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Moment-angle complexes from simplicial posets

Research Article

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Abstract: We extend the construction of moment-angle complexes to simplicial posets by associating a certain T^m-space $\mathcal{Z}_{\mathcal{S}}$ to an arbitrary simplicial poset $\mathcal{S}$ on m vertices. Face rings $\mathbb{Z}[\mathcal{S}]$ of simplicial posets generalise those of simplicial complexes, and give rise to new classes of Gorenstein and Cohen–Macaulay rings. Our primary motivation is to study the face rings $\mathbb{Z}[\mathcal{S}]$ by topological methods. The space $\mathcal{Z}_{\mathcal{S}}$ has many important topological properties of the original moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ associated to a simplicial complex $\mathcal{K}$. In particular, we prove that the integral cohomology algebra of $\mathcal{Z}_{\mathcal{S}}$ is isomorphic to the Tor-algebra of the face ring $\mathbb{Z}[\mathcal{S}]$. This leads directly to a generalisation of Hochster's theorem, expressing the algebraic Betti numbers of the ring $\mathbb{Z}[\mathcal{S}]$ in terms of the homology of full subposets in $\mathcal{S}$. Finally, we estimate the total amount of homology of $\mathcal{Z}_{\mathcal{S}}$ from below by proving the toral rank conjecture for the moment-angle complexes $\mathcal{Z}_{\mathcal{S}}$.

MSC: 57S15, 14M25

Keywords: Moment-angle complex • Simplicial poset • Simplicial complex • Torus action • Face ring • Stanley-Reisner ring • Toral rank

1. Introduction

Simplicial posets describe the combinatorial structures underlying "generalised simplicial complexes" whose faces are still simplices, but two faces are allowed to intersect in any subcomplex of their boundary, rather than just in a single face. These are also known as "ideal triangulations" in low-dimensional topology, or as "simplicial cell complexes".

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Simplicial posets also attract a lot of interest in algebraic combinatorics and *combinatorial commutative algebra*. Stanley [17] introduced the *face ring* $\mathbb{Z}[\mathcal{S}]$ of a simplicial poset $\mathcal{S}$ as a quotient of certain graded polynomial ring by a homogeneous ideal determined by the poset relation in $\mathcal{S}$ (see Definition 2.1 below). The ring $\mathbb{Z}[\mathcal{S}]$ generalises the *Stanley–Reisner face ring* $\mathbb{Z}[\mathcal{K}]$ of a simplicial complex $\mathcal{K}$. The rings $\mathbb{Z}[\mathcal{S}]$ have remarkable algebraic and homological properties, albeit they are much more complicated than the Stanley–Reisner rings $\mathbb{Z}[\mathcal{K}]$. Unlike $\mathbb{Z}[\mathcal{K}]$, the ring $\mathbb{Z}[\mathcal{S}]$ is not generated in the lowest positive degree. (For topological reasons it is convenient to double the grading making it even; so that $\mathbb{Z}[\mathcal{K}]$ is generated by degree-two elements, but $\mathbb{Z}[\mathcal{S}]$ is not.) Face rings of simplicial posets were further studied by Duval [7] and Maeda–Masuda–Panov [12, 13], among others. *Cohen–Macaulay* and *Gorenstein** face rings are particularly important; both properties are topological, that is, depend only on the topological type of the geometric realisation $|\mathcal{S}|$. Gorenstein* simplicial posets also feature in toric topology, as combinatorial structures associated to orbit quotients of *torus manifolds* with special cohomological properties [13].

Here we suggest an approach to studying the face rings of simplicial posets by topological methods. We associate to $\mathcal{S}$ a certain space $\mathcal{Z}_{\mathcal{S}}$, called the *moment-angle complex*, which is glued from products of discs and circles (Definition 3.1). The original moment-angle complex was introduced by Buchstaber and Panov in [3] as a disc-circle decomposition of the Davis–Januszkiewicz universal space $\mathcal{Z}_{\mathcal{K}}$ associated to a simplicial complex $\mathcal{K}$ [6]; this decomposition was used in the calculation of the cohomology ring of $\mathcal{Z}_{\mathcal{K}}$ in terms of the face ring of $\mathcal{K}$ [3].

