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Modulus of continuity of some conditionally sub-Gaussian fields, application to stable random fields

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In this paper, we study modulus of continuity and rate of convergence of series of conditionally sub-Gaussian random fields. This framework includes both classical series representations of Gaussian fields and LePage series representations of stable fields. We enlighten their anisotropic properties by using an adapted quasi-metric instead of the classical Euclidean norm. We specify our assumptions in the case of shot noise series where arrival times of a Poisson process are involved. This allows us to state unified results for harmonizable (multi)operator scaling stable random fields through their LePage series representation, as well as to study sample path properties of their multistable analogous.

Keywords: Hölder regularity; operator scaling property; stable and multistable random fields; sub-Gaussian

1. Introduction

In recent years, lots of new random fields have been defined to propose new models for rough real data. To cite a few of them, let us mention the (multi)fractional Brownian fields (see, e.g., [6]), the linear and harmonizable (multi)fractional stable processes [12,36] and some anisotropic fields such as the (multi)fractional Brownian and stable sheets [3,4] and the (multi)operator scaling Gaussian and stable fields [8,9]. In the Gaussian setting, sample path regularity relies on mean square regularity. To study finer properties such as modulus of continuity, a powerful technique consists in considering a representation of the field as a series of random fields, using for instance Karhunen Loeve decomposition (see [1], Chapter 3), Fourier or wavelet series (as in [5,17]). This also allows generalizations to non-Gaussian framework using for instance LePage series [24,25] for stable distributions (see, e.g., [20]). Actually, following previous works of LePage [24] and Marcus and Pisier [29], Kôno and Maejima proved in [21] that, for $\alpha \in (0,2)$, an isotropic complex-valued α-stable random variable may be represented as a convergent shot noise series of the form

$$\sum_{n=1}^{+\infty} T_n^{-1/\alpha} X_n, \tag{1}$$

with $(T_n)_{n\geq 1}$ the sequence of arrival times of a Poisson process of intensity 1, and $(X_n)_{n\geq 1}$ a sequence of independent identically distributed (i.i.d.) isotropic complex-valued random variables, which is assumed to be independent of $(T_n)_{n\geq 1}$ and such that $\mathbb{E}(|X_1|^\alpha) < +\infty$. When $X_n = V_n g_n$ with $(g_n)_{n\geq 1}$ a sequence of i.i.d. Gaussian random variables independent of $(V_n, T_n)_{n\geq 1}$, the series may be considered as a conditional Gaussian series. This is one of the main argument used in [7,8,12,20] to study the sample path regularity of some stable random fields. Another classical representation consists in choosing $X_n = V_n\varepsilon_n$ with $(\varepsilon_n)_{n\geq 1}$ a sequence of i.i.d. Rademacher random variables that is, such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. Both g_n and ε_n are sub-Gaussian random variables. Sub-Gaussian random variables have first been introduced in [17] for the study of random Fourier series. Their main property is that their tail distributions behave like the Gaussian ones and then sample path properties of sub-Gaussian fields may be set as for Gaussian ones (see Theorem 12.16 of [23], e.g.). In particular, they also rely on their mean square regularity.

In this paper, we study the sample path regularity of the complex-valued series of conditionally sub-Gaussian fields defined as

$$S(x) = \sum_{n=1}^{+\infty} W_n(x) g_n, \qquad \text{for } x \in K_d \subset \mathbb{R}^d, \tag{2}$$

with $(g_n)_{n\geq 1}$ a sequence of independent symmetric sub-Gaussian complex random variables, which is assumed independent of $(W_n)_{n\geq 1}$. In this setting, we give sufficient assumptions on the sequence $(W_n)_{n\geq 1}$ to get an upper bound of the modulus of continuity of S as well as a uniform rate of convergence. Then, we focus on shot noises series

$$S(\alpha, u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_n(\alpha, u) g_n, \qquad x = (\alpha, u) \in K_{d+1} \subset (0, 2) \times \mathbb{R}^d,$$

with $(T_n)_{n\geq 1}$ the sequence of arrival times of a Poisson process. Assuming the independence of $(T_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$, we state some more convenient conditions based on moments of V_n to ensure that the main assumptions of this paper are fulfilled. In particular when $V_n(\alpha, u) := X_n$ is a symmetric random variable, one of our main result gives a uniform rate of convergence of the shot noise series (1) in α on any compact $K_1 = [a, b] \subset (0, 2)$, which improves the results obtained in [11] on the convergence of such series. In the framework of LePage random series, which are particular examples of shot noise series, we also establish that to improve the upper bound of the modulus of continuity of S, one has the opportunity to use an other series representation of S. On the one hand, our framework allows to include in a general setting some sample path regularity results already obtained in [7,8] for harmonizable (multi)operator scaling stable random fields. On the other hand, considering α as a function of $u \in \mathbb{R}^d$, we also investigate sample path properties of multistable random fields that have been introduced in [13]. To illustrate our results, we focus on harmonizable random fields.

The paper falls into the following parts. In Section 2, we recall definition and properties of sub-Gaussian random variables and state our first assumption needed to ensure that the random field S is well-defined by (2). We also introduce a notion of anisotropic local regularity, which is obtained by replacing the isotropic Euclidean norm of $\mathbb{R}^d$ by a quasi-metric that can reveal the anisotropy of the random fields. Section 3 is devoted to our main results concerning both local modulus of continuity of the random field S defined by the series (2) and rate of convergence of this series. Section 4 deals with the particular setting of shot noise series, the case of LePage series being treated in Section 4.3. Then Section 5 is devoted to the study of the sample path regularity of stable or even multistable random fields. Technical proofs are postponed to Appendix for reader convenience.

2. Preliminaries

2.1. Sub-Gaussian random variables

Real-valued sub-Gaussian random variables have been defined by [17]. The structure of the class of these random variables and some conditions for continuity of real-valued sub-Gaussian random fields have been studied in [10]. In this paper, we focus on conditionally complex-valued sub-Gaussian random fields, where a complex sub-Gaussian random variable is defined as follows.

Definition 2.1. *A complex-valued random variable Z is sub-Gaussian if there exists $s \in [0, +\infty)$ such that*

$$\forall z \in \mathbb{C}, \qquad \mathbb{E}\big(e^{\Re(\overline{z}Z)}\big) \le e^{(s^2|z|^2)/2}. \tag{3}$$

Remark 2.1. This definition coincides also with complex sub-Gaussian random variables as defined in [15] in the more general setting of random variables with values in a Banach space. Moreover, for a real-valued random variable Z, it also coincides with the definition in [17]. Kahane [17] called the smallest s such that (3) holds the Gaussian shift of the sub-Gaussian variable Z. In this paper, if (3) is fulfilled, we say that Z is sub-Gaussian with parameter s.

Remark 2.2. A complex-valued random variable Z is sub-Gaussian if and only if $\Re(Z)$ and $\Im(Z)$ are real sub-Gaussian random variables. Note that if Z is sub-Gaussian with parameter s then $\mathbb{E}(\Re(Z)) = \mathbb{E}(\Im(Z)) = 0$ and $\mathbb{E}(\Re(Z)^2) \le s^2$ as well as $\mathbb{E}(\Im(Z)^2) \le s^2$.

The main property of sub-Gaussian random variables is that their tail distributions decrease exponentially as the Gaussian ones (see Lemma A.1). Moreover, considering convergent series of independent symmetric sub-Gaussian random variables, a uniform rate of decrease is available and the limit remains a sub-Gaussian random variable. This result, stated below, is one of the main tool we use to study sample path properties of conditionally sub-Gaussian random fields.

Proposition 2.1. *Let $(g_n)_{n\ge1}$ be a sequence of independent symmetric sub-Gaussian random variables with parameter $s = 1$. Let us consider a complex-valued sequence $a = (a_n)_{n\ge1}$ such that*

$$\|a\|_{\ell^2}^2 = \sum_{n=1}^{+\infty} |a_n|^2 < +\infty.$$

1. *Then, for any* $t \in (0, +\infty)$, $\mathbb{P}(\sup_{N\in\mathbb{N}\setminus\{0\}} |\sum_{n=1}^{N} a_n g_n| > t\|a\|_{\ell^2}) \leq 8\mathrm{e}^{-t^2/8}$.
2. *Moreover, the series* $\sum a_n g_n$ *converges almost surely, and its limit* $\sum_{n=1}^{+\infty} a_n g_n$ *is a sub-Gaussian random variable with parameter* $\|a\|_{\ell^2}$.

Proof. See Appendix A. □

Remark 2.3. In the previous proposition, assuming that the parameter $s = 1$ is not restrictive since a_n can be replaced by $a_n s_n$ and g_n by g_n/s_n when g_n is sub-Gaussian with parameter $s_n > 0$.

2.2. Conditionally sub-Gaussian series

In the whole paper, for $d \geq 1$, $K_d = \prod_{j=1}^{d} [a_j, b_j] \subset \mathbb{R}^d$ is a compact d-dimensional interval and for each integer $N \in \mathbb{N}$, we consider

$$S_N(x) = \sum_{n=1}^{N} W_n(x) g_n, \qquad x \in K_d, \tag{4}$$

where $\sum_{n=1}^{0} = 0$ by convention and where the sequence $(W_n, g_n)_{n\geq 1}$ satisfies the following assumption.

Assumption 1. *Let* $(g_n)_{n\geq 1}$ *and* $(W_n)_{n\geq 1}$ *be independent sequences of random variables.*

1. $(g_n)_{n\geq 1}$ *is a sequence of independent symmetric complex-valued sub-Gaussian random variables with parameter* $s = 1$.
2. $(W_n)_{n\geq 1}$ *is a sequence of complex-valued continuous random fields defined on* K_d *and such that*

$$\forall x \in K_d, \qquad \textit{almost surely} \sum_{n=1}^{+\infty} \left|W_n(x)\right|^2 < +\infty.$$

Under Assumption 1, conditionally on $(W_n)_{n\geq 1}$, each S_N is a sub-Gaussian random field defined on K_d. Moreover, for each x, Proposition 2.1 and Fubini theorem lead to the almost sure convergence of $S_N(x)$ as $N \to +\infty$. The limit field S defined by

$$S(x) = \sum_{n=1}^{+\infty} W_n(x) g_n, \qquad x \in K_d \subset \mathbb{R}^d, \tag{5}$$

is then a conditionally sub-Gaussian random field. In the sequel, we study almost sure uniform convergence and rate of uniform convergence of $(S_N)_{N\in\mathbb{N}}$ as well as the sample path properties of S.

Assume first that each g_n is a Gaussian random variable and that each W_n is a deterministic random field, which implies that S is a Gaussian centered random field. Then, it is well known that its sample path properties are given by the behavior of

$$s(x,y) := \left(\sum_{n=1}^{+\infty} \left|W_n(x) - W_n(y)\right|^2\right)^{1/2}, \qquad x, y \in K_d, \tag{6}$$

since s^2 is proportional to the variogram $(x,y) \mapsto v(x,y) := \mathbb{E}[|S(x) - S(y)|^2]$. In the following, we see that under Assumption 1, the behavior of S is still linked with the behavior of the parameter s. In this more general framework, a key tool is to remark that conditionally on $(W_n)_{n\geq 1}$, S is a sub-Gaussian random field and the random variable $S(x) - S(y)$ is sub-Gaussian with parameter $s(x,y)$.

We are particularly interested in anisotropic random fields S (and then anisotropic parameters s). Therefore, next section deals with an anisotropic generalization of the classical Hölder regularity, that is, with a notion of regularity which takes into account the anisotropy of the fields under study.

2.3. Anisotropic local regularity

Let us first recall the notion of quasi-metric (see, e.g., [32]), which is more adapted to our framework.

Definition 2.2. *A continuous function* $\rho : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ *is called a quasi-metric on* $\mathbb{R}^d$ *if*

1. ρ *is faithful*, *that is*, $\rho(x,y) = 0$ *iff* $x = y$;
2. ρ *is symmetric*, *that is*, $\rho(x,y) = \rho(y,x)$;
3. ρ *satisfies a quasi-triangle inequality*: *there exists a constant* $\kappa \geq 1$ *such that*

$$\forall x, y, z \in \mathbb{R}^d, \qquad \rho(x,z) \leq \kappa\big(\rho(x,y) + \rho(y,z)\big).$$

Observe that a continuous function ρ is a metric on $\mathbb{R}^d$ if and only if ρ is a quasi-metric on $\mathbb{R}^d$ which satisfies assertion 3 with $\kappa = 1$. In particular, the Euclidean distance is an isotropic quasi-metric and its following anisotropic generalization

$$(x,y) \mapsto \rho(x,y) := \left(\sum_{i=1}^{d} |x_i - y_i|^{p/a_i}\right)^{1/p}, \qquad \text{where } p > 0 \text{ and } a_1, \dots, a_d > 0,$$

is also a quasi-metric. Such quasi-metrics are particular cases of the following general example.

Example 2.1. Let us consider E a real $d \times d$ matrix whose eigenvalues have positive real parts and $\tau_E : \mathbb{R}^d \to \mathbb{R}^+$ a continuous even function such that

(i) for all $x \neq 0$, $\tau_E(x) > 0$;
(ii) for all $r > 0$ and all $x \in \mathbb{R}^d$, $\tau_E(r^E x) = r\tau_E(x)$ with $r^E = \exp((\ln r)E)$.

The classical example of such a function is the radial part of polar coordinates with respect to E introduced in Chapter 6 of [31]. Other examples have been given in [9].

Let us consider the continuous function ρ_E, defined on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$\rho_E(x,y) = \tau_E(x-y).$$

Then, by definition of τ_E, ρ_E is faithful and symmetric. Moreover, by Lemma 2.2 of [9], ρ_E also satisfies a quasi-triangle inequality. Hence, ρ_E is a quasi-metric on $\mathbb{R}^d$ and it is adapted to study operator scaling random fields (see, [7,9], e.g.).

Let us remark that since ρ_E^β defines a quasi-metric for E/β whatever $\beta > 0$ is, we may restrict our study to matrix E whose eigenvalues have real parts greater than one. Then, by Proposition 3.5 of [8], there exist $0 < \underline{H} \leq \overline{H} \leq 1$ and two constants $c_{2,1}, c_{2,2} \in (0,\infty)$ such that for all $x, y \in \mathbb{R}^d$,

$$c_{2,1} \min\bigl(\|x-y\|^{\overline{H}}, \|x-y\|^{\underline{H}}\bigr) \leq \rho_E(x,y) \leq c_{2,2} \max\bigl(\|x-y\|^{\overline{H}}, \|x-y\|^{\underline{H}}\bigr),$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. In [7,8], this comparison is one of the main tool in the study of the regularity of some stable anisotropic random fields. Therefore, throughout the paper, we consider a quasi-metric ρ such that there exist $0 < \underline{H} \leq \overline{H} \leq 1$ and two constants $c_{2,1}, c_{2,2} \in (0,\infty)$ such that for all $x, y \in \mathbb{R}^d$, with $\|x-y\| \leq 1$,

$$c_{2,1} \|x-y\|^{\overline{H}} \leq \rho(x,y) \leq c_{2,2} \|x-y\|^{\underline{H}}. \tag{7}$$

Before we introduce the anisotropic regularity used in the following, let us briefly comment this assumption.

Remark 2.4.

1. The upper bound is needed in the sequel to construct a particular 2^{-k} net for ρ, whose cardinality can be estimated using the lower bound.
2. Using the quasi-triangle inequality satisfied by ρ and its continuity, one deduces from (7) that for any non-empty compact set $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$, there exist two finite positive constants $c_{2,1}(K_d)$ and $c_{2,2}(K_d)$ such that for all $x, y \in K_d$,

$$c_{2,1}(K_d) \|x-y\|^{\overline{H}} \leq \rho(x,y) \leq c_{2,2}(K_d) \|x-y\|^{\underline{H}}. \tag{8}$$

3. It is not restrictive to assume that $\overline{H} \leq 1$ since for any $c > 0$, ρ^c is also a quasi-metric.

We will consider the following anisotropic local and uniform regularity property.

Definition 2.3. *Let $\beta \in (0,1]$ and $\eta \in \mathbb{R}$. Let $x_0 \in K_d$ with $K_d \subset \mathbb{R}^d$. A real-valued function f defined on K_d belongs to $\mathcal{H}_{\rho,K_d}(x_0, \beta, \eta)$ if there exist $\gamma \in (0,1)$ and $C \in (0,+\infty)$ such that*

$$\bigl|f(x) - f(y)\bigr| \leq C\rho(x,y)^\beta \bigl|\log\bigl(\rho(x,y)\bigr)\bigr|^\eta$$

for all $x, y \in B(x_0, \gamma) \cap K_d = \{z \in K_d; \|z - x_0\| \le \gamma\}$. *Moreover* f *belongs to* $\mathcal{H}_\rho(K_d, \beta, \eta)$ *if there exists* $C \in (0, +\infty)$ *such that*

$$\forall x, y \in K_d, \qquad \left|f(x) - f(y)\right| \le C\rho(x, y)^\beta \left[\log\left(1 + \rho(x, y)^{-1}\right)\right]^\eta.$$

Remark 2.5.

