(F)$ is odd, $F$ is $\mu$-stable as soon as it is $\mu$-semistable.

The following theorem is well known. Up to the authors' knowledge, there is no proof of this result, other than a case-by-case analysis. We refer e.g. to [Mad00] for $g = 3$, [Mad02] for $g = 4, 5$, [Gus82] for $g = 6$, [IM04], [IM07c], [Kuz05], for $g = 7$, [Gus83], [Gus92], [Muk89] for $g = 8$, [IR05] for $g = 9$, [Muk89] for $g = 10$, [Kuz96] (see also [Sch01], [Fae07]) for $g = 12$.

**Theorem 3.1.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g$, and set $m_g = \left\lceil \frac{g+2}{2} \right\rceil$. Then $\mathsf{M}_X(2, 1, m_g)$ is not empty.*

The following result appears in [BF07, Proposition 3.4], and is essentially due to Iliev and Manivel.

**Proposition 3.2.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g$ and set $m_g$ as in (2.7). Then any sheaf $F$ in $\mathsf{M}_X(2,1,m_g)$ is stable, locally free and it satisfies:*

$$\begin{aligned} &\mathrm{H}^k(X, F(-1)) = 0, && \textit{for all } k \in \mathbb{Z}, \\ &\mathrm{H}^k(X, F) = 0, && \textit{for all } k \geq 1, \\ &\mathrm{h}^0(X, F) = g - m_g + 3. && \end{aligned}$$

*Moreover i f $g \geq 6$, then any sheaf $F$ in $\mathsf{M}_X(2,1,m_g)$ is globally generated and ACM.*

In the next lemmas we review some results concerning the moduli space $\mathsf{M}_X(2,1,m_g)$. Though all the statements are essentially well-known, we present a proof in some cases where we outline a more direct argument, or when no explicit reference is available in the literature.

**Lemma 3.3.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g \geq 6$, and let $F$ and $F'$ be sheaves in $\mathsf{M}_X(2,1,m_g)$. Then we have $\mathrm{Ext}^2_X(F,F') = 0$. In particular the space $\mathsf{M}_X(2,1,m_g)$ is smooth. If $g$ is even, this implies that $\mathsf{M}_X(2,1,m_g)$ consists of a single point.*

*Proof.* Recall by Proposition 3.2 that $F$ and $F'$ are stable locally free globally generated ACM sheaves. We can thus write the natural evaluation exact sequence:

$$0 \to K \to \mathrm{H}^0(X, F') \otimes \mathscr{O}_X \to F' \to 0, \tag{3.1}$$

where the sheaf $K$ is locally free, and we have:

$$\mathrm{rk}(K) = g - m_g + 1, \qquad c_1(K) = -1.$$

Note that $K$ is a stable bundle by Hoppe's criterion, see for instance [AO94, Theorem 1.2], [Hop84, Lemma 2.6]. Indeed, note that $\mathrm{H}^0(X,K) = 0$, and we have $-1 < \mu(\wedge^p K) < 0$. By the inclusion:

$$\wedge^p K \hookrightarrow \wedge^{p-1} K \otimes \mathrm{H}^0(X, F),$$

we obtain recursively $\mathrm{H}^0(X, \wedge^p K) = 0$ for all $p \geq 0$.

Note that, by stability and by Proposition 3.2, we have $\mathrm{H}^k(X, F^*) = 0$ all $k$. Thus, tensoring (3.1) by $F^*$, we obtain:

$$\mathrm{ext}^2_X(F,F') = \mathrm{h}^2(X, F^* \otimes F') = \mathrm{h}^3(X, F^* \otimes K) = \mathrm{h}^0(X, K^* \otimes F^*) = 0,$$

where the last equality holds by stability, indeed $c_1(K^* \otimes F^*) = m_g - g + 1 < 0$ for $g \geq 6$.

Note that, when $g$ is even, we have $\chi(F,F') = 1$. This gives $\mathrm{Hom}_X(F,F') \neq 0$. But a nonzero morphism $F \to F'$ has to be an isomorphism. This concludes the proof. □

**Lemma 3.4.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g \geq 6$ and $g$ odd. Then the space $\mathsf{M}_X(2,1,m_g)$ is fine and isomorphic to a smooth irreducible curve.*

*Proof.* Given a sheaf $F$ in $\mathsf{M}_X(2,1,m_g)$, by Lemma 3.2, it is an ACM globally generated vector bundle. By Lemma 3.3, the moduli space is smooth and, since $\chi(F,F) = 0$, we have $\mathrm{ext}^1_X(F,F) = \hom_X(F,F) = 1$. Thus $\mathsf{M}_X(2,1,m_g)$ is a nonsingular curve.

Recall now that the obstruction to the existence of a universal sheaf on $X \times \mathsf{M}_X(2,1,m_g)$ corresponds to an element of the Brauer group of $\mathsf{M}_X(2,1,m_g)$, see for instance [Căl00], [Căl02]. But this group vanishes as soon as the variety $\mathsf{M}_X(2,1,m_g)$ is a nonsingular curve (see again [Căl02]). Hence we have a universal vector bundle on $X \times \mathsf{M}_X(2,1,m_g)$. We consider a component $\mathsf{M}$ of $\mathsf{M}_X(2,1,m_g)$ and we let $\mathscr{E}$ be the restriction of the universal sheaf to $X \times \mathsf{M}$. We let $p$ and $q$ be the projections of $X \times \mathsf{M}$ respectively to $X$ and $\mathsf{M}$.

We prove now the irreducibility of $\mathsf{M}_X(2,1,m_g)$. Denoting by $\mathscr{E}_y$ the restriction of $\mathscr{E}$ to $X \times \{y\}$, we have $\mathscr{E}_y \cong \mathscr{E}_z$ if and only if $y = z$, for $y, z \in \mathsf{M}$. Moreover, for any sheaf $F$ in $\mathsf{M}_X(2,1,m_g)$, we have:

$$\mathrm{Ext}^k_X(\mathscr{E}_y, F) = 0, \qquad \text{for } k = 2, 3, \text{ and for all } k \text{ if } F \ncong \mathscr{E}_y,$$

where the vanishing for $k = 2$ follows from Lemma 3.3. Hence we have:

$$\begin{aligned} &\mathbf{R}^k q_*(\mathscr{E}^* \otimes p^*(F)) = 0, && \text{for } k \neq 1, \\ &\mathbf{R}^1 q_*(\mathscr{E}^* \otimes p^*(F)) \cong \mathscr{O}_y, && \text{for } F \cong \mathscr{E}_y. \end{aligned}$$

In particular, for any sheaf $F$ in $\mathsf{M}_X(2,1,m_g)$, the sheaf $\mathbf{R}^1 q_*(\mathscr{E}^* \otimes p^*(F))$ has rank zero and we have $\chi(\mathbf{R}^1 q_*(\mathscr{E}^* \otimes p^*(F))) = 1$, for this value can be computed by Grothendieck-Riemann-Roch. Thus there must be a point $y \in \mathsf{M}$ such that $\mathrm{Ext}^1_X(\mathscr{E}_y, F) \neq 0$, hence $\mathrm{Hom}_X(\mathscr{E}_y, F) \neq 0$, so $F \cong \mathscr{E}_y$. This implies that $F$ belongs to $\mathsf{M}$. □

**Proposition 3.5.** *Let $X$ be a smooth ordinary non-hyperelliptic prime Fano threefold of genus $g$ and let $F$ be a sheaf in $\mathsf{M}_X(2,1,m_g)$. Then $F$ is locally free and ACM. Furthermore, the moduli space $\mathsf{M}_X(2,1,m_g)$ contains a sheaf $F$ satisfying the conditions:*

(3.2) $\quad \mathrm{Ext}^2_X(F,F) = 0,$

(3.3) $\quad F \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(1), \qquad$ *for some line $L$ with $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$.*

*Moreover, for each value of $g$, the following table summarizes the properties of $F$ and $\mathsf{M}_X(2,1,m_g)$.*

| $g$ | $H^3_X$ | $m_g$ | $F$ *g.g.* | *The space* $\mathsf{M}_X(2,1,m_g)$ | $\chi(F,F)$ |
|---|---|---|---|---|---|
| *3* | *4* | *3* | $\nexists$ | $\mathscr{H}^0_1(X)$ | *0* |
| *4* | *6* | *3* | $\exists$ | *two points* | *1* |
| *5* | *8* | *4* | $\exists$ | *Hesse septic curve* | *0* |
| *6* | *10* | *4* | $\forall$ | *one point* | *1* |
| *7* | *12* | *5* | $\forall$ | *smooth curve of genus 7* | *0* |
| *8* | *14* | *5* | $\forall$ | *one point* | *1* |
| *9* | *16* | *6* | $\forall$ | *smooth plane quartic* | *0* |
| *10* | *18* | *6* | $\forall$ | *one point* | *1* |
| *12* | *22* | *7* | $\forall$ | *one point* | *1* |

*Proof.* If $g \geq 6$, the moduli space $\mathsf{M}_X(2,1,m_g)$ is smooth by Lemma 3.3, and irreducible if $g$ is even by the same lemma. If $g$ is odd, irreducibility follows from Lemma 3.4. From Proposition 3.2 it follows that any sheaf $F$ in $\mathsf{M}_X(2,1,m_g)$ is a globally generated vector bundle which is ACM. The sheaf $F$ in this case will also satisfy (3.3), indeed $F$ has degree 1 and is globally generated on $L$.

Note that (3.2) is equivalent to the moduli space $\mathsf{M}_X(2,1,m_g)$ being smooth of expected dimension $1-\chi(F,F)$ at the point $[F]$. On the other hand, (3.3) holds if there exists a global section of $F$ whose vanishing locus does not meet $L$.

Let us now review in detail the cases $3 \le g \le 5$.

If $g = 3$, the threefold $X$ is a smooth quartic hypersurface in $\mathbb{P}^4$, since $X$ is not hyperelliptic. By [Mad00], the moduli space $\mathsf{M}_X(2,1,3)$ is birational to the Hilbert scheme $\mathscr{H}_1^0(X)$ of lines contained in $X$, which has a reduced component for $X$ is ordinary. Indeed, a general plane $\Lambda$ containing a line $L \subset X$ defines a plane cubic $C$ as the residual curve of $L$ in $X \cap \Lambda$. The curve $C$ is thus projectively normal, and by Hartshorne-Serre's construction (see Proposition 2.2 and Lemma 2.3), this provides an ACM vector bundle $F$ lying in $\mathsf{M}_X(2,1,3)$. Choosing a general line $L'$ which does not meet $C$, we get (3.3). One can easily prove that $F$ cannot be globally generated, for it has only three independent global sections.

