MODULUS OF CURVE FAMILIES AND EXTREMALITY OF SPIRAL-STRETCH MAPS

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Abstract. We develop a method by modulus of curve families to study minimisation problems for the mean distortion functional in the class of finite distortion homeomorphisms. We apply our method to prove extremality of the spiral-stretch mappings defined on annuli in the complex plane. This generalises results by Gutlyanskiĭ, Martio [GM01] and Strebel [Str62].

1. Introduction

The study of extremal problems of mappings of finite distortion has been initiated by Astala, Iwaniec, Martin and Onninen in [AIMO05]. These are generalisations of quasiconformal mappings without uniform bound on their distortion. Whereas for quasiconformal mappings, one is usually interested in minimisers of the maximal distortion, for mappings of finite distortion a mean distortion functional is to be minimised. This viewpoint was clearly indicated in the recent monograph [AIM09b]. From the discussion in [AIM09b] it becomes evident that the problem of minimising the mean distortion is different from the one for the maximal distortion and new techniques are needed. For example, Martin showed in [Mar09], that in contrast to the classical case, the Teichmüller problem for the mean distortion does not have a solution in the class of mappings with finite distortion.

In the present paper we pursue another approach, complementing the techniques in [AIM09b, AIMO05], to identify the extremal mapping within a given class of mappings of finite distortion. This procedure is by modulus estimates for curve families.

The *modulus* of a curve family is a geometric quantity which is quasi-invariant under quasiconformal mappings and therefore yields a lower bound on the maximal distortion for an appropriate choice of curve family. Similarly, it was shown in [MRSY05] that this technique can be used for mappings of finite distortion. We use this approach also in our work as a general tool to treat the minimisation problem for a weighted mean distortion with fixed boundary data.

We will illustrate how to apply this general method in the case of mappings of finite distortion on the annulus with boundary values given by a composition of logarithmic spiral and radial stretching map.

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More precisely, we look at the following problem:

> Consider "all" self-maps of an annulus in the plane which leave the outer boundary pointwise fixed and rotate the inner boundary by a given angle. Identify a mapping in this class which is "as conformal as possible"!

This problem has been studied first for the class of quasiconformal mappings, where a natural way to describe how close such a mapping is to a conformal map is to consider its *maximal distortion.* Strebel proved in [Str62] that the *logarithmic spiral map* is a solution to the above problem in this class, that is, a minimiser for the maximal distortion. More recently, Gutlyanskiĭ and Martio [GM01] have shown that it is also a minimiser for a *weighted mean distortion* in the same class. This leads to estimates for the rotation angle and a sharp solution to John's angle distortion problem [Joh61]. For a survey on results related to the logarithmic spiral map and its importance in applications, see [GM01].

One of the main results in the present paper is the solution to the above problem in the more general setting of mappings with finite distortion and also for mappings between two annuli of different radii. This boundary value minimisation problem has a $\mathcal{C}^\infty$-smooth solution: the *spiral-stretch map* – the spiral map composed with the well-known radial stretching. Unlike in the quasiconformal case, the existence of an extremal map of finite distortion cannot be a priori guaranteed by the standard normal family argument. Because of the lack of this type of convergence, results obtained by the curve families technique will be used. The curve family we shall employ extensively in this paper is a family of spirals which was used recently by Brakalova [Bra10a, Bra10b] in the characterisation of conformality at a point.

As a second application of the modulus method we prove the extremality of the radial-stretch map among all finite distortion homeomorphisms between two annuli which preserve inner and outer boundary. Related results have been obtained recently in [AIM09a] by invariant integral methods.

The paper is organised as follows. In Section 2 we provide the necessary definitions and notations and state our main results. The proofs are given in Section 3 and 4. In addition, we discuss an example which illustrates the different nature of the minimisation problem for the maximal and the mean distortion in terms of the uniqueness of the extremal mapping. The last section is devoted to additional remarks and comments on related results.

2. Preliminaries and statements of the main results

2.1. **Quasiconformal mappings and mappings of finite distortion.** Let Ω and Ω' be domains in $\mathbb{C}$ and consider a sense-preserving homeomorphism $f : \Omega \to \Omega'$ of class $W^{1,2}(\Omega)$. Assume that there exists a measurable function $z \mapsto K(z)$, finite a.e., with

$$|Df(z)|^2 \leq K(z)J(z,f) \quad \text{a.e.},$$

where $|Df(z)| = \max\{|Df(z)v| : |v| = 1\}$ is the norm of the formal differential $Df(z)$ and $J(z,f) = \det Df(z)$ is its Jacobian determinant. The map is called *quasiconformal* if there is a constant $K \geq 1$ such that $K(z) \leq K$ holds almost everywhere.

We are interested in the larger class of the so-called *mappings of finite distortion*, see e.g. [KKM01a, IKO01, KKM01b]. Unlike quasiconformal maps, these mappings are not required to be homeomorphisms and there needs to be no uniform bound on $K(z)$. We will, however, only consider mappings of finite distortion which are homeomorphisms.

Definition 1. A function $f : \Omega \to \mathbb{C}$ on a domain $\Omega \subset \mathbb{C}$ is called *mapping of finite distortion* if

(i) $f \in W^{1,1}_{loc}(\Omega, \mathbb{C})$;
(ii) $J(z, f)$ is locally integrable; and
(iii) there exists a measurable function $z \mapsto K(z)$, finite a.e., with

$$|Df(z)|^2 \le K(z) J(z, f) \quad \text{a.e.}$$

The *(linear) distortion function* is defined as

$$K(z,f) := \begin{cases} \frac{|Df(z)|^2}{J(z,f)} & \text{if } J(z,f) > 0 \\ 1 & \text{otherwise} \end{cases} = \begin{cases} \frac{|f_z(z)|+|f_{\bar z}(z)|}{|f_z(z)|-|f_{\bar z}(z)|} & \text{if } J(z,f) > 0 \\ 1 & \text{otherwise} \end{cases} .$$

The *maximal distortion* is given by

$$K_f := \text{ess sup}_{z\in\Omega} K(z, f),$$

which is always finite for quasiconformal maps but may be infinite for mappings of finite distortion.

The minimisation of K_f in a given class of quasiconformal mappings is a classical problem; yet, the finite distortion class is naturally considered while minimising the following *mean distortion functional*

$$f \mapsto \int_\Omega \Psi(K(z, f))\rho^2(z) \, \mathrm{d}\mathcal{L}^2(z),$$

where ρ is a suitable density and Ψ a continuous, convex, non-decreasing function on $[1, \infty)$ with $\Psi(1) = 1$.

2.2. **Main results.** We concentrate on the minimisation problem for the above functional in the class of mappings between annuli with certain boundary conditions. This is in some sense the first non-trivial case for which the extremal problem has an interesting solution. Remember that in the case of bounded simply connected domains without boundary data, the existence of a *conformal* solution is always guaranteed by the Riemann mapping theorem.

For $0 < a < b$, we denote the annulus with centre at the origin, inner radius a and outer radius b by

$$A(a, b) := \{z \in \mathbb{C} : \ a < |z| < b\}.$$

We are considering mappings between two annuli of identical outer radius. For simplicity we may assume that the mappings are defined on an annulus $A(q, 1)$, $0 < q < 1$.

We consider the following class of mappings

Definition 2. Assume that $q \in (0,1)$, $\theta \in [-\pi, \pi]$ and $k_1 > 0$ are arbitrary constants. We define the class $\mathcal{F}$ of $W^{1,2}_{loc}$-homeomorphisms of finite distortion

$$f : A(q,1) \to A(q^{k_1}, 1)$$

with integrable distortion function $z \mapsto K(z,f)$ and extension to the boundary

$$f(z) = \begin{cases} z & |z| = 1 \\ q^{k_1-1} z e^{\mathrm{i}\theta} & |z| = q \end{cases} .$$

Remark 3. Note that we have restricted the class $\mathcal{F}$ under consideration to the subclass comprising of mappings of finite distortion which are also sense-preserving $W^{1,2}_{loc}$-homeomorphisms with integrable distortion. These additional regularity assumptions are needed to apply the modulus estimate from Theorem 9 on which our proof is based. The conditions are known to be the optimal assumptions to ensure that

(1) $J(\cdot, f) > 0$ a.e.; and
(2) f satisfies Lusin's condition (N).

There is no conformal mapping in the class $\mathcal{F}$ unless $\theta = 0$ and $k_1 = 1$. Still, one can identify a mapping that is *as conformal as possible* in the sense that it minimises a weighted mean distortion. Candidates for a solution to this extremal problem are the *spiral-stretch maps* $f_N \in \mathcal{F}$, $N \in \mathbb{Z}$, which are defined as follows.

Definition 4. Let $N \in \mathbb{Z}$. The *N-th spiral-stretch map* f_N, is defined by

$$f_N(z) := z|z|^{k_1-1} e^{\mathrm{i}k_2 \log|z|}, \quad z \in A(q,1),$$

where $k_2 := k_2(N) := \frac{\theta + 2\pi N}{\log q}$.