1. If $f \in \mathcal{H}_{\rho,K_d}(x_0, \beta, \eta)$, then f is continuous at x_0. Moreover, since $h^\beta[\log(1 + h^{-1})]^\eta \sim_{h\to 0_+} h^\beta|\log(h)|^\eta$ and since ρ satisfies equation (7), $f \in \mathcal{H}_{\rho,K_d}(x_0, \beta, \eta)$ if and only if for some $\gamma > 0$, $f \in \mathcal{H}_\rho(B(x_0, \gamma) \cap K_d, \beta, \eta)$.
2. If $f \in \mathcal{H}_\rho(K_d, \beta, \eta)$, then $f \in \mathcal{H}_{\rho,K_d}(x_0, \beta, \eta)$ for all $x_0 \in K_d$. The converse is also true since K_d is a compact. This follows from the Lebesgue's number lemma and the boundedness of the continuous function f on the compact set K_d (see Lemma B.2 stated in the Appendix for an idea of the proof).
3. A function in $\mathcal{H}_\rho(K_d, \beta, 0)$ may be view as a Lipschitz function on an homogeneous space [28]. Note also that when ρ is the Euclidean distance, for any $\beta \le 1$ and $\eta \le 0$, the set $\mathcal{H}_\rho(K_d, \beta, \eta)$ (resp., $\mathcal{H}_{\rho,K_d}(x_0, \beta, \eta)$) is included in the set of Hölder functions of order β on K_d (resp., around x_0).
4. Assuming $\beta \le 1$ is not restrictive since, for any $c > 0$, ρ^c is also a quasi-metric.

The introduction of the logarithmic term appears naturally when considering Gaussian random fields. Actually, [6] proves that for all $\beta \in (0, 1]$, a large class of elliptic Gaussian random fields X_β, including the famous fractional Brownian fields, belongs a.s. to $\mathcal{H}_{\rho,K_d}(x_0, \beta, 1/2)$ with ρ the Euclidean distance (see Theorem 1.3 in [6]). Moreover, Xiao [38] also gives some anisotropic examples of Gaussian fields belonging a.s. to $\mathcal{H}_{\rho,K_d}(x_0, 1, 1/2)$ for some anisotropic quasi-distance $\rho = \rho_E$ associated with E a diagonal matrix (see Theorem 4.2 of [37]). Finally, in [8], we construct stable and Gaussian random fields belonging a.s. to $\mathcal{H}_{\rho_{x_0},K_d}(x_0, 1 - \varepsilon, 0)$ for some convenient ρ_{x_0} (see Theorem 4.6 in [8]).

3. Main results on conditionally sub-Gaussian series

3.1. Local modulus of continuity

In this section, we first give an upper bound of the local modulus of continuity of S defined by (4) under the following local assumption on the conditional parameter (6).

Assumption 2. *Let* $x_0 \in K_d$ *with* $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$. *Let us consider* ρ *a quasi-metric on* $\mathbb{R}^d$ *satisfying equation* (7). *Assume that there exist an almost sure event* Ω' *and some random variables* $\gamma > 0$, $\beta \in (0, 1]$, $\eta \in \mathbb{R}$ *and* $C \in (0, +\infty)$ *such that on* Ω'

$$\forall x, y \in B(x_0, \gamma) \cap K_d, \qquad s(x, y) \le C\rho(x, y)^\beta \left|\log\left(\rho(x, y)\right)\right|^\eta,$$

where we recall that the conditional parameter s *is given by* (6).

Note that the event Ω', the random variables γ, β, η, C and the quasi-metric ρ may depend on x_0.

Let us now state the main result of this section on the modulus of continuity. The main difference with [7,8,21] is that we do not only consider the limit random field S but obtain a uniform upper bound in N for the modulus of continuity of S_N.

Theorem 3.1. *Assume that Assumptions 1 and 2 are fulfilled. Then, almost surely, there exist $\gamma^* \in (0, \gamma)$ and $C \in (0, +\infty)$ such that for all $x, y \in B(x_0, \gamma^*) \cap K_d$,*

$$\sup_{N \in \mathbb{N}} \left| S_N(x) - S_N(y) \right| \le C\rho(x, y)^\beta \left| \log \rho(x, y) \right|^{\eta+1/2}.$$

Moreover, almost surely $(S_N)_{N\in\mathbb{N}}$ converges uniformly on $B(x_0, \gamma^*) \cap K_d$ to S and the limit S belongs to $\mathcal{H}_{\rho,K_d}(x_0, \beta, \eta + 1/2)$. In particular, almost surely S is continuous at x_0.

Proof. See Appendix B.1. □

Strengthening Assumption 2, the uniform convergence and the upper bound for the modulus of continuity are obtained on deterministic set. Next corollary is obtained using some covering argument.

Corollary 3.2. *Assume that Assumption 1 is fulfilled.*

1. *Assume that Assumption 2 holds for any $x_0 \in K_d$ with the same almost sure event Ω', the same random variables β and η, and the same quasi-metric ρ. Then Theorem 3.1 holds replacing $B(x_0, \gamma^*) \cap K_d$ by all the set K_d and almost surely S belongs to $\mathcal{H}_\rho(K_d, \beta, \eta + 1/2)$.*
2. *Assume now that Assumption 2 holds with a deterministic γ. Then Theorem 3.1 holds replacing $B(x_0, \gamma^*) \cap K_d$ by $B(x_0, \gamma) \cap K_d$ and almost surely S belongs to $\mathcal{H}_\rho(B(x_0, \gamma) \cap K_d, \beta, \eta + 1/2)$.*

Proof. See Appendix B.1. □

When considering S an operator scaling Gaussian random field, note that Li *et al.* [27] proves that the upper bound obtained by Corollary 3.2 is optimal. Moreover for some Gaussian anisotropic random fields, Xiao [37] also obtains a sample path regularity in the stronger L^p-sense on whole the compact K_d. This follows from an extension of the Garsia–Rodemich–Rumsey continuity lemma Garsia *et al.* [16] or the minorization metric method of Kwapień and Rosiński [22]. This would be interesting to study if these results still hold when considering a quasi-metric ρ (and not a metric) and if they can be applied to obtain the sample path regularity of S in the stronger L^p-sense on whole the compact K_d, strengthening the assumption on the parameter s.

3.2. Rate of almost sure uniform convergence

This section is concerned with the rate of uniform convergence of the series $(S_N)_{N\in\mathbb{N}}$ defined by (4). Under Assumption 1, this series converges to S and, for any integer N, we consider the rest

$$R_N(x) = S(x) - S_N(x) = \sum_{n=N+1}^{+\infty} W_n(x) g_n, \qquad x \in K_d \subset \mathbb{R}^d.$$

Then, conditionally on $(W_n)_{n\geq 1}$, $R_N(x) - R_N(y)$ is a sub-Gaussian random variable with parameter

$$r_N(x,y) = \left(\sum_{n=N+1}^{+\infty} \left|W_n(x) - W_n(y)\right|^2\right)^{1/2}, \qquad x, y \in K_d. \tag{9}$$

Observe that $R_0 = S$ and that $r_0(x,y) = s(x,y)$. To obtain a rate of uniform convergence for the sequence $(S_N)_{N\in\mathbb{N}}$, the general assumption relies on a rate of convergence for the sequence $(r_N)_{N\in\mathbb{N}}$.

Assumption 3. *Let $x_0 \in K_d$ with $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$ and let ρ be a quasi-metric on $\mathbb{R}^d$ satisfying (7). Assume that there exist an almost sure event Ω', some random variables $\gamma > 0$, $\beta \in (0,1]$, $\eta \in \mathbb{R}$ and a positive random sequence $(b(N))_{N\in\mathbb{N}}$ such that on Ω',*

$$\forall N \in \mathbb{N}, \forall x, y \in B(x_0, \gamma) \cap K_d, \qquad r_N(x,y) \leq b(N)\rho(x,y)^\beta \left|\log\big(\rho(x,y)\big)\right|^\eta. \tag{10}$$

Note that Ω', ρ and the random variables γ, β, η and $b(N)$ may depend on x_0. Note also that since Assumption 3 implies Assumption 2, according to Theorem 3.1, almost surely, there exists $\gamma^* \in (0,\gamma)$ such that $R_N = S - S_N$ is continuous on $B(x_0, \gamma^*)$. The following theorem precises the modulus of continuity of R_N with respect to N and a rate of uniform convergence.

Theorem 3.3. *Assume that Assumptions 1 and 3 are fulfilled.*

1. *Then, almost surely, there exists $\gamma^* \in (0,\gamma)$ and $C \in (0,+\infty)$ such that for*

$$\left|R_N(x) - R_N(y)\right| \leq C b(N)\sqrt{\log(N+2)}\rho(x,y)^\beta \left|\log \rho(x,y)\right|^{\eta+1/2}$$

for all $N \in \mathbb{N}$ and all $x, y \in B(x_0, \gamma^*) \cap K_d$.

2. *Moreover, if almost surely, for all $N \in \mathbb{N}$,*

$$\left|R_N(x_0)\right| \leq b(N)\sqrt{\log(N+2)}, \tag{11}$$

then, almost surely, there exists $\gamma^* \in (0,\gamma)$ and $C \in (0,+\infty)$ such that

$$\left|R_N(x)\right| \leq C b(N)\sqrt{\log(N+2)}$$

for all $N \in \mathbb{N}$ and all $x \in B(x_0, \gamma^*) \cap K_d$.

Proof. See Appendix B.2. □

An analogous of Corollary 3.2 holds for strengthening the previous local theorem to get uniform results on K_d or on $B(x_0, \gamma) \cap K_d$ when γ is deterministic.

Corollary 3.4. *Assume that Assumptions 1 is fulfilled.*

1. *Assume that Assumption 3 holds for any $x_0 \in K_d$ with the same almost sure event Ω', the same random variables β and η, the same sequence $(b(N))_{N\in\mathbb{N}}$ and the same quasi-metric ρ. Then assertion 1 of Theorem 3.3 holds replacing $B(x_0, \gamma^*) \cap K_d$ by all the set K_d. If moreover, equation (11) is fulfilled for some $x_0 \in K_d$, then*

$$\sup_{N\in\mathbb{N}} \sup_{x\in K_d} \frac{|R_N(x)|}{b(N)\sqrt{\log(N+2)}} < +\infty \qquad \textit{almost surely.}$$

2. *Assume now that Assumption 3 holds with a deterministic γ. Then Theorem 3.3 holds replacing $B(x_0, \gamma^*) \cap K_d$ by $B(x_0, \gamma) \cap K_d$.*

4. Shot noise series

4.1. Preliminaries

In this section, we consider the sequence of shot noise series defined by

$$\forall N \in \mathbb{N}, \forall \alpha \in K_1 = [a, b] \subset (0, 2), \qquad S_N^*(\alpha) = \sum_{n=1}^{N} T_n^{-1/\alpha} X_n,$$

where for all $n \geq 1$, the random variable T_n is the nth arrival time of a Poisson process with intensity 1 and $(X_n)_{n\geq 1}$ is a sequence of i.i.d. symmetric random variables, which is assumed independent of $(T_n)_{n\geq 1}$. Let us first recall that $S_N^*(\alpha)$ converges almost surely to $S^*(\alpha)$ an α-stable random variable as soon as $X_n \in L^\alpha$ (see, [35], e.g.). Under a strengthened assumption on the integrability of X_n, rate of pointwise almost sure convergence and rate of absolute convergence have also been given in Theorems 2.1 and 2.2 of [11].

Since $(X_n)_{n\geq 1}$ may not be a sequence of sub-Gaussian random variables, we cannot apply Section 3 to the sequence $(S_N^*)_{N\in\mathbb{N}}$. However, due to symmetry of $(X_n)_{n\geq 1}$,

$$(X_n)_{n\geq 1} \stackrel{(d)}{=} (X_n g_n)_{n\geq 1}$$

with $(g_n)_{n\geq 1}$ a Rademacher sequence independent of $(X_n, T_n)_{n\geq 1}$ and Section 3 allows to study

$$S_N(\alpha) = \sum_{n=1}^{N} W_n(\alpha) g_n \qquad \text{with } W_n(\alpha) := T_n^{-1/\alpha} X_n.$$

Moreover, in Theorems 3.1 and 3.3, S_N (resp., $R_N = S - S_N$) can be replaced by S_N^* (resp., $R_N^* = S^* - S_N^*$), see the proof of next theorem for details. Then, assuming that X_n is sufficiently integrable, we obtain the uniform convergence of S_N^* on a deterministic compact interval $K_1 = [a,b] \subset (0,2)$ and a rate of uniform convergence. These results, stated in the following theorem, strengthen Theorem 2.1 of [11] which deals with the pointwise rate of convergence.

Theorem 4.1. *For any integer $n \geq 1$, let T_n be the nth arrival time of a Poisson process with intensity 1. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. symmetric random variables, which is assumed independent of $(T_n)_{n\geq 1}$. Furthermore assume that $\mathbb{E}(|X_1|^{2p}) < +\infty$ for some $p > 0$.*

1. *Then, almost surely, for all $b \in (0, \min(2, 2p))$ and for all $a \in (0, b]$, the sequence of partial sums $(S_N^*)_{N\in\mathbb{N}}$ converges uniformly on $[a,b]$.*
2. *Moreover, almost surely, for all $b \in (0, \min(2, 2p))$ and for all $a \in (0, b)$, for all $p' > 0$ with $1/p' \in (0, 1/b - 1/\min(2p, 2))$,*

$$\sup_{N\in\mathbb{N}} \sup_{\alpha\in[a,b]} N^{1/p'} \left| \sum_{n=N+1}^{+\infty} T_n^{-1/\alpha} X_n \right| < +\infty.$$

Proof. See Appendix C.1. □

4.2. Modulus of continuity and rate of convergence of shot noise series

In this section, we focus on some shot noise series, which are particular examples of conditionally sub-Gaussian series. For this purpose, we assume that the following assumption is fulfilled.

Assumption 4. *Let $(T_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ be independent sequences satisfying the following conditions.*

1. *$(g_n)_{n\geq 1}$ is a sequence of independent complex-valued symmetric sub-Gaussian random variables with parameter $s = 1$.*
2. *T_n is the nth arrival time of a Poisson process with intensity 1.*
3. *$(V_n)_{n\geq 1}$ is a sequence of i.i.d. complex-valued random fields defined on $K_{d+1} \subset (0,2) \times \mathbb{R}^d$.*
4. *For any $(\alpha, u) \in K_{d+1}$, $V_n(\alpha, u) \in L^\alpha$.*

For any integer $n \geq 1$, we consider the complex-valued random field W_n defined by

$$W_n(\alpha, u) := T_n^{-1/\alpha} V_n(\alpha, u), \qquad (\alpha, u) \in K_{d+1} \subset (0,2) \times \mathbb{R}^d. \tag{12}$$

Since $|V_n(\alpha, u)|^2 \in L^{\alpha/2}$ and $\alpha/2 \in (0,1)$, according to Theorem 1.4.5 of [35],

$$\sum_{n=1}^{+\infty} \left| W_n(\alpha, u) \right|^2 = \sum_{n=1}^{+\infty} T_n^{-2/\alpha} \left| V_n(\alpha, u) \right|^2 < +\infty \qquad \text{almost surely.}$$

Therefore, the independent sequences $(W_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ satisfy Assumption 1. Then, S and $(S_N)_{N\in\mathbb{N}}$ are well-defined on $K_{d+1}\subset(0,2)\times\mathbb{R}^d\subset\mathbb{R}^{d+1}$ by (5) and (4). Before we study, the modulus of continuity of S and the rate of convergence of $(S_N)_{N\in\mathbb{N}}$, let us state some remarks.

Remark 4.1. Assume that conditions 1–3 of Assumption 4 are fulfilled with $(g_n)_{n\geq 1}$ a sequence of i.i.d. random variables. Then Remark 2.6 of [34] proves that condition 4 is a necessary and sufficient condition for the almost sure convergence of $(S_N(\alpha,u))_{N\in\mathbb{N}}$ for each $(\alpha,u)\in K_{d+1}$. Note that by Itô–Nisio theorem (see, e.g., Theorem 6.1 of [23]), it is also a necessary and sufficient condition for the convergence in distribution of the sequence $(S_N(\alpha,u))_{N\in\mathbb{N}}$. Then, condition 4 is not a strong assumption and is clearly essential to ensure that $S(\alpha,u)$ is well-defined.