For $g = 4$, the threefold $X$ must be the complete intersection of a quadric $Q$ and a cubic in $\mathbb{P}^5$. The result follows from [Mad02, Section 3.2], and we only have to check (3.2). If the quadric $Q$ is nonsingular, one considers the bundles $F_1$ and $F_2$ obtained restricting to $X$ the two nonisomorphic spinor bundles $\mathscr{S}_1$ and $\mathscr{S}_2$ on $Q$. Note that $F_1$ and $F_2$ are not isomorphic, with $\mathrm{Ext}^k_X(F_i,F_i) = 0$ for $k \ge 1$. One can check this computing the vanishing of $\mathrm{H}^k(Q, \mathscr{S}_i \otimes \mathscr{S}_i^*(-3))$ for all $k$, which in turn follows from Bott's theorem. Thus the moduli space $\mathsf{M}_X(2,1,4)$ consists of two reduced points. Each of the bundles $F_i$ is globally generated, hence (3.3) follows. By Bott's theorem each $\mathscr{S}_i$ is ACM, hence each $F_i$ is ACM too. We leave the remaining case to the reader, namely, when the rank of the quadric $Q$ is 5.

For $g = 5$, the threefold $X$ is obtained as the complete intersection of three quadrics in $\mathbb{P}^6$. One considers thus the net generated by the three quadrics as a projective plane $\Pi$. The set of singular quadrics in this net is the so-called Hesse curve $\Delta \subset \Pi$. It is a plane septic with only ordinary double points, see for instance [Bea77]. Following [Mad02, Section 3.2], one can prove that the moduli space $\mathsf{M}_X(2,1,4)$ is birational to the curve $\Delta$, so (3.2) holds. More precisely, a singular quadric corresponding to a point of $\Delta$ contains a $\mathbb{P}^3$ which cuts on $X$ a projectively normal elliptic quartic $C$. The Hartshorne-Serre correspondence thus provides an ACM vector bundle $F$. Since the ideal sheaf of $C$ is generated by three linear forms, the vector bundle $F$ is globally generated and (3.3) follows. $\square$

3.2. **Moduli of ACM 2-bundles with higher $c_2$.** The main result of this section is the following existence theorem.

**Theorem 3.6.** *Let $X$ be an ordinary smooth non-hyperelliptic prime Fano threefold of genus $g$. Then there exists an ACM vector bundle $F$ of rank 2 with $c_1(F) = 1$ and $c_2(F) = d$ for any $m_g \le d \le g+3$. The bundle $F$ lies in a reduced component of dimension $2d-g-2$ of $\mathsf{M}_X(2,1,d)$.*

We will need a series of lemmas. The following one is proved in [BF07, Theorem 3.7].

**Lemma 3.7.** *Let $X$ be as in Theorem 3.6 and let $L$ be a general line in $X$. Then, for any integer $d \geq m_g$, there exists a rank 2 stable locally free sheaf $F$ with $c_1(F) = 1$, $c_2(F) = d$, and satisfying:*

$$\operatorname{Ext}^2_X(F, F) = 0, \tag{3.4}$$

$$\mathrm{H}^1(X, F(-1)) = 0, \tag{3.5}$$

$$F \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(1), \tag{3.6}$$

*where $L$ is a line with $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$.*

As shown in the proof of [BF07, Theorem 3.7], we recall that given a stable 2-bundle $F_{d-1}$ with $c_1(F_{d-1}) = 1$ and $c_2(F_{d-1}) = d - 1$, satisfying the properties (3.4), (3.5) and (3.6) and a general $L$ line contained in $X$, we have the following exact sequence:

$$0 \to \mathscr{S}_d \to F_{d-1} \xrightarrow{\sigma} \mathscr{O}_L \to 0, \tag{3.7}$$

where $\sigma$ is the natural surjection and $\mathscr{S}_d = \ker(\sigma)$ is a nonreflexive sheaf in $\mathsf{M}_X(2, 1, d)$.

**Definition 3.8.** Let $\mathsf{M}(m_g)$ be a component of $\mathsf{M}_X(2, 1, m_g)$ containing a stable locally free sheaf $F$ satisfying the three conditions (3.4), (3.5) and (3.6). This exists by Proposition 3.5, and coincides with $\mathsf{M}_X(2, 1, m_g)$ for $g \geq 6$. For each $d \geq m_g + 1$, we recursively define $\mathsf{N}(d)$ as the set of nonreflexive sheaves $\mathscr{S}_d$ fitting as kernel in an exact sequence of the form (3.7), with $F_{d-1} \in \mathsf{M}(d - 1)$, and $\mathsf{M}(d)$ as the component of the moduli scheme $\mathsf{M}_X(2, 1, d)$ containing $\mathsf{N}(d)$. We have:

$$\dim(\mathsf{M}(d)) = 2\,d - g - 2.$$

**Lemma 3.9.** *For each $m_g \leq d \leq g + 2$, the general element $F_d$ of $\mathsf{M}_X(d)$ satisfies:*

$$\mathrm{h}^0(X, F_d) = g + 3 - d,$$
$$\mathrm{H}^k(X, F_d) = 0, \qquad \textit{for } k \geq 1.$$

*Proof.* Note that $\mathrm{H}^3(X, F_d) = 0$ for all $d$ by Serre duality and stability, and by Riemann-Roch we have $\chi(F_d) = g + 3 - d$.

We prove the remaining claim by induction on $d$. The first step of the induction corresponds to $d = m_g$, and it follows from Proposition 3.5. Assume now that the statement holds for $F_{d-1}$ with $d \leq g + 2$, and let us prove it for a general element $\mathscr{S}_d$ of $\mathsf{N}_X(d)$. By semicontinuity the claim will follow for the general element $F_d \in \mathsf{M}(d)$. So let $F_{d-1}$ be a locally free sheaf in $\mathsf{M}_X(d - 1)$. By induction we know that $\mathrm{h}^0(X, F_d) = g + 3 - d + 1 \geq 2$. A nonzero global section $s$ of $F_{d-1}$ gives the exact sequence:

$$0 \to \mathscr{O}_X \xrightarrow{s} F_{d-1} \to \mathcal{I}_C(1) \to 0, \tag{3.8}$$

where $C$ is a curve of degree $d - 1$ and arithmetic genus 1. We have $\mathrm{h}^0(X, \mathcal{I}_C(1)) = g + 3 - d \geq 1$, so $C$ is contained in some hyperplane section surface $S$ given by a global section $t$ of $\mathcal{I}_C(1)$. Let $L$ be a general line such that $F_{d-1} \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(1)$ and $L$ meets $S$ at a single point $x$. We may assume the latter condition because there exists a line in $X$ not contained in $S$. Indeed, the lines contained in $X$ sweep a divisor of degree greater than one, see section 2.3.

We can write down the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathscr{O}_X \arrow[r, equal] \arrow[d, "t"] & \mathscr{O}_X \arrow[d, "t"] & \\
0 \arrow[r] \mathscr{O}_X \arrow[r, "s"] \arrow[d, equal] & F_{d-1} \arrow[r, "s^\top"] \arrow[d, "t^\top"] & \mathcal{I}_C(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] \mathscr{O}_X \arrow[r, "s"] \arrow[d] & \mathcal{I}_D(1) \arrow[r] \arrow[d] & \mathscr{O}_S(H_S - C) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

which in turn yields the exact sequence:

$$0 \to \mathscr{O}_X^2 \xrightarrow{\binom{s}{t}} F_{d-1} \to \mathscr{O}_S(H_S - C) \to 0,$$

and dualizing we obtain:

$$0 \to F_{d-1}^* \xrightarrow{(s^\top\, t^\top)} \mathscr{O}_X^2 \to \mathscr{O}_S(C) \to 0. \tag{3.9}$$

Thus the curve $C$ moves in a pencil without base points in the surface $S$, and each member $C'$ of this pencil corresponds to a global section $s'$ of $F_{d-1}$ which vanishes on $C'$. Therefore we can choose $s$ so that $C$ does not contain $x$. In particular we get $\mathscr{O}_L \otimes \mathcal{I}_C(1) \cong \mathscr{O}_L(1)$. Now let $\sigma$ be the natural surjection $F_{d-1} \to \mathscr{O}_L$ and $\mathscr{S}_d = \ker(\sigma)$. We have thus the exact sequence (3.7). Taking global sections we obtain $\mathrm{H}^2(X, \mathscr{S}_d) = 0$.

By tensoring (3.8) by $\mathscr{O}_L$ we see that the composition $\sigma \circ s$ must be nonzero (in fact it is surjective). Thus the section $s$ does not lift to $\mathscr{S}_d$ for $d \leq g+2$, so $\mathrm{h}^0(X, \mathscr{S}_d) \leq \mathrm{h}^0(X, F_{d-1}) - 1$. We have thus:

$$\mathrm{h}^0(\mathscr{S}_d) \geq \chi(\mathscr{S}_d) = \chi(F_{d-1}) - 1 = \mathrm{h}^0(F_{d-1}) - 1 \geq \mathrm{h}^0(\mathscr{S}_d),$$

and our claim follows. □

Using Lemma 2.3 we deduce the following corollary.