We obtain the minimising property of f_N in the subclass $\mathcal{F}_N \subset \mathcal{F}$ of mappings in $\mathcal{F}$ that are homotopic to f_N with respect to the boundary.

Theorem 5. *Let $q \in (0,1)$, $\theta \in [-\pi,\pi]$ and $k_1 > 0$. For an arbitrary $N \in \mathbb{Z}$*

$$\int_{A(q,1)} K(z, f_N) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z) \le \int_{A(q,1)} K(z,f) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z) \quad \textit{for all } f \in \mathcal{F}_N.$$

One can show that f_0 is extremal among all f_N, $N \in \mathbb{Z}$, which immediately yields the following main result.

Theorem 6. *Let $q \in (0,1)$, $\theta \in [-\pi,\pi]$, $k_1 > 0$. Then for any continuous, non-decreasing, convex function $\Psi : [1,\infty) \to [1,\infty)$ with $\Psi(1) = 1$,*

$$\int_{A(q,1)} \Psi(K(z,f_0)) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z) \le \int_{A(q,1)} \Psi(K(z,f)) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z) \quad \textit{for all } f \in \mathcal{F},$$

where $f_0(z) = z|z|^{k_1-1} e^{\mathrm{i}\frac{\theta}{\log q}\log|z|}$.

Theorem 6 generalises a result due to Gutlyanskiĭ and Martio [GM01] who considered the analogous problem for quasiconformal self-maps of the annulus $A(q,1)$.
Indeed, by setting $k_1 = 1$, we obtain as a Corollary to Theorem 6 the extremality of the logarithmic spiral map $f_0(z) = ze^{\mathrm{i}k_2 \log|z|}$, $k_2 = \frac{\theta}{\log q}$, in the smaller class of quasiconformal self-maps of $A(q,1)$ which rotate the inner boundary by θ and leave the outer boundary pointwise fixed.
In the case $\theta = 0$, the extremal map is the radial stretching $f_0(z) = z|z|^{k_1-1}$. This map plays a particular role in the theory of quasiconformal mappings, as it is the conjectured extremal mapping in Gehring's problem (solved by Astala [Ast94] in the planar case) of higher integrability of the Jacobian [Geh73].
An application of Theorem 6 to the case $\theta = 0$ shows that the radial stretching minimises the mean distortion in the respective class of mappings from $A(q,1)$ to $A(q^{k_1},1)$ (without rotation of the inner boundary). However, the same map turns out to be extremal for the mean distortion even in the larger class of finite distortion maps $A(q,1) \to A(q^{k_1},1)$ which send the inner and outer boundary of $A(q,1)$ to the inner and outer boundary of $A(q^{k_1},1)$, respectively, but do not necessarily leave the boundary pointwise fixed, see Theorem 8 below.

Definition 7. Assume that $q \in (0,1)$ and $k_1 > 0$ are arbitrary constants. We define the class $\mathcal{G}$ of $W^{1,2}_{loc}$-homeomorphisms of finite distortion

$$f : A(q,1) \to A(q^{k_1},1)$$

with integrable distortion function $z \mapsto K(z,f)$ and extension to the boundary

$$|f(z)| = q^{k_1} \text{ for } |z| = q \quad \text{and} \quad |f(z)| = 1 \text{ for } |z| = 1$$

Theorem 8. *Let $q \in (0,1)$ and $k_1 > 0$. Then for any continuous, non-decreasing, convex function $\Psi : [1,\infty) \to [1,\infty)$ with $\Psi(1) = 1$,*

$$\int_{A(q,1)} \Psi(K(z,f_0)) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z) \le \int_{A(q,1)} \Psi(K(z,f)) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z) \quad \text{for all } f \in \mathcal{G},$$

where $f_0(z) = z|z|^{k_1-1}$.

A similar result was obtained in [AIM09a] for the case of the non-linear distortion function

$$\mathbb{K}(z,f) := \begin{cases} \frac{\|Df(z)\|^2}{J(z,f)} & \text{if } J(z,f) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

The authors prove that for any mapping f of finite distortion between annuli with the same boundary conditions as in Definition 7 the following estimate holds

$$\int_{A(q,1)} \mathbb{K}(z,f_0) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z) \le \int_{A(q,1)} \mathbb{K}(z,f) \frac{1}{|z|^2} \, \mathrm{d}\mathcal{L}^2(z).$$

and the extremal map f_0 is unique up to rotation of the annuli.
This is more general than the statement in Theorem 8 in the sense that the mappings under consideration are not required to be of class $W^{1,2}_{loc}$ with integrable distortion. On

the other hand, we consider $\Psi(K(z,f))$ for an arbitrary convex non-decreasing function Ψ instead of $\mathbb{K}(z,f)$. Note that the non-linear distortion can be written as

$$\mathbb{K}(z,f) = \frac{1}{2}\left(K(z,f) + \frac{1}{K(z,f)}\right) = \Psi(K(z,f)),$$

where Ψ is the convex non-decreasing function $\Psi(x) := \frac{1}{2}\left(x + \frac{1}{x}\right)$. The result for the mean $\mathbb{K}(z,f)$-distortion can thus be obtained as a particular case of Theorem 8.

3. Extremality of the spiral-stretch map. Proof of Theorem 6

In the subsequent discussion, we will always assume that $q \in (0,1)$, $\theta \in [-\pi,\pi]$ and $k_1 > 0$ are arbitrary constants. Moreover, we assume in this section that $\theta \neq 0$ (and hence $k_2 \neq 0$). The case $\theta = 0$ can be obtained as a corollary to Theorem 8, where the extremality of f_0 is proved even within a larger class of mappings than $\mathcal{F}$.

3.1. **Strategy of the proof.** The proof of Theorem 6 consists of several steps.

Step 1. In Section 3.2.1, we recall the definition of the modulus $M(\Gamma)$ of a curve family Γ and cite the following modulus estimate for mappings of finite distortion, [MRSY05], which will be one of the key tools in the proof

$$M(f(\Gamma)) \leq \int_\Omega K(z,f)\rho^2(z)\,\mathrm{d}\mathcal{L}^2(z) \quad \text{for all } \rho \in \mathrm{adm}(\Gamma). \tag{1}$$

In Section 3.2.2, we formulate sufficient conditions on a curve family $\Gamma_0 \subset \Omega$ and a density $\rho_0 \in \mathrm{adm}(\Gamma_0)$ to obtain equality in (1) for a quasiconformal map f with constant distortion.

Step 2. (Section 3.3) The aim is to show that for a fixed $N \in \mathbb{Z}$ the spiral-stretch map f_N is extremal in its subclass $\mathcal{F}_N$. Using the previous result, we identify a family Γ_0 of spirals and a density $\rho_0(z) = \frac{c}{|z|}$ such that

$$\int_\Omega K(z,f_N)\rho_0^2(z)\,\mathrm{d}\mathcal{L}^2(z) = M(f_N(\Gamma_0)), \tag{2}$$

with $\Omega = A(q,1)$. Then, we introduce a larger curve family Γ which contains Γ_0 as a subfamily and satisfies

$$\rho_0 \in \mathrm{adm}(\Gamma) \tag{3}$$

and

$$M(f_N(\Gamma_0)) \leq M(f(\Gamma)) \quad \text{for all } f \in \mathcal{F}_N. \tag{4}$$

The first two steps together yield the following estimate

$$\int_\Omega K(z,f_N)\rho_0^2(z)\,\mathrm{d}\mathcal{L}^2(z) \overset{(2)}{=} M(f_N(\Gamma_0)) \overset{(4)}{\leq} M(f(\Gamma)) \overset{(3),(1)}{\leq} \int_\Omega K(z,f)\rho_0^2(z)\,\mathrm{d}\mathcal{L}^2(z)$$

for all $f \in \mathcal{F}_N$. Since $\rho_0(z)$ is a constant multiple of $\frac{1}{|z|}$, we obtain the statement of Theorem 5:

$$\int_{A(q,1)} K(z, f_N)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \le \int_{A(q,1)} K(z, f)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \quad \text{for all } f \in \mathcal{F}_N. \tag{5}$$

Step 3. Finally, in Section 3.4 we show that the spiral-stretch map f_0 is extremal among all admissible spiral-stretch maps

$$K_{f_0} \le K_{f_N} \quad \text{for all } N \in \mathbb{Z}. \tag{6}$$

Hence,

$$\int_{A(q,1)} K(z, f_0)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \overset{(6)}{\le} \int_{A(q,1)} K(z, f_N)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \overset{(5)}{\le} \int_{A(q,1)} K(z, f)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z)$$

for all $f \in \mathcal{F}$. We exploit the monotonicity and the convexity of the given function Ψ and conclude the proof of Theorem 6 by applying Jensen's inequality.

3.2. **Step 1: General approach to the minimisation problem by modulus methods.** It is the aim of this section to establish expressions for the modulus of the image of a curve family in terms of an integral average of the distortion. This is a general approach which can be used in specific cases to identify a candidate mapping as an extremal mapping of finite distortion in a given class.