We therefore continue here the unifying theme of toric topology which links several aspects of equivariant topology to combinatorial commutative algebra. Motivated by the categorical constructions in toric topology [15] we describe the face ring $\mathbb{Z}[\mathcal{S}]$ as the (inverse) limit of a certain diagram of polynomial rings over the opposite face category $\mathrm{cat}^{\mathrm{op}}(\mathcal{S})$ of $\mathcal{S}$ (Lemma 2.5). This generalises the limit description [16, (4.7)] for the Stanley–Reisner face ring $\mathbb{Z}[\mathcal{K}]$ of a simplicial complex, and leads to an important functorial property (Proposition 2.6).

The face ring $\mathbb{Z}[\mathcal{S}]$ of a simplicial poset $\mathcal{S}$ with m vertices is naturally an algebra over the polynomial ring $\mathbb{Z}[v_1, \dots, v_m]$. We show that the corresponding Tor-*algebra* $\mathrm{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{S}], \mathbb{Z})$ is isomorphic to the integral cohomology ring of $\mathcal{Z}_{\mathcal{S}}$ with the appropriately refined grading (Theorem 3.5), therefore extending the result of [2] and [14] to simplicial posets. The Koszul complex calculating the Tor splits into the sum of subcomplexes corresponding to the *full subposets* $\mathcal{S}_{\boldsymbol{a}}$ of $\mathcal{S}$; the cohomology of these subcomplexes can be identified with the cellular cohomology of $|\mathcal{S}_{\boldsymbol{a}}|$ after a shift of dimension. This leads to a generalisation of Hochster's theorem calculating the algebraic Betti numbers of $\mathbb{Z}[\mathcal{S}]$ (Corollary 3.10).

Recently a lot of work has been done on generalising the construction of moment-angle complex $\mathcal{Z}_{\mathcal{K}} = \mathcal{Z}_{\mathcal{K}}(D^2, S^1)$ to pairs of spaces (X, W) different from (D^2, S^1), and studying the resulting spaces from the homotopy-theoretical perspective. See [11] and [1] for important advances in this direction. Among examples of these "generalised moment-angle complexes" we mention those corresponding to the pairs (D^1, S^0) (the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_{\mathcal{K}}$), $(\mathbb{C}, \mathbb{C}\setminus 0)$ and $(\mathbb{R}, \mathbb{R}\setminus 0)$ (the complex and real *coordinate subspace arrangement complements* respectively), and $(\mathbb{C}P^\infty, \mathrm{pt})$ (the so-called *Davis–Januszkiewicz space*, whose cohomology is the face ring $\mathbb{Z}[\mathcal{K}]$), see [2, Chapter 6].

Here we follow a different route: instead of replacing the pair (D^2, S^1) in $\mathcal{Z}_{\mathcal{K}} = \mathcal{Z}_{\mathcal{K}}(D^2, S^1)$ by a different pair, we extend the "indexing structure" from a simplicial complex $\mathcal{K}$ to a simplicial poset $\mathcal{S}$. One of the main reasons to keep the pair (D^2, S^1) intact is that the space $\mathcal{Z}_{\mathcal{S}} = \mathcal{Z}_{\mathcal{S}}(D^2, S^1)$ supports a T^m-action, like the original moment-angle complex $\mathcal{Z}_{\mathcal{K}}$. Moreover, if the dimension of $|\mathcal{S}|$ is $n-1$, then there is always an $(m-n)$-dimensional subtorus in T^m acting on $\mathcal{Z}_{\mathcal{S}}$ almost freely (Corollary 4.3). A choice of such subtorus is equivalent to a choice of a linear system of parameters in the $\mathbb{Q}$-face ring $\mathbb{Q}[\mathcal{S}]$ (Theorem 4.2). It has been shown recently by Cao–Lü [5] and Ustinovsky [18] that the total dimension of the rational cohomology of $\mathcal{Z}_{\mathcal{K}}$ is at least 2^{m-n}. Here we extend this result to $\mathcal{Z}_{\mathcal{S}}$, thereby settling Halperin's *toral rank conjecture* for moment-angle complexes corresponding to simplicial posets (Corollary 4.7).