**Corollary 3.10.** *For $d \leq g+2$, let $D$ be the zero locus of a nonzero global section of a general element $F$ of $\mathsf{M}_X(d)$. Then we have:*

$$\mathrm{h}^0(X, \mathcal{I}_D(1)) = g + 2 - d,$$

**Lemma 3.11.** *For $d \geq g+3$, the general element $F_d$ of $\mathsf{M}_X(d)$ satisfies:*

$$\begin{aligned} &\mathrm{h}^1(X, F_d) = d - g - 3, \\ &\mathrm{H}^k(X, F_d) = 0. && \textit{for } k \neq 1. \end{aligned}$$

*Proof.* By the proof of Lemma (3.9), we can choose a sheaf $F_{d-1} \in \mathsf{M}(d-1)$, a line $L \subset X$, a projection $\sigma : F_{d-1} \to \mathscr{O}_L$, and a global section $s \in$

$\mathrm{H}^0(X, F_{d-1})$, $\mathscr{S}_d = \ker(\sigma)$ such that $\sigma \circ s$ is surjective. Therefore we have the following exact diagram:

(3.10)
$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathcal{I}_L \arrow[r] \arrow[d] & \mathscr{O}_X \arrow[r] \arrow[d, "s"] & \mathscr{O}_L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{S}_d \arrow[r] \arrow[d] & F_{d-1} \arrow[r, "\sigma"] \arrow[d] & \mathscr{O}_L \arrow[r] & 0 \\ & \mathcal{I}_C(1) \arrow[r, equal] \arrow[d] & \mathcal{I}_C(1) \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

The leftmost column induces, for all $d$, an isomorphism $\mathrm{H}^0(X, \mathscr{S}_d) \cong \mathrm{H}^0(X, \mathcal{I}_C(1))$. Assume now $d = g + 3$. Then Corollary 3.10 implies $\mathrm{H}^0(X, \mathscr{S}_{g+3}) = 0$. Taking cohomology of the middle row of (3.10), by induction we get $\mathrm{H}^2(X, \mathscr{S}_{g+3}) = \mathrm{H}^3(X, \mathscr{S}_{g+3}) = 0$. By semicontinuity, we also have $\mathrm{H}^k(X, F_{g+3}) = 0$, for $k = 0, 2, 3$. For higher $d$ the claim is now obvious. $\square$

Here is the proof of the main result of this section.

*Proof of Theorem 3.6.* We work by induction on $d \geq m_g$. Note that by Proposition 3.5, the statement holds for $d = m_g$.

Let $F$ be a general sheaf in $\mathsf{M}(d)$ and recall that $F$ is obtained as a general deformation of a sheaf $\mathscr{S}_d$ fitting into an exact sequence of the form (3.7), where $F_{d-1}$ is a vector bundle in $\mathsf{M}(d)$. By induction we assume that $F_{d-1}$ is ACM. Let $C$ be the zero locus of a nonzero global section of $F_{d-1}$. By the proof of Lemma 3.11 we have thus an exact sequence:

$$0 \to \mathcal{I}_L \to \mathscr{S}_d \to \mathcal{I}_C(1) \to 0.$$

Since $L$ is projectively normal, this implies $\mathrm{H}^1_*(X, \mathscr{S}_d) \subset \mathrm{H}^1_*(X, \mathcal{I}_C(1))$. By Lemma 2.3 we have $\mathrm{H}^1_*(X, \mathcal{I}_C(1)) \cong \mathrm{H}^1_*(X, F_{d-1})$, and this module vanishes by the induction hypothesis.

So we obtain $\mathrm{H}^1_*(X, \mathscr{S}_d) = 0$, hence by semicontinuity the module $\mathrm{H}^1_*(X, F)$ is zero as well. Then by Serre duality the vector bundle $F$ is ACM. $\square$

3.3. **Elliptic and half-canonical curves.** Choose a general element $F$ of $\mathsf{M}(m_g)$, and consider the zero locus $C_{m_g}$ of a general global section of $F$. We can assume that $C_{m_g}$ is a smooth elliptic curve of degree $m$. For $d \geq m_g + 1$ we consider the subsets $\mathscr{K}^1_d$, $\mathscr{L}^1_d$ of $\mathscr{H}^1_d(X)$ defined as follows:

(3.11) $$\mathscr{K}^1_{m_g} = \text{the component of } \mathscr{H}^1_{m_g}(X) \text{ containing } [C_{m_g}],$$

(3.12) $$\mathscr{L}^1_d = \{[C \cup L] \mid \mathrm{len}(C \cap L) = 1, [C] \in \mathscr{K}^1_{d-1}, L \in \mathscr{H}^0_1(X)\},$$

(3.13) $$\mathscr{K}^1_d = \text{the component of } \mathscr{H}^1_d(X) \text{ containing } \mathscr{L}^1_d.$$

**Lemma 3.12.** *For each $m_g \leq d \leq g+1$, let $F_d$ be a general element of $\mathsf{M}(d)$ and $L$ be a general line contained in $X$. Then there exists a global section $s$*

*of $F_d$ whose zero locus $C$ meets $L$ at a single point and we have:*

$$\mathrm{h}^0(X, \mathcal{I}_{C\cup L}(1)) = \mathrm{h}^0(X, \mathcal{I}_C(1)) - 1.$$

*Proof.* Reasoning as in the proof of the previous lemma, we choose a hyperplane section surface $S$ containing the zero locus $C$ of a given global section of $F_d$ and meeting $L$ at a single point $x$.

Note that the line bundle $\mathscr{O}_S(C)$ has a global section which vanishes at the point $x$. Indeed by (3.9) we have $\mathrm{h}^0(S, \mathscr{O}_S(C)) = 2$, which implies $\mathrm{h}^0(S, \mathcal{I}_{x,S} \otimes \mathscr{O}_S(C)) = 1$. The corresponding global section $s$ of $F_d$ satisfies our statement. Indeed, since $L$ is not contained in $S$, we have $\mathrm{h}^0(X, \mathcal{I}_{C\cup L}(1)) \leq \mathrm{h}^0(X, \mathcal{I}_C(1)) - 1$. So the claim follows taking global sections of the natural exact sequence:

$$0 \to \mathcal{I}_{C\cup L}(1) \to \mathcal{I}_C(1) \to \mathscr{O}_L \to 0.$$

Note that the sections satisfying the claim form a vector subspace of $\mathrm{H}^0(X, F_d)$ of codimension 1. $\square$

**Lemma 3.13.** *Let $D$ be the zero locus of a global section of a sheaf $F$ lying in $\mathsf{M}_X(2, 1, d)$ satisfying* (3.4), (3.5) *and such that* $\mathrm{H}^1(X, \mathcal{I}_D(1)) = 0$. *Then we have* $\mathrm{Ext}^2_X(\mathcal{I}_D, \mathcal{I}_D) = 0$ *and we obtain an exact sequence:*

$$0 \to \mathrm{H}^0(X, \mathcal{I}_D(1)) \to \mathrm{Ext}^1_X(\mathcal{I}_D, \mathcal{I}_D) \to \mathrm{Ext}^1_X(F, F) \to 0, \tag{3.14}$$

*so* $\mathrm{ext}^1_X(\mathcal{I}_D, \mathcal{I}_D) = d$.

*Proof.* We apply the functor $\mathrm{Hom}_X(F, -)$ to the exact sequence (2.6). It is easy to check that $\mathrm{Ext}^k_X(F, \mathscr{O}_X) = 0$ for any $k$, thus we obtain for each $k$ an isomorphism:

$$\mathrm{Ext}^k_X(F, \mathcal{I}_D(1)) \cong \mathrm{Ext}^k_X(F, F).$$

Therefore, applying $\mathrm{Hom}_X(-, \mathcal{I}_D(1))$ to (2.6), we get the vanishing $\mathrm{Ext}^2_X(\mathcal{I}_D, \mathcal{I}_D) = 0$ and, since $F$ is a stable sheaf, we obtain the exact sequence (3.14). The value of $\mathrm{ext}^1_X(\mathcal{I}_D, \mathcal{I}_D)$ can now be computed by Riemann-Roch. $\square$

**Proposition 3.14.** *Let $X$ be an ordinary smooth non-hyperelliptic prime Fano threefold of genus $g$ and assume $X$ to be general for $g \leq 5$. For $m_g \leq d \leq g + 2$, the subset $\mathscr{K}^1_d$ of $\mathscr{H}^1_d(X)$ is reduced of dimension $d$, and it contains a dense open subset consisting of smooth integral projectively normal curves. The subset $\mathscr{L}^1_d$ of $\mathscr{H}^1_d(X)$ is a divisor in $\mathscr{K}^1_d$.*

*Proof.* Let $D$ be the zero locus of a nonzero global section of a vector bundle $F$ in $\mathsf{M}(d)$. We can assume that $F$ is ACM, so by Lemma 2.3 we have $\mathrm{H}^1_*(X, \mathcal{I}_D) = 0$ so $D$ is projectively normal. We would like to prove that $D$ can be chosen smooth integral.

We work by induction on $d \geq m_g$. Set $d = m_g$. For $g \geq 4$, by Proposition 3.5 we can assume that $F$ is globally generated so $D$ is smooth integral if the global section of $F$ is general. For $g = 3$, this was proved in [Mad00].

Assume thus that $C$ is a smooth projectively normal curve, given by a section $s_{d-1}$ of a vector bundle $F_{d-1}$ in $\mathsf{M}(d-1)$ which satisfies (3.4). Suppose that $s_{d-1}$ is a global section provided by Lemma 3.12. Then by construction we have the commutative exact diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathscr{O}_X \arrow[r, equal] \arrow[d, "s_d"] & \mathscr{O}_X \arrow[d, "s_{d-1}"] & & \\
0 \arrow[r] & \mathscr{S}_d \arrow[r] \arrow[d] & F_{d-1} \arrow[r] \arrow[d] & \mathscr{O}_L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{I}_{C\cup L}(1) \arrow[r] \arrow[d] & \mathcal{I}_C(1) \arrow[r] \arrow[d] & \mathscr{O}_L \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.15}$$

By Lemma 3.12 we have $\mathrm{H}^1(X, \mathcal{I}_{C\cup L}(1)) = 0$ and by the proof of Theorem 3.7 we have $\mathrm{Ext}^2_X(\mathscr{S}_d, \mathscr{S}_d) = 0$. Thus the variety $\mathscr{K}^1_d$ is smooth of dimension $d$ at $[C \cup L]$ by Lemma 3.13. Moreover, the subvariety $\mathscr{L}^1_d$ has codimension 1 in $\mathscr{K}^1_d$, i.e. $\dim(\mathscr{L}^1_d) = d - 1$. Indeed, by Lemma 3.12, the sections $s_{d-1}$ of $F_{d-1}$ which give rise to (3.15) form a vector subspace $W_0 \subset \mathrm{H}^0(X, F_{d-1})$ of codimension 1. Projecting $W_0 \cap \mathrm{H}^0(X, \mathscr{S}_d)$ to $\mathrm{H}^0(X, \mathcal{I}_{C\cup L}(1))$ via the map induced by (2.6) we obtain a subspace $W$ of codimension 1. That is, $\dim(W) = g - d + 1$. Thus the tangent space $T$ to $\mathscr{L}^1_d$ at $[C \cup L]$ fits into an exact sequence analogous to (3.14) of the form:

$$0 \to W \to T \to \mathrm{Ext}^1_X(\mathscr{S}_d, \mathscr{S}_d) \to 0.$$

Since $\mathrm{ext}^1_X(\mathscr{S}_d, \mathscr{S}_d) = 2d - g - 2$, and $\dim(W) = g - d + 1$, we conclude that $T$ has dimension $d - 1$.