3.2.1. *Modulus of a curve family. General estimate.* We start by recalling the definition of the modulus of a curve family; see e.g. [Mar08, MRSY09] for more details.
A *curve* γ in a domain $\Omega \subseteq \mathbb{C}$ is the image of a continuous map $\gamma : [a, b] \to \Omega$. We denote both the map and its image by γ. A *family of curves* in Ω will be denoted by Γ. A Borel function $\rho : \Omega \to [0, \infty]$ is called *admissible metric density* for Γ if

$$\int_\gamma \rho(z)|\mathrm{d}z| \ge 1 \tag{7}$$

for all $\gamma \in \Gamma$ which are locally rectifiable. The set of admissible metric densities for Γ will be denoted by $\mathrm{adm}(\Gamma)$.
For a Borel function ρ and a rectifiable curve $\gamma : [a, b] \to \Omega$ the curve integral in (7) is given by

$$\int_\gamma \rho(z)|\mathrm{d}z| = \int_0^{l(\gamma)} \rho(\bar\gamma(s))\,\mathrm{d}s,$$

where $\bar\gamma : [0, l(s)] \to \Omega$ is the arc-length parametrisation of γ. If γ is only locally rectifiable, then we set

$$\int_\gamma \rho(z)|\mathrm{d}z| = \sup \int_{\gamma'} \rho(z)|\mathrm{d}z|,$$

where the supremum is taken over all rectifiable subcurves $\gamma' : [a', b'] \to \Omega$ of γ. If γ is absolutely continuous, then

$$\int_\gamma \rho(z)|\mathrm{d}z| = \int_a^b \rho(\gamma(t))|\dot\gamma(t)|\,\mathrm{d}t.$$

Note that the value of the curve integral is independent of the parametrisation. The *modulus* of a curve family Γ is defined as

$$M(\Gamma) := \inf_{\rho\in\mathrm{adm}(\Gamma)} \int_\Omega \rho^2(z)\,\mathrm{d}\mathcal{L}^2(z).$$

Let $\Gamma_{\mathrm{rect}} \subseteq \Gamma$ be the family of curves $\gamma \in \Gamma$ which are (locally) rectifiable. Then $M(\Gamma \setminus \Gamma_{\mathrm{rect}}) = 0$ since $\rho \equiv 0$ is admissible for $\Gamma \setminus \Gamma_{\mathrm{rect}}$.
Also observe that $M(\cdot)$ is an outer measure on the set of curves in Ω. It follows that

$$M(\Gamma_{\mathrm{rect}}) \le M(\Gamma) \le M(\Gamma \setminus \Gamma_{\mathrm{rect}}) + M(\Gamma_{\mathrm{rect}}) = M(\Gamma_{\mathrm{rect}})$$

thus

$$M(\Gamma) = M(\Gamma_{\mathrm{rect}}). \tag{8}$$

To prove that a given mapping $f_0 \in \mathcal{F}$ is a minimiser for the mean distortion among all mappings in the class $\mathcal{F}$, we should establish a lower bound for the mean distortion of mappings in $\mathcal{F}$ and show that this bound is achieved by our candidate map f_0. The essential modulus estimate will be the following particular case of the estimate given in [MRSY05, Theorem 2.19] for $Q(x)$-quasiconformal mappings, see also [MRSY09, Theorem 4.1].

Theorem 9 ([MRSY05]). *Let Ω be a domain in $\mathbb{C}$, Γ a curve family in Ω and let $f : \Omega \to \mathbb{C}$ be a homeomorphism onto its image. Assume further that f belongs to the Sobolev class $W^{1,2}_{loc}(\Omega)$ and that it is of finite distortion with $K(\cdot, f) \in L^1_{loc}(\Omega)$, then*

$$M(f(\Gamma)) \le \int_\Omega K(z,f)\rho^2(z)\,\mathrm{d}\mathcal{L}^2(z) \quad \textit{for all } \rho \in \mathrm{adm}(\Gamma). \tag{9}$$

Remark 10. A modulus estimate of this type was first established for quasiconformal mappings, see [Ahl66] and [LV73, p. 221].

3.2.2. *Curve family for an optimal modulus estimate.* In the subsequent proposition, we formulate sufficient conditions to have equality in (9). This will be the case if the family consists of curves which are tangential to directions of the largest shrinking of the candidate mapping g. In other words, $|\mathrm{d}g| = (|g_z| - |g_{\bar z}|)|\mathrm{d}z|$ should hold infinitesimally along the respective curves.

Theorem 11. *Let Ω and Ω' be two domains in $\mathbb{C}$ and let $g : \Omega \to \Omega'$ be a quasiconformal map with $K(z,g) \equiv K_g$. Consider a family Γ_0 of absolutely continuous curves γ in Ω with the following properties:*

(i) For all $\gamma \in \Gamma_0$

$$\frac{g_{\bar z}(\gamma(s))\dot{\bar\gamma}(s)}{g_z(\gamma(s))\dot\gamma(s)} < 0 \quad \textit{for almost every } s\textit{; and} \tag{10}$$

(ii) there exists a $\rho_0 \in \operatorname{adm}(\Gamma_0)$ *such that* $M(\Gamma_0) = \int_\Omega \rho_0^2(z)\, \mathrm{d}\mathcal{L}^2(z)$.
Then,

$$M(g(\Gamma_0)) = \int_\Omega K(z,g)\rho_0^2(z)\, \mathrm{d}\mathcal{L}^2(z).$$

Remark 12. Note that a statement of this type always holds, even without the requirement of an extremal density ρ_0. In fact, it can be shown that extremal densities exist almost everywhere: For any given curve family Γ_0 there is a subfamily $\hat\Gamma_0$ with $M(\Gamma_0 \setminus \hat\Gamma_0) = 0$ and $\hat\rho_0 \in \operatorname{adm}(\hat\Gamma_0)$ such that

$$M(\Gamma_0) = M(\hat\Gamma_0) = \int_\Omega \hat\rho_0^2(z)\, \mathrm{d}\mathcal{L}^2(z),$$

see e.g. [Fug57, p. 182].
If a curve family Γ_0 satisfies the first assumption of the above theorem, the smaller curve family $\hat\Gamma_0$ will satisfy it as well and consequently, one has

$$M(g(\hat\Gamma_0)) = \int_\Omega K(z,g)\hat\rho_0^2(z)\, \mathrm{d}\mathcal{L}^2(z).$$

From the fact that g is quasiconformal and $M(\Gamma_0 \setminus \hat\Gamma_0) = 0$ we may conclude that the modulus $M(g(\Gamma_0 \setminus \hat\Gamma_0))$ vanishes and thus

$$M(g(\Gamma_0)) = M(g(\hat\Gamma_0)) = \int_\Omega K(z,g)\hat\rho_0^2(z)\, \mathrm{d}\mathcal{L}^2(z).$$

Proof of Theorem 11. Let Γ_0 be such that it satisfies the assumptions (i) and (ii) of the theorem. Note that we may assume without loss of generality that the curves in $g(\Gamma_0)$ are absolutely continuous; the function g belongs to ACL^2, that is, its coordinate functions are absolutely continuous on almost every line segment parallel to the coordinate axes and the first partial derivatives exist almost everywhere and belong to L^2_{loc}. It follows from Fuglede's theorem [Fug57] (see e.g. [Väi71, page 95] or [MRSY09, Theorem 2.7]) that g is absolutely continuous on almost every curve in Γ_0, meaning that the sub-family of locally rectifiable curves in Γ_0 on which g is absolutely continuous has the same modulus as Γ_0.
By (9) we have

$$M(g(\Gamma_0)) \le \int_\Omega K(z,g)\rho^2(z)\, \mathrm{d}\mathcal{L}^2(z) \quad \text{for all } \rho \in \operatorname{adm}(\Gamma_0),$$

in particular

$$M(g(\Gamma_0)) \le \int_\Omega K(z,g)\rho_0^2(z)\, \mathrm{d}\mathcal{L}^2(z).$$

We will prove that the above inequalities are in fact equalities.