We work over $\mathbb{Z}$ throughout most of the paper, as this is the most natural coefficient ring from topologist's point of view. All our statements are readily generalised to an arbitrary commutative associative ring with unit.

There is a clash of terminology between combinatorialists and homotopy theorists about using the term "simplicial". We do not use simplicial homotopy theory in this paper, so that our simplicial posets and simplicial cell complexes *do not mean* simplicial objects in the appropriate categories.

2. Simplicial posets and their face rings

A poset (partially ordered set) $\mathcal{S}$ with the order relation $\leqslant$ is called *simplicial* if it has an initial element $\hat{0}$ and for each $\sigma \in \mathcal{S}$ the lower segment $[\hat{0}, \sigma]$ is a boolean lattice (the face poset of a simplex). We assume all our posets to be finite and sometimes refer to elements $\sigma \in \mathcal{S}$ as *simplices*. The *rank function* $|\cdot|$ on $\mathcal{S}$ is defined by setting $|\sigma| = k$ for $\sigma \in \mathcal{S}$ if $[\hat{0}, \sigma]$ is the face poset of a $(k-1)$-dimensional simplex. The rank of $\mathcal{S}$ is the maximum of ranks of its elements, and the *dimension* of $\mathcal{S}$ is its rank minus one. The *vertices* of $\mathcal{S}$ are elements of rank one. We assume that $\mathcal{S}$ has m vertices and denote the vertex set by $V(\mathcal{S}) = [m] = \{1, \dots, m\}$. Similarly we denote by $V(\sigma)$ the vertex set of σ, that is the set of i with $i \leqslant \sigma$.

The face poset of a simplicial complex is a simplicial poset, but there are many simplicial posets that do not arise in this way. We identify a simplicial complex with its face poset, thereby regarding simplicial complexes as particular cases of simplicial posets.

To each $\sigma \in \mathcal{S}$ we assign a geometric simplex Δ^σ whose face poset is $[\hat{0}, \sigma]$, and glue these geometric simplices together according to the order relation in $\mathcal{S}$. We get a regular cell complex in which the closure of each cell is identified with a simplex preserving the face structure, and all attaching maps are inclusions. We call it a *simplicial cell complex* and denote its underlying space by $|\mathcal{S}|$.

Using a more formal categorical language, we consider the *face category* $\mathrm{cat}(\mathcal{S})$ whose objects are elements $\sigma \in \mathcal{S}$ and there is a morphism from σ to τ whenever $\sigma \leqslant \tau$. Then we may write

$$|\mathcal{S}| = \operatorname{colim} \Delta^{\mathcal{S}},$$

where $\Delta^{\mathcal{S}}$ is a diagram (covariant functor) from $\mathrm{cat}(\mathcal{S})$ sending every morphism $\sigma \leqslant \tau$ to the inclusion of geometric simplices $\Delta^\sigma \hookrightarrow \Delta^\tau$, and the colimit is taken in the category of (good) topological spaces.

For every simplicial poset $\mathcal{S}$ there is the associated simplicial complex $\mathcal{K}_{\mathcal{S}}$ on the same vertex set $V(\mathcal{S})$, whose simplices are the sets $V(\sigma)$, $\sigma \in \mathcal{S}$. There is a *folding map* of simplicial posets

$$\mathcal{S} \to \mathcal{K}_{\mathcal{S}}, \qquad \sigma \mapsto V(\sigma). \tag{1}$$

The corresponding geometric folding $|\mathcal{S}| \to |\mathcal{K}_{\mathcal{S}}|$ is a "branched combinatorial covering" in the sense of [4]; it is the identity on the vertices, and every simplex in $\mathcal{K}_{\mathcal{S}}$ is covered by a certain positive number of simplices of $\mathcal{S}$.

For any two simplices $\sigma, \tau \in \mathcal{S}$, denote by $\sigma \vee \tau$ the set of their least common upper bounds (*joins*), and by $\sigma \wedge \tau$ the set of their greatest common lower bounds (*meets*). Since $\mathcal{S}$ is a simplicial poset, $\sigma \wedge \tau$ consists of a single simplex whenever $\sigma \vee \tau$ is non-empty. It is easy to observe that $\mathcal{S}$ is a simplicial complex if and only if for any $\sigma, \tau \in \mathcal{S}$ the set $\sigma \vee \tau$ is either empty or consists of a single simplex [12, Proposition 5.1]. In this case $\mathcal{S}$ coincides with $\mathcal{K}_{\mathcal{S}}$.