We have thus proved that a general deformation $D$ of the curve $C \cup L$ in $\mathscr{K}^1_d$ cannot contain a line, hence by semicontinuity, it must be an integral curve. More than that, this curve must be nonsingular. Indeed, assume the contrary, and consider the open neighborhood $[D] \in \Omega$ consisting of smooth points of $\mathscr{K}^1_d$. For each point $b \in \Omega$, consider the curve $C_b$, and its (unique) singular point $x_b \in C_b$. Blowing up the trace of $x_b$ in $X \times \Omega$, we get a flat family of smooth rational curves $\tilde{C}$ in $\tilde{X}$, parametrised by $\Omega$. But for $b \in \Omega \cap \mathscr{L}^1_d$ the curve $\tilde{C}_b$ is disconnected, a contradiction by [Har77, Chapter III, exercise 11.4]. □

**Proposition 3.15.** *The general element of* $\mathsf{M}(g+3)$ *corresponds to a locally free sheaf* $F$ *such that* $F(1)$ *is globally generated and ACM with* $c_1(F(1)) = 3$, $c_2(F(1)) = 5\,g - 1$. *In particular, a general global section of* $F(1)$ *vanishes along a smooth half-canonical curve of genus* $5\,g$.

*Proof.* By Lemma 3.11, $F$ satisfies $\mathrm{H}^0(X, F) = \mathrm{H}^1(X, F) = 0$. By stability we have $\mathrm{H}^3(X, F) = 0$ so Riemann-Roch implies $\mathrm{H}^2(X, F) = 0$. If the sheaf $F$ is general, we have thus:

$$\begin{aligned}
&\mathrm{h}^1(X, F) = 0, \\
&\mathrm{h}^2(X, F(-1)) = \mathrm{h}^1(X, F(-1)) = 0 && \text{by (3.5),} \\
&\mathrm{h}^3(X, F(-2)) = \mathrm{h}^0(X, F) = 0.
\end{aligned}$$

So $F(1)$ is globally generated by Castelnuovo-Mumford regularity. □

## 4. Bundles with even first Chern class

In this section we will prove the existence of the remaining case of Madonna's list, namely case (d) of Remark 2.5, see Theorem 4.10. We will need a preliminary analysis of the moduli space $\mathsf{M}_X^{\mu}(2,0,2)$, which turns out to be in bijection with $\mathscr{H}_2^0(X)$. The investigation of the moduli space of ACM bundles with $c_1 = 0$, $c_2 = 4$ is carried out via deformations in section 4.2. Finally, we show some further properties of $\mathsf{M}_X(2,0,4)$ for threefolds of high genus.

4.1. **Conics and 2-bundles with $c_2 = 2$.** Here we study rank 2 sheaves on $X$ with $c_1 = 0$, $c_2 = 2$, and their relation with the Hilbert scheme $\mathscr{H}_2^0(X)$ of conics contained in $X$. We rely on well-known properties of the Hilbert scheme $\mathscr{H}_2^0(X)$, see section 2.3.

**Lemma 4.1.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g$. Then any Cohen-Macaulay curve $C$ of degree 2 has $p_a(C) \leq 0$. Moreover if $C$ is nonreduced it must be a Gorenstein double structure on a line $L$ defined by the exact sequence:*

$$0 \to \mathcal{I}_C \to \mathcal{I}_L \to \mathscr{O}_L(t) \to 0, \tag{4.1}$$

*where $t \geq -1$ and we have $p_a(C) = -1 - t$ and $\omega_C \cong \mathscr{O}_C(-2-t)$.*

*Proof.* If $C$ is reduced, clearly it must be a conic (then $p_a(C) = 0$) or the union of two skew lines (then $p_a(C) = -1$). So assume that $C$ is nonreduced, hence a double structure on a line $L$. By [Man86, Lemma 2] we have the exact sequences:

$$0 \to \mathcal{I}_C/\mathcal{I}_L^2 \to \mathcal{I}_L/\mathcal{I}_L^2 \to \mathscr{O}_L(t) \to 0,$$
$$0 \to \mathscr{O}_L(t) \to \mathscr{O}_C \to \mathscr{O}_L \to 0, \tag{4.2}$$

and $C$ is a Gorenstein structure given by Ferrand's doubling (see [Man94], [BF81]).

Recall that $\mathcal{I}_L/\mathcal{I}_L^2 \cong N_L^*$. By [Isk78, Lemma 3.2] we have either $N_L^* \cong \mathscr{O}_L \oplus \mathscr{O}_L(1)$, or $N_L^* \cong \mathscr{O}_L(-1) \oplus \mathscr{O}_L(2)$. It follows that $t \geq -1$ and we obtain (4.1). We compute that $c_3(\mathcal{I}_L) = -1$ and $c_3(\mathscr{O}_L(t)) = 1 + 2t$, hence $c_3(\mathcal{I}_C) = -2 - 2t$, so $p_a(C) = -1 - t$.

Dualizing (4.1), by the fundamental local isomorphism we obtain the exact sequence:

$$0 \to \mathscr{O}_L(-2) \to \omega_C \to \mathscr{O}_L(-2-t) \to 0, \tag{4.3}$$

which by functoriality is obtained twisting by $\mathscr{O}_X(-2-t)$ the exact sequence (4.2). This concludes the proof. □

**Corollary 4.2.** *Let $X$ a smooth non-hyperelliptic prime Fano threefold. Then any conic contained in $X$ is reduced if and only if $\mathscr{H}_0^1(X)$ is smooth. This takes place if $X$ is general.*

*Proof.* From [IP99, Proposition 4.2.2], the Hilbert scheme $\mathscr{H}_0^1(X)$ is smooth if and only if we have $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$ for any line $L$ in $X$. By the previous lemma this is equivalent to the fact that any conic contained in $X$ is reduced. Notice that if $X$ is general, by [IP99, Theorem 4.2.7], we have that $\mathscr{H}_0^1(X)$ is smooth. □

We will need the following lemma.

**Lemma 4.3.** *Let $X$ be a smooth prime non-hyperelliptic Fano threefold of genus $g$ and $F$ be a $\mu$-semistable sheaf in $\mathsf{M}_X(2,0,c_2,c_3)$, with $c_2 < m_g$. Then we have:*

$$\mathrm{H}^2(X,F) = 0, \tag{4.4}$$

$$\mathrm{H}^2(X,F(1)) = 0. \tag{4.5}$$

*Proof.* Let us first prove (4.5). Assume by contradiction that $\mathrm{H}^2(X,F(1)) \neq 0$. Thus a nonzero element of $\mathrm{Ext}^1(F,\mathscr{O}_X(-2))$ provides an exact sequence of the form:

$$0 \to \mathscr{O}_X(-2) \to \tilde{F} \to F \to 0, \tag{4.6}$$

where the sheaf $\tilde{F}$ has rank 3 and Chern classes $c_1(\tilde{F}) = -2$, $c_2(\tilde{F}) = c_2$.

The sheaf $\tilde{F}$ cannot be semistable by Bogomolov's inequality, see (2.3). Thus we consider the Harder-Narasimhan filtration $0 \subset K_1 \subset \cdots \subset \tilde{F}$, which provides the following exact sequence:

$$0 \to K_1 \to \tilde{F} \to Q \to 0,$$

where the semistable sheaf $K_1$ destabilizes $\tilde{F}$, and the sheaf $Q$ is torsionfree.

Consider first the case $\mathrm{rk}(K_1) = 2$. This implies either $c_1(K_1) = 0$ or $c_1(K_1) = -1$. In the former case, the composition $K_1 \hookrightarrow \tilde{F} \to F$ must be injective, for its kernel would be a degree-zero subsheaf of $\mathscr{O}_X(-2)$. This implies that $\mathscr{O}_X(-2)$ sits into $Q$, which is a rank 1 torsionfree sheaf with $c_1(Q) = -2$. It follows that $Q \cong \mathscr{O}_X(-2)$ and so we have $K_1 \cong F$ and the sequence (4.6) splits, a contradiction. In the latter case, we have $c_2(K_1) \geq m_g$ since $K_1$ and its restriction to a general hyperplane section must satisfy (2.4) by Maruyama's theorem [Mar80]. On the other hand, $Q$ is a rank 1 torsionfree sheaf hence $c_2(Q) \geq 0$. Since $c_1(Q) = -1$, we get $m_g > c_2(\tilde{F}) = c_2(K_1) + c_2(Q) + 2g - 2 \geq 2g - 2 + m_g$, a contradiction.

Consider now the case $\mathrm{rk}(K_1) = 1$. Note that $c_1(K_1)$ equals 0 and $c_2(K_1)$ is nonnegative. It follows that $\mathrm{rk}(Q) = 2$, $c_1(Q) = -2$ and $c_2(Q) \leq c_2 < m_g$. Notice that $Q$ is not $\mu$-semistable Bogomolov's inequality.

Then, a Harder-Narasimhan filtration of $Q$ gives the exact sequence:

$$0 \to P \to Q \to R \to 0,$$

where $P, R$ are rank 1 torsionfree sheaves with nonnegative $c_2$, and $c_1(P) \geq 0$. Notice that (4.6) induces an exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[d] & \\
0 \arrow[r] & \mathscr{O}_X(-2) \arrow[r] \arrow[d, equal] & \tilde{F} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_X(-2) \arrow[r] & Q \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

which in turn induces an injective map either $\mathscr{O}_X(-2) \hookrightarrow P$, or $\mathscr{O}_X(-2) \hookrightarrow R$. In the former case, we obtain a surjective map $M \to R$, hence $R$ is a destabilizing quotient for $F$, contradiction. In the latter case, since $c_1(R) \leq -2$, we have $R \cong \mathscr{O}_X(-2)$. Then $Q$ contains $\mathscr{O}_X(-2)$ as direct summand, so (4.6) splits, a contradiction. This finishes the proof of (4.5).