Since $J(z,g) > 0$ a.e., we have that $|g_z| - |g_{\bar z}| > 0$ a.e. Then the *push-forward density*

$$\rho'(\zeta) := \frac{\rho(g^{-1}(\zeta))}{|g_z(g^{-1}(\zeta))| - |g_{\bar z}(g^{-1}(\zeta)|}, \quad \zeta \in g(\Omega)$$

can be defined for any $\rho \in \mathrm{adm}(\Gamma_0)$. Note that the curves in Γ_0 and $g(\Gamma_0)$ are absolutely continuous. It can be shown (see e.g. [Ahl66]) that for all $\tilde\gamma \in g(\Gamma_0)$ and all $\rho \in \mathrm{adm}(\Gamma_0)$

$$\int_{\tilde\gamma} \rho'(\zeta)|\mathrm{d}\zeta| \geq \int_{g^{-1}\circ\tilde\gamma} \rho(z)|\mathrm{d}z| \geq 1.$$

Hence $\rho' \in \mathrm{adm}(g(\Gamma_0))$ for $\rho \in \mathrm{adm}(\Gamma_0)$, i.e. the inclusion

$$\{\rho' :\ \rho \in \mathrm{adm}(\Gamma_0)\} \subseteq \mathrm{adm}(g(\Gamma_0))$$

always holds. Using (10), the reverse inclusion can be established as well:
For $\tilde\rho \in \mathrm{adm}(g(\Gamma_0))$ and $\gamma : [a,b] \to \Omega$ in Γ_0, we find

$$\begin{aligned} 1 \leq \int_{g\circ\gamma} \tilde\rho(\zeta)|\mathrm{d}\zeta| &= \int_a^b \tilde\rho(g(\gamma(s))|g_z(\gamma(s))\dot\gamma(s) + g_{\bar z}(\gamma(s))\dot{\bar\gamma}(s)|\ \mathrm{d}s \\ &\overset{(10)}{=} \int_a^b \tilde\rho(g(\gamma(s)))\left(|g_z(\gamma(s))| - |g_{\bar z}(\gamma(s))|\right)|\dot\gamma(s)|\ \mathrm{d}s \\ &= \int_a^b \rho(\gamma(s))|\dot\gamma(s)|\ \mathrm{d}s \\ &= \int_\gamma \rho(z)|\mathrm{d}z|, \end{aligned}$$

where we have set $\rho(z) := \tilde\rho(g(z))(|g_z(z)| - |g_{\bar z}(z)|)$ (then $\rho'(\zeta) = \tilde\rho(\zeta)$). The above computation shows that $\rho \in \mathrm{adm}(\Gamma_0)$.
We conclude $\mathrm{adm}(g(\Gamma_0)) \subseteq \{\rho' : \rho \in \mathrm{adm}(\Gamma_0)\}$. Altogether, we obtain for a curve family Γ_0 which fulfils the conditions of the theorem that

$$\mathrm{adm}(g(\Gamma_0)) = \{\rho' : \rho \in \mathrm{adm}(\Gamma_0)\} \tag{11}$$

This implies

$$\begin{aligned} M(g(\Gamma_0)) &= \inf_{\tilde\rho\in\mathrm{adm}(g(\Gamma_0))} \int_{g(\Omega)} \tilde\rho^2(\zeta)\ \mathrm{d}\mathcal{L}^2(\zeta) \\ &= \inf_{\rho\in\mathrm{adm}(\Gamma_0)} \int_{g(\Omega)} (\rho'(\zeta))^2\ \mathrm{d}\mathcal{L}^2(\zeta) \\ &= \inf_{\rho\in\mathrm{adm}(\Gamma_0)} \int_{g(\Omega)} \left(\frac{\rho(g^{-1}(\zeta))}{|g_z(g^{-1}(\zeta))| - |g_{\bar z}(g^{-1}(\zeta))|}\right)^2\ \mathrm{d}\mathcal{L}^2(\zeta). \end{aligned}$$

Since g is of class $W^{1,1}_{loc}$ and satisfies Lusin's condition (N), we can apply the change of variables formula to $\zeta = g(z)$ for the measurable function

$$u(\cdot) = \left(\frac{\rho(g^{-1}(\cdot))}{|g_z(g^{-1}(\cdot))| - |g_{\bar z}(g^{-1}(\cdot))|}\right)^2 \geq 0.$$

Hence

$$\begin{aligned}
&\int_{g(\Omega)} \left(\frac{\rho(g^{-1}(\zeta))}{|g_z(g^{-1}(\zeta))| - |g_{\bar z}(g^{-1}(\zeta))|} \right)^2 \, \mathrm{d}\mathcal{L}^2(\zeta) \\
&= \int_\Omega \frac{\rho^2(z)}{(|g_z(z)| - |g_{\bar z}(z)|)^2} |J(z,g)| \, \mathrm{d}\mathcal{L}^2(z) \\
&= \int_\Omega \frac{\rho^2(z)}{(|g_z(z)| - |g_{\bar z}(z)|)^2} (|g_z(z)|^2 - |g_{\bar z}(z)|^2) \, \mathrm{d}\mathcal{L}^2(z) \\
&= \int_\Omega \frac{|g_z(z)| + |g_{\bar z}(z)|}{|g_z(z)| - |g_{\bar z}(z)|} \rho^2(z) \, \mathrm{d}\mathcal{L}^2(z) = \int_\Omega K(z,g)\rho^2(z) \, \mathrm{d}\mathcal{L}^2(z).
\end{aligned}$$

This yields

$$\begin{aligned}
M(g(\Gamma_0)) &= \inf_{\rho \in \mathrm{adm}(\Gamma_0)} \int_\Omega K(z,g)\rho^2(z) \, \mathrm{d}\mathcal{L}^2(z) = K_g \cdot \inf_{\rho \in \mathrm{adm}(\Gamma_0)} \int_\Omega \rho^2(z) \, \mathrm{d}\mathcal{L}^2(z) \\
&= K_g \cdot M(\Gamma_0) = K_g \int_\Omega \rho_0^2(z) \, \mathrm{d}\mathcal{L}^2(z) = \int_\Omega K(z,g)\rho_0^2(z) \, \mathrm{d}\mathcal{L}^2(z),
\end{aligned}$$

which concludes the proof. □

3.3. **Step 2: Minimisation in a given homotopy class.** The general results from the previous section will be applied to the case where the domains $\Omega = A(q,1)$ and $\Omega' = A(q^{k_1},1)$ are annuli in the complex plane. Throughout this section, $N \in \mathbb{Z}$ shall be an arbitrary but fixed integer. We consider the spiral-stretch map

$$f_N(z) = z|z|^{k_1 - 1} e^{\mathrm{i}k_2 \log|z|}, \quad \text{with } k_2 = k_2(N) = \frac{\theta + 2\pi N}{\log q}.$$

Recall that we assume $k_2 \neq 0$. The case for $k_2 = 0$ will be obtained as a corollary to Theorem 8.

In this section, we will assign to each $f \in \mathcal{F}$ a winding number N_f which measures the amount of twisting. The aim is to show that the spiral-stretch map f_N is a minimiser for a weighted mean distortion in the class

$$\mathcal{F}_N := \{f \in \mathcal{F} : \ N_f = N\}.$$

First, we use Theorem 11 to identify a curve family Γ_0 and a density ρ_0 for which equality holds in (9) with $g = f_N$ and $\Omega = A(q,1)$, i.e.

$$\int_{A(q,1)} K(z,f_N)\rho_0^2(z) \, \mathrm{d}\mathcal{L}^2(z) = M(f_N(\Gamma_0)).$$

Note that f_N is a quasiconformal map with $K(z,f_N) = K_{f_N}$ for all $z \in A(q,1)$. The aim is to find a family $\Gamma_{0,N}$ of absolutely continuous curves $\gamma : [q,1] \to \overline{A(q,1)}$ such that

$$\frac{f_{N\bar z}(\gamma(s))\dot{\bar\gamma}(s)}{f_{Nz}(\gamma(s))\dot\gamma(s)} < 0 \quad \text{for almost every } s. \tag{12}$$

By inserting the explicit formulas

$$f_{N_z}(z) = \frac{1}{2}(k_1 + 1 + \mathrm{i}k_2)|z|^{k_1-1}e^{\mathrm{i}k_2 \log|z|}, \text{ and}$$
$$f_{N_{\bar{z}}}(z) = \frac{1}{2}(k_1 - 1 + \mathrm{i}k_2)\left(\frac{z}{\bar{z}}\right)|z|^{k_1-1}e^{\mathrm{i}k_2 \log|z|}$$

in inequality (12), one can see that $\Gamma_{0,N}$ can be taken to be a family of spirals of a certain curvature.

Definition 13. For $t \in \mathbb{R}$ and $\phi \in [0, 2\pi)$, define

$$\gamma_\phi^t(s) = se^{\mathrm{i}(t\log s+\phi)}, \quad s \in [q, 1].$$

We consider the curve family

$$\Gamma_{0,N} := \{\gamma_\phi^{c_N} : \phi \in [0, 2\pi)\}$$

with

$$c_N := c(k_1, k_2) = \frac{-k_1^2 - k_2^2 + 1 - \sqrt{(k_1^2 + k_2^2 - 1)^2 + 4k_2^2}}{2k_2}, \quad \text{where } k_2 = k_2(N) \neq 0. \tag{13}$$

Lemma 14. *The curve family $\Gamma_{0,N}$ has the following properties:*

(i) f_N is absolutely continuous on each $\gamma \in \Gamma_{0,N}$; and

(ii) for all $\gamma \in \Gamma_{0,N}$

$$\frac{f_{N_{\bar{z}}}(\gamma(s))\dot{\bar{\gamma}}(s)}{f_{N_z}(\gamma(s))\dot{\gamma}(s)} < 0 \quad \textit{for all } s \in [q, 1]. \tag{14}$$

Proof. The first part of the statement is trivial. Any curve γ in the family $\Gamma_{0,N}$ is of the form $\gamma = \gamma_\phi^{c_N}$ for a $\phi \in [0, 2\pi)$. Clearly, the map

$$s \mapsto f_N \circ \gamma(s) = s^{k_1}e^{\mathrm{i}((c_N+k_2)\log s+\phi)}, \quad s \in [q, 1]$$

is absolutely continuous.