Now consider the graded polynomial ring $\mathbb{Z}[v_\sigma : \sigma \in \mathcal{S}]$ with one generator v_σ of degree $\deg v_\sigma = 2|\sigma|$ for every $\sigma \in \mathcal{S}$.

Definition 2.1 ([17]).

The *face ring* of a simplicial poset $\mathcal{S}$ is the quotient

$$\mathbb{Z}[\mathcal{S}] = \mathbb{Z}[v_\sigma : \sigma \in \mathcal{S}] \,/\, \mathcal{I}_{\mathcal{S}},$$

where $\mathcal{I}_{\mathcal{S}}$ is the ideal generated by the elements $v_{\hat{0}} - 1$ and

$$v_\sigma v_\tau - v_{\sigma\wedge\tau} \cdot \sum_{\eta \in \sigma\vee\tau} v_\eta. \tag{2}$$

The sum over the empty set is assumed to be zero, so we have $v_\sigma v_\tau = 0$ in $\mathbb{Z}[\mathcal{S}]$ if $\sigma \vee \tau = \varnothing$.

The grading may be refined to a $\mathbb{Z}^m$-grading by setting $\operatorname{mdeg} v_\sigma = 2V(\sigma)$. Here $V(\sigma)$ is a subset of $[m]$, and we identify such subsets $\boldsymbol{a} \subset [m]$ with vectors in $\{0,1\}^m \subset \mathbb{Z}^m$ in the standard way: the unit coordinates of a vector correspond to the elements in a subset. In particular, $\operatorname{mdeg} v_i = 2\boldsymbol{e}_i$ (two times the ith basis vector).

Remark 2.2.

The definition above extends the notion of the face ring of a simplicial complex (also known as the *Stanley–Reisner ring*) to simplicial posets. In the case when $\mathcal{S}$ is a simplicial complex we may rewrite (2) as $v_\sigma v_\tau - v_{\sigma\wedge\tau}v_{\sigma\vee\tau}$ (because $\sigma\vee\tau$ is either empty or consists of a single simplex), and use the latter relation to express any v_σ as

$$v_\sigma = \prod_{i\in V(\sigma)} v_i .$$

The relations between the v_i coming from (2) can now be written as

$$v_{i_1}\cdots v_{i_k} = 0 \qquad \text{if } \{i_1,\dots,i_k\} \text{ does not span a simplex of } \mathcal{S}. \tag{3}$$

The face ring $\mathbb{Z}[\mathcal{S}]$ is therefore isomorphic to the quotient of the polynomial ring $\mathbb{Z}[v_1,\dots,v_m]$ by (3), where $\deg v_i = 2$. This is the standard way of describing the face ring of a simplicial complex.

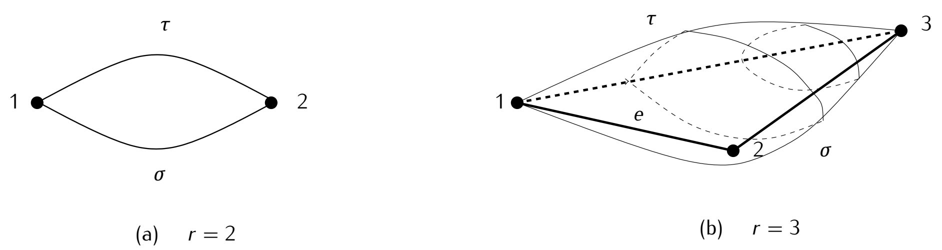

Figure 1. Simplicial cell complexes.

Example 2.3.