We shall now prove $\mathrm{H}^1(S, F_S(1)) = 0$, where $S$ is a general hyperplane section surface. Note that, together with (4.5), this easily implies (4.4).

Assume by contradiction that $\mathrm{H}^1(S, F(1)) \neq 0$. By Serre duality, this provides a nontrivial extension of the form:

$$0 \to \mathscr{O}_S(-1) \to \widetilde{F_S} \to F_S \to 0, \tag{4.7}$$

where the sheaf $\widetilde{F_S}$ has rank 3 and Chern classes $c_1(\widetilde{F_S}) = -1$, $c_2(\widetilde{F_S}) = c_2 < m_g$. Note that $\widetilde{F_S}$ cannot be stable, for the space $\mathsf{M}_S(3, -1, c_2)$ is empty by the dimension count (2.4). Then, the Jordan-Hölder filtration gives:

$$0 \to K_1 \to \widetilde{F_S} \to Q \to 0,$$

where this time the destabilizing sheaf $K_1$ and the torsionfree sheaf $Q$ are defined on $S$. To show that this leads to a contradiction, we repeat the above argument, where $\mathscr{O}_X(-2)$ is replaced with $\mathscr{O}_S(-1)$. The only difference is that, if the destabilising subsheaf $K_1$ of $\widetilde{F_S}$ has rank 1, then the quotient $Q = \widetilde{F_S}/K_1$ has rank 2 and $c_1(Q) = -1$, $c_2(Q) < m_g$. Thus $Q$ is not stable by the dimension count (2.4). Taking again a Harder-Narasimhan filtration of $Q$ as above we can show that the extension (4.7) should split, which is again a contradiction. □

**Remark 4.4.** Note that in the previous lemma, the hypothesis $c_2(F) < m_g$ cannot be dropped. In fact, Iliev and Markushevich in [IM00b, Proposition 1.4, arXiv version] proved that there exists a component of $\mathsf{M}_X(2, 0, 4)$, where a general element $F$ satisfies $\mathrm{h}^1(X, F(1)) = 1$, and this contradicts either $\mathrm{H}^1(S, F_S(1)) = 0$ or $\mathrm{H}^2(X, F) = 0$.

**Remark 4.5.** Let $X$ be as in the previous lemma, and let $P$ a torsionfree sheaf of rank 1 with $c_1(P) = 0$, $c_2(P) \leq m_g$. Taking $F = \mathscr{O}_X \oplus P$ in Lemma 4.3, we get $\mathrm{H}^2(X, P) = 0$. Note that, unless $P$ is isomorphic to $\mathscr{O}_X$, we also have $\mathrm{H}^0(X, P) = 0$, hence, $\chi(P)$ must be nonpositive. Now, by Riemann-Roch, we have $\chi(P) = \frac{2 - c_2(P) + c_3(P)}{2}$. This implies:

$$c_3(P) \leq c_2(P) - 2. \tag{4.8}$$

**Proposition 4.6.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold. Then the moduli space $\mathsf{M}_X(2, 0, 2)$ is empty. Moreover any sheaf $F$ in $\mathsf{M}^{\mu}_X(2, 0, 2)$ is strictly $\mu$-semistable, and the moduli space $\mathsf{M}^{\mu}_X(2, 0, 2)$ is in bijection with the Hilbert scheme $\mathscr{H}^0_2(X)$.*

*Proof.* By Lemma 4.3, we have $\mathrm{H}^2(X, F) = 0$. Since $\chi(F) = 1$, we obtain $\mathrm{H}^0(X, F) \neq 0$. So $\mathscr{O}_X$ is a destabilizing subsheaf of $F$ in the sense of Gieseker. Therefore, an element $F$ of $\mathsf{M}^{\mu}_X(2, 0, 2)$ must be strictly semistable. A Jordan-Hölder filtration of $F$ amounts to an exact sequence:

$$0 \to P \to F \to R \to 0, \tag{4.9}$$

where $P, R$ are torsionfree sheaves of rank 1. The graded associated to $F$ is $\mathrm{gr}(F) \cong P \oplus R$, and we want to prove that $\mathrm{gr}(F) \cong \mathscr{O}_X \oplus \mathcal{I}_C$ for a conic $C \in \mathscr{H}_2^0(X)$.

By semistability of $F$ we have:

$$c_1(P) = c_1(R) = 0, \qquad c_2(P) + c_2(R) = 2, \qquad c_3(P) = -c_3(R),$$

and since $P$ is torsionfree, we must have $c_2(P) \geq 0$ and by the same reason $c_2(R) \geq 0$. Summing up, we must take into account the three possibilities: $c_2(P) = 2$, $c_2(P) = 1$ or $c_2(P) = 0$. Note that the third case is equivalent to the first one.

In the first case we have $c_2(R) = 0$, and thus we have $c_3(R) \leq 0$ and $c_3(P) \geq 0$. Then the singular locus of $P$ contains a Cohen-Macaulay curve $C$ of degree 2, so we have an exact sequence

$$0 \to P \to \mathcal{I}_C \to Q \to 0,$$

where $Q$ is a torsion sheaf with $c_1(Q) = c_2(Q) = 0$ and $c_3(Q) \geq 0$. On the other hand $c_3(\mathcal{I}_C) = 2p_a(C) \leq 0$ by Lemma 4.1. Then $c_3(P) = c_3(\mathcal{I}_C) - c_3(Q) \leq 0$. Hence we obtain $c_3(P) = 0$, moreover we get $c_3(Q) = 0$ and $P \cong \mathcal{I}_C$ where $p_a(C) = 0$. Finally since $c_3(R) = 0$ we get $R \cong \mathscr{O}_X$. It follows $\mathrm{gr}(F) \cong \mathcal{I}_C \oplus \mathscr{O}_X$.

In the second case, since $c_2(P) = 1$, the singular locus of $P$ contains a line and the quotient $Q = \mathcal{I}_L/P$ is a torsion sheaf with $c_1(Q) = c_2(Q) = 0$ and $c_3(Q) \geq 0$. Hence $c_3(P) = -1 - c_3(Q) < 0$. Analogously since $c_2(R) = 1$, we get $c_3(R) < 0$, a contradiction. □

**Remark 4.7.** Note that the previous argument shows that a rank 1 torsionfree sheaf $P$ with $c_1(P) = 0$, $c_2(P) = 2$ satisfies $c_3(P) \leq 0$. One proves analogously that if $c_2(P) = 1$, then $c_3(P) \leq -1$.

**Lemma 4.8.** *Given $X$ as above, any element in $\mathsf{M}_X^\mu(2,0,2)$ is represented by a locally free sheaf $F^C$ satisfying $\mathrm{h}^0(X, F^C) = 1$, $\mathrm{H}^k(X, F^C) = 0$ for $k \geq 1$.*

*Proof.* Given a class $[F]$ in $\mathsf{M}_X^\mu(2,0,2)$, by Proposition 4.6, $\mathrm{gr}(F)$ is isomorphic to $\mathscr{O}_X \oplus \mathcal{I}_C$, for some conic $C$ contained in $X$. Note that $C$ is subcanonical: this is clear if it is reduced, and it follows by Lemma 4.1 otherwise. By the Hartshorne-Serre correspondance, we get that $F$ is equivalent to an extension of the form:

$$0 \to \mathscr{O}_X \to F^C \to \mathcal{I}_C \to 0, \tag{4.10}$$

and $F^C$ is locally free. Note that $\mathrm{H}^k(X, \mathcal{I}_C) = 0$ for all $k$: this follows by (4.1) if $C$ is nonreduced. Thus the cohomology of $F^C$ behaves as desired. □

The following lemma will be needed later on.

**Lemma 4.9.** *Given $X$ as above, let $F$ be a locally free sheaf in $\mathsf{M}_X(2,0,2\,d)$, $C$ be a smooth conic contained in $X$ with normal bundle $N_C \cong \mathscr{O}_C^2$, and $x$ be a point of $C$. Assume that $F \otimes \mathscr{O}_C \cong \mathscr{O}_C^2$ and that $\mathrm{Ext}_X^2(F,F) = 0$. Let $\mathscr{S}$ be a sheaf fitting into an exact sequence of the form:*

$$0 \to \mathscr{S} \to F \to \mathscr{O}_C \to 0. \tag{4.11}$$

*Then we have $\mathrm{H}^0(C, \mathscr{S}(-x)) = 0$ and $\mathrm{Ext}_X^2(\mathscr{S}, \mathscr{S}) = 0$.*

*Proof.* To prove the vanishing of $\mathrm{H}^0(C, \mathscr{S}(-x))$, we tensor (4.11) by $\mathscr{O}_C$ and we get the following exact sequence of sheaves on $C$:

$$0 \to \mathscr{T}or_1^X(\mathscr{O}_C, \mathscr{O}_C) \to \mathscr{S} \otimes \mathscr{O}_C \to F \otimes \mathscr{O}_C \to \mathscr{O}_C \to 0. \tag{4.12}$$

Recall that $\mathscr{T}or_1^X(\mathscr{O}_C, \mathscr{O}_C)$ is isomorphic to $N_C^* \cong \mathscr{O}_C \oplus \mathscr{O}_C$. Now, twisting (4.12) by $\mathscr{O}_C(-x)$ and taking global sections, we easily get $\mathrm{H}^0(C, \mathscr{S}(-x)) = 0$.