Remark 4.2. Assume that Assumption 4 is fulfilled with $(g_n)_{n\geq 1}$ a sequence of i.i.d. random variables. Then, it is well known that for each $\alpha\in(0,2)$, $S(\alpha,\cdot)$ is an α-stable symmetric random field, as field in variable u. In Section 5.1, we will focus on α-stable random fields defined through a stochastic integral and see that, up to a multiplicative constant, such a random field X_α has the same finite distributions as $S(\alpha,\cdot)$ for a suitable choice of $(g_n,V_n)_{n\geq 1}$. The sample path regularity of S in its variable α is not needed to obtain an upper bound of the modulus of continuity of X_α. Nevertheless, this regularity is useful to deal with multistable random fields (see Section 5.2).

The sequel of this section is devoted to simple criteria, based on some moments of V_n, which ensure that Assumption 3 (and then Assumption 2) is fulfilled. More precisely, the results given below help us to give simple conditions in order to get Assumption 3 and (11) satisfied with $b(N)=(N+1)^{-1/p'}$ for some convenient $p'>0$. Then, all the results of Section 3 hold.

Theorem 4.2. *Assume that Assumption 4 is fulfilled with* $K_{d+1}=[a,b]\times\prod_{j=1}^d[a_j,b_j]\subset(0,2)\times\mathbb{R}^d$ *and let* ρ *be a quasi-metric on* $\mathbb{R}^{d+1}$ *satisfying equation (7). Assume also that for some* $x_0\in K_{d+1}$*, there exist* $r\in(0,+\infty)$, $\beta\in(0,1]$, $\eta\in\mathbb{R}$ *and* $p\in(b/2,+\infty)$ *such that* $\mathbb{E}(|V_1(x_0)|^{2p})<+\infty$ *and*

$$\mathbb{E}\left(\left[\sup_{\substack{x,y\in K_{d+1}\\0<\|x-y\|\leq r}}\frac{|V_1(x)-V_1(y)|}{\rho(x,y)^\beta|\log\rho(x,y)|^\eta}\right]^{2p}\right)<\infty. \tag{13}$$

Let us recall that S *and* S_N *are defined by (5) and (4) with* W_n *given by (12).*

1. *Then, almost surely* $(S_N)_{N\in\mathbb{N}}$ *converges uniformly on* K_{d+1} *and its limit* S *belongs almost surely to* $\mathcal{H}_\rho(K_{d+1},\beta,\max(\eta,0)+1/2)$.
2. *Moreover, when* $p'>0$ *is such that* $1/p'\in(0,1/b-1/\min(2p,2))$*, almost surely*

$$\sup_{N\in\mathbb{N}}N^{1/p'}\sup_{x\in K_{d+1}}\left|S(x)-S_N(x)\right|<+\infty.$$

Proof. See Appendix C.2. □

Example 4.1. Assume that V_1 is a fractional Brownian field on $\mathbb{R}^d$ with Hurst parameter H. Then (13) is satisfied for all $p > 0$ with $\rho(x, y) = \|x - y\|$, $\beta = H$ and $\eta = 1/2$ (see, e.g., Theorem 1.3 of [6]).

Let us now present a method (similar to those used in [7,8,20] to bound some conditional variance) to establish (13).

Proposition 4.3. *Let $x_0 = (\alpha_0, u_0) \in K_{d+1}$ with $K_{d+1} = [a, b] \times \prod_{j=1}^{d}[a_j, b_j] \subset (0, 2) \times \mathbb{R}^d$. Let V_1 be a complex-valued random field defined on K_{d+1}. Assume that there exists a random field $(\mathcal{G}(h))_{h\in[0,+\infty)}$ with values in $[0, \infty)$ and such that*

(i) *there exists ρ a quasi-metric on $\mathbb{R}^{d+1}$ satisfying equation (7) such that almost surely,*

$$\forall x, y \in K_{d+1}, \qquad \left|V_1(x) - V_1(y)\right| \le \mathcal{G}\bigl(\rho(x, y)\bigr);$$

(ii) *there exists $h_0 \in (0, 1]$ such that almost surely, the function $h \mapsto \mathcal{G}(h)$ is monotonic on $[0, h_0]$;*

(iii) *there exist $p > b/2$ and some constants $\beta \in (0, 1]$, $\eta \in \mathbb{R}$ and $C \in (0, \infty)$ such that for some $\varepsilon > 0$ and for $h > 0$ small enough,*

$$I(h) := \mathbb{E}\bigl(\mathcal{G}(h)^{2p}\bigr) \le Ch^{2p\beta}|\log h|^{2p(\eta-1/2p-\varepsilon)}. \tag{14}$$

Then, equation (13) holds for $r > 0$ small enough.

Proof. See Appendix C.2. □

Remark 4.3. If $(V_n)_{n\ge1}$ is a sequence of independent symmetric random variables, Theorem 4.2 still holds replacing $S_N(\alpha, u)$ (resp., $S(\alpha, u)$) by

$$S_N^*(\alpha, u) = \sum_{n=1}^{N} T_n^{-1/\alpha} V_n(\alpha, u) \qquad \left(\text{resp., by } S^*(\alpha, u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_n(\alpha, u)\right).$$

In particular, following Example 4.1, when Assumption 4 is fulfilled with V_1 a fractional Brownian field on $\mathbb{R}^d$ with Hurst parameter H, assumptions of Theorem 4.2 are fulfilled with ρ_{d+1} the Euclidean distance on $\mathbb{R}^{d+1}$, $\beta = H$ and $\eta = 1/2$, on any compact $(d + 1)$-dimensional interval K_{d+1}. Especially, this leads to an upper bound of the modulus of continuity of S^* on any compact $(d + 1)$-dimensional interval K_{d+1}. Then for any fixed $\alpha_0 \in (0, 2)$, we also obtain that the α_0-stable random field $(S^*(\alpha_0, u))_{u\in\mathbb{R}^d}$ is in $\mathcal{H}_{\rho_d}(K_d, H, 1)$ for ρ_d the Euclidean distance on $\mathbb{R}^d$ and for any compact set $K_d \subset \mathbb{R}^d$.

4.3. LePage random series representation

Representations in random series of infinitely divisible laws have been studied in [24,25]. Such representations have been successfully used to study sample path properties of some symmetric α_0-stable random processes ($d = 1$) and fields (see, e.g., [7,8,12,20]).

Let us be more precise on the assumptions on the LePage series under study.

Assumption 5. *Let $(T_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ be as in Assumption 4. Let $(\xi_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with common law*

$$\mu(\mathrm{d}\xi) = m(\xi)\nu(\mathrm{d}\xi)$$

equivalent to a σ-finite measure ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ (that is such that $m(\xi) > 0$ for ν-almost every ξ). This sequence is independent from $(g_n, T_n)_{n\geq 1}$. Moreover, we consider

$$V_n(\alpha, u) := f_\alpha(u, \xi_n) m(\xi_n)^{-1/\alpha},$$

where for any $\alpha \in K_1 \subset (0, 2)$, $f_\alpha : K_d \times \mathbb{R}^d \to \mathbb{C}$ is a deterministic function such that

$$\forall u \in K_d \subset \mathbb{R}^d, \qquad \int_{\mathbb{R}^d} \left| f_\alpha(u, \xi)\right|^\alpha \nu(\mathrm{d}\xi) < +\infty.$$

Under this assumption, Assumption 4 is fulfilled with $K_{d+1} = K_1 \times K_d$. Then, emphasizing the dependence on the function m, $S_{m,N}$ and S_m are well defined on K_{d+1} by (4) and (5) with W_n given by (12). In particular,

$$\begin{aligned} S_m(\alpha, u) &= \sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_\alpha(u, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \\ (\alpha, u) &\in K_{d+1} := K_1 \times K_d \subset (0, 2) \times \mathbb{R}^d. \end{aligned} \tag{15}$$

Under appropriate assumptions on f_α and m, the previous sections state the uniform convergence of the series, give a rate of convergence and some results on regularity for S_m. Precise results on regularity of S_m may be obtained using the following proposition, which states that the finite distributions of S_m does not depend on the choice of the ν-density m.

Proposition 4.4. *Assume that Assumption 5 is fulfilled and let S_m be defined by (15). Let $(\tilde{\xi}_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with common law $\tilde{\mu}(\mathrm{d}\xi) = \tilde{m}(\xi)\nu(\mathrm{d}\xi)$ equivalent to ν. Assume that the sequences $(\tilde{\xi}_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ are independent.*

1. *Then, $S_m \overset{fdd}{=} S_{\tilde{m}}$, where $\overset{fdd}{=}$ means equality of finite distributions. In other words,*

$$\left(S_m(\alpha, u)\right)_{(\alpha,u)\in K_{d+1}} \overset{fdd}{=} \left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_\alpha(u, \tilde{\xi}_n) \tilde{m}(\tilde{\xi}_n)^{-1/\alpha} g_n\right)_{(\alpha,u)\in K_{d+1}}.$$

2. *Assume moreover that for ν-almost every $\xi \in \mathbb{R}^d$, the map $(\alpha, u) \mapsto f_\alpha(u, \xi)$ is continuous on the compact set $K_{d+1} \subset (0, 2) \times \mathbb{R}^d$. Let us consider ρ a quasi-metric on $\mathbb{R}^{d+1}$, $\beta \in (0, 1]$ and $\eta \in \mathbb{R}$. Then, S_m belongs almost surely in $\mathcal{H}_\rho(K_{d+1}, \beta, \eta)$ if and only if $S_{\tilde{m}}$ does.*

Proof. See Appendix C.3. □

In particular, when studying the sample path properties of S_m, this result allows us to replace m by an other function $\tilde{m}$ so that the regularity of S_m may be deduced from the regularity of $S_{\tilde{m}}$.

For example, replacing m by m_{x_0} depending on x_0 this may lead to a more precise bound for the modulus of continuity of S_m around x_0 (see, e.g., Example 5.3).

5. Applications

5.1. α-stable isotropic random fields

Let us fix $\alpha = \alpha_0 \in (0, 2)$ and assume that Assumption 5 is fulfilled with g_n some isotropic complex random variables. Then, the proof of Proposition 4.4 (see Section C.3) allows to compute the characteristic function of the isotropic α_0-stable random field $S_m(\alpha_0, \cdot) = (S_m(\alpha_0, u))_{u \in K_d}$, which leads to

$$S_m(\alpha_0, \cdot) \overset{\text{fdd}}{=} d_{\alpha_0} \left(\int_{\mathbb{R}^d} f_{\alpha_0}(u, \xi) M_{\alpha_0}(\mathrm{d}\xi) \right)_{u \in K_d},$$

with M_{α_0} a complex isotropic α_0-stable random measure on $\mathbb{R}^d$ with control measure ν and

$$d_{\alpha_0} = \mathbb{E}\big(\big|\Re(g_1)\big|^{\alpha_0}\big)^{1/\alpha_0} \left(\frac{1}{2\pi} \int_0^{2\pi} \big|\cos(\theta)\big|^{\alpha_0} \,\mathrm{d}\theta \right)^{-1/\alpha_0} \left(\int_0^{+\infty} \frac{\sin(\theta)}{\theta^{\alpha_0}} \,\mathrm{d}\theta \right)^{1/\alpha_0}. \tag{16}$$

When ν is a finite measure (resp., the Lebesgue measure), this stochastic integral representation of $S_m(\alpha_0, \cdot)$ has been provided in [29,35] (resp., [7,20]).

Let us note that assumptions of Theorem 4.2 and Proposition 4.3 can be stated in term of the deterministic kernel f_{α_0} to obtain an upper bound of the modulus of continuity of S_m. In general, well-choosing m_{u_0} and applying Proposition 4.4, we obtain a more precise upper bound of the modulus of continuity of $S_m(\alpha_0, \cdot)$ around u_0, which also holds for a modification of the random field

$$X_{\alpha_0} = \left(\int_{\mathbb{R}^d} f_{\alpha_0}(u, \xi) M_{\alpha_0}(\mathrm{d}\xi) \right)_{u \in K_d}. \tag{17}$$

To illustrate how the previous sections can be applied to study the field X_{α_0}, which is defined through a stochastic integral and not a series, let us focus on the case of harmonizable stable random fields. More precisely, we consider

$$f_{\alpha_0}(u, \xi) = \big(\mathrm{e}^{\mathrm{i}\langle u, \xi\rangle} - 1\big)\psi_{\alpha_0}(\xi), \qquad \forall (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{18}$$

with $\psi_{\alpha_0} : \mathbb{R}^d \to \mathbb{C}$ a Borelian function such that

$$\int_{\mathbb{R}^d} \min\big(1, \|\xi\|^{\alpha_0}\big) \big|\psi_{\alpha_0}(\xi)\big|^{\alpha_0} \nu(\mathrm{d}\xi) < +\infty.$$

Note that, since this assumption does not depend on u, the random field X_{α_0} may be defined on the whole space $\mathbb{R}^d$. For the sake of simplicity, in the sequel, we consider the case where ν is the Lebesgue measure and first focus on a random field X_{α_0} which behaves as operator scaling random fields studied in [9].

Proposition 5.1. *Let $\alpha_0 \in (0,2)$ and let X_{α_0} be defined by* (17) *with ν the Lebesgue measure on $\mathbb{R}^d$. Let E be a real matrix of size $d \times d$ whose eigenvalues have positive real parts. Let τ_E and τ_{E^t} be functions as introduced in Example* 2.1 *and let us set $q(E) = \operatorname{trace}(E)$ and $a_1 = \min_{\lambda \in \operatorname{Sp}(E)} \Re(\lambda)$ with $\operatorname{Sp}(E)$ the spectrum of E, that is, the set of the eigenvalues of E. Assume that there exist some finite positive constants c_ψ, A and $\beta \in (0, a_1)$ such that*

$$\left|\psi_{\alpha_0}(\xi)\right| \le c_\psi \tau_{E^t}(\xi)^{-\beta - q(E)/\alpha_0}, \qquad \text{for almost every } \|\xi\| > A. \tag{19}$$

Then, there exists a modification $X^*_{\alpha_0}$ of X_{α_0} such that almost surely, for any $\varepsilon > 0$, for any non-empty compact set $K_d \subset \mathbb{R}^d$,

$$\sup_{\substack{u,v \in K_d \\ u \neq v}} \frac{|X^*_{\alpha_0}(u) - X^*_{\alpha_0}(v)|}{\tau_E(u-v)^\beta [\log(1 + \tau_E(u-v)^{-1})]^{\varepsilon + 1/2 + 1/\alpha_0}} < +\infty.$$

Remark 5.1. The quasi-metric $(x,y) \mapsto \tau_E(x-y)$ may not fulfill equation (7) since the eigenvalues of E may not be greater than 1. Nevertheless, the quasi-metric $(x,y) \mapsto \tau_{E/a_1}(x-y)$ does and the conclusion with τ_E in the previous proposition then follows from the comparison

$$\forall \xi \in \mathbb{R}^d, \qquad c_1 \tau_E(\xi)^{a_1} \le \tau_{E/a_1}(\xi) \le c_2 \tau_E(\xi)^{a_1}$$

with c_1, c_2 two finite positive constants.

Proof. See Appendix D.1. □

An upper bound for the modulus of continuity of such harmonizable random fields is also obtained in [38]. This upper bound is given in term of the Euclidean norm and then does not take into account the anisotropic behavior of X_{α_0}. Even when τ_E is the Euclidean norm, our result is a little more precise than the one of [38]. The difference is only in the power of the logarithmic term.

Let us now give some examples. We keep the notation of the previous proposition and the eigenvalues of the matrix E have always positive real parts.

Example 5.1 (Operator scaling random fields [9]). Let $\psi : \mathbb{R}^d \to [0,\infty)$ be an E^t-homogeneous function, which means that

$$\forall c \in (0,+\infty), \forall \xi \in \mathbb{R}^d, \qquad \psi\left(c^{E^t}\xi\right) = c\psi(\xi),$$

where $c^{E^t} = \exp(E^t \log c)$. Let us assume that ψ is a continuous function such that $\psi(\xi) \neq 0$ for $\xi \neq 0$. Then we consider the function $\psi_{\alpha_0} : \mathbb{R}^d \to [0,+\infty]$ defined by

$$\psi_{\alpha_0}(\xi) = \psi(\xi)^{-H - q(E)/\alpha_0}.$$

The random field X_{α_0}, associated with ψ_{α_0} by (17) and (18), is well-defined and is stochastically continuous if and only if $H \in (0, a_1)$. Then, let us now fix $H \in (0, a_1)$. Since ψ_{α_0} is

E^t-homogeneous, one easily checks that there exists $c_\psi \in (0, +\infty)$ such that

$$\forall \xi \in \mathbb{R}^d, \qquad \psi_{\alpha_0}(\xi) \leq c_\psi \tau_{E^t}(\xi)^{-H-q(E)/\alpha_0}.$$

Then, the assumptions of Proposition 5.1 are fulfilled with $\beta = H$. The corresponding conclusion was stated in Theorem 5.1 of [7] when $H = 1$ and $a_1 > 1$, which is enough to cover the general case using Remark 2.1 of [7].