1. The simplicial cell complex shown in Figure 1 (a) is obtained by gluing two segments along their boundaries and has rank 2. The vertices are 1, 2 and we denote the 1-dimensional simplices by σ and τ. Then the face ring $\mathbb{Z}[\mathcal{S}]$ is the quotient of the graded polynomial ring

$$\mathbb{Z}[v_1,v_2,v_\sigma,v_\tau], \qquad \deg v_1 = \deg v_2 = 2, \quad \deg v_\sigma = \deg v_\tau = 4$$

by the two relations

$$v_1v_2 = v_\sigma + v_\tau, \qquad v_\sigma v_\tau = 0.$$

2. The simplicial cell complex in Figure 1 (b) is obtained by gluing two triangles along their boundaries and has rank 3. The vertices are 1, 2, 3 and we denote the 1-dimensional simplices (edges) by e, f and g, and the 2-dimensional simplices by σ and τ. The face ring $\mathbb{Z}[\mathcal{S}]$ is isomorphic to the quotient of the graded polynomial ring

$$\mathbb{Z}[v_1,v_2,v_3,v_\sigma,v_\tau], \qquad \deg v_1 = \deg v_2 = \deg v_3 = 2, \qquad \deg v_\sigma = \deg v_\tau = 6$$

by the two relations

$$v_1v_2v_3 = v_\sigma + v_\tau, \qquad v_\sigma v_\tau = 0.$$

The generators corresponding to the edges can be excluded because of the relations $v_e = v_1v_2$, etc.

The following lemma gives another perspective on the algebraic structure of the ring $\mathbb{Z}[\mathcal{S}]$.

Lemma 2.4 ([13, Lemma 5.4]).

Every element of $\mathbb{Z}[\mathcal{S}]$ can be uniquely written as a linear combination of monomials $v_{\tau_1}^{\alpha_1}v_{\tau_2}^{\alpha_2}\cdots v_{\tau_k}^{\alpha_k}$ corresponding to chains of totally ordered elements $\tau_1<\tau_2<\ldots<\tau_k$ of $\mathcal{S}\backslash\hat{0}$.

In other words, the monomials $v_{\tau_1}^{\alpha_1}v_{\tau_2}^{\alpha_2}\cdots v_{\tau_k}^{\alpha_k}$ with $\tau_1<\tau_2<\ldots<\tau_k$ constitute a basis for the graded free abelian group $\mathbb{Z}[\mathcal{S}]$. We refer to the expansion of an element $x\in\mathbb{Z}[\mathcal{S}]$ in terms of this basis as the *chain decomposition* of x. The proof of the above lemma uses the *straightening relation* (2) inductively, which allows one to express a product of two elements via products of elements in order. This can be restated by saying that $\mathbb{Z}[\mathcal{S}]$ is an example of an *algebra with straightening law* (see discussion in [17, p. 323]).

As was observed in [16, (4.7)], the face ring $\mathbb{Z}[\mathcal{K}]$ of a simplicial complex can be realised as the limit of a diagram of polynomial algebras over $\mathrm{cat}^{\mathrm{op}}(\mathcal{K})$. A similar description exists for the face ring $\mathbb{Z}[\mathcal{S}]$.

$\mathbb{Z}[\mathcal{S}]$ as a limit

We consider the diagram (covariant functor) $\mathbb{Z}[\cdot]_{\mathcal{S}}$ from the opposite face category $\mathrm{cat}^{\mathrm{op}}(\mathcal{S})$ to the category cgr of commutative associative graded rings with unit. Its value on $\sigma\in\mathcal{S}$ is the polynomial ring $\mathbb{Z}[\sigma]=\mathbb{Z}[v_i:i\in V(\sigma)]$, and its value on the morphism $\sigma\leqslant\tau$ is the surjection $\mathbb{Z}[\tau]\to\mathbb{Z}[\sigma]$ sending each v_i with $i\notin V(\sigma)$ to zero.

Lemma 2.5.

We have

$$\mathbb{Z}[\mathcal{S}]=\lim\mathbb{Z}[\cdot]_{\mathcal{S}},$$

where the (inverse) limit is taken in the category cgr.