Now let us prove the vanishing of $\mathrm{Ext}_X^2(\mathscr{S}, \mathscr{S})$. Applying the functor $\mathrm{Hom}_X(-, \mathscr{S})$ to (4.11) we obtain the exact sequence:

$$\mathrm{Ext}_X^2(F, \mathscr{S}) \to \mathrm{Ext}_X^2(\mathscr{S}, \mathscr{S}) \to \mathrm{Ext}_X^3(\mathscr{O}_C, \mathscr{S}).$$

We will prove that both the first and the last terms of the above sequence vanish. Consider the former, and apply $\mathrm{Hom}_X(F, -)$ to (4.11). We get the exact sequence:

$$\mathrm{Ext}_X^1(F, \mathscr{O}_C) \to \mathrm{Ext}_X^2(F, \mathscr{S}) \to \mathrm{Ext}_X^2(F, F).$$

By assumption we have $\mathrm{Ext}_X^2(F, F) = 0$ and $\mathrm{Ext}_X^1(F, \mathscr{O}_C) \cong \mathrm{H}^1(C, F) = 0$. We obtain $\mathrm{Ext}_X^2(F, \mathscr{S}) = 0$. To show the vanishing of the group $\mathrm{Ext}_X^3(\mathscr{O}_C, \mathscr{S})$, we apply Serre duality, obtaining:

$$\mathrm{Ext}_X^3(\mathscr{O}_C, \mathscr{S})^* \cong \mathrm{Hom}_X(\mathscr{S}, \mathscr{O}_C(-1)) \cong \mathrm{H}^0(X, \mathscr{H}om_X(\mathscr{S}, \mathscr{O}_C(-1))).$$

To show that this group is zero, apply the functor $\mathscr{H}om_X(-, \mathscr{O}_C)$ to the sequence (4.11) to get:

$$0 \to \mathscr{O}_C \to F^* \otimes \mathscr{O}_C \to \mathscr{H}om_X(\mathscr{S}, \mathscr{O}_C) \to N_C \to 0,$$

which implies $\mathscr{H}om_X(\mathscr{S}, \mathscr{O}_C(-1)) \cong \mathscr{O}_C(-1)^3$, and this sheaf has no global sections. □

4.2. **Rank 2 bundles with $c_2 = 4$.** In this section, we begin the study of semistable sheaves $F$ with Chern classes $c_1(F) = 0$, $c_2(F) = 4$, $c_3(F) = 0$ on smooth non-hyperelliptic prime Fano threefolds, and we prove the existence of case (d) of Madonna's list. The main result of this part is the following.

**Theorem 4.10.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold. Then there exists a rank 2 ACM stable locally free sheaf $F$ with $c_1(F) = 0$, $c_2(F) = 4$. The bundle $F$ lies in a reduced component of dimension 5 of the space $\mathsf{M}_X(2, 0, 4)$.*

We will need the following lemma.

**Lemma 4.11.** *Let $X$ as above and let $C$ and $D$ be smooth disjoint conics contained in $X$ with trivial normal bundle. Then a sheaf $\mathscr{S}_2$ fitting into a nontrivial extension of the form:*

$$0 \to \mathcal{I}_C \to \mathscr{S}_2 \to \mathcal{I}_D \to 0, \tag{4.13}$$

*is simple.*

*Proof.* In order to prove the simplicity, apply $\mathrm{Hom}_X(\mathscr{S}_2, -)$ to (4.13) and get

$$\mathrm{Hom}_X(\mathscr{S}_2, \mathcal{I}_C) \to \mathrm{Hom}_X(\mathscr{S}_2, \mathscr{S}_2) \to \mathrm{Hom}_X(\mathscr{S}_2, \mathcal{I}_D).$$

The first term vanishes, indeed applying $\mathrm{Hom}_X(-,\mathcal{I}_C)$ to (4.13) and since $C\cap D=\emptyset$ we get

$$0\to \mathrm{Hom}_X(\mathscr{S}_2,\mathcal{I}_C)\to \mathrm{Hom}_X(\mathcal{I}_C,\mathcal{I}_C)\xrightarrow{\delta} \mathrm{Ext}^1_X(\mathcal{I}_D,\mathcal{I}_C).$$

Clearly the map $\delta:\mathbb{C}\to\mathbb{C}$ is nonzero, hence $\mathrm{Hom}_X(\mathscr{S}_2,\mathcal{I}_C)=0$. On the other hand, applying $\mathrm{Hom}_X(-,\mathcal{I}_D)$ to (4.13) we get:

$$\mathrm{Hom}_X(\mathscr{S}_2,\mathcal{I}_D)\cong \mathrm{Hom}_X(\mathcal{I}_D,\mathcal{I}_D)\cong\mathbb{C},$$

and we deduce $\hom_X(\mathscr{S}_2,\mathscr{S}_2)=1$, i.e. the sheaf $\mathscr{S}_2$ is simple. □

*Proof of Theorem 4.10.* Choose two smooth conics $C$ and $D$ in $X$ with trivial normal bundle. This is possible in view of well know results, see section 2.3. Let us check that we can assume that $C$ and $D$ are disjoint.

Let $S$ be a hyperplane section surface containing $C$. A general conic $D$ intersects $S$ at 2 points. Since $X$ is covered by conics, moving $D$ in $\mathscr{H}_2^0(X)$, these two points sweep out $S$. Thus, a general conic $D$ meets $C$ at most at a single point. This gives a rational map $\varphi:\mathscr{H}_2^0(X)\to C$. Note that, for any point $x\in C$, we have $\mathrm{H}^0(C,N_C(-x))=0$. So there are only finitely many conics contained in $X$ through $x$, for this space parametrises the deformations of $C$ which pass through $x$. Thus the general fiber of $\varphi$ is finite, which is a contradiction.

Let $F^D$ be the vector bundle in $\mathsf{M}^\mu_X(2,0,2)$ fitting into:

$$0\to\mathscr{O}_X\xrightarrow{\iota} F^D\to\mathcal{I}_D\to 0. \tag{4.14}$$

One can easily prove the vanishing $\mathrm{Ext}^2_X(F^D,F^D)=0$, since the normal bundle to $D$ is trivial. Tensoring by $\mathscr{O}_C$ the exact sequence (4.14), we obtain $F^D_C\cong\mathscr{O}_C^2$. Then we have $\hom_X(F^D,\mathscr{O}_D)=2$, and for any morphism $F^D_C\to\mathscr{O}_C$ we denote by $\sigma$ the surjective composition $\sigma: F^D\to F^D_C\to\mathscr{O}_C$. We can choose $\sigma$ such that the composition $\sigma\circ\iota:\mathscr{O}_X\to\mathscr{O}_C$ is nonzero, i.e. such that:

$$\ker(\sigma)\not\supset \mathrm{Im}(\iota)\otimes\mathscr{O}_C. \tag{4.15}$$

We denote by $\mathscr{S}_2$ the kernel of $\sigma$ and we have the exact sequence:

$$0\to\mathscr{S}_2\to F^D\xrightarrow{\sigma}\mathscr{O}_C\to 0, \tag{4.16}$$

and $\mathscr{S}_2$ fits into (4.13). Clearly, the sheaf $\mathscr{S}_2$ represents an element of $\mathsf{M}_X(2,0,4)$, and by (4.13) we get $\mathrm{H}^k(X,\mathscr{S}_2)=0$ for all $k$. More than that, since a smooth conic is projectively normal, again by (4.13) we obtain:

$$\mathrm{H}^1_*(X,\mathscr{S}_2)=0. \tag{4.17}$$

Note that the sheaf $\mathscr{S}_2$ is strictly semistable, and simple by Lemma 4.11.

Now we would like to flatly deform $\mathscr{S}_2$ to a stable ACM vector bundle $F$. We perform this deformation in the moduli space of simple sheaves $\mathsf{Spl}_X(2,0,4)$, see section 2.2.

Note that Lemma 4.9 gives $\mathrm{Ext}^2_X(\mathscr{S}_2,\mathscr{S}_2)=0$. Hence by Riemann-Roch and by the simplicity of $\mathscr{S}_2$ we get that $\mathrm{ext}^1(\mathscr{S}_2,\mathscr{S}_2)=5$. This implies that $\mathsf{Spl}_X(2,0,4)$ is smooth and locally of dimension 5 around the point $[\mathscr{S}_2]$.

**Claim 4.12.** *The set of sheaves in* $\mathsf{Spl}_X(2,0,4)$ *fitting into an exact sequence of the form* (4.16) *forms a subset of codimension* 1 *in* $\mathsf{Spl}_X(2,0,4)$.

Let us postpone the proof of the above claim, and assume it from now on. We may thus choose a deformation $G$ of $\mathscr{S}_2$ in $\mathsf{Spl}_X(2,0,4)$ which does not fit into (4.16), and, setting $E = G^{**}$, we write the double dual sequence:

$$0 \to G \to E \to T \to 0. \tag{4.18}$$

By semicontinuity, we may assume $\hom_X(G,G) = 1$, and $\mathrm{H}^1(X, G(-1)) = 0$. We may also assume that, for any given line $L$ contained in $X$, we have the vanishing $\mathrm{Ext}^1_X(\mathscr{O}_L(t), G) = 0$ for all $t \in \mathbb{Z}$. Indeed, applying $\mathrm{Hom}_X(\mathscr{O}_L(t), -)$ to (4.16) we get:

$$\mathrm{Hom}_X(\mathscr{O}_L(t), \mathscr{O}_C) \to \mathrm{Ext}^1_X(\mathscr{O}_L(t), \mathscr{S}_2) \to \mathrm{Ext}^1_X(\mathscr{O}_L(t), F^D),$$

and observe that the leftmost term vanishes as soon as $L$ is not contained in $C$ (but $C$ is irreducible), while the rightmost does for $F^D$ is locally free.

Clearly $E$ is a semistable sheaf, so $\mathrm{H}^1(X, G(-1)) = 0$ implies $\mathrm{H}^0(X, T(-1)) = 0$, hence $T$ must be a pure sheaf supported on a Cohen-Macaulay curve $B \subset X$. Summing up, we have $c_1(T) = 0$ and $c_2(T) < 0$.

**Claim 4.13.** *We may assume that $c_2(T) \geq -2$. Hence in particular we have $-c_2(T) \in \{1, 2\}$.*

We postpone the proof of the last claim and we prove now that $B$ must be a smooth conic. For otherwise, either $T \cong \mathscr{O}_N(a)$ for some line $L_1 \subset X$ and some $a_1 \in \mathbb{Z}$ (if $c_2(T) = -1$), or, if $c_2(T) = -2$ in view of Lemma 4.1, there must be a second line $L_2 \subset X$ (possibly coincident with $L_1$), and $a_2 \in \mathbb{Z}$ such that $T$ fits into:

$$0 \to \mathscr{O}_{L_1}(a_1) \to T \to \mathscr{O}_{L_2}(a_2) \to 0. \tag{4.19}$$

But we have seen that $\mathrm{Ext}^1_X(\mathscr{O}_L(t), \mathscr{S}_2) = 0$ for all $t \in \mathbb{Z}$, and for any line $L \subset X$. By semicontinuity we get $\mathrm{Ext}^1_X(\mathscr{O}_L(t), G) = 0$, for all $t \in \mathbb{Z}$ and any line $L \subset X$. In particular $\mathrm{Ext}^1_X(\mathscr{O}_{L_i}(a_i), G) = 0$, for $i = 1, 2$, so $\mathrm{Ext}^1_X(T, G) = 0$ and (4.18) should split, contradicting semistability of $E$.