Example 5.2 (Anisotropic Riesz–Bessel α-stable random fields). Let us consider

$$\psi_{\alpha_0}(\xi) = \frac{1}{\tau_{E^t}(\xi)^{2\beta_1/\alpha_0}(1 + \tau_{E^t}(\xi)^2)^{\beta_2/\alpha_0}}, \qquad \xi \in \mathbb{R}^d \setminus \{0\}$$

with two real numbers β_1 and β_2. Assuming that

$$\frac{q(E)}{2} < \beta_1 + \beta_2 \quad \text{and} \quad \beta_1 < \frac{q(E)}{2} + \frac{\alpha_0 a_1}{2},$$

the random field X_{α_0} is well-defined by (17). When τ_{E^t} is the Euclidean norm, this random field has been introduced in [38] to generalize the Gaussian fractional Riesz–Bessel motion [2].

We distinguish two cases. If $\beta_1 + \beta_2 < \frac{q(E)}{2} + \frac{\alpha_0 a_1}{2}$, Proposition 5.1 can be applied with $\beta = \frac{2(\beta_1+\beta_2)-q(E)}{\alpha_0}$. Otherwise, Proposition 5.1 can be applied for any $\beta \in (0, a_1)$.

Random fields defined by (17) have stationary increments so that their regularity on K_d does not depend on the compact set K_d. To avoid this feature, one can consider non-stationary generalizations by substituting ψ_{α_0} by a function that also depends on $u \in K_d$. More precisely, we can consider

$$X_{\alpha_0} = \left(\int_{\mathbb{R}^d} \left(e^{i\langle u,\xi\rangle} - 1\right) \psi_{\alpha_0}(u, \xi) M_{\alpha_0}(d\xi) \right)_{u \in K_d} \tag{20}$$

with M_{α_0} a complex isotropic α_0-stable random measure with Lebesgue control measure and ψ_{α_0} a Borelian function such that, for all $u \in K_d$,

$$\int_{\mathbb{R}^d} \left|e^{i\langle u,\xi\rangle} - 1\right|^{\alpha_0} \left|\psi_{\alpha_0}(u, \xi)\right|^{\alpha_0} d\xi < +\infty.$$

Under some conditions on ψ_{α_0}, when considering the local behavior of X_{α_0} around a point u_0 one can conveniently choose a Lebesgue density m_{u_0} to obtain an upper bound of the modulus of continuity of the shot noise series $S_{m_{u_0}}(\alpha_0, \cdot)$ given by (15) with

$$f_{\alpha_0}(u, \xi) = \left(e^{i\langle u,\xi\rangle} - 1\right) \psi_{\alpha_0}(u, \xi).$$

For the sake of conciseness, let us illustrate this with multi-operator random fields, which have already been studied in [8].

Example 5.3 (Multi-operator scaling α-stable random fields). In [8], we consider E a function defined on $\mathbb{R}^d$ with values in the set of real matrix of size $d \times d$ whose eigenvalues have real parts greater than 1 and $\psi : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ a continuous function such that for any $u \in \mathbb{R}^d$, $\psi(u, \cdot)$ is homogeneous with respect to $E(u)^t$, that is,

$$\psi\big(u, c^{E(u)^t}\xi\big) = c\psi(u, \xi), \qquad \forall \xi \in \mathbb{R}^d, \forall c > 0.$$

Under convenient regularity assumptions on ψ and E, the α_0-stable random field X_{α_0} is well-defined by (20) setting

$$\psi_{\alpha_0}(u, \xi) = \psi(u, \xi)^{-1-q(E(u))/\alpha_0} \qquad \text{with } q\big(E(u)\big) = \operatorname{trace}\big(E(u)\big).$$

Let $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$ and $u_0 \in K_d$. Let us set $K_{d+1} = \{\alpha_0\} \times K_d$ and consider the quasi-metric ρ defined on $\mathbb{R}^{d+1}$ by

$$\rho\big((\alpha, u), \big(\alpha', v\big)\big) = \big|\alpha - \alpha'\big| + \tau_{E(u_0)}(u - v)$$

for all $(\alpha, u), (\alpha', v) \in \mathbb{R} \times \mathbb{R}^d$, which clearly satisfies equation (7). Then, under assumptions of [8], there exists a Lebesgue density $m_{u_0} > 0$ a.e. such that Assumption 2 holds for $S_{m_{u_0}}$ on K_{d+1} with $\eta = 0$ and all $\beta \in (0, 1)$, adapting similar arguments as in Proposition 5.1 (see Lemma 4.7 of [8]). Therefore, following a part of the proof of Proposition 5.1, there exists a modification $X^*_{\alpha_0}$ of X_{α_0} such that almost surely,

$$\lim_{r \downarrow 0} \sup_{\substack{u, v \in B(u_0, r) \cap K_d \\ u \neq v}} \frac{|X^*_{\alpha_0}(u) - X^*_{\alpha_0}(v)|}{\tau_{E(u_0)}(u - v)^{1-\varepsilon}} < +\infty$$

for any $\varepsilon \in (0, 1)$. This is Theorem 4.6 of [8].

For the sake of conciseness, we do not develop other examples. Nevertheless, let us mention that our results can also be applied to harmonizable fractional α-stable sheets or even to operator stable sheets. In particular, this improves the result stated in [30] for fractional α-stable sheets. Note that we can also deal with real symmetric measure W_α.

5.2. Multistable random fields

Multistable random fields have first been introduced in [13] and then studied in [14]. Each marginal $X(u)$ of such a random field is a stable random variable but its stability index is allowed to depend on the position u.

Generalizing the class of multistable random fields introduced in [26], we consider a multi-stable random field defined by a LePage series. More precisely, under Assumption 5, we consider

$$\tilde{S}_m(u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha(u)} f_{\alpha(u)}(u, \xi_n) m(\xi_n)^{-1/\alpha(u)} g_n, \qquad u \in K_d, \tag{21}$$

where $\alpha : K_d \to (0,2)$ is a function. Then since $\tilde{S}_m(u) = S_m(\alpha(u), u)$ with S_m defined by (15), we deduce from Section 4 an upper bound for the modulus of continuity of $\tilde{S}$. In particular, assuming that α is smooth enough, we obtain the following theorem.

Proposition 5.2. *Let $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$. Let us choose $u_0 \in K_d$. Let $\tilde{\rho}$ be a quasi-metric on $\mathbb{R}^d$ satisfying equation (7) and let $\alpha : K_d \to (0,2)$ belongs to $\mathcal{H}_{\tilde{\rho}}(K_d, 1, 0)$. Let us set*

$$a = \min_{K_d} \alpha, \qquad b = \max_{K_d} \alpha \quad \textit{and} \quad K_1 = [a, b] \subset (0,2)$$

and consider the quasi-metric ρ defined on $\mathbb{R} \times \mathbb{R}^d$ by

$$\rho\big((\alpha, u), (\alpha', v)\big) = |\alpha - \alpha'| + \tilde{\rho}(u, v).$$

Assume that Assumption 5 is fulfilled and that equation (13) holds on $K_{d+1} = [a, b] \times K_d$ for some $p > b/2$, $\beta \in (0,1]$ and $\eta \in \mathbb{R}$. Assume also that

$$\mathbb{E}\big(\big|V_1\big(\alpha(u_0), u_0\big)\big|^{2p}\big) = \int_{\mathbb{R}^d} \big|f_{\alpha(u_0)}(u_0, \xi)\big|^{2p} m(\xi)^{1-2p/\alpha(u_0)}\, \mathrm{d}\xi < +\infty.$$

Let $S_{m,N}$ be defined by (4) with $W_n(\alpha, u) = T_n^{-1/\alpha} f_\alpha(u, \xi_n) m(\xi_n)^{-1/\alpha}$ and let $\tilde{S}_{m,N}(u) = S_{m,N}(\alpha(u), u)$.

1. *Then, almost surely, $(\tilde{S}_{m,N})_{N\in\mathbb{N}}$ converges uniformly on K_d to $\tilde{S}_m$ and almost surely the limit $\tilde{S}_m$ belongs to $\mathcal{H}_{\tilde{\rho}}(K_d, \beta, \max(\eta, 0) + 1/2)$.*
2. *Moreover, for all $p' > 0$ such that $1/p' \in (0, 1/b - 1/\min(2p, 2))$,*

$$\sup_{N\in\mathbb{N}} N^{1/p'} \sup_{u\in K_d} \big|\tilde{S}_m(u) - \tilde{S}_{m,N}(u)\big| < +\infty.$$

Proof. See Appendix D.2. □

Remark 5.2. Let us recall that $\tilde{S}_m \in \mathcal{H}_{\tilde{\rho}}(K_d, \beta, \max(\eta, 0) + 1/2)$ if and only if $\tilde{S}_{\tilde{m}} \in \mathcal{H}_{\tilde{\rho}}(K_d, \beta, \max(\eta, 0) + 1/2)$, with $\tilde{m}$ an other ν-density equivalent to ν, by Proposition 4.4.

To illustrate the previous proposition, we only focus on multistable random fields obtained replacing in a LePage series representation of an harmonizable operator scaling stable random field the index α by a function. Many other examples can be given, such as multistable anisotropic Riesz–Bessel random fields or the class of linear multistable random fields defined in [14].

Corollary 5.3 (Multistable versions of harmonizable operator scaling random fields). *Let E be a real matrix of size $d \times d$ such that $\min_{\lambda\in \mathrm{Sp}\, E} \Re(\lambda) > 1$. Let us consider ρ_E and τ_E as defined in Example 2.1. Let us also consider $\psi : \mathbb{R}^d \to [0, \infty)$ a continuous, E^t-homogeneous function such that $\psi(\xi) \neq 0$ for $\xi \neq 0$. Then we set*

$$f_\alpha(u, \xi) = \big(\mathrm{e}^{\mathrm{i}\langle u, \xi\rangle} - 1\big)\psi(\xi)^{-1-q(E)/\alpha}$$

with $q(E)=\operatorname{trace}(E)$. *Let* m *be a Lebesgue density a.e. positive on* $\mathbb{R}^d$, $(\xi_n,T_n,g_n)_{n\ge 1}$ *be as in Assumption* 5 *with* ν *the Lebesgue measure and consider a function* $\alpha:\mathbb{R}^d\to(0,2)$. *Therefore, the multistable random field* $\tilde{S}_m$ *is well-defined by* (21) *on the whole space* $\mathbb{R}^d$. *Moreover if* $\alpha\in\mathcal{H}_{\rho_E}(\mathbb{R}^d,1,0)$, *then for any* $u_0\in\mathbb{R}^d$ *and* $\varepsilon>0$, *there exists* $r\in(0,1]$ *such that almost surely*

$$\sup_{\substack{u,v\in B(u_0,r)\\ u\neq v}}\frac{|\tilde{S}_m(u)-\tilde{S}_m(v)|}{\tau_E(u-v)|\log\tau_E(u-v)|^{1/\alpha(u_0)+1/2+\varepsilon}}<+\infty.$$

Proof. See Appendix D.2. □

Remark 5.3. In particular, when $E=\mathrm{Id}$, τ_E is the Euclidean norm and we obtain an upper bound of the modulus of continuity of multistable versions of fractional harmonizable stable fields.

Appendix A: Proof of Proposition 2.1

The proof of Proposition 2.1 is based on the following lemma.

Lemma A.1. *If* Z *is a complex-valued sub-Gaussian random variable with parameter* $s\in(0,+\infty)$, *then for all* $t\in(0,+\infty)$, $\mathbb{P}(|Z|>t)\le 4\mathrm{e}^{-t^2/(8s^2)}$.

Proof. Let $t\in(0,+\infty)$. Since Z is sub-Gaussian with parameter s, $\Re(Z)$ and $\Im(Z)$ are real-valued sub-Gaussian random variables with parameter s. Then applying Proposition 4 of [17],

$$\mathbb{P}\big(|Z|>t\big)\le\mathbb{P}\biggl(\bigl|\Re(Z)\bigr|>\frac{t}{2}\biggr)+\mathbb{P}\biggl(\bigl|\Im(Z)\bigr|>\frac{t}{2}\biggr)\le 4\exp\biggl(-\frac{t^2}{8s^2}\biggr),$$

which concludes the proof. □

Let us now prove Proposition 2.1.

Proof of Proposition 2.1. Let $t\in(0,+\infty)$. Since Proposition 2.1 is straightforward if $a=0$, we assume that $a\neq 0$. Since the sequence $(g_n)_{n\ge1}$ is symmetric, by the Lévy inequalities (see Proposition 2.3 in [23]), for any $M\in\mathbb{N}\setminus\{0\}$,

$$\mathbb{P}\Biggl(\sup_{1\le P\le M}\Biggl|\sum_{n=1}^{P}a_ng_n\Biggr|>t\|a\|_{\ell^2}\Biggr)\le 2\mathbb{P}\Biggl(\Biggl|\sum_{n=1}^{M}a_ng_n\Biggr|>t\|a\|_{\ell^2}\Biggr).$$

We now prove that $\sum_{n=1}^{M}a_ng_n$ is sub-Gaussian. By independence of the random variables g_n and since each g_n is sub-Gaussian with parameter $s=1$,

$$\forall z\in\mathbb{C},\qquad \mathbb{E}\bigl(\mathrm{e}^{\Re(\overline{z}\sum_{n=1}^{M}a_ng_n)}\bigr)=\prod_{n=1}^{M}\mathbb{E}\bigl(\mathrm{e}^{\Re(\overline{z}a_ng_n)}\bigr)\le\prod_{n=1}^{M}\mathrm{e}^{(|z|^2|a_n|^2)/2}=\mathrm{e}^{(|z|^2s_M^2)/2}\tag{22}$$

with $s_M = (\sum_{n=1}^{M} |a_n|^2)^{1/2} \le \|a\|_{\ell^2}$. Hence, for any $M \in \mathbb{N} \setminus \{0\}$, $\sum_{n=1}^{M} a_n g_n$ is sub-Gaussian with parameter s_M. Since $a \ne 0$, for M large enough, $s_M \ne 0$ and then applying Lemma A.1,

$$\forall t > 0, \qquad \mathbb{P}\left(\sup_{1 \le P \le M} \left|\sum_{n=1}^{P} a_n g_n\right| > t\|a\|_{\ell^2}\right) \le 8 \exp\left(-\frac{t^2\|a\|_{\ell^2}^2}{8 s_M^2}\right) \le 8\mathrm{e}^{-t^2/8}.$$

Assertion 1 follows letting $M \to +\infty$.

Let us now prove assertion 2. If there exists $N \in \mathbb{N} \setminus \{0\}$, such that

$$\forall n \ge N, \qquad a_n = 0,$$

then, assertion 2 is fulfilled since $\sum_{n=1}^{+\infty} a_n g_n = \sum_{n=1}^{N} a_n g_n$ is a sub-Gaussian random variable with parameter $s_N = (\sum_{n=1}^{N} |a_n|^2)^{1/2} = \|a\|_{\ell^2}$. Therefore to prove assertion 2, we now assume that

$$\forall N \in \mathbb{N} \setminus \{0\}, \exists n \ge N, \qquad a_n \ne 0,$$

so that $\sum_{n=N}^{+\infty} |a_n|^2 \ne 0$ for any integer $N \ge 1$. Then, applying assertion 1 replacing a_n by $a_n \mathbf{1}_{n \ge N}$, we have

$$\forall \varepsilon > 0, \forall N \in \mathbb{N} \setminus \{0\}, \qquad \mathbb{P}\left(\sup_{P \ge N} \left|\sum_{n=N}^{P} a_n g_n\right| > \varepsilon\right) \le 8\mathrm{e}^{-\varepsilon^2/8\sum_{n=N}^{+\infty} |a_n|^2}.$$

Since $\|a\|_{\ell^2}^2 = \sum_{n=1}^{+\infty} |a_n|^2 < +\infty$, this implies that $(\sum_{n=1}^{N} a_n g_n)_N$ is a Cauchy sequence in probability. Then, by Lemma 3.6 in [18], the series $\sum_{n=1}^{+\infty} a_n g_n$ converges in probability. By Itô–Nisio theorem (see, [23], e.g.), this series also converges almost surely, since the random variables g_n, $n \ge 1$, are independent. Moreover, since $\sup_{M \ge 1} s_M^2 = \|a\|_{\ell_2}^2 < +\infty$, equation (22) implies the uniform integrability of the sequence $(\mathrm{e}^{\Re(\overline{z}\sum_{n=1}^{M} a_n g_n)})_{M \ge 1}$ for any $z \in \mathbb{C}$. Then, letting $M \to +\infty$ in (22), we obtain that $\sum_{n=1}^{+\infty} a_n g_n$ is sub-Gaussian with parameter $\|a\|_{\ell^2}$. Moreover, we conclude the proof noting that

$$\forall t > 0, \qquad \mathbb{P}\left(\left|\sum_{n=1}^{+\infty} a_n g_n\right| > t\|a\|_{\ell^2}\right) \le \mathbb{P}\left(\sup_{P \ge 1} \left|\sum_{n=1}^{P} a_n g_n\right| > t\|a\|_{\ell^2}\right) \le 8\mathrm{e}^{-t^2/8}.$$

□

Appendix B: Main results on conditionally sub-Gaussian series

B.1. Local modulus of continuity

This section is devoted to the proofs of the results stated in Section 3.1.