Proof. We enumerate the elements of $\mathcal{S}$ so that the rank function does not decrease, and proceed by induction. We therefore may assume the statement is proved for a simplicial poset $\mathcal{T}$, and need to prove it for $\mathcal{S}$ which is obtained from $\mathcal{T}$ by adding one element σ. Note that $\mathcal{S}_{<\sigma}=\{\tau\in\mathcal{S}:\tau<\sigma\}$ is the face poset of the boundary of the simplex Δ^σ. Geometrically, we may think of $|\mathcal{S}|$ as obtained from $|\mathcal{T}|$ by attaching one simplex Δ^σ along its boundary (if $|\sigma|=1$, then Δ^σ is a single point, so $|\mathcal{S}|$ is a disjoint union of $|\mathcal{T}|$ and a point). We therefore need to prove that the following is a pullback diagram:

$$\begin{array}{ccc}\mathbb{Z}[\mathcal{S}] & \longrightarrow & \mathbb{Z}[\sigma]\;=\;\mathbb{Z}[\mathcal{S}_{\leqslant\sigma}]\\ \big\downarrow & & \big\downarrow\\ \mathbb{Z}[\mathcal{T}] & \longrightarrow & \mathbb{Z}[\mathcal{S}_{<\sigma}].\end{array}\tag{4}$$

Here the vertical arrows are obtained by mapping v_σ to 0, while the horizontal ones are obtained by mapping v_τ to 0 for $\tau\nleqslant\sigma$. Denote the pullback of (4) by A; we need to show that $\mathbb{Z}[\mathcal{S}]\to A$ is an isomorphism.

Since the limits in cgr are created in the underlying category cgg of graded abelian groups (graded $\mathbb{Z}$-modules), the underlying group of A is the direct sum of $\mathbb{Z}[\mathcal{T}]$ and $\mathbb{Z}[\sigma]$ with the pieces $\mathbb{Z}[\mathcal{S}_{<\sigma}]$ identified in both groups. In other words,

$$A=T\oplus\mathbb{Z}[\mathcal{S}_{<\sigma}]\oplus S,\tag{5}$$

where T is the complement to $\mathbb{Z}[\mathcal{S}_{<\sigma}]$ in $\mathbb{Z}[\mathcal{T}]$, and S is the complement to $\mathbb{Z}[\mathcal{S}_{<\sigma}]$ in $\mathbb{Z}[\sigma]$. By Lemma 2.4, the group $\mathbb{Z}[\mathcal{S}_{<\sigma}]$ has basis of monomials $v_{\tau_1}^{\alpha_1}v_{\tau_2}^{\alpha_2}\cdots v_{\tau_k}^{\alpha_k}$ with $\tau_k<\sigma$. Similarly, S has basis of those monomials with $\tau_k=\sigma$ and $\alpha_k>0$, while T has basis of those monomials with $\tau_k\nleqslant\sigma$ and $\alpha_k>0$. Yet another application of Lemma 2.4 gives a decomposition of $\mathbb{Z}[\mathcal{S}]$ identical to (5): a basis element $v_{\tau_1}^{\alpha_1}v_{\tau_2}^{\alpha_2}\cdots v_{\tau_k}^{\alpha_k}$ with $\alpha_k>0$ has either $\tau_k\nleqslant\sigma$, or $\tau_k<\sigma$, or $\tau_k=\sigma$. These three possibilities map to T, $\mathbb{Z}[\mathcal{S}_{<\sigma}]$ and S respectively. It follows that $\mathbb{Z}[\mathcal{S}]\to A$ is a group isomorphism. Since it is a ring map, it is also a ring isomorphism, thus finishing the proof. □

The description of $\mathbb{Z}[\mathcal{S}]$ as a limit has the following corollary, describing the functorial properties of the face ring.

Proposition 2.6.

Let $f\colon\mathcal{S}\to\mathcal{T}$ be a rank-preserving map of simplicial posets. Define a homomorphism

$$f^*\colon\mathbb{Z}[w_\tau:\tau\in\mathcal{T}]\;\to\;\mathbb{Z}[v_\sigma:\sigma\in\mathcal{S}]$$

by $f^*(w_\tau)=\sum_{\sigma\in f^{-1}(\tau)}v_\sigma$. Then f^* descends to a ring homomorphism $\mathbb{Z}[\mathcal{T}]\to\mathbb{Z}[\mathcal{S}]$, which we continue to denote by f^*.