Therefore $T$ must be of the form $\mathscr{O}_B(a\,x)$, for some integer $a$, and for some point $x$ of a smooth conic $B \subset X$. By [Har80, Proposition 2.6], we have $c_3(E) = c_3(T) = 2\,a \geq 0$, while $\mathrm{H}^0(X, T(-1)) = 0$ implies $a - 2 < 0$.

We are left with the cases $a = 0$ or $a = 1$. We may exclude the former by Claim 4.12. The latter can be excluded too, by proving that $\mathrm{Ext}^1_X(\mathscr{O}_B(x), G)$ is zero for any conic $B \subset X$. That this is indeed the case can be checked by semicontinuity applying $\mathrm{Hom}_X(\mathscr{O}_B(x), -)$ to (4.16), obtaining:

$$\mathrm{Hom}_X(\mathscr{O}_B(x), \mathscr{O}_C) \to \mathrm{Ext}^1_X(\mathscr{O}_B(x), \mathscr{S}_2) \to \mathrm{Ext}^1_X(\mathscr{O}_B(x), F^D).$$

The rightmost term in the above sequence vanishes because $F^D$ is locally free. The leftmost term is isomorphic to $\mathrm{H}^0(X, \mathscr{H}om(\mathscr{O}_B(x), \mathscr{O}_C))$, and the sheaf $\mathscr{H}om(\mathscr{O}_B(x), \mathscr{O}_C)$ is zero for $B \neq C$. On the other hand, if $B = C$ we have $\mathrm{Hom}_X(\mathscr{O}_C(x), \mathscr{O}_C) \cong \mathrm{H}^0(C, \mathscr{O}_C(-x)) = 0$.

Summing up, we have proved that $T$ must be zero, so $G$ is isomorphic to $E$, and thus locally free. Since $\mathrm{H}^0(X, G) = 0$, the sheaf $G$ must be stable. By (4.17) and semicontinuity $\mathrm{H}^1_*(X, G(t)) = 0$, so by Serre duality we get that $G$ is ACM. □

*Proof of Claim 4.12.* We have proved that a sheaf $\mathscr{S}_2$ fitting into an exact sequence of the form (4.16), for some disjoint conics $C, D \subset X$, fits also into

(4.13). So, we need only prove that the set of sheaves fitting into (4.13) is a closed subset of dimension 4 of $\mathsf{Spl}_X(2,0,4)$. Since $C$ and $D$ belong to the surface $\mathscr{H}_2^0(X)$, it is enough to prove that there is in fact a unique (up to isomorphism) nontrivial such extension, i.e. $\mathrm{ext}^1_X(\mathcal{I}_D,\mathcal{I}_C) = 1$.

Note that $\mathrm{Hom}_X(\mathcal{I}_D,\mathcal{I}_C) = \mathrm{Ext}^3_X(\mathcal{I}_D,\mathcal{I}_C) = 0$, hence by Riemann-Roch it suffices to prove $\mathrm{Ext}^2_X(\mathcal{I}_D,\mathcal{I}_C) = 0$. Applying $\mathrm{Hom}_X(-,\mathcal{I}_C)$ to the exact sequence defining $D$ in $X$, we are reduced to prove the vanishing of $\mathrm{Ext}^3_X(\mathscr{O}_D,\mathcal{I}_C)$. But this group is dual to:

$$\mathrm{Hom}_X(\mathcal{I}_C,\mathscr{O}_D(-1)) \cong \mathrm{H}^0(X,\mathscr{H}om_X(\mathcal{I}_C,\mathscr{O}_D(-1))) \cong \mathrm{H}^0(X,\mathscr{O}_C(-1)) = 0.$$

□

*Proof of Claim 4.13.* We have already proved that $\mathrm{H}^0(X,T(-1)) = 0$ and this implies that $\chi(T(t)) = -\mathrm{h}^1(X,T(t))$ for any negative integer $t$.

Recall that, by [Har80, Remark 2.5.1], the reflexive sheaf $E$ satisfies $\mathrm{H}^1(X,E(t)) = 0$ for all $t \ll 0$. Thus, tensoring by $\mathscr{O}_X(t)$ the exact sequence (4.18) and taking cohomology, we obtain $\mathrm{h}^1(X,T(t)) \le \mathrm{h}^2(X,G(t))$ for all $t \ll 0$. Further, for any integer $t$, we can easily compute the following Chern classes:

$$c_1(T(t)) = 0, \quad c_2(T(t)) = c_2(T) = 4 - c_2(E), \quad c_3(T(t)) = c_3(E) - 2tc_2(T),$$

hence by Riemann-Roch formula we have

$$\chi(T(t)) = -tc_2(T) + \frac{1}{2}(c_3(E) - c_2(T)).$$

Since $G$ is a general deformation of the sheaf $\mathscr{S}_2$, we also have, by semicontinuity, $\mathrm{h}^2(X,G(t)) \le \mathrm{h}^2(X,\mathscr{S}_2(t))$. On the other hand, by (4.16) we have $\mathrm{h}^2(X,\mathscr{S}_2(t)) = \mathrm{h}^1(X,\mathscr{O}_C(t)) = -\chi(\mathscr{O}_C(t)) = -2t-1$. Summing up we have, for all $t \ll 0$, the following inequality:

$$-tc_2(T) + \frac{1}{2}(c_3(E) - c_2(T)) \ge 2t+1,$$

which implies $c_2(T) \ge -2$. □

The following result provides the existence of a well-behaved component of $\mathsf{M}_X(2,0,2d)$, for each $d \ge 2$.

**Theorem 4.14.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold. Let $C$ be a general conic in $X$, and let $x$ be a point of $C$. For any integer $d \ge 2$, there exists a rank 2 stable locally free sheaf $F_d$ with $c_1(F_d) = 0$, $c_2(F_d) = 2d$, and satisfying:*

$$\mathrm{Ext}^2_X(F_d,F_d) = 0, \tag{4.20}$$

$$\mathrm{H}^1(X,F_d(-1)) = 0, \tag{4.21}$$

$$\mathrm{H}^0(C,F_d(-x)) = 0. \tag{4.22}$$

*The bundle $F_d$ lies in a reduced component of dimension $4d-3$ of the space $\mathsf{M}_X(2,0,2d)$.*

*Proof.* We work by induction on $d \ge 2$. In Theorem 4.10 we constructed a stable vector bundle as a deformation of a sheaf $\mathscr{S}_2$. We denote it by $F_2$. By Lemma 4.9, the sheaf $\mathscr{S}_2$ satisfies (4.20), (4.22) and by the proof Theorem 4.10, it also satisfies (4.21). The same hols for $F_2$ by semicontinuity.

By induction we can choose a stable locally free sheaf $F_{d-1}$ in $\mathsf{M}_X(2,0,2d-2)$, satisfying conditions (4.20), (4.22) and (4.21). Choose a smooth conic $C \subset X$ with $N_C \cong \mathscr{O}_C \oplus \mathscr{O}_C$. By (4.22) we deduce that $F_{d-1} \otimes \mathscr{O}_C \cong \mathscr{O}_C^2$, then we get $\hom(F_{d-1}, \mathscr{O}_C) = 2$.

To any surjective morphism $\sigma : F_{d-1} \to F_{d-1} \otimes \mathscr{O}_C \to \mathscr{O}_C$ we associate the kernel $\mathscr{S}_d$ and we have the exact sequence:

$$0 \to \mathscr{S}_d \to F_{d-1} \xrightarrow{\sigma} \mathscr{O}_C \to 0. \tag{4.23}$$

Clearly $\mathscr{S}_d$ is a stable torsionfree sheaf with rank 2 and Chern classes $c_1(\mathscr{S}_d) = 0$ and $c_2(\mathscr{S}_d) = 2d$. In order to prove properties (4.20) and (4.22) we apply Lemma 4.9, while property (4.21) follows immediately from (4.23) and from the inductive hypothesis.

Now, we would like to flatly deform the sheaf $\mathscr{S}_d$ to a stable vector bundle $F_d$. Notice that the set of sheaves in $\mathsf{M}_X(2,0,2d)$ fitting into an exact sequence of the form (4.23) is a closed subset of dimension $4\,d-4$. Indeed by Riemann-Roch we know that the irreducible component $\mathsf{M}$ of $\mathsf{M}_X(2,0,2d-2)$ containing $[F_{d-1}]$ is smooth at $[F_{d-1}]$ and has locally dimension $4\,d-7$ and recall that $\mathscr{H}_2^0(X)$ is a surface. Since $\chi(F_{d-1}, \mathscr{O}_C) = 2$, the fact that $\hom_X(F_{d-1}, \mathscr{O}_C) = 2$ takes place for general $[C] \in \mathscr{H}_2^0(X)$ and for general $[F_{d-1}] \in \mathsf{M}$. Hence given a general element $([F_{d-1}], C) \in \mathsf{M} \times \mathscr{H}_2^0(X)$, to any map in $\operatorname{Hom}(F_{d-1}, \mathscr{O}_C)$ it corresponds a sheaf $\mathscr{S}_d$ lying in $\mathsf{M}_X(2,0,2d)$. This correspondence is injective by simplicity of $F_{d-1}$, hence the dimension of the set of sheaves in $\mathsf{M}_X(2,0,2d)$ fitting in (4.23) has dimension $4\,d-7+2+1 = 4\,d-4$.