Proof of Theorem 3.1. Let us recall that $x_0 \in K_d = \prod_{j=1}^{d} [a_j, b_j] \subset \mathbb{R}^d$. We assume, without loss of generality, that

$$\forall 1 \le j \le d, \qquad a_j < b_j.$$

Actually, if some $a_j = b_j$, we may identify $(S_N)_{N\in\mathbb{N}}$ and its limits S as random fields defined on $K_{d'} \subset \mathbb{R}^{d'}$ for $d' < d$. Note that if $a_j = b_j$ for all $1 \le j \le d$, there is nothing to prove.

We also assume that $\gamma(\omega) \in (0,1)$, which is not restrictive and allows us to apply equation (7) as soon as $\|x-y\| \le \gamma(\omega)$ (with $c_{2,1}$ and $c_{2,2}$ which do not depend on γ).

First step. We first introduce a convenient sequence $(\mathcal{D}_{\nu_k})_{k\ge 1}$ of countable sets included on dyadics, which is linked to the quasi-metric ρ. It allows to follow some arguments of the proof of the Kolmogorov's lemma to obtain an upper bound for the modulus of continuity of S.

Let us first introduce some notation. For any $k \in \mathbb{N}\setminus\{0\}$ and $j \in \mathbb{Z}^d$, we set

$$x_{k,j} = \frac{j}{2^k}, \qquad \mathcal{D}_k = \{x_{k,j} : j \in \mathbb{Z}^d\} \quad \text{and} \quad \nu_k = \min\{n \in \mathbb{N}\setminus\{0\} : c_{2,2} d^{\overline{H}/2} 2^{-n\underline{H}} \le 2^{-k}\}$$

with $c_{2,2}$ the constant given by equation (7). Then, choosing $c_{2,2}$ large enough (which is not restrictive), one checks that $(\nu_k)_{k\ge1}$ is an increasing sequence. In particular, the sequence $(\mathcal{D}_{\nu_k})_{k\ge1}$ is increasing and $\mathcal{D} = \bigcup_{k=1}^{+\infty} \mathcal{D}_k = \bigcup_{k=1}^{+\infty} \mathcal{D}_{\nu_k}$. Moreover, $\mathcal{D} \cap K_d$ is dense in K_d since $a_j < b_j$ for any $1 \le j \le d$. Then, as done in Step 1 of the proof of Theorem 5.1 of [7], one also checks that for k large enough, $\mathcal{D}_{\nu_k} \cap K_d$ is a 2^{-k} net of K_d for ρ, which means that for any $x \in K_d$, there exists $j \in \mathbb{Z}^d$ such that $\rho(x, x_{\nu_k,j}) \le 2^{-k}$, with $x_{\nu_k,j} = j/2^{\nu_k} \in K_d$.

Second step. This step is inspired from Step 2 of [7,8]. The main difference is that we use Proposition 2.1 to obtain a uniform control in N.

For $k \in \mathbb{N}\setminus\{0\}$ and $(i,j) \in \mathbb{Z}^d$, we consider

$$E_{i,j}^k = \Big\{\omega : \sup_{N\in\mathbb{N}} |S_N(x_{\nu_k,i}) - S_N(x_{\nu_k,j})| > s(x_{\nu_k,i}, x_{\nu_k,j})\varphi\big(\rho(x_{\nu_k,i}, x_{\nu_k,j})\big)\Big\}$$

with, following [19],

$$\varphi(t) = \sqrt{8Ad\log\frac{1}{t}}, \qquad t > 0, \tag{23}$$

for $A > 0$ conveniently chosen later. We choose $\delta \in (0,1)$ and set for $k \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned} &\delta_k = 2^{-(1-\delta)k}, \\ &I_k = \{(i,j) \in (\mathbb{Z}^d \cap 2^{\nu_k} K_d)^2 : \rho(x_{\nu_k,i}, x_{\nu_k,j}) \le \delta_k\} \quad \text{and} \quad E_k = \bigcup_{(i,j)\in I_k} E_{i,j}^k. \end{aligned} \tag{24}$$

Since φ is a decreasing function and $s \ge 0$, for any $k \in \mathbb{N}\setminus\{0\}$ and for any $(i,j) \in I_k$

$$\mathbb{P}(E_{i,j}^k) \le \mathbb{P}\Big(\sup_{N\in\mathbb{N}} |S_N(x_{\nu_k,i}) - S_N(x_{\nu_k,j})| > s(x_{\nu_k,i}, x_{\nu_k,j})\varphi(\delta_k)\Big).$$

Since $(g_n)_{n\ge1}$ is a sequence of symmetric independent sub-Gaussian random variables with parameter $s = 1$, conditioning to $(W_n)_{n\ge1}$ and applying assertion 1 of Proposition 2.1, one has

$$\forall k \in \mathbb{N}\setminus\{0\}, \forall (i,j) \in I_k, \qquad \mathbb{P}(E_{i,j}^k) \le 8\mathrm{e}^{-\varphi(\delta_k)^2/8} = 8\mathrm{e}^{-Ad(1-\delta)k\log 2}$$

by definition of s, S_N, φ and δ_k. Moreover, since $K_d \subset \mathbb{R}^d$ is a compact set, using equation (8) and the definition of ν_k, one easily proves that there exists a finite positive constant $c_1 \in (0, +\infty)$ such that for any $k \in \mathbb{N} \setminus \{0\}$, card $I_k \le c_1 2^{(2kd)/\underline{H}} \delta_k^{d/\overline{H}}$. Hence,

$$\sum_{k=1}^{+\infty} \mathbb{P}(E_k) \le \sum_{k=1}^{+\infty} \sum_{(i,j)\in I_k} \mathbb{P}\big(E_{i,j}^k\big) \le c_1 \sum_{k=1}^{+\infty} \mathrm{e}^{-(A(1-\delta)-2/\underline{H}+(1-\delta)/\overline{H})kd\log 2} < +\infty$$

choosing $A > \frac{2}{\underline{H}} - \frac{1}{\overline{H}}$ and δ small enough. Then, setting

$$\Omega'' = \Omega' \cap \left(\bigcup_{k=1}^{+\infty} \bigcap_{\ell=k}^{+\infty} E_\ell^c \right)$$

with Ω' the almost sure event introduced by Assumption 2, the Borel–Cantelli lemma leads to $\mathbb{P}(\Omega'') = 1$. Moreover, by Assumption 2, for any $\omega \in \Omega''$ there exists $k^*(\omega)$ such that for every $k \ge k^*(\omega)$ and for all $x, y \in \mathcal{D}_{\nu_k}$ with $x, y \in B(x_0, \gamma(\omega)) \cap K_d$ and $\rho(x, y) \le \delta_k = 2^{-(1-\delta)k}$,

$$\sup_{N\in\mathbb{N}} \big|S_N(x) - S_N(y)\big| \le C\rho(x, y)^\beta \big|\log\big(\rho(x, y)\big)\big|^{\eta+1/2}. \tag{25}$$

Third step. In this step, we prove that (25) holds, up to a multiplicative constant, for any $x, y \in \mathcal{D}$ closed enough to x_0. This step is adapted from Step 4 of the proof of Theorem 5.1 in [7], taking care that (25) only holds for some $x, y \in \mathcal{D}_{\nu_k} \cap K_d$ randomly closed enough of x_0. Let us mention that this step has been omitted in the proof of the main result of [8] but is not trivial. We then decide to provide a proof here for the sake of completeness and clearness.

Let us now fix $\omega \in \Omega''$ and denote by $\kappa \ge 1$ the constant appearing in the quasi-triangle inequality satisfied by ρ. We also consider the function F defined on $(0, +\infty)$ by

$$F(h) := h^\beta \big|\log(h)\big|^{\eta+1/2}.$$

Observe that F is a random function since β and η are random variables. Then, we choose $k_0 = k_0(\omega) \in \mathbb{N}$ such that the three following assertions are fulfilled:

(a) F is increasing on $(0, \delta_{k_0}]$, where δ_k is given by (24),
(b) for all $k \ge k_0(\omega)$, $\mathcal{D}_{\nu_k} \cap K_d$ is a 2^{-k} net of K_d for ρ,
(c) $2^{k_0}\delta_{k_0+1} > 3\kappa^2$.

Even if it means to choose $k^*(\omega)$ larger, we can assume that $k^*(\omega) \ge k_0$ and that

$$\gamma(\omega) \ge \left(\frac{\delta_{k^*(\omega)}}{3\kappa^2 c_{2,2}} \right)^{1/\underline{H}} := 2\gamma^*(\omega), \tag{26}$$

where $\underline{H}$ and $c_{2,2}$ are defined in equation (7).

Let us now consider $x, y \in \mathcal{D} \cap K_d \cap B(x_0, \gamma^*(\omega))$ such that $x \neq y$. Let us first note that $x, y \in B(x_0, \gamma(\omega))$. Moreover, since $\|x-y\| \leq 2\gamma^*(\omega) \leq \gamma(\omega) \leq 1$, the upper bound of equation (7) leads to

$$3\kappa^2 \rho(x,y) \leq 3\kappa^2 c_{2,2} \|x-y\|^{\underline{H}} \leq \delta_{k^*(\omega)}$$

by definition of $\gamma^*(\omega)$. Then, there exists a unique $k \geq k^*(\omega)$ such that

$$\delta_{k+1} < 3\kappa^2 \rho(x,y) \leq \delta_k. \tag{27}$$

Furthermore, since $x, y \in \mathcal{D} \cap K_d$, there exists $n \geq k+1$ such that $x, y \in \mathcal{D}_{\nu_n} \cap K_d$ and for $j = k, \dots, n-1$, there exist $x^{(j)} \in \mathcal{D}_{\nu_j} \cap K_d$ and $y^{(j)} \in \mathcal{D}_{\nu_j} \cap K_d$ such that

$$\rho\big(x, x^{(j)}\big) \leq 2^{-j} \quad \text{and} \quad \rho\big(y, y^{(j)}\big) \leq 2^{-j}. \tag{28}$$

Let us now fix $N \in \mathbb{N}$ and focus on $S_N(x) - S_N(y)$. Then, setting $x^{(n)} = x$ and $y^{(n)} = y$,

$$\begin{aligned} S_N(x) - S_N(y) = {} & \big(S_N\big(x^{(k)}\big) - S_N\big(y^{(k)}\big)\big) + \sum_{j=k}^{n-1} \big(S_N\big(x^{(j+1)}\big) - S_N\big(x^{(j)}\big)\big) \\ & - \sum_{j=k}^{n-1} \big(S_N\big(y^{(j+1)}\big) - S_N\big(y^{(j)}\big)\big). \end{aligned} \tag{29}$$

The following lemma, whose proof is given below for the sake of clearness, allows to apply (25) for each term of the right-hand side of the last inequality.

Lemma B.1. *Choosing $k^*(\omega)$ large enough, the sequences $(x^{(j)})_{k \leq j \leq n}$ and $(y^{(j)})_{k \leq j \leq n}$ satisfy the three following assertions.*

(a) $x^{(j)}, y^{(j)} \in B(x_0, \gamma(\omega))$ *for any* $j = k, \dots, n$,
(b) *for any* $j = k, \dots, n-1$, $\max(\rho(x^{(j+1)}, x^{(j)}), \rho(y^{(j+1)}, y^{(j)})) \leq \delta_{j+1}$,
(c) $\rho(x^{(k)}, y^{(k)}) \leq \delta_k$.

Therefore, even if it means to choose $k^*(\omega)$ larger, applying this lemma and equations (25) and (29), we obtain

$$\big|S_N(x) - S_N(y)\big| \leq C\left(F\big(\rho\big(x^{(k)}, y^{(k)}\big)\big) + 2\sum_{j=k}^{n-1} F(\delta_{j+1})\right)$$

since F is increasing on $(0, \delta_{k_0}]$ and since $j \geq k_0$. This implies, by definition of F that

$$\big|S_N(x) - S_N(y)\big| \leq C\big(F\big(\rho\big(x^{(k)}, y^{(k)}\big)\big) + 2\tilde{C} F(\delta_{k+1})\big),$$

where

$$\tilde{C}(\omega) = 2\sum_{j=0}^{+\infty} \delta_j^{\beta(\omega)} (j+1)^{\max(\eta(\omega)+1/2,0)} < +\infty$$

since $\beta > 0$ and $\delta_j = 2^{-(1-\delta)j}$ with $\delta < 1$. Then, since F is increasing on $(0, \delta_0)$, by assertion 3 of Lemma B.1 and equation (27), we get

$$\left|S_N(x) - S_N(y)\right| \leq C(1 + 2\tilde{C})F\left(3\kappa^2\rho(x, y)\right),$$

for every $N \in \mathbb{N}$ and $x, y \in \mathcal{D} \cap B(x_0, \gamma^*(\omega)) \cap K_d$. Therefore, by continuity of ρ and each S_N and by density of $\mathcal{D} \cap K_d$ in K_d

$$\left|S_N(x) - S_N(y)\right| \leq C(1 + 2\tilde{C})F\left(3\kappa^2\rho(x, y)\right), \tag{30}$$

for every $N \in \mathbb{N}$ and $x, y \in B(x_0, \gamma^*(\omega)) \cap K_d$.

Fourth step: Uniform convergence of S_N. Let us now consider

$$\tilde{\Omega} = \bigcap_{u \in \mathcal{D}} \left\{\lim_{N \to +\infty} S_N(u) = S(u)\right\} \cap \Omega''.$$

Observe that $\mathbb{P}(\tilde{\Omega}) = 1$. Let us now fix $\omega \in \tilde{\Omega}$. Hence, by equation (30), the sequence $(S_N(\cdot)(\omega))_{N \in \mathbb{N}}$, which converges pointwise on $\mathcal{D} \cap B(x_0, \gamma^*(\omega)) \cap K_d$ is uniformly equicontinuous on $B(x_0, \gamma^*(\omega))$. Since $\mathcal{D} \cap B(x_0, \gamma^*(\omega)) \cap K_d$ is dense in $B(x_0, \gamma^*(\omega)) \cap K_d$, by Theorem I.26 and adapting Theorem I.27 in [33], $(S_N(\cdot)(\omega))_{N \in \mathbb{N}}$ converges uniformly on $B(x_0, \gamma^*(\omega)) \cap K_d$. Therefore, its limit S is continuous on $B(x_0, \gamma^*(\omega)) \cap K_d$. Moreover, letting $N \to +\infty$ in (30) (which holds since $\omega \in \tilde{\Omega}$), we get

$$\left|S(x) - S(y)\right| \leq C(1 + 2\tilde{C})F\left(3\kappa^2\rho(x, y)\right), \tag{31}$$

for every $x, y \in B(x_0, \gamma^*(\omega)) \cap K_d$, which concludes the proof. □

Let us now prove Lemma B.1.

Proof of Lemma B.1. Let us first observe that $x^{(n)} = x \in B(x_0, \gamma(\omega)) \cap K_d$ and $y^{(n)} = y \in B(x_0, \gamma(\omega)) \cap K_d$. Let us now fix $j \in \{k, \dots, n-1\}$. The lower bound of equation (7) leads to

$$\left\|x^{(j)} - x_0\right\| \leq \left\|x^{(j)} - x\right\| + \|x - x_0\| \leq \frac{\rho(x^{(j)}, x)^{1/\overline{H}}}{c_{2,1}^{1/\overline{H}}} + \|x - x_0\|.$$

Since $x \in B(x_0, \gamma^*(\omega))$ with γ^* satisfying equation (26) and since $\rho(x^{(j)}, x) \leq 2^{-j}$ with $j \geq k \geq k^*(\omega)$,

$$\left\|x^{(j)} - x_0\right\| \leq \frac{2^{-k^*(\omega)/\overline{H}}}{c_{2,1}^{1/\overline{H}}} + \frac{\gamma(\omega)}{2}.$$

Then, choosing $k^*(\omega)$ large enough, $x^{(j)} \in B(x_0, \gamma(\omega))$ for $j = k, \dots, n-1$. The same holds for $y^{(j)}$. Assertion 1 is then proved.