Proof. The poset map f gives rise to a functor $f\colon \mathrm{cat}^{\mathrm{op}}(\mathcal{S}) \to \mathrm{cat}^{\mathrm{op}}(\mathcal{T})$ and therefore to

$$f^*\colon \left[\mathrm{cat}^{\mathrm{op}}(\mathcal{T}), \mathrm{cgr}\right] \to \left[\mathrm{cat}^{\mathrm{op}}(\mathcal{S}), \mathrm{cgr}\right],$$

where $[\mathrm{cat}^{\mathrm{op}}(\mathcal{S}), \mathrm{cgr}]$ denotes the functors from $\mathrm{cat}^{\mathrm{op}}(\mathcal{S})$ to cgr. It is easy to see that $f^*\mathbb{Z}[\cdot]_{\mathcal{T}} = \mathbb{Z}[\cdot]_{\mathcal{S}}$ (see Lemma 2.5), so we have the induced map of limits $f^*\colon \mathbb{Z}[\mathcal{T}] \to \mathbb{Z}[\mathcal{S}]$. We also have that $f^*(w_\tau) = \sum_{\sigma \in f^{-1}(\tau)} v_\sigma$ by the construction of lim in cgr. □

Example 2.7.
The folding map (1) induces a monomorphism $\mathbb{Z}[\mathcal{K}_{\mathcal{S}}] \to \mathbb{Z}[\mathcal{S}]$, which embeds $\mathbb{Z}[\mathcal{K}_{\mathcal{S}}]$ in $\mathbb{Z}[\mathcal{S}]$ as the subring generated by the elements v_i.

Remark 2.8.
The functoriality property for the face ring $\mathbb{Z}[\mathcal{K}]$ of a simplicial complex was observed in [2, Proposition 3.4]. However, an attempt to prove Proposition 2.6 directly from the definition, by showing that $f^*(\mathcal{I}_{\mathcal{T}}) \subset \mathcal{I}_{\mathcal{S}}$, runs into a complicated combinatorial analysis of the poset structure. This is an example of a situation where the use of an abstract categorical description of $\mathbb{Z}[\mathcal{S}]$ proves to be beneficial.

The lim-construction of $\mathbb{Z}[\mathcal{S}]$ also opens the way to further generalisations of the face ring, to more general posets and maybe to simplicial sets. Whether these rings would have a nice algebraic description like that of Definition 2.1 is questionable though.

3. Moment-angle complexes

Let D^2 denote the standard unit 2-disc and S^1 its boundary circle. We further consider the unit *polydisc* $\left(D^2\right)^m$ in the complex space $\mathbb{C}^m$:

$$\left(D^2\right)^m = \left\{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| \leqslant 1,\ i = 1, \dots, m\right\}.$$

For every $\sigma \in \mathcal{S}$, consider the following subset in $(D^2)^m$:

$$B_\sigma = \left\{(z_1, \dots, z_m) \in \left(D^2\right)^m : |z_j| = 1 \text{ if } j \nleqslant \sigma\right\}.$$

Then B_σ is homeomorphic to a product of $|\sigma|$ discs and $m - |\sigma|$ circles. We have an inclusion $B_\tau \subset B_\sigma$ whenever $\tau \leqslant \sigma$. It follows that the assignment $\sigma \mapsto B_\sigma$ defines a diagram from $\mathrm{cat}(\mathcal{S})$ to top, which we denote $\left(D^2, S^1\right)^{\mathcal{S}}$.

Definition 3.1.
The *moment-angle complex* corresponding to a simplicial poset $\mathcal{S}$ is

$$\mathcal{Z}_{\mathcal{S}} = \operatorname{colim}\left(D^2, S^1\right)^{\mathcal{S}}. \tag{6}$$

The space $\mathcal{Z}_{\mathcal{S}}$ is glued from the "moment-angle blocks" B_σ according to the poset relation in $\mathcal{S}$. When $\mathcal{S}$ is a simplicial complex $\mathcal{K}$ it becomes the standard moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ of [2, §6.2].

Remark 3.2.
The definition of $\mathcal{Z}_{\mathcal{S}}$ is readily generalised to an arbitrary pair of spaces (X, W) as $\mathcal{Z}_{\mathcal{S}}(X, W) = \operatorname{colim}(X, W)^{\mathcal{S}}$. An easy argument similar to [14, Proposition 3.5] shows that

$$H^*\left(\operatorname{colim}(\mathbb{C}P^\infty, \mathrm{pt})^{\mathcal{S}}; \mathbb{Z}\right) \cong \mathbb{Z}[\mathcal{S}].$$

Example 3.3.