On the other hand we have that the irreducible component of $\mathsf{M}_X(2,0,2d)$ containing $[\mathscr{S}_d]$ is smooth at $[\mathscr{S}_d]$ and has locally dimension $4\,d-3$. Therefore we may choose a deformation $F_d$ of $\mathscr{S}_d$ which does not fit into (4.23), and we write the double dual sequence:

$$0 \to F_d \to F_d^{**} \to T \to 0.$$

Following the argument used in the proof of Theorem 4.10, we conclude that $T$ must be zero, and so $F$ is locally free. $\square$

4.3. **The space $\mathsf{M}_X(2,0,4)$ on threefolds of high genus.** Here we will give a more detailed description of the moduli space of rank 2 ACM bundles with $c_1 = 0$ and $c_2 = 4$ on threefolds of high genus. The first result relates the existence of the bundle provided by Theorem 4.10 with that of a smooth canonical curve contained in $X$, when $g$ is at least 5. The second one classifies the sheaves in $\mathsf{M}_X(2,0,4)$ for $g$ at least 7.

**Proposition 4.15.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g \geq 5$, and let $F$ be a general element of the component of $\mathsf{M}_X(2,0,4)$ given by Theorem 4.14. Then $F(1)$ is globally generated. In particular, a general section of $F(1)$ vanishes along a smooth projectively normal canonical curve of genus $g+2$ and degree $2g+2$.*

*Proof.* We prove that the sheaf $\mathscr{S}_2(1)$ given by (4.16) is globally generated. Note that this implies that a general element in our component of $\mathsf{M}_X(2,0,4)$ is globally generated. Indeed, let $\Omega$ be an open subset of $\mathsf{M}_X(2,0,4)$, and let $\mathcal{F}$ be a universal sheaf over $X \times \Omega$ with $[\mathcal{F}_{b_0}] = [\mathscr{S}_2]$.

Observe that $\mathrm{h}^0(X,\mathcal{F}_b(1)) = \mathrm{h}^0(X,\mathscr{S}_2(1)) = 2\,g-2$ for $b$ general in $\Omega$, since $\mathrm{H}^k(X,\mathscr{S}_2(1)) = 0$ for $k \geq 1$ by (4.13). Then the natural evaluation map $\mathscr{O}_{X\times\Omega}^{2\,g-2} \to \mathcal{F}(1)$ is surjective at $b_0$, hence at a general point $b$.

By the exact sequence (4.13), it is enough to prove that, for any conic $C \subset X$, the sheaf $\mathcal{I}_C(1)$ is globally generated. Following Iliev-Manivel [IM07a, Proposition 5.4], see also [BF07, Proposition 3.4], one reduces to show that $\mathcal{I}_{C\cap S}(1)$ is globally generated on $S$ for any smooth hyperplane section $S$ of $X$ with $\mathrm{Pic}(S) \cong \mathbb{Z}$.

The subscheme $Z = C \cap S$ has length 2, and note that $\mathrm{H}^1(S,\mathcal{I}_Z(1)) = 0$. Assuming now that $\mathcal{I}_Z(1)$ is not globally generated, we get a subscheme $Z' \supset Z$ of length 3 and $\mathrm{h}^0(S,\mathcal{I}_{Z'}(1)) = \mathrm{h}^0(S,\mathcal{I}_Z(1)) = g-1$, therefore $\mathrm{h}^1(S,\mathcal{I}_{Z'}(1)) = 1$, a contradiction by [BF07, Lemma 3.3] for $g \geq 5$. Our result now follows by a Chern class computation since a general deformation of $\mathscr{S}_2$ is an ACM vector bundle. $\square$

In the following proposition, we give a classification of sheaves lying in $\mathsf{M}_X(2,0,4)$, for threefolds of genus at least 7.

**Proposition 4.16.** *Let $X$ be a smooth non-hyperelliptic prime Fano threefold of genus $g \geq 7$ and $F$ be a rank 2 semistable sheaf with $c_1(F) = 0$, $c_2(F) = 4$.*

*Then the sheaf $F$ is stable unless it fits into an exact sequence of the form (4.13), with $[C],[D] \in \mathcal{H}_2^0(X)$. If $F$ is stable, then $F$ is locally free unless it fits into one of the following exact sequences:*

(4.24) $\quad 0 \to \mathcal{I}_C \to F \to \mathcal{I}_L \to 0, \quad$ *with* $[C] \in \mathcal{H}_3^0(X)$ *and* $[L] \in \mathcal{H}_1^0(X)$,

(4.25) $\quad 0 \to \mathcal{I}_C \to F \to \mathcal{I}_x \to 0, \quad$ *with* $[C] \in \mathcal{H}_4^0(X)$ *and* $x \in C$,

(4.26) $\quad 0 \to \mathcal{I}_C \to F \to \mathscr{O}_X \to 0, \quad$ *with* $[C] \in \mathcal{H}_4^{-1}(X)$,

*Proof.* Assume first that the sheaf $F$ is strictly semistable, and consider its Jordan-Holder filtration. This amounts to an exact sequence of the form (4.9), where $P$ and $R$ lie in $\mathsf{M}(1,0,2,0)$. So $P$ and $R$ must be isomorphic to the ideal sheaf of a conic contained in $X$.

We may thus assume that the sheaf $F$ is stable, and let us assume that $F$ is not locally free. Consider the double dual exact sequence:

$$0 \to F \to F^{**} \to T \to 0,$$

where $T$ is a torsion sheaf supported on a subscheme of codimension at least 2. We have $c_2(T) \leq 0$ so $c_2(F^{**}) \leq 4$. Note that the sheaf $F^{**}$ is $\mu$-semistable, hence $c_2(F^{**}) \geq 0$ by Bogomolov's inequality.

Since $g \geq 7$ implies $m_g \geq 5$, we may apply Lemma 4.3, to obtain $\mathrm{H}^2(X,F^{**}) = 0$. On the other hand $c_3(F^{**}) \geq 0$ gives $\chi(F^{**}) \geq 0$, so $\mathrm{H}^0(X,F^{**}) \neq 0$. We have thus an exact sequence:

$$0 \to \mathscr{O}_X \to F^{**} \to Q \to 0. \tag{4.27}$$

Assume first $c_2(F^{**}) > 0$. Then $F^{**}$ is not semistable, since $\mathscr{O}_X$ is a destabilizing subsheaf, and we may consider the Harder-Narasimhan filtration for

$F^{**}$. This induces an exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & F & \longrightarrow & R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P & \longrightarrow & F^{**} & \longrightarrow & R & \longrightarrow & 0, \end{array} \tag{4.28}$$

where the vertical arrows are injective, and we have $\mathrm{rk}(P') = \mathrm{rk}(R') = \mathrm{rk}(P) = \mathrm{rk}(R) = 1$ and $c_1(P') = c_1(R') = c_1(P) = c_1(R) = 0$.

We obtain the inequalities:

$$2 = \frac{c_2(F)}{2} \geq c_2(R') \geq c_2(R) \geq \frac{c_2(F^{**})}{2} > 0, \tag{4.29}$$

and $c_2(R') = 2$ forces $c_3(R') > 0$, which is impossible by Remark 4.7. It remains to exclude $c_2(R') = c_2(R) = 1$. Note that this implies $c_2(P) = 1$. By Remark 4.7 we have $c_3(R) \leq -1$, and $c_3(P) \leq -1$. Since $0 \leq c_3(F^{**}) = c_3(P) + c_3(R)$, we get a contradiction.

We have thus proved $c_2(F^{**}) = 0$. Let us show that this implies $F^{**} \cong \mathscr{O}_X^2$. Indeed, taking the double dual of (4.27), we may assume that $Q$ is torsionfree. Then $c_2(Q) = 0$ and we have $0 \geq c_3(Q) = c_3(F^{**}) \geq 0$. One gets $Q \cong \mathscr{O}_X$ and $F^{**} \cong \mathscr{O}_X^2$.

We may consider thus a diagram of the form (4.28), where this time $P = R = \mathscr{O}_X$, and $c_2(R')$ is either 0 or 1.

Now, if $c_2(R')$ is zero, then $c_3(R') \leq 0$. So $c_2(P') = 4$ and $c_3(P') \geq 0$. Applying (4.8) to $P'$, we obtain $c_3(P') = 2$ or 0. These cases correspond respectively to (4.25) and (4.26).

On the other hand, in case $c_2(R') = 1$, then $c_3(R') \leq -1$. Hence $c_2(P') = 3$ and $c_3(P') \geq 1$. This time (4.8) implies $c_3(P') = 1$, i.e. case (4.24).

Finally, let us see that, in case (4.25), the point $x$ must lie in $C$. Observe that otherwise one easily gets the isomorphism $\mathscr{H}om_X(\mathcal{I}_x, \mathcal{I}_C) \cong \mathcal{I}_C$, and $\mathscr{E}xt^1_X(\mathcal{I}_x, \mathcal{I}_C) = 0$. Since $\mathrm{H}^2(X, \mathcal{I}_C) = 0$, by Riemann-Roch one obtains $\mathrm{H}^1(X, \mathcal{I}_C) = 0$, so that the extension group $\mathrm{Ext}^1_X(\mathcal{I}_x, \mathcal{I}_C)$ vanishes (by the local-to-global spectral sequence). Hence the sheaf $F$ would not be semistable. $\square$

**Remark 4.17.** By the previous proposition, the open subset of stable sheaves $\mathsf{M}^s_X(2,0,4)$ of the moduli space $\mathsf{M}_X(2,0,4)$ consists of two disjoint subschemes $\mathsf{M}^{\ell f}_X(2,0,4)$ and $\mathsf{N}^s_X(2,0,4) = \mathsf{M}^s_X(2,0,4) \setminus \mathsf{M}^{\ell f}_X(2,0,4)$, where $\mathsf{M}^{\ell f}_X(2,0,4)$ is the subset of $\mathsf{M}_X(2,0,4)$ consisting of locally free sheaves, while any sheaf $F$ in $\mathsf{N}^s_X(2,0,4)$ satisfies $F^{**} \cong \mathscr{O}_X^2$.

The scheme $\mathsf{N}^s_X(2,0,4)$ contains a subset consisting of the extensions of the form (4.25). This subset has dimension at least 5, since $\dim(\mathscr{H}^0_4(X)) \geq 4$ and we have a 1-dimensional parameter given by the choice of $x$ in $C$.

The closure of $\mathsf{M}^{\ell f}_X(2,0,4)$ and the closure of $\mathsf{N}^s_X(2,0,4)$ in $\mathsf{M}_X(2,0,4)$ meet along a subset containing the $S$-equivalence classes of sheaves fitting into (4.13).

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