Let us now observe that since $j \geq k_0$ and since $\kappa \geq 1$,

$$2^j\delta_{j+1} \geq 2^{k_0}\delta_{k_0+1} > 3\kappa^2 \geq 3\kappa \tag{32}$$

by definition of k_0 (see the third step of the proof of Theorem 3.1). Then, using the quasi-triangle inequality fulfilled by ρ and (28), we obtain that

$$\rho\big(x^{(j+1)}, x^{(j)}\big) \le 3\kappa 2^{-(j+1)} \le \frac{\delta_{j+1}}{2} \le \delta_{j+1}.$$

Since the same holds for $\rho(y^{(j+1)}, y^{(j)})$, assertion 2 is fulfilled. Moreover, applying twice the quasi-triangle inequality fulfilled by ρ and equations (27), (28) and (32) (with $j = k$), we obtain

$$\rho\big(x^{(k)}, y^{(k)}\big) \le \kappa^2\big(2^{1-k} + \rho(x, y)\big) \le 3\kappa^2\rho(x, y) \le \delta_k,$$

which is assertion 3. □

Let us now focus on Corollary 3.2. Its proof is based on the following technical lemma.

Lemma B.2. *Let $K_d = \prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$ be a compact d-dimensional interval, $\beta \in (0, 1)$, $\eta \in \mathbb{R}$ and ρ be a quasi-metric on $\mathbb{R}^d$ satisfying equation (7). Let $(f_n)_{n\in\mathbb{N}}$ be sequence of functions defined on K_d and let $(\mathring{B}(x_i, r_i))_{1\le i\le p}$ be a finite covering of K_d by open balls with $x_i \in K_d$ and $r_i > 0$. Assume that for each $1 \le i \le p$, there exists a finite positive constant C_i such that*

$$\forall x, y \in \mathring{B}(x_i, r_i) \cap K_d, \qquad \sup_{n\in\mathbb{N}} \big|f_n(x) - f_n(y)\big| \le C_i \rho(x, y)^\beta \big[\log\big(1 + \rho(x, y)^{-1}\big)\big]^\eta.$$

Then there exists a finite positive constant C such that

$$\forall x, y \in K_d, \qquad \sup_{n\in\mathbb{N}} \big|f_n(x) - f_n(y)\big| \le C\rho(x, y)^\beta \big[\log\big(1 + \rho(x, y)^{-1}\big)\big]^\eta. \tag{33}$$

Proof. By the Lebesgue's number lemma, there exists $r > 0$ such that

$$\forall x \in K_d, \exists 1 \le i \le p, \qquad \mathring{B}(x, r) \subset \mathring{B}(x_i, r_i).$$

Let us first note that since the map $F_\rho : (u, v) \mapsto \rho^\beta(u, v)[\log(1 + \rho(u, v)^{-1})]^\eta$ is positive and continuous on the compact set $\tilde{K} = \{(u, v) \in K \times K / \|u - v\| \ge r\}$,

$$m := \inf_{\tilde{K}} F_\rho \in (0, +\infty).$$

Then distinguishing the cases $\|x - y\| < r$ and $\|x - y\| \ge r$, one easily sees that

$$\sup_{n\in\mathbb{N}} \big|f_n(x) - f_n(y)\big| \le \max\left(\max_{1\le i\le p} C_i, \frac{M}{m}\right)\rho(x, y)^\beta \big[\log\big(1 + \rho(x, y)^{-1}\big)\big]^\eta,$$

where $M = \sup_{n\in\mathbb{N}} \sup_{x,y\in K_d} |f_n(x) - f_n(y)|$. It remains to prove that $M < +\infty$. Note that

$$\sup_{\substack{x,y\in K_d \\ \|x-y\|<r}} \big|f_n(x) - f_n(y)\big| \le c \max_{1\le i\le p} C_i,$$

where $c = \sup_{K_d \times K_d} F_\rho < +\infty$ by continuity of F_ρ on the compact set $K_d \times K_d$. Then since K_d is a compact convex set, using a chaining argument, one easily obtains that $M < +\infty$, which concludes the proof. □

Proof of Corollary 3.2. We only prove assertion 1. Actually, assertion 2 is proved using the same arguments but replacing K_d by $B(x_0, \gamma) \cap K_d$.

Assume that for any $x_0 \in K_d$, Assumption 2 holds with Ω', β, η and the quasi-metric ρ independent of x_0. Following the proof of Theorem 3.1 and keeping its notation, let us quote that γ^* and $\tilde{\Omega}$ do not depend on x_0. Let us now fix $\omega \in \tilde{\Omega}$. From the third step of the proof of Theorem 3.1 and Lemma B.2, we deduce that equation (30) still holds for any $x, y \in K_d$. This allows to replace $B(x_0, \gamma^*(\omega))$ by K_d in the fourth step of the proof of Theorem 3.1, which leads to assertion 1. □

B.2. Rate of almost sure uniform convergence

Proof of Theorem 3.3. Let us first observe that Theorem 3.1 holds. Then, for almost ω, even if it means to choose γ smaller, the sequence of continuous functions $(S_N(\cdot)(\omega))_{N\in\mathbb{N}}$ converges uniformly on $B(x_0, \gamma(\omega)) \cap K_d$, which implies that each $R_N(\cdot)(\omega)$ is continuous on $B(x_0, \gamma(\omega)) \cap K_d$. As in the proof of Theorem 3.1, we assume without loss of generality that $K_d = \prod_{j=1}^d [a_j, b_j]$ with $a_j < b_j$.

Proof of assertion 1. Since it is quite similar to the proof of equation (30), we only sketch it.

For $k \in \mathbb{N} \setminus \{0\}$, $N \in \mathbb{N}$ and $(i, j) \in \mathbb{Z}^d$, we consider

$$E_{i,j}^{k,N} = \{\omega : |R_N(x_{\nu_k,i}) - R_N(x_{\nu_k,j})| > \sqrt{\log(N+2)}\, r_N(x_{\nu_k,i}, x_{\nu_k,j})\varphi(\rho(x_{\nu_k,i}, x_{\nu_k,j}))\}$$

with r_N defined by (9), φ by (23) and $(\nu_k)_{k\geq 1}$ by Step 1 of the proof of Theorem 3.1. Then, we proceed as in Step 1 of the proof of Theorem 3.1 replacing the set E_k by

$$E'_k = \bigcup_{N=0}^{+\infty} \bigcup_{(i,j)\in I_k} E_{i,j}^{k,N},$$

with I_k and δ_k defined by (24), and applying assertion 2 of Proposition 2.1 instead of assertion 1. Then, choosing the constant A, which appears in the definition of φ, and $\delta \in (0, 1)$ such that

$$A(1-\delta) - \frac{2}{\underline{H}} + \frac{1-\delta}{\overline{H}} > 0 \quad \text{and} \quad A(1-\delta)\log 2 > 1$$

we obtain that

$$\sum_{k=1}^{+\infty} \mathbb{P}(E'_k) \leq c_2 \sum_{N=2}^{+\infty} 2^{-A(1-\delta)\log N} = c_2 \sum_{N=2}^{+\infty} N^{-A(1-\delta)\log 2} < +\infty$$

with c_2 a finite positive constant. Then, by Borel–Cantelli lemma, the definition of φ and Assumption 3, almost surely there exists an integer $k^*(\omega)$ such that for every $k \geq k^*(\omega)$, for all $N \in \mathbb{N}$, and for all $x, y \in \mathcal{D}_{\nu_k}$ with $x, y \in B(x_0, \gamma(\omega)) \cap K_d$ and $\rho(x, y) \leq \delta_k = 2^{-(1-\delta)k}$

$$\left|R_N(x) - R_N(y)\right| \leq Cb(N)\sqrt{\log(N+2)}\rho(x, y)^\beta \left|\log\bigl(\rho(x, y)\bigr)\right|^{\eta+1/2}.$$

In addition, replacing in Step 2 of the proof of Theorem 3.1, S_N by R_N (which still be, for almost all ω, continuous on $B(x_0, \gamma(\omega)) \cap K_d$), we obtain that for almost all ω, there exists $\gamma^* \in (0, \gamma)$, such that

$$\left|R_N(x) - R_N(y)\right| \leq Cb(N)\sqrt{\log(N+2)}\rho(x, y)^\beta \left|\log\bigl(\rho(x, y)\bigr)\right|^{\eta+1/2} \tag{34}$$

for every $N \in \mathbb{N}$ and $x, y \in B(x_0, \gamma^*(\omega)) \cap K_d$. This establishes assertion 1.

Proof of assertion 2. This assertion follows from equations (34) and (11), the continuity of ρ on the compact set $B(x_0, \gamma(\omega)) \cap K_d$ and

$$\left|R_N(x)\right| \leq \left|R_N(x) - R_N(x_0)\right| + \left|R_N(x_0)\right|.$$

The proof of Theorem 3.3 is then complete. □

Proof of Corollary 3.4. We only prove assertion 1. Actually, assertion 2 is proved using the same arguments but replacing K_d by $B(x_0, \gamma) \cap K_d$.

Let us assume that Assumption 3 holds with Ω', β, η and the quasi-metric ρ independent of x_0. Note first that the almost sure event $\tilde{\Omega}$ under which (34) holds does not depend on x_0. Then applying Lemma B.2 to $f_n = R_n/(b(n)\sqrt{\log(n+2)})$, we obtain that equation (34) still holds for $x, y \in K_d$. If moreover for some x_0, equation (11) is fulfilled, then following the proof of assertion 2 of Theorem 3.3, we also have: there exists C a finite positive random variable such that for all $N \in \mathbb{N}$,

$$\sup_{x \in K_d} \left|R_N(x)\right| \leq Cb(N)\sqrt{\log(N+2)},$$

which concludes the proof. □

Appendix C: Shot noise series

C.1. Proof of Theorem 4.1

Let $(g_n)_{n\geq 1}$ be a Rademacher sequence, that is, a sequence of i.i.d. random variables with symmetric Bernoulli distribution. This Rademacher sequence is assumed to be independent of $(T_n, X_n)_{n\geq 1}$. Then, by independence and also by symmetry of the sequence $(X_n)_{n\geq 1}$, $(X_n g_n)_{n\geq 1}$ has the same distribution as $(X_n)_{n\geq 1}$ and is independent of the sequence $(T_n)_{n\geq 1}$.

Let us now set

$$W_n(\alpha) = T_n^{-1/\alpha} X_n \quad \text{and} \quad S_N(\alpha) = \sum_{n=1}^{N} W_n(\alpha) g_n,$$

so that $\{S_N^*(\alpha), N \geq 1\}$ has the same finite distribution as $\{S_N(\alpha), N \geq 1\}$. Moreover, since

$$\sum_{n=1}^{N} \left| W_n(\alpha) \right|^2 = \sum_{n=1}^{N} T_n^{-2/\alpha} |X_n|^2$$

with $X_n \in L^{2p}$ (with $p > 0$), Assumption 1 is fulfilled on any $K_1 = [a, b] \subset (0, \min(2, 2p))$ (see, e.g., [35]).

Let us now fix $a, b \in (0, \min(2, 2p))$ such that $a < b$, $a' \in (0, a)$ and $b' \in (b, \min(2, 2p))$.

Proof of assertion 1. By the Mean Value Inequality, we get that for any $\alpha, \alpha' \in [a, b]$ and $n \geq 1$,

$$\left| T_n^{-1/\alpha} - T_n^{-1/\alpha'} \right| \leq c \left| \alpha - \alpha' \right| \max \left(T_n^{-1/b'}, T_n^{-1/a'} \right) \tag{35}$$

with c a finite positive constant. It follows that, almost surely, for all $\alpha, \alpha' \in [a, b]$,

$$s\left(\alpha, \alpha'\right) := \left(\sum_{n=1}^{+\infty} \left| W_n(\alpha) - W_n\left(\alpha'\right) \right|^2 \right)^{1/2} \leq C \left| \alpha - \alpha' \right|, \tag{36}$$

with $C = c(\sum_{n=1}^{+\infty} T_n^{-2/b'} |X_n|^2 + \sum_{n=1}^{+\infty} T_n^{-2/a'} |X_n|^2)^{1/2} < +\infty$ since $|X_n|^2 \in L^p$ with $2p > b' \geq a'$ and $a', b' \in (0, 2)$. Therefore, the assumptions of assertion 1 of Corollary 3.2 hold. Let us now remark that for all $\alpha, \alpha' \in [a, b]$,

$$\left\{ \left(S_N^*(\alpha) - S_N^*\left(\alpha'\right), s\left(\alpha, \alpha'\right) \right); N \geq 1 \right\} \overset{\text{fdd}}{=} \left\{ \left(S_N(\alpha) - S_N\left(\alpha'\right), s\left(\alpha, \alpha'\right) \right); N \geq 1 \right\}.$$

This allows us to replace S_N by S_N^* in the second step of the proof of Theorem 3.1. Then, the third and the fourth step of this proof still hold replacing S_N by S_N^* and the limit S by the limit S^* since each S_N^* is continuous (as S_N is) and since S_N^* converges pointwise to S^*. This allows us to also replace (S_N, S) by (S_N^*, S^*) in the proof of assertion 1 of Corollary 3.2. It follows that almost surely, $(S_N^*)_{N \in \mathbb{N}}$ converges uniformly on $[a, b]$ to S^*. Since this holds for any $0 < a < b < \min(2, 2p)$, assertion 1 of Theorem 4.1 is established.

Proof of assertion 2. Since almost surely the sequence of continuous random fields $(S_N^*)_{N \in \mathbb{N}}$ converges uniformly on $[a, b]$, for all $N \in \mathbb{N}$ the rest R_N^*, defined by

$$R_N^*(\alpha) := \sum_{n=N+1}^{+\infty} T_n^{-1/\alpha} X_n,$$

is also continuous on $[a,b]$. Remark also that we have, for all $\alpha,\alpha'\in[a,b]$ and $N\in\mathbb{N}$,

$$\left(R_N^*(\alpha)-R_N^*(\alpha'),r_N(\alpha,\alpha')\right)\stackrel{d}{=}\left(R_N(\alpha)-R_N(\alpha'),r_N(\alpha,\alpha')\right),$$

where $R_N(\alpha)=\sum_{n=N+1}^{+\infty}T_n^{-1/\alpha}X_ng_n=S(\alpha)-S_N(\alpha)$ and

$$r_N(\alpha,\alpha')=\left(\sum_{n=N+1}^{+\infty}|X_n|^2\left|T_n^{-1/\alpha}-T_n^{-1/\alpha'}\right|^2\right)^{1/2}.$$

As done for S_N, the previous lines allow to replace R_N by R_N^* in the proof of Theorem 3.3. Moreover, by equation (35), almost surely, for all $N\in\mathbb{N}$, and $\alpha,\alpha'\in[a,b]$,

$$r_N(\alpha,\alpha')\le c|\alpha-\alpha'|\left(\sum_{n=N+1}^{+\infty}T_n^{-2/b'}|X_n|^2+\sum_{n=N+1}^{+\infty}T_n^{-2/a'}|X_n|^2\right)^{1/2}. \tag{37}$$

Let us now fix $p'>0$ such that $1/p'\in(0,1/b-1/\min(2p,2))$. Choosing if necessary $b'>b$ smaller, we assume without loss of generality that $1/p'\in(0,1/b'-1/\min(2p,2))\subset(0,1/a'-1/\min(2p,2))$. Then, by Theorem 2.2 in [11], almost surely, for all $\alpha,\alpha'\in[a,b]$,

$$\sup_{N\in\mathbb{N}}N^{2/p'}\left(\sum_{n=N+1}^{+\infty}T_n^{-2/b'}|X_n|^2+\sum_{n=N+1}^{+\infty}T_n^{-2/a'}|X_n|^2\right)<+\infty$$

since $X_n^2\in L^p$ with $p>b'/2>a'/2$ and $a',b'\in(0,2)$. Note also that by Theorem 2.1 in [11], for all $x_0=\alpha_0\in[a,b]$, almost surely

$$\sup_{N\in\mathbb{N}}N^{1/p'}\left|\sum_{n=N+1}^{+\infty}T_n^{-1/\alpha_0}X_n\right|<+\infty.$$

Therefore, the assumptions of assertion 1 of Corollary 3.2 hold with $b(N)=(N+1)^{-1/p'}$ for any p' such that $1/p'\in(0,1/b-1/\min(2p,2))$. And then, substituting in its proof R_N by R_N^*, almost surely

$$\sup_{N\in\mathbb{N}}\sup_{\alpha\in[a,b]}N^{1/p'}\left|R_N^*(\alpha)\right|<+\infty,$$

which concludes the proof.

C.2. Modulus of continuity and rate of convergence

This section is devoted to the proofs of the results stated in Section 4.2. First, let us establish Theorem 4.2.

Proof of Theorem 4.2. Let us fix $x_0=(\alpha_0,u_0)\in K_{d+1}=[a,b]\times\prod_{j=1}^d[a_j,b_j]\subset(0,2)\times\mathbb{R}^d$.