Inequality (14) can be verified by a direct computation. For $\gamma(s) = se^{\mathrm{i}(c_N \log s+\phi)}$, one obtains

$$\frac{f_{N_{\bar{z}}}(\gamma(s))\dot{\bar{\gamma}}(s)}{f_{N_z}(\gamma(s))\dot{\gamma}(s)} = \frac{1 - \mathrm{i}c_N}{1 + \mathrm{i}c_N}\frac{k_1 - 1 + \mathrm{i}k_2}{k_1 + 1 + \mathrm{i}k_2}.$$

This last expression is negative exactly when

$$(1 - \mathrm{i}c_N)^2(k_1 - 1 + \mathrm{i}k_2)(k_1 + 1 - \mathrm{i}k_2) < 0.$$

Using the definition of c_N, we compute

$$\begin{aligned}&(1 - \mathrm{i}c_N)^2(k_1 - 1 + \mathrm{i}k_2)(k_1 + 1 - \mathrm{i}k_2)\\ &\qquad = -\frac{k_1^2 + k_2^2 - 1 + \sqrt{(k_1^2 + k_2^2 - 1)^2 + 4k_2^2}}{2k_2^2}((k_1^2 + k_2^2 - 1)^2 + 4k_2^2).\end{aligned}$$

Note that this last expression is negative since

$$k_1^2 + k_2^2 - 1 + \sqrt{(k_1^2 + k_2^2 - 1)^2 + 4k_2^2} > 0.$$

This concludes the proof. □

Associated to the curve family $\Gamma_{0,N}$, we consider the following density:

Definition 15. Define the non-negative Borel function

$$\rho_{0,N}(z) := \frac{1}{-\log q \cdot \sqrt{1+c_N{}^2}} \frac{1}{|z|}, \quad z \in A(q,1).$$

Lemma 16. *Let $\Gamma_{0,N}$ and $\rho_{0,N}$ be as above. Then the following holds*

$$\rho_{0,N} \in \operatorname{adm}(\Gamma_{0,N}) \quad \text{and} \quad M(\Gamma_{0,N}) = \int_{A(q,1)} \rho_{0,N}^2(z)\, \mathrm{d}\mathcal{L}^2(z) = \frac{-2\pi}{(1+c_N{}^2)\log q}.$$

Proof. First, we show that the density $\rho_{0,N}$ is admissible for $\Gamma_{0,N}$. In order to do so, let us note the following fact, which will also be used in what follows. An absolutely continuous curve $\gamma : [q,1] \to \overline{A(q,1)}$ can be written as $\gamma(s) = |\gamma(s)|e^{\mathrm{i}\varphi(s)}$, where $\varphi : [q,1] \to \mathbb{R}$ is an absolutely continuous angle function, which is unique up to addition of a constant multiple of 2π. Using the well-known properties of absolutely continuous functions, we find

$$\begin{aligned}
\int_\gamma \rho_{0,N}(z)|\mathrm{d}z| &= \frac{1}{-\log q\sqrt{1+c_N^2}} \int_\gamma \frac{1}{|z|}|\mathrm{d}z| = \frac{1}{-\log q\sqrt{1+c_N^2}} \int_q^1 \frac{|\dot\gamma(s)|}{|\gamma(s)|}\, \mathrm{d}s \\
&\geq \frac{1}{-\log q\sqrt{1+c_N^2}} \left| \int_q^1 \frac{\dot\gamma(s)}{\gamma(s)}\, \mathrm{d}s \right| \\
&= \frac{1}{-\log q\sqrt{1+c_N^2}} \left| \int_q^1 \frac{\frac{d}{ds}|\gamma(s)|}{|\gamma(s)|} + \mathrm{i}\dot\varphi(s)\, \mathrm{d}s \right|.
\end{aligned}$$

Hence, for every absolutely continuous curve $\gamma : [q,1] \to \overline{A(q,1)}$, the following holds

$$\int_\gamma \rho_{0,N}(z)|\mathrm{d}z| \geq \frac{1}{-\log q\sqrt{1+c_N^2}} \left| \log\left(\frac{|\gamma(1)|}{|\gamma(q)|}\right) + \mathrm{i}(\varphi(1)-\varphi(q)) \right|. \tag{15}$$

In particular, we get $\int_{\gamma_\phi^{c_N}} \rho_{0,N}(z)|\mathrm{d}z| \geq 1$ for all $\phi \in [0,2\pi)$, which proves that $\rho_{0,N}$ is admissible for $\Gamma_{0,N}$.

Secondly, we prove that the density $\rho_{0,N}$ is extremal for $\Gamma_{0,N}$: On one hand, we have

$$M(\Gamma_{0,N}) \leq \int_{A(q,1)} \rho_{0,N}^2(z)\, \mathrm{d}\mathcal{L}^2(z) = \frac{1}{(-\log q)^2(1+c_N^2)} \int_0^{2\pi}\int_q^1 \frac{1}{r^2} r\, \mathrm{d}r\mathrm{d}\phi = \frac{-2\pi}{(1+c_N^2)\log q}.$$

On the other hand, let $\rho \in \operatorname{adm}(\Gamma_{0,N})$ be an arbitrary admissible density for $\Gamma_{0,N}$, hence

$$1 \leq \int_{\gamma_\phi^{c_N}} \rho(z)|\mathrm{d}z| \quad \text{for all } \phi \in [0,2\pi).$$

Then, for all $\phi \in [0,2\pi)$, the following holds

$$1 \leq \int_q^1 \rho(se^{\mathrm{i}(c_N \log s+\phi)})|(1+\mathrm{i}c_N)e^{\mathrm{i}(c_N\log s+\phi)}|\, \mathrm{d}s = \int_q^1 \rho(se^{\mathrm{i}(c_N\log s+\phi)})\sqrt{1+c_N^2}\, \mathrm{d}s.$$

Integrating both sides of the inequality with respect to ϕ from 0 to 2π, applying Cauchy-Schwarz inequality, Fubini's theorem and the transformation formula yields

$$\begin{aligned}
2\pi &\leq \sqrt{1+c_N^2}\int_0^{2\pi}\int_q^1 \rho(se^{\mathrm{i}(c_N\log s+\phi)})\sqrt{s}\frac{1}{\sqrt{s}}\,\mathrm{d}s\mathrm{d}\phi\\
&\leq \sqrt{1+c_N^2}\left(\int_0^{2\pi}\int_q^1 (\rho(se^{\mathrm{i}(c_N\log s+\phi)}))^2 s\,\mathrm{d}s\mathrm{d}\phi\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\int_q^1 \frac{1}{s}\,\mathrm{d}s\mathrm{d}\phi\right)^{\frac{1}{2}}\\
&= \sqrt{1+c_N^2}\left(\int_{A(q,1)}\rho^2\circ f(z)\,\mathrm{d}\mathcal{L}^2(z)\right)^{\frac{1}{2}}(-2\pi\log q)^{\frac{1}{2}},
\end{aligned}$$

where $f(z) := ze^{\mathrm{i}c_N\log|z|}$. Since $J(z,f)=1$ and $f(A(q,1))=A(q,1)$, we obtain by applying the change of variables formula

$$\begin{aligned}
2\pi &\leq \sqrt{1+c_N^2}\left(\int_{A(q,1)}\rho^2(f(z))|J(z,f)|\,\mathrm{d}\mathcal{L}^2(z)\right)^{\frac{1}{2}}(-2\pi\log q)^{\frac{1}{2}}\\
&= \sqrt{1+c_N^2}\left(\int_{A(q,1)}\rho^2(\zeta)\,\mathrm{d}\mathcal{L}^2(\zeta)\right)^{\frac{1}{2}}(-2\pi\log q)^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\int_{A(q,1)}\rho^2(\zeta)\,\mathrm{d}\mathcal{L}^2(\zeta) \geq \frac{-2\pi}{(1+c_N^2)\log q}\quad \text{for all } \rho\in\mathrm{adm}(\Gamma_{0,N}),$$

and therefore

$$M(\Gamma_{0,N}) \geq \frac{-2\pi}{(1+c_N^2)\log q}.$$

We conclude

$$M(\Gamma_{0,N}) = \int_{A(q,1)}\rho_{0,N}^2(z)\,\mathrm{d}\mathcal{L}^2(z) = \frac{-2\pi}{(1+c_N^2)\log q}.$$

□

Proposition 17. *Using the same notation as before, the following holds*

$$M(f_N(\Gamma_{0,N})) = \frac{1}{(-\log q)^2(1+c_N^2)}\int_{A(q,1)}K(z,f_N)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z). \tag{16}$$

Proof. The result follows by applying Theorem 11 to $g=f_N$, $\Gamma_0=\Gamma_{0,N}$ with $c_N=c(k_1,k_2)$ and $\rho_0=\rho_{0,N}$. The necessary assumptions have been established in Lemmas 14 and 16. □

3.3.1. *Modulus for homotopic curves and maps.* Proposition 17 allows us to express the modulus of the family of spirals $f_N(\Gamma_{0,N})$ with respect to the distortion of the spiral-stretch map f_N. Ideally, this value $M(f_N(\Gamma_{0,N}))$ would serve as a lower bound for the modulus of the image family for an arbitrary $f\in\mathcal{F}$. However, the family $f(\Gamma_{0,N})$ does not in general contain the spiral family $f_N(\Gamma_{0,N})$.
To remedy this, we enlarge the curve family $\Gamma_{0,N}$ in such a way that $\rho_{0,N}$ is still admissible for this larger family: We are going to consider all curves which are homotopic to arcs of spirals in $\Gamma_{0,N}$.

Recall that two curves, $\gamma_0 : [q,1] \to \overline{A(q,1)}$ and $\gamma_1 : [q,1] \to \overline{A(q,1)}$, with $\gamma_0(q) = \gamma_1(q)$ and $\gamma_0(1) = \gamma_1(1)$ are called *homotopic* (with respect to the endpoints), $\gamma_0 \simeq \gamma_1$, if there exists a continuous map

$$H : [q,1] \times [0,1] \to \overline{A(q,1)}, \quad (s,t) \mapsto H(s,t),$$

such that

$$H(s,0) = \gamma_1(s) \quad \text{and} \quad H(s,1) = \gamma_2(s) \quad \text{for all } s \in [q,1]$$

and

$$H(q,t) = \gamma_1(q) = \gamma_2(q) \quad \text{and} \quad H(1,t) = \gamma_1(1) = \gamma_2(1) \quad \text{for all } t \in [0,1].$$

Definition 18. We define

$$\Gamma_N := \{\gamma : [q,1] \to \overline{A(q,1)} \text{ absolutely continuous}, \, \gamma \simeq \gamma_\phi^{c_N}, \; \phi \in [0,2\pi)\}.$$

Lemma 19. *The density $\rho_{0,N}$ is admissible for Γ_N.*

Proof. Let $\gamma \in \Gamma_N$. By definition, such a curve is absolutely continuous and homotopic to $\gamma_\phi^{c_N}$, where $\phi \in [0,2\pi)$ is such that $\gamma(1) = e^{\mathrm{i}\phi}$. Hence it can be written as $\gamma(s) = |\gamma(s)|e^{\mathrm{i}\varphi(s)}$, $s \in [q,1]$, for an absolutely continuous function $\varphi : [q,1] \to \mathbb{R}$ with

$$\varphi(1) = \phi \quad \text{and} \quad \varphi(q) = \phi + c_N \log q \pmod{2\pi}.$$

Using the homotopy between γ and $\gamma_\phi^{c_N}$ one can show that in fact $\varphi(q) = \phi + c_N \log q$ and thus by (15),

$$\int_\gamma \rho_{0,N}(z)|\mathrm{d}z| \geq \frac{1}{-\log q\sqrt{1+c_N^2}} |-\log q - \mathrm{i}c_N \log q| = 1.$$

As we have chosen $\gamma \in \Gamma_N$ arbitrarily, this shows that $\rho_{0,N}$ is admissible for Γ_N. □

It is now our goal to compare the modulus of $f_N(\Gamma_{0,N})$ with that of $f(\Gamma_N)$. However, the value of $M(f(\Gamma_N))$ is not the same for all f since not all mappings in $\mathcal{F}$ twist the annulus by the same amount.

To classify the mappings according to their rotation behaviour, we assign to each mapping f in $\mathcal{F}$ a winding number N_f which measures the amount of twisting. Here we use the boundary conditions for mappings in the class $\mathcal{F}$.

Definition 20. For $f \in \mathcal{F}$, we define

$$N_f = N_f(\phi,t) := \frac{\mathrm{Im}[\mathrm{Log}(f \circ \gamma_\phi^t(q)) - \mathrm{Log}(f \circ \gamma_\phi^t(1)) - \mathrm{i}(t\log q + \theta)]}{2\pi}, \quad (\phi,t) \in [0,2\pi)\times\mathbb{R},$$

where θ denotes the rotation angle from Definition 2.

We can write $f \circ \gamma_\phi^{q,t}(s) = r(s)e^{\mathrm{i}\varphi(s)}$, where $r : [q,1] \to [q^{k_1},1]$ is a continuous function with $r(q) = q^{k_1}$, $r(1) = 1$ and $\varphi : [q,1] \to \mathbb{R}$ is a continuous argument with

$$\varphi(q) = \varphi(1) + t\log q + \theta \pmod{2\pi}.$$

We leave as an exercise to the reader to verify that

$$(\phi,t) \mapsto N_f(\phi,t) = \frac{\varphi(q) - \varphi(1) - t\log q - \theta}{2\pi}, \quad (\phi,t) \in [0,2\pi) \times \mathbb{R}$$

is a continuous integer-valued function and as such it must be constant.

The modulus $M(f(\Gamma_N))$ can now be compared to $M(f_N(\Gamma_{0,N}))$ for all f in the subclass

$$\mathcal{F}_N := \{f \in \mathcal{F} : \ N_f = N\}.$$

Essentially, we would like to show for a given $f \in \mathcal{F}_N$ that $f_N(\Gamma_{0,N}) \subseteq f(\Gamma_N)$, i.e. that any curve $\gamma' \in f_N(\Gamma_{0,N})$ can be written as $\gamma' = f \circ \gamma$, where γ belongs to Γ_N. This is not true, though, since the curve $\gamma = f^{-1} \circ \gamma'$ will usually not be absolutely continuous. However, since a map in $W^{1,2}_{loc}$ belongs to the space ACL^2, it is possible to apply Fuglede's theorem [Fug57] and we can still conclude $M(f_N(\Gamma_{0,N})) \leq M(f(\Gamma_N))$.

Proposition 21. *With the same notation as before, we have*

$$M(f_N(\Gamma_{0,N})) \leq M(f(\Gamma_N)) \quad \textit{for all } f \in \mathcal{F}_N. \tag{17}$$

Proof. Let γ' be an arbitrary curve in $f_N(\Gamma_{0,N})$. Then, there exists a $\phi \in [0, 2\pi)$ such that $\gamma' = f_N \circ \gamma_\phi^{c_N}$. Let f be an arbitrary map in $\mathcal{F}_N$. Since f is a homeomorphism, we may write

$$f_N \circ \gamma_\phi^{c_N} = f \circ \gamma,$$

with

$$\gamma := f^{-1} \circ \gamma' = f^{-1} \circ f_N \circ \gamma_\phi^{c_N}.$$

Using that f belongs to $\mathcal{F}_N$, one can show in addition that

$$f \circ \gamma = f_N \circ \gamma_\phi^{c_N} \simeq f \circ \gamma_\phi^{c_N}, \tag{18}$$

which proves that γ is homotopic to $\gamma_\phi^{c_N}$. We leave the details to the reader.
In order to conclude that the curve $\gamma := f^{-1} \circ \gamma'$ lies in Γ_N (and thus γ' is an element of $f(\Gamma_N)$ as desired), we must have that γ is absolutely continuous, which is generally not the case. This problem can be resolved by applying Fuglede's Theorem. Note that we have assumed our mappings to be homeomorphisms of integrable distortion and therefore $f^{-1} \in W^{1,2}_{loc}$, which guarantees that Fuglede's Theorem can be applied to the inverse map f^{-1}. Together with (8) this results in the following identity

$$M(f_N(\Gamma_{0,N})) = M(\{\gamma' \in f_N(\Gamma_{0,N}) \text{ rectifiable } : \ f^{-1} \text{ is absolutely continuous on } \gamma'\}).$$

Any rectifiable curve γ' in $f_N(\Gamma_{0,N})$ on which f^{-1} is absolutely continuous belongs to $f(\Gamma_N)$ as explained above. Hence

$$M(\{\gamma' \in f_N(\Gamma_{0,N}) \text{ rectifiable } : \ f^{-1} \text{ is absolutely continuous on } \gamma'\}) \leq M(f(\Gamma_N)).$$

This concludes the proof. $\square$

3.3.2. *Extremality in a subclass: Proof of Theorem 5.* A combination of the previous results yields the extremality of the spiral-stretch map f_N in its subclass:

Proof of Theorem 5. Let $N \in \mathbb{Z}$ be fixed. An arbitrary $f \in \mathcal{F}_N$ satisfies

$$\int_{A(q,1)} K(z, f_N)\rho_{0,N}^2(z)\, \mathrm{d}\mathcal{L}^2(z) \overset{(16)}{=} M(f_N(\Gamma_{0,N})) \overset{(17)}{\leq} M(f(\Gamma_N))$$

$$\overset{(9)}{\leq} \int_{A(q,1)} K(z, f)\rho_{0,N}^2(z)\, \mathrm{d}\mathcal{L}^2(z),$$

where we have used in the last step the fact that $\rho_{0,N}$ is admissible for Γ_N. Since this density is a constant multiple of $\frac{1}{|z|}$, the result follows. ☐