Proof of assertion 1. Let us assume that $p > b/2$ and consider s the conditional parameter defined by (6). Then, for any $x = (\alpha, u) \in K_{d+1}$ and $y = (\alpha', v) \in K_{d+1}$,

$$s(x, y) \le s_1(x, y) + s_2(x, y), \tag{38}$$

where

$$s_1(x, y) = \left(\sum_{n=1}^{+\infty} T_n^{-2/\alpha} \left| V_n(x) - V_n(y) \right|^2 \right)^{1/2}$$

and

$$s_2(x, y) = \left(\sum_{n=1}^{+\infty} \left(T_n^{-1/\alpha} - T_n^{-1/\alpha'} \right)^2 \left| V_n(y) \right|^2 \right)^{1/2}.$$

First, let us focus on s_1. Note that for any $x, y \in K_{d+1}$,

$$s_1(x, y) \le C_1 \rho(x, y)^\beta \log\left(1 + \rho(x, y)^{-1}\right)^\eta,$$

with $C_1 = (\sum_{n=1}^{+\infty} T_n^{-2/b} |Y_n|^2 + \sum_{n=1}^{+\infty} T_n^{-2/a} |Y_n|^2)^{1/2}$, where we have set

$$Y_n = \sup_{\substack{x, y \in K_{d+1} \\ x \ne y}} \frac{|V_n(x) - V_n(y)|}{\rho(x, y)^\beta \log(1 + \rho(x, y)^{-1})^\eta}.$$

Since K_d is a convex compact set, applying a chaining argument and using the continuity of ρ, one checks that equation (13) implies that $Y_n \in L^{2p}$. Then, since $2p > b \ge a$ and since the random variables Y_n, $n \ge 1$, are i.i.d., Theorem 1.4.5 of [35] ensures that $C_1 < +\infty$ almost surely.

Let us now focus on s_2. Observe that $|V_n(y)| \le X_n$, with

$$X_n = \left| V_n(x_0) \right| + c_1 Y_n$$

for $c_1 = \sup_{z \in K_{d+1}} \rho(x_0, z)^\beta |\log(1 + \rho(x_0, z)^{-1})|^\eta$. Let us remark that $c_1 < +\infty$, by continuity of ρ on the compact set $\{x_0\} \times K_{d+1}$. Moreover, since $V_n(x_0) \in L^{2p}$, $(X_n)_{n \ge 1}$ is still a sequence of i.i.d. variables in L^{2p} and following the same lines as for equation (36), we obtain that, almost surely, for any $x, y \in K_{d+1}$,

$$s_2(x, y) \le C_2 \left| \alpha - \alpha' \right|$$

with C_2 a finite positive random variable. Let us also note that by equation (8), there exist finite positive constants c_2 and c_3 such that for any $x = (\alpha, u) \in K_{d+1}$ and any $y = (\alpha', v) \in K_{d+1}$,

$$\left| \alpha - \alpha' \right| \le c_2 \rho(x, y)^{1/\overline{H}} \le c_3 \rho(x, y)$$

since $\overline{H} \le 1$. Hence, since $\beta \in (0, 1]$, almost surely, for any $x, y \in K_{d+1}$,

$$s(x, y) \le C \rho(x, y)^\beta \log\left(1 + \rho(x, y)^{-1}\right)^{\max(\eta, 0)}$$

with C a finite positive random variable. Then assertion 1 follows from Corollary 3.2.

Proof of assertion 2. Let us choose $p' > 0$ such that $1/p' \in (0, 1/b - 1/\min(2, 2p))$. Then, replacing in the previous lines s by the parameter r_N and Theorem 1.4.5 of [35] by Theorem 2.2 of [11], we obtain: there exists C a finite positive random variable such that almost surely, for any $x, y \in K_{d+1}$, and for any $N \in \mathbb{N}$,

$$r_N(x, y) \leq C(N+1)^{-1/p'} \rho(x, y)^\beta \log\bigl(1 + \rho(x, y)^{-1}\bigr)^{\max(\eta, 0)}.$$

Note also that by Theorem 2.1 in [11], almost surely

$$\sup_{N \in \mathbb{N}} N^{1/p'} \bigl|R_N(x_0)\bigr| < +\infty.$$

Therefore, by Corollary 3.4, almost surely,

$$\sup_{N \in \mathbb{N}} N^{1/p'} \sup_{x \in K_{d+1}} \bigl|R_N(x)\bigr| < +\infty,$$

which concludes the proof. □

Let us now prove Proposition 4.3.

Proof of Proposition 4.3. Since equation (7) is fulfilled, there exists $r \in (0, 1)$ such that $\rho(x, y) \leq h_0$ for all $x, y \in K_{d+1}$ with $\|x - y\| \leq r$. Then, the assumptions done imply that

$$X_1 := \sup_{\substack{x, y \in K_{d+1} \\ 0 < \|x - y\| \leq r}} \frac{|V_1(x) - V_1(y)|}{\rho(x, y)^\beta |\log \rho(x, y)|^\eta} \leq \sup_{h \in (0, h_0]} \frac{\mathcal{G}(h)}{F(h)} := G,$$

where $F(h) := h^\beta |\log h|^\eta$. We assume without loss of generality that $h_0 = 2^{-k_0}$ with an integer $k_0 \geq 1$ is such that F is increasing on $(0, h_0]$ and equation (14) holds for $h \in (0, h_0]$. Then, using the monotonicity of $\mathcal{G}$ and F,

$$G^{2p} \leq \sum_{k=k_0}^{+\infty} \sup_{h \in (2^{-k-1}, 2^{-k}]} \left(\frac{\mathcal{G}(h)}{F(h)}\right)^{2p} \leq \max\bigl(2^\eta, 1\bigr)^{2p} \sum_{k=k_0}^{+\infty} \left(\frac{\mathcal{G}(2^{-k})}{F(2^{-k})}\right)^{2p}.$$

Therefore, by equation (14) and definition of F,

$$\mathbb{E}\bigl(X_1^{2p}\bigr) \leq \mathbb{E}\bigl(G^{2p}\bigr) \leq \max\bigl(2^\eta, 1\bigr)^{2p} \sum_{k=k_0}^{+\infty} |k \log 2|^{-1-2p\varepsilon} < +\infty,$$

which concludes the proof. □

C.3. Proof of Proposition 4.4

Let $K_{d+1} = [a, b] \times \prod_{j=1}^{d} [a_j, b_j] \subset (0, 2) \times \mathbb{R}^d$.

Proof of assertion 1. Let us fix an integer $p \geq 1$ and consider $x^{(j)} = (\alpha_j, u^{(j)}) \in K_{d+1}$ for each integer $1 \leq j \leq p$. Then, we set $\vec{x} = (x^{(1)}, \dots, x^{(p)})$. Choosing $\mathcal{S} = \{\xi \in \mathbb{R}^d; m(\xi) > 0\}$ we define $H_{\vec{x}} : (0, +\infty) \times \mathcal{S} \times \mathbb{C} \to \mathbb{C}^p$ by

$$H_{\vec{x}}(r, \xi, g) = \big(r^{-1/\alpha_1} f_{\alpha_1}\big(u^{(1)}, \xi\big) m(\xi)^{-1/\alpha_1} g, \dots, r^{-1/\alpha_p} f_{\alpha_p}\big(u^{(p)}, \xi\big) m(\xi)^{-1/\alpha_p} g\big). \tag{39}$$

Let us note that almost surely

$$\sum_{n=1}^{N} H_{\vec{x}}(T_n, \xi_n, g_n) = \big(S_{m,N}\big(x^{(1)}\big), \dots, S_m\big(x^{(p)}\big)\big),$$

where $S_{m,N}$ is defined by (4) with W_n given by (12). Then this series converges almost surely to $(S_m(x^{(1)}), \dots, S_m(x^{(p)}))$. Since g_1 is symmetric, applying Theorem 2.4 of [34] and using a simple change of variables ($t = rm(\xi)$) and $\nu(\mathbb{R}^d \setminus \mathcal{S}) = 0$, we obtain that

$$\forall \lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{C}^p, \forall z \in \mathbb{C}, \qquad \mathbb{E}\big(e^{i\Re(\bar{z} \sum_{j=1}^{p} \lambda_j S_m(x^{(j)}))}\big) = \exp\big(I_{\vec{x},\lambda}(z)\big),$$

where

$$I_{\vec{x},\lambda}(z) = \int_{(0,+\infty)\times\mathbb{R}^d\times\mathbb{C}} \Big(e^{i\Re(\bar{z}\langle\lambda, J_{\vec{x}}(t,(\xi,g))\rangle)} - 1 - i\Re\big(\bar{z}\big\langle\lambda, J_{\vec{x}}\big(t, (\xi, g)\big)\big\rangle\mathbf{1}_{|\Re(\bar{z}\langle\lambda, J_{\vec{x}}(t,(\xi,g))\rangle)|\leq 1}\big)\Big) \, dt \, \nu(d\xi) \mathbb{P}_g(dg)$$

with $\mathbb{P}_g$ the distribution of g_1 and

$$J_{\vec{x}}\big(t, (\xi, g)\big) = \big(t^{-1/\alpha_1} f_{\alpha_1}\big(u^{(1)}, \xi\big) g, \dots, t^{-1/\alpha_p} f_{\alpha_p}\big(u^{(p)}, \xi\big) g\big).$$

Therefore, $I_{\vec{x},\lambda}$ does not depend on the function m, and then neither does the distribution of the vector $(S_m(x^{(1)}), \dots, S_m(x^{(p)}))$. Since this holds for any p and $\vec{x}$, assertion 1 is established.

Proof of assertion 2. Let us now consider the space $B = \mathcal{C}(K_{d+1}, \mathbb{C})$ of complex-valued continuous functions defined on the compact set K_{d+1}. This space is endowed with the topology of the uniform convergence, so that it is a Banach space.

Let us assume that $S_{\tilde{m}}$ belongs almost surely to $\mathcal{H}_\rho(K_{d+1}, \beta, \eta) \subset B$. For any $\vec{x} = (x^{(1)}, \dots, x^{(p)}) \in K_{d+1}^p$, in view of its characteristic function, the vector $(S_{\tilde{m}}(x^{(1)}), \dots, S_{\tilde{m}}(x^{(p)}))$ is infinitely divisible and its Lévy measure is given by

$$F_{\vec{x}}(A) = \int_{(0,+\infty)\times\mathcal{S}\times\mathbb{C}} \mathbf{1}_{A\setminus\{0\}}\big(H_{\vec{x}}(r, \xi, g)\big) m(\xi) \, dr \, \nu(d\xi) \mathbb{P}_g(dg)$$

for any Borel set $A \in \mathcal{B}(\mathbb{C}^p)$. We first assume that $(\alpha, u) \mapsto f_\alpha(u, \xi)$ belongs to B for all $\xi \in \mathbb{R}^d$ so that the function

$$H : (0, +\infty) \times \mathcal{S} \times \mathbb{C} \longrightarrow B,$$
$$(r, \xi, g) \mapsto \bigl((\alpha, u) \mapsto r^{-1/\alpha} f_\alpha(u, \xi) m(\xi)^{-1/\alpha}\bigr)$$

is well-defined. Since $H_{\vec{x}}$ is defined by (39), one checks that $(S_{\tilde{m}}(x))_{x \in K_{d+1}}$ is a B-valued infinitely divisible random variable with Lévy measure defined by

$$F(A) = \int_{(0,+\infty)\times\mathcal{S}\times\mathbb{C}} \mathbf{1}_{A\setminus\{0\}}\bigl(H(r, \xi, g)\bigr) m(\xi)\,\mathrm{d}r\,\nu(\mathrm{d}\xi)\mathbb{P}_g(\mathrm{d}g), \qquad A \in \mathcal{B}(B).$$

Then, by Theorem 2.4 of [34], $\sum_{n=1}^N H(T_n, (\xi_n, g_n))$ converges almost surely in B as $N \to +\infty$. Then, by definition of H, the sequence $(S_{m,N})_{N\in\mathbb{N}}$ converges in B almost surely. Therefore, its limit S_m is almost surely continuous on K_{d+1}.

Let us now consider $\mathcal{D} \subset K_{d+1}$ a countable dense set in K_{d+1}. Then, since almost surely $S_{\tilde{m}} \in \mathcal{H}_\rho(K_{d+1}, \beta, \eta)$ and since $S_m \overset{\mathrm{fdd}}{=} S_{\tilde{m}}$, we get that almost surely

$$\sup_{\substack{x,y\in\mathcal{D}\\ x\neq y}} \frac{|S_m(x) - S_m(y)|}{\rho(x, y)^\beta[\log(1 + \rho(x, y)^{-1})]^\eta} < +\infty.$$

Then, by continuity of ρ, by almost sure continuity of S_m and by density of $\mathcal{D}$ on the compact set K_{d+1},

$$\sup_{\substack{x,y\in K_{d+1}\\ x\neq y}} \frac{|S_m(x) - S_m(y)|}{\rho(x, y)^\beta[\log(1 + \rho(x, y)^{-1})]^\eta} < +\infty$$

almost surely, that is, S_m belongs almost surely to $\mathcal{H}_\rho(K_{d+1}, \beta, \eta)$. This establishes assertion 2 when $(\alpha, u) \mapsto f_\alpha(u, \xi)$ is continuous for all $\xi \in \mathbb{R}^d$.

Assume now that $(\alpha, u) \mapsto f_\alpha(u, \xi)$ is continuous for $\xi \in \mathbb{R}^d \setminus \mathcal{N}$ with $\nu(\mathcal{N}) = 0$ and set

$$g_\alpha(u, \xi) := f_\alpha(u, \xi)\mathbf{1}_{\mathbb{R}^d\setminus\mathcal{N}}(\xi).$$

Then, almost surely, for all $x = (\alpha, u) \in (0, 2) \times \mathbb{R}^d$ and all $N \geq 1$,

$$S_{m,N}(x) = \sum_{n=1}^N T_n^{-1/\alpha} g_\alpha(u, \xi_n) m(\xi_n)^{-1/\alpha} g_n,$$

and the conclusion follows from the previous lines since $(\alpha, u) \mapsto g_\alpha(u, \xi)$ is continuous on K_{d+1} for all $\xi \in \mathbb{R}^d$. The proof of Proposition 4.4 is then complete.

Appendix D: Applications

D.1. Proof of Proposition 5.1

Let us first note that using Remark 5.1, we can and may assume without loss of generality that $a_1 = 1$, up to replace E by E/a_1 and τ_E by $\tau_{E/a_1}^{1/a_1}$.

Let us choose $\zeta > 0$ arbitrarily small and consider the Borel function $\tilde{m}$ defined on $\mathbb{R}^d$ by

$$\tilde{m}(\xi) = \|\xi\|^{\alpha_0}\mathbf{1}_{\|\xi\|\le A} + \tau_{E^t}(\xi)^{-q(E)}\left|\log \tau_{E^t}(\xi)\right|^{-1-\zeta}\mathbf{1}_{\|\xi\|>A}.$$

Observe that $\tilde{m}$ is positive on $\mathbb{R}^d \setminus \{0\}$. Then,

$$0 < c = \int_{\mathbb{R}^d} \tilde{m}(\xi)\,\mathrm{d}\xi = c_1 + c_2$$

with

$$c_1 = \int_{\|\xi\|\le A} \|\xi\|^{\alpha_0}\,\mathrm{d}\xi \quad \text{and} \quad c_2 = \int_{\|\xi\|>A} \tau_{E^t}(\xi)^{-q(E)}\left|\log \tau_{E^t}(\xi)\right|^{-1-\zeta}\,\mathrm{d}\xi.$$

Let us first observe that $c_1 < \infty$ since $\alpha_0 > 0$. To prove that c_2 is also a finite constant, we need some tools given in [9,31]. As in Chapter 6 of [31], let us consider the norm $\|\cdot\|_{E^t}$ defined by

$$\|x\|_{E^t} = \int_0^1 \left\|\theta^{E^t}x\right\|\frac{\mathrm{d}\theta}{\theta}, \qquad \forall x \in \mathbb{R}^d. \tag{40}$$

According to the change of variables in polar coordinates (see [9]) there exists a finite positive Radon measure σ_{E^t} on $S_{E^t} = \{\xi \in \mathbb{R}^d : \|\xi\|_{E^t} = 1\}$ such that for all measurable function φ non-negative or in $L^1(\mathbb{R}^d, \mathrm{d}\xi)$,

$$\int_{\mathbb{R}^d} \varphi(\xi)\,\mathrm{d}\xi = \int_0^{+\infty}\int_{S_{E^t}} \varphi\left(r^{E^t}\theta\right)\sigma_{E^t}(\mathrm{d}\theta)r^{q(E)-1}\,\mathrm{d}r.$$

Applying this change of variables, it follows that $c_2 < \infty$ since $\zeta > 0$. Hence, $m = \tilde{m}/c$ is well-defined and $\mu(\mathrm{d}\xi) = m(\xi)\,\mathrm{d}\xi$ is a probability measure equivalent to the Lebesgue measure. Then we may consider $S_m(\alpha_0, u)$ defined by (15) for $u \in \mathbb{R}^d$ so that $X_{\alpha_0} \overset{\text{fdd}}{=} d_{\alpha_0} S_m(\alpha_0, \cdot)$ with d_{α_0} given by (16).

To study the sample path regularity of $S_m(\alpha_0, \cdot)$ on $K_d = \prod_{j=1}^d [a_j, b_j]$, we apply Proposition 4.3 on $K_{d+1} = \{\alpha_0\} \times K_d \subset (0,2) \times \mathbb{R}^d$ for

$$V_1(\alpha_0, u) = f_{\alpha_0}(u, \xi_1)m(\xi_1)^{-1/\alpha_0}$$

with f_{α_0} defined by (18). We recall that here ξ_1 is a random vector of $\mathbb{R}^d$ with density m. Therefore let us now check that assumptions of Proposition 4.3 are fulfilled.

For $h > 0$ and $\xi \in \mathbb{R}^d$ we consider

$$g(h, \xi) = \min\left(c_{E^t}\left\|h^{E^t}\xi\right\|_{E^t}, 1\right)\left|\psi_{\alpha_0}(\xi)\right|,$$

where $c_{E^t} > 0$ is chosen such that $|e^{i\langle u,\xi\rangle} - 1| \le c_{E^t} \|\tau_E(u)^{E^t}\xi\|_{E^t}$. We consider the quasi-metric defined on $\mathbb{R}^{d+1}$ by

$$\rho\big((\alpha, u), (\alpha', v)\big) = |\alpha - \alpha'| + \rho_E(u, v), \qquad \forall (\alpha, u), (\alpha', v) \in \mathbb{R} \times \mathbb{R}^d,$$

which clearly satisfies equation (7). By definition of V_1, g and $\|\cdot\|_{E^t}$, the random field $\mathcal{G} = (g(h, \xi_1))_{h\in[0,+\infty)}$ satisfies (i) and (ii) of Proposition 4.3. It remains to consider assumption (iii). Let

$$I(h) = \mathbb{E}\big(\mathcal{G}(h)^2\big) = \int_{\mathbb{R}^d} g(h, \xi)^2 m(\xi)^{1-2/\alpha_0}\,\mathrm{d}\xi. \tag{41}$$

Since ψ_{α_0} satisfies (19),

$$I(h) = \int_{\mathbb{R}^d} m(\xi)^{1-2/\alpha_0} \min\big(c_{E^t}\|h^{E^t}\xi\|_{E^t}, 1\big)^2 \big|\psi_{\alpha_0}(\xi)\big|^2\,\mathrm{d}\xi \le I_1(h) + I_2(h)$$

with

$$I_1(h) = c^{2/\alpha_0-1} c_{E^t}^2 \int_{\|\xi\|\le A} \|h^{E^t}\xi\|_{E^t}^2 \|\xi\|^{\alpha_0-2} \big|\psi_{\alpha_0}(\xi)\big|^2\,\mathrm{d}\xi,$$

where A is given by the condition (19), and

$$I_2(h) = c^{2/\alpha_0-1} c_\psi \int_{\mathbb{R}^d} \min\big(c_{E^t}\|h^{E^t}\xi\|_{E^t}, 2\big)^2 \tau_{E^t}(\xi)^{-q(E)-2\beta} \big|\log \tau_{E^t}(\xi)\big|^{(1+\zeta)(2/\alpha_0-1)}\,\mathrm{d}\xi.$$

From Lemma 3.2 of [7] there exists a finite constant $C_1 > 0$ such that for all $h \in (0, e^{-1}]$

$$I_1(h) \le C_1 h^{2a_1} \big|\log(h)\big|^{2(d-1)}.$$

Moreover, using again the change of variables in polar coordinates, there exists a finite constant $C_2 > 0$ such that for all $h \in (0, e^{-1}]$,

$$I_2(h) \le C_2 h^{2\beta} \big|\log(h)\big|^{(1+\zeta)(2/\alpha_0-1)}.$$

Since $\beta < a_1$, one find a finite constant $C_3 > 0$ such that

$$I(h) \le C_3 h^{2\beta} \big|\log(h)\big|^{2(1+\zeta)(1/\alpha_0-1/2)}. \tag{42}$$

Hence, assumption (iii) of Proposition 4.3 is also fulfilled and applying this proposition, it follows that (13) is satisfied with β and $\eta = 1/\alpha_0 + \varepsilon$, for all $\varepsilon > 0$. Then, by Theorem 4.2, almost surely $S_m \in \mathcal{H}_\rho(K_{d+1}, \beta, 1/\alpha_0 + 1/2 + \varepsilon)$. By definition of ρ and K_{d+1}, this means that

$$S_m(\alpha_0, \cdot) \in \mathcal{H}_{\rho_E}(K_d, \beta, 1/\alpha_0 + 1/2 + \varepsilon).$$

In particular, $S_m(\alpha_0, \cdot)$ is continuous on K_d. Then, since $d_{\alpha_0} S_m(\alpha_0, \cdot) \overset{\text{fdd}}{=} X_{\alpha_0}$, X_{α_0} is stochastically continuous and almost surely

$$C := \sup_{u,v\in\mathcal{D}, u\neq v} \frac{|X_{\alpha_0}(u) - X_{\alpha_0}(v)|}{\tau_E(u-v)^\beta [\log(1 + \tau_E(u-v)^{-1})]^{1/\alpha_0+1/2+\varepsilon}} < +\infty,$$

where $\mathcal{D} \subset K_d$ is a countable dense in $K_d = \prod_{j=1}^{d}[a_j, b_j]$. So let us write Ω^* this event and let us define a modification of X_{α_0} on K_d.

First, if $\omega \notin \Omega^*$, we set $X^*_{\alpha_0}(u)(\omega) = 0$ for all $u \in K_d$. Let us now fix $\omega \in \Omega^*$. Then, we set

$$X^*_{\alpha_0}(u)(\omega) = X_{\alpha_0}(u)(\omega), \qquad \forall u \in \mathcal{D}.$$

Let us now consider $u \in K_d$. Then, there exists $u^{(n)} \in \mathcal{D}$ such that $\lim_{n\to+\infty} u^{(n)} = u$. It follows that,

$$\begin{aligned}&\bigl|X^*_{\alpha_0}\bigl(u^{(n)}\bigr)(\omega) - X^*_{\alpha_0}\bigl(u^{(m)}\bigr)(\omega)\bigr| \\ &\quad\le C(\omega)\tau_E\bigl(u^{(n)} - u^{(m)}\bigr)^{\beta}\bigl[\log\bigl(1 + \tau_E\bigl(u^{(n)} - u^{(m)}\bigr)^{-1}\bigr)\bigr]^{1/\alpha_0+1/2+\varepsilon},\end{aligned}$$

so that $(X^*_{\alpha_0}(u^{(n)})(\omega))_n$ is a Cauchy sequence and hence converges. We set

$$X^*_{\alpha_0}(u)(\omega) = \lim_{n\to+\infty} X^*_{\alpha_0}\bigl(u^{(n)}\bigr)(\omega).$$

Remark that this limit does not depend on the choice of $(u^{(n)})_n$ and that $X^*_{\alpha_0}(\cdot)(\omega)$ is then well-defined on K_d. Observe also that, by stochastic continuity of X_{α_0}, $X^*_{\alpha_0}$ is a modification of X_{α_0}. Moreover, by continuity of τ_E,

$$C(\omega) = \sup_{u,v\in K_d, u\neq v} \frac{|X^*_{\alpha_0}(u)(\omega) - X^*_{\alpha_0}(v)(\omega)|}{\tau_E(u-v)^{\beta}[\log(1+\tau_E(u-v)^{-1})]^{1/\alpha_0+1/2+\varepsilon}} < +\infty$$

for all $\omega \in \Omega$ and $X^*_{\alpha_0}$ is continuous on K_d. This concludes the proof.

D.2. Multistable random fields

This section is devoted to the proofs of the results stated in Section 5.2. Let us first establish Proposition 5.2.

Proof of Proposition 5.2. Since $\tilde{\rho}$ satisfies equation (7), so does ρ. Then, assumptions of Theorem 4.2 are fulfilled, which implies that $(S_{m,N})_{N\in\mathbb{N}}$ converges uniformly to S_m on $K_{d+1} = [a, b] \times K_d$. Therefore, $(\tilde{S}_{m,N})_{N\in\mathbb{N}}$ converges uniformly to $\tilde{S}_m$ on K_d since $\tilde{S}_{m,N}(u) = S_{m,N}(\alpha(u), u)$ and $\tilde{S}_m(u) = S_m(\alpha(u), u)$ and α is continuous.

Moreover, by Theorem 4.2 there exists a finite positive random variable C such that for any $u, v \in K_d$,

$$\bigl|\tilde{S}_m(u) - \tilde{S}_m(v)\bigr| \le C\rho\bigl(x(u), x(v)\bigr)^{\beta}\bigl[\log\bigl(1 + \rho\bigl(x(u), x(v)\bigr)^{-1}\bigr)\bigr]^{\max(\eta,0)+1/2},$$

where $x(w) = (\alpha(w), w)$. Moreover, by definition of ρ and since $\alpha \in \mathcal{H}_{\tilde{\rho}}(K_d, 1, 0)$, there exists a finite positive constant c_1 such that

$$\forall u, v \in K_d, \qquad \rho\bigl(x(u), x(v)\bigr) \le c_1\tilde{\rho}(u, v).$$

Let us now recall that since $\tilde{\rho}$ is continuous on the compact set $K_d \times K_d$, $M = \sup_{u,v\in K_d} \tilde{\rho}(u, v) < +\infty$. Then, up to change C, for all $u, v \in K_d$,

$$\left|\tilde{S}_m(u) - \tilde{S}_m(v)\right| \le C\tilde{\rho}(u, v)^\beta \left[\log\left(1 + \tilde{\rho}(u, v)^{-1}\right)\right]^{\max(\eta,0)+1/2}$$

since $h \mapsto h^\beta \log(1 + h^{-1})^{\max(\eta,0)+1/2}$ is increasing around 0 and bounded on $[0, M]$. Assertion 1 is then proved. Moreover, assertion 2 is a direct consequence of assertion 2 of Theorem 4.2. The proof is then complete. □

Let us conclude this paper by the proof of Corollary 5.3.

Proof of Corollary 5.3. Let $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$ and $u_0 \in K_d$. Let us set

$$a = \min_{K_d} \alpha, \qquad b = \max_{K_d} \alpha \quad \text{and} \quad K_{d+1} = [a, b] \times K_d \subset (0, 2) \times \mathbb{R}^d.$$

Let us first note that Assumption 5 is fulfilled with $K_1 = [a, b]$ and then $\tilde{S}_m$ is well-defined. Let us now consider ρ_E and τ_E as defined in Example 2.1. Then we set

$$\tilde{m}(\xi) = \frac{c_\zeta}{\tau_{E^t}(\xi)^{q(E)} |\log \tau_{E^t}(\xi)|^{1+\zeta}},$$

with $\zeta > 0$ a parameter chosen arbitrarily small. Therefore, let us consider

$$\tilde{V}_n(\alpha, u) = f_\alpha(u, \tilde{\xi}_n)\tilde{m}(\tilde{\xi}_n)^{-1/\alpha},$$

where $(\tilde{\xi}_n)_{n\ge1}$ is a sequence of i.i.d. random variables with common distribution $\tilde{\mu}(\mathrm{d}\xi) = \tilde{m}(\xi)\,\mathrm{d}\xi$. The sequence $(\tilde{\xi}_n)_{n\ge1}$ is assumed to be independent from $(T_n, g_n)_{n\ge1}$. Then, Assumption 4 is fulfilled. Moreover,

$$\left|\tilde{V}_n(\alpha, u) - \tilde{V}_n(\alpha', v)\right| \le \left|\tilde{V}_n(\alpha, u) - \tilde{V}_n(\alpha, v)\right| + \left|\tilde{V}_n(\alpha, v) - \tilde{V}_n(\alpha', v)\right|.$$

Let us set

$$C_1 = \sup_{\substack{u,v\in K_d \\ 0<\|u-v\|\le r}} \sup_{\alpha\in[a,b]} \frac{|\tilde{V}_1(\alpha, u) - \tilde{V}_1(\alpha, v)|}{\rho_E(u, v)|\log \rho_E(u, v)|^\eta}$$

and

$$C_2 = \sup_{\substack{\alpha,\alpha'\in[a,b] \\ \alpha\ne\alpha'}} \sup_{u\in K_d} \frac{|\tilde{V}_1(\alpha, u) - \tilde{V}_1(\alpha', u)|}{|\alpha - \alpha'|},$$

where $r > 0$ and the choice of $\eta \in \mathbb{R}$ is given below. Then, for any $x = (\alpha, u) \in K_{d+1}$ and any $y = (\alpha', v) \in K_{d+1}$ such that $\|x - y\| \le r$,

$$\begin{aligned}\left|\tilde{V}_1(x) - \tilde{V}_1(y)\right| &\le (C_1 + C_2)\left(\rho_E(u, v)\left|\log \rho_E(u, v)\right|^\eta + \left|\alpha - \alpha'\right|\right) \\ &\le c_1(C_1 + C_2)\rho(x, y)\left|\log \rho(x, y)\right|^\eta,\end{aligned}$$

where $c_1 \in (0, +\infty)$ is a finite constant and $\rho(x, y) = \rho_E(u, v) + |\alpha - \alpha'|$. Then, to apply assertion 1 of Proposition 5.2 with $\tilde{\rho} = \rho_E$ and $\beta = 1$, it suffices to establish that C_1, C_2 and $\tilde{V}_1(\alpha(u_0), u_0) \in L^2$ (since $b < 2$).

Let us first deal with $\tilde{V}_1(\alpha(u_0), u_0)$. Using polar coordinates associated with E^t (see [31]),

$$\mathbb{E}\left(\left|\tilde{V}_1\left(\alpha(u_0), u_0\right)\right|^2\right) \le c_2 \int_0^{+\infty} \min\left(\left\|t^{E^t}\right\|, 1\right)^2 t^{-3} |\log t|^{2(1+\zeta)/\alpha(u_0)-1}\, \mathrm{d}t$$

with $c_2 \in (0, +\infty)$. Hence, Lemma 2.1 of [9] proves that $V_1(\alpha(u_0), u_0) \in L^2$ for any choice of ζ.

Let us now consider the random variable C_1. By homogeneity and continuity of ψ, there exists a finite positive constant c_3 such that for any $u, v \in K_d$,

$$\sup_{\alpha \in [a,b]} \left|\tilde{V}_1(\alpha, u) - \tilde{V}_1(\alpha, v)\right| \le c_3 \left|\mathrm{e}^{\mathrm{i}\langle u-v, \tilde{\xi}_1\rangle} - 1\right| Z_1$$

with

$$Z_1 = \tau_{E^t}(\tilde{\xi}_1)^{-1} \max\left(\left|\log \tau_{E^t}(\tilde{\xi}_1)\right|^{(1+\zeta)/a}, \left|\log \tau_{E^t}(\tilde{\xi}_1)\right|^{(1+\zeta)/b}\right).$$

Combining the proofs of Propositions 4.3 and 5.1, we obtain that for any $\varepsilon > 0$, choosing r small enough,

$$\mathbb{E}\left(\left[\sup_{\substack{u,v \in K_d \\ 0 < \|u-v\| \le r}} \frac{|\mathrm{e}^{\mathrm{i}\langle u-v, \tilde{\xi}_1\rangle} - 1| Z_1}{\rho_E(u, v) |\log \rho_E(u, v)|^{1/a+\varepsilon}}\right]^2\right) < +\infty.$$

This implies that for any $\varepsilon > 0$, $C_1 \in L^2$ for $\eta = 1/a + \varepsilon$ and ζ well-chosen.

Let us now study C_2. Since K_d is a compact set, using polar coordinates and the Mean Value theorem, we have

$$\sup_{v \in K_d} \left|\tilde{V}_1(\alpha, v) - \tilde{V}_1\left(\alpha', v\right)\right| \le c_4 \left|\alpha - \alpha'\right| Z_2$$

with $Z_2 = \min(\|\tau_{E^t}(\xi_n)^{E^t}\|, 1) Z_1 |\log \tau_{E^t}(\xi_n) + c_5|$ and c_4 and c_5 two finite positive constants. Using polar coordinates, one checks that $Z_2 \in L^2$, which implies that $C_2 \in L^2$.

Therefore, for any $\varepsilon > 0$, assumptions of assertion 1 of Proposition 5.2 are fulfilled for a well-chosen ζ. This implies that almost surely, for any $\varepsilon > 0$, $\tilde{S}_m \in \mathcal{H}_{\rho_E}(K_d, 1, 1/a + 1/\varepsilon)$ with $a = \min_{K_d} \alpha$. Hence, for any $\varepsilon > 0$, $\tilde{S}_m \in \mathcal{H}_{\rho_E, B(u_0,r)}(u_0, 1, 1/\alpha(u_0) + 1/2 + 1/\varepsilon)$ for r small enough. This concludes the proof. □

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