3.4. **Step 3: Comparing homotopy classes.**

Proof of Theorem 6. In order to derive Theorem 6 from Theorem 5, we show that the spiral-stretch map f_0 is minimal among all admissible spiral-stretch maps:

$$K_{f_0} \leq K_{f_N} \quad \text{for all } N \in \mathbb{Z},$$

where $f_N(z) = z|z|^{k_1-1}e^{\mathrm{i}k_2(N)\log|z|}$.
Note that the spiral-stretch map f_N has constant distortion

$$K_{f_N} = K(z, f_N) = \frac{\sqrt{(k_1+1)^2 + k_2(N)^2} + \sqrt{(k_1-1)^2 + k_2(N)^2}}{\sqrt{(k_1+1)^2 + k_2(N)^2} - \sqrt{(k_1-1)^2 + k_2(N)^2}}. \tag{19}$$

To prove the extremality of f_0 among all f_N, $N \in \mathbb{Z}$, we consider the function

$$x \mapsto \kappa(x) := \frac{\sqrt{(k_1+1)^2 + x^2} + \sqrt{(k_1-1)^2 + x^2}}{\sqrt{(k_1+1)^2 + x^2} - \sqrt{(k_1-1)^2 + x^2}}.$$

Note that $\kappa(k_2(N)) = K_{f_N}$. A direct computation shows that $\kappa(x)$ is an even function and monotone increasing as $|x| \to \infty$. Now, since by assumption $|\theta| \leq \pi$, it follows that $|k_2(0)\log q| = |\theta| \leq |\theta + 2\pi N| = |k_2(N)\log q|$ and thus $|k_2(0)| \leq |k_2(N)|$ for all $N \in \mathbb{Z}$. This implies $K_{f_0} \leq K_{f_N}$ for all $N \in \mathbb{Z}$.
Let $N \in \mathbb{Z}$ be arbitrary. By Theorem 5 the map f_N is extremal in the subclass $\mathcal{F}_N \subset \mathcal{F}$. More precisely:

$$\int_{A(q,1)} K(z, f_N)\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z) \leq \int_{A(q,1)} K(z, f)\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z) \quad \text{for all } f \in \mathcal{F}_N. \tag{20}$$

Since the map f_0 is extremal among all the spiral-stretch maps f_N, $N \in \mathbb{Z}$, we may conclude

$$\int_{A(q,1)} K(z, f_0)\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z) \leq \int_{A(q,1)} K(z, f_N)\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z) \quad \text{for all } N \in \mathbb{Z}. \tag{21}$$

A combination of (20) and (21) yields

$$\int_{A(q,1)} K(z,f_0)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \leq \int_{A(q,1)} K(z,f)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \quad \text{for all } f\in\mathcal{F}. \tag{22}$$

For any convex and non-decreasing function Ψ, by applying Jensen's inequality in (22), we obtain

$$\begin{aligned}
\Psi(K(z,f_0)) =& \Psi\left(\frac{1}{\int_{A(q,1)}\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z)}\int_{A(q,1)} K(z,f_0)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z)\right)\\
\overset{(22)}{\leq}& \Psi\left(\frac{1}{\int_{A(q,1)}\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z)}\int_{A(q,1)} K(z,f)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z)\right)\\
\leq& \frac{1}{\int_{A(q,1)}\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z)}\int_{A(q,1)} \Psi(K(z,f))\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z),
\end{aligned}$$

which yields the result

$$\int_{A(q,1)} \Psi(K(z,f_0))\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \leq \int_{A(q,1)} \Psi(K(z,f))\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) \quad \text{for all } f\in\mathcal{F}.$$

□

4. Extremality of the radial stretching. Proof of Theorem 8

In the subsequent discussion, we will always assume that $q\in(0,1)$ and $k_1>0$ are arbitrary constants.

Proof of Theorem 8. The arguments used in this proof are similar as in the proof of Theorem 6. Therefore, we only give a sketch of the proof, illustrating how to apply the modulus method based on Theorem 9, and leave the details to the interested reader.
The crucial point in the proof is to find an appropriate curve family Γ_0 and a metric density ρ_0 which is extremal for Γ_0 such that the assumptions of Theorem 11 hold with

$$g(z)=f_0(z)=z|z|^{k_1-1}.$$

The geometric interpretation of Γ_0 is that it contains curves which are tangential to directions of largest shrinking (least stretching) of f_0. We consider two cases when $k_1>1$ and when $k_1<1$.
In the first case $k_1>1$, we employ the curve family

$$\Gamma_0 := \{\gamma_r:\ r\in(q,1)\},$$

which contains the circles

$$\gamma_r(s)=re^{\mathrm{i}s},\ s\in[0,2\pi].$$

The associated extremal density is given by

$$\rho_0(z)=\frac{1}{2\pi|z|},\ z\in A(q,1).$$

One can verify that

$$M(\Gamma_0) = \int_{A(q,1)} \rho_0^2(z)\, \mathrm{d}\mathcal{L}^2(z) = \frac{-\log q}{2\pi}.$$

Since Γ_0, ρ_0 and f_0 satisfy the assumptions in Theorem 11, we conclude

$$\int_{A(q,1)} K(z, f_0) \frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z) = (2\pi)^2 M(f_0(\Gamma_0)). \tag{23}$$

Again, the idea is to enlarge the curve family Γ_0 such that $\Gamma \supseteq \Gamma_0$ satisfies

(i) $\rho_0 \in \mathrm{adm}(\Gamma)$; and
(ii) $M(f_0(\Gamma_0)) \leq M(f(\Gamma))$ for all $f \in \mathcal{G}$.

Prior to the definition of this curve family, we recall the concept of the *winding number* of a closed curve. For any continuous curve $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$, a continuous function ("polar angle function") $\varphi : [a, b] \to \mathbb{R}$ can be defined such that

$$\gamma(t) = |\gamma(t)| e^{\mathrm{i}\varphi(t)}.$$

The difference $\varphi(b) - \varphi(a)$ is independent of the choice of the angle function and it can be used to define the winding number about the origin

$$n(\gamma, 0) := \frac{1}{2\pi}(\varphi(b) - \varphi(a)) \in \mathbb{Z}.$$

If γ is a closed piecewise $\mathcal{C}^1$-curve, then

$$n(\gamma, 0) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{z}\, \mathrm{d}z.$$

We consider the following curve family

$$\Gamma := \{\gamma : [0, 2\pi] \to A(q, 1) \text{ absolutely continuous}, \ \gamma(0) = \gamma(2\pi), \ n(\gamma, 0) \neq 0\}.$$

The density ρ_0 is admissible for this larger family, too:

$$\int_\gamma \rho_0(z)|\mathrm{d}z| = \frac{1}{2\pi} \int_0^{2\pi} \frac{|\dot\gamma(s)|}{|\gamma(s)|}\, \mathrm{d}s \geq \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{\dot\gamma(s)}{\gamma(s)}\, \mathrm{d}s \right| = |n(\gamma, 0)| \geq 1.$$

By Fuglede's theorem and (8), we have

$$M(f_0(\Gamma_0)) = M(\{\gamma' \in f_0(\Gamma_0) : \ \gamma' \text{ rectifiable}, \ f^{-1} \text{ absolutely continuous on } \gamma'\}).$$

For a given map $f \in \mathcal{G}$, any such curve $\gamma' \in f_0(\Gamma_0)$ can be written as $\gamma' = f \circ \gamma$, where $\gamma := f^{-1} \circ \gamma'$ is absolutely continuous by assumption and $n(\gamma, 0) \neq 0$ since f is a homeomorphism, hence $\gamma \in \Gamma$ and $\gamma' \in f(\Gamma)$. We conclude

$$M(f_0(\Gamma_0)) \leq M(f(\Gamma)). \tag{24}$$

Eventually, this yields the desired estimate

$$\begin{aligned}\int_{A(q,1)} K(z,f_0)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z) &\overset{(23)}{=} (2\pi)^2 M(f_0(\Gamma_0))\\ &\overset{(24)}{\leq} (2\pi)^2 M(f(\Gamma))\\ &\overset{(9)}{\leq} \int_{A(q,1)} K(z,f)\frac{1}{|z|^2}\,\mathrm{d}\mathcal{L}^2(z).\end{aligned}$$

In the second case $k_1 < 1$, we employ the family

$$\Gamma_0 := \{\gamma_\phi :\ \phi \in [0,2\pi)\}$$

of radial segments

$$\gamma_\phi(s) = se^{\mathrm{i}\phi}, \quad s \in [q,1].$$

The associated extremal density is given by

$$\rho_0(z) = \frac{1}{-\log q}\frac{1}{|z|}.$$

Notice that these are exactly the same curve family and density as in Definition 13 and 15 respectively for $c_N = 0$. The reasoning of the proof of the extremal property of f_0 is therefore very similar to that in Section 3. To obtain extremality not only in the class $\mathcal{F}$ of mappings with pointwise boundary conditions but in the larger class $\mathcal{G}$, one has to consider not just curves homotopic to γ_ϕ, $\phi \in [0,2\pi)$, but instead work with the larger family Γ of all absolutely continuous curves that connect the two boundary components of $A(q,1)$. One can show that

$$M(f_0(\Gamma_0)) \leq M(f(\Gamma)) \quad \text{for all } f \in \mathcal{G}$$

and proceed as before.

$\square$

Remark 22. An important question related to the minimisation of the distortion concerns the uniqueness of the solution. Even in the case of quasiconformal mappings and maximal distortion, the solution of the extremal problem for given boundary values is in general not unique, as can be seen by Strebel's famous chimney example [Str62]. But there are uniqueness results for quasiconformal mappings with a Beltrami coefficient of a particular form in a certain subclass [Str86].

Less is known about the uniqueness of extremal mappings of finite distortion. In [AIMO05], the authors consider a variant of the classical Grötzsch extremal problem for finite distortion mappings between rectangles. They prove the existence of a unique minimiser for an integrated non-linear distortion function $\mathbb{K}(z,f)$. However, the uniqueness is lost if one replaces the non-linear distortion by the usual linear distortion $K(z,f)$.

We stress here the fact that minimising the maximal distortion and minimising the mean distortion are indeed two problems of a different nature.

As before, we fix constants $k_1 > 1$ and $q \in (0,1)$ and let $\mathcal{G}$ be as in Theorem 8 (see Definition 7). It is a classical result that the radial stretching

$$f_0 : A(q,1) \to A(q^{k_1},1), \quad z \mapsto z|z|^{k_1-1}$$

is the unique solution for the minimisation of the maximal distortion

$$f \mapsto K_f$$

within the subclass of quasiconformal maps in the class $\mathcal{G}$.
According to Theorem 8, it is also a minimiser for the mean distortion functional

$$f \mapsto \int_{A(q,1)} \Psi(K(z,f)) \frac{1}{|z|^2} \,\mathrm{d}\mathcal{L}^2(z)$$

in the class $\mathcal{G}$.
A direct computation yields

$$K(z,f_0) = K_{f_0} = k_1 \quad \text{and} \quad \int_{A(q,1)} \Psi(K(z,f_0)) \frac{1}{|z|^2} \,\mathrm{d}\mathcal{L}^2(z) = -2\pi \cdot \log q \cdot \Psi(k_1).$$

It is possible to construct another mapping $f_1 \in \mathcal{G}$ which has a larger maximal distortion but the same weighted mean distortion as f_0 – at least for a specific choice of Ψ. In fact, there are infinitely many such mappings, which are equal to the identity map on an outer ring and they stretch an inner ring by a larger amount than f_0. This will clearly increase the maximal distortion, but it can be achieved that the mean distortion stays the same, as we are integrating a larger distortion over a smaller area. More precisely, we consider

$$f_1(z) := \begin{cases} z\left(\frac{|z|}{\tilde{q}}\right)^{\tilde{k}-1} & z \in A(q,\tilde{q}) \\ z & z \in A(\tilde{q},1). \end{cases}$$

Here, $\tilde{k} > k$ and $\tilde{q} \in (q,1)$ are chosen so that $f_1 \in \mathcal{G}$, i.e.

$$|f_1(z)| = |z| \left(\frac{|z|}{\tilde{q}}\right)^{\tilde{k}-1} = q \left(\frac{q}{\tilde{q}}\right)^{\tilde{k}-1} \stackrel{!}{=} q^{k_1} \quad \text{for } |z| = q.$$

This is equivalent to

$$\tilde{k}(\log q - \log \tilde{q}) + \log \tilde{q} = k_1 \log q. \tag{25}$$

We compute

$$\mu_{f_1}(z) = \begin{cases} \frac{\tilde{k}-1}{\tilde{k}+1}\frac{z}{\bar{z}} & z \in A(q,\tilde{q}) \\ 0 & z \in A(\tilde{q},1), \end{cases} \qquad K(z,f_1) = \begin{cases} \tilde{k} & z \in A(q,\tilde{q}) \\ 1 & z \in A(\tilde{q},1), \end{cases} \qquad K_{f_1} = \tilde{k},$$

and

$$\int_{A(q,1)} \Psi(K(z,f_1)) \frac{1}{|z|^2} \,\mathrm{d}\mathcal{L}^2(z) = 2\pi(\Psi(\tilde{k})(\log \tilde{q} - \log q) - \log \tilde{q}).$$

This shows

$$K_{f_0} < K_{f_1}.$$

Yet the mean distortion for $\Psi(t) = t$ is equal

$$\int_{A(q,1)} K(z, f_0)\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z) \overset{(25)}{=} \int_{A(q,1)} K(z, f_1)\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z).$$

On the other hand, if Ψ is strictly convex, we have

$$\Psi(k_1) \overset{(25)}{=} \Psi\left(\tilde{k}\left(1 - \frac{\log \tilde{q}}{\log q}\right) + \frac{\log \tilde{q}}{\log q}\right) < \left(1 - \frac{\log \tilde{q}}{\log q}\right)\Psi(\tilde{k}) + \frac{\log \tilde{q}}{\log q}\Psi(1)$$

and thus

$$\int_{A(q,1)} \Psi(K(z, f_0))\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z) < \int_{A(q,1)} \Psi(K(z, f_1))\frac{1}{|z|^2}\, \mathrm{d}\mathcal{L}^2(z).$$

5. Final remarks

The modulus method by curve families that was used in this paper can be applied to other minimisation problems for a weighted mean distortion with fixed boundary values and a conjectured extremal mapping.

Remark 23. Bishop, Gutlyanskiĭ, Martio and Vuorinen studied in [BGMV03] modulus estimates for quasiconformal mappings on domains in $\mathbb{R}^n$, $n \geq 2$. We believe that an analogue of our approach can be used to treat higher dimensional extremal problems for mappings of finite distortion, too.

As explained before, our method can be used to identify a quasiconformal map f_0 as minimiser for the mean distortion functional in the class of finite distortion maps. In all the examples above, this conjectured extremal f_0 had a Beltrami coefficient of the form

$$\mu(z) = k|\varphi_0|/\varphi_0, \quad \text{where } 0 \leq k < 1 \tag{26}$$

and φ_0 was a holomorphic function on the annulus, i.e. a *Teichmüller differential.* Moreover, μ was related to the extremal density ρ_0 associated to f_0 and the curve family Γ_0 as in Theorem 11 via the identity $\rho_0(z) = |\varphi_0(z)|^{\frac{1}{2}}$. In the classical Teichmüller theory, quasiconformal maps with a Beltrami coefficient of the form (26) are shown to be unique extremals for their boundary values [Tei40, Tei43], see e.g. [Str86, GL00]. The proof of the uniqueness is based on the length-area principle, using the trajectories of the so-called quadratic differential φ_0. *Horizontal trajectories* of φ_0 are lines of largest stretching, lines along which $\varphi_0(z)\mathrm{d}z^2 > 0$, whereas lines of largest shrinking are called *vertical trajectories* and they are defined by the condition $\varphi_0(z)\mathrm{d}z^2 < 0$.

Remark 24. We find that if the Beltrami coefficient of a quasiconformal map g on a domain Ω in $\mathbb{C}$ is a Teichmüller differential (i.e. it is of the form (26)), then the family Γ_0 of curves satisfying condition (10) in Theorem 11 consists exactly of the vertical trajectories of φ_0. Notice further that a Teichmüller map obviously always has constant distortion $K(z, g) = K_g$, hence is of the form required in Theorem 11.

Although related to the classical quasiconformal theory in this sense, the problems under consideration in this paper are of different nature, as the goal is to minimise the mean distortion and the method yields extremal mappings in the larger class of finite distortion maps.

Remark 25. Notice that minimisers obtained by our method will always be quasiconformal with constant distortion, as Theorem 11 can only be applied if the conjectured extremal is of this form. As shown in [AIMO05, Theorem 11.27], the minimiser of the mean distortion

$$f \mapsto \int_\Omega \mathbb{K}(z, f)\mathrm{d}\mathcal{L}^2(z) \tag{27}$$

with fixed boundary data f_0 and Ω being the unit disk is seldom quasiconformal, in fact only if f_0 is bi-Lipschitz. It would be an interesting subject for further research to see if an analogous result holds for the *weighted* mean distortion $f \mapsto \int_\Omega \Psi(K(z, f))\mathrm{d}\mu(z)$, for a convex non-decreasing function Ψ and $\mathrm{d}\mu(z) = \rho_0^2(z)\mathrm{d}\mathcal{L}^2(z)$. If this is the case, it will be of particular importance to develop further techniques for the identification of *non-quasiconformal* extremal maps of finite distortion.

Remark 26. We note that the minimisation of the integral mean (27) is a problem of different nature than the minimisation of the weighted mean distortion functional

$$f \mapsto \int_\Omega \Psi(K(z, f))\mathrm{d}\mu(z),$$

where $\mathrm{d}\mu(z) = \rho_0^2(z)\mathrm{d}\mathcal{L}^2(z)$ with the density ρ_0 coming from the conditions in Theorem 11.

It has been shown recently in [AIM09a] that the infimum of (27) in the class of finite distortion homeomorphisms between two annuli A and A' which map the inner and the outer boundary of A to the inner and outer boundary of A', respectively, is attained only when the image annulus A' is not too fat compared to A. This condition on the moduli can be explicitly described by the so-called Nitsche bound appearing in Nitsche's conjecture [Nit62]. The result is in strong contrast to the analogous minimisation problem for the weighted mean distortion, where one can always identify a minimiser. It would be interesting to study further the relation between the choice of the density and the existence of a minimiser for the associated mean distortion functional.

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