Let $\mathcal{S}$ be the simplicial poset of Figure 1 (a). Then $\mathcal{Z}_{\mathcal{S}}$ is obtained by gluing two copies of $D^2 \times D^2$ along their boundary $S^3 = D^2\times S^1 \cup S^1\times D^2$. Therefore, $\mathcal{Z}_{\mathcal{S}} \cong S^4$. Here, $\mathcal{K}_{\mathcal{S}} = \Delta^1$ (a segment), and the moment-angle complex map induced by (1) folds S^4 onto D^4. Similarly, if $\mathcal{S}$ is of Figure 1 (b), then $\mathcal{Z}_{\mathcal{S}} \cong S^6$. Note that even-dimensional spheres do not appear as moment-angle complexes $\mathcal{Z}_{\mathcal{K}}$ for simplicial complexes $\mathcal{K}$.

The polydisc $(D^2)^m$ has the natural coordinatewise action of the m-torus T^m, with quotient the m-cube I^m. Since every inclusion $B_\tau \subset B_\sigma$ is T^m-equivariant, the moment-angle complex $\mathcal{Z}_{\mathcal{S}}$ acquires a T^m-action.

The *join* of simplicial posets $\mathcal{S}_1$ and $\mathcal{S}_2$ is the simplicial poset $\mathcal{S}_1{*}\mathcal{S}_2$ whose elements are pairs (σ_1, σ_2), with $(\sigma_1, \sigma_2) \leqslant (\tau_1, \tau_2)$ whenever $\sigma_1 \leqslant \tau_1$ in $\mathcal{S}_1$ and $\sigma_2 \leqslant \tau_2$ in $\mathcal{S}_2$.

The following properties of $\mathcal{Z}_{\mathcal{S}}$ are similar to those of $\mathcal{Z}_{\mathcal{K}}$ and can be proved in a very much similar fashion; see [2, Chapter 6].

Proposition 3.4.

(a) $\mathcal{Z}_{\mathcal{S}_1*\mathcal{S}_2} \cong \mathcal{Z}_{\mathcal{S}_1}\times\mathcal{Z}_{\mathcal{S}_2}$;

(b) *the quotient $\mathcal{Z}_{\mathcal{S}}/T^m$ is homeomorphic to the cone over $|\mathcal{S}|$;*

(c) *if $|\mathcal{S}| \cong S^{n-1}$, then $\mathcal{Z}_{\mathcal{S}}$ is a manifold of dimension $m+n$.*

An important series of examples of simplicial posets $\mathcal{S}$ with $|\mathcal{S}| \cong S^{n-1}$ comes from the inverse face posets of *face-acyclic manifolds with corners* in the sense of [13]. These manifolds with corners Q provide decompositions of an n-dimensional ball into faces, generalising those face decompositions coming from *simple n-polytopes* P. We therefore obtain *moment-angle manifolds* $\mathcal{Z}_Q$ generalising the manifolds $\mathcal{Z}_P$ corresponding to simple polytopes.

Construction (cell decomposition).

The disc D^2 decomposes in the standard way into three cells of dimensions 0, 1 and 2, which we denote $*$, T and D respectively. The polydisc $(D^2)^m$ then acquires the product cell decomposition, with each $B_\tau \subset B_\sigma$ being an inclusion of cellular subcomplexes for $\tau \leqslant \sigma$. We therefore obtain a cell decomposition of $\mathcal{Z}_{\mathcal{S}}$. Each cell in $\mathcal{Z}_{\mathcal{S}}$ is determined by an element $\sigma \in \mathcal{S}$ and a subset $\omega \in V(\mathcal{S})$ with $V(\sigma) \cap \omega = \varnothing$. Such a cell is a product of $|\sigma|$ cells of D-type, $|\omega|$ cells of T-type and the rest of $*$-type. We denote this cell by $\kappa(\omega, \sigma)$.

The resulting cellular cochain complex $C^*(\mathcal{Z}_{\mathcal{S}})$ has an additive basis consisting of cochains $\kappa(\omega, \sigma)^*$ dual to the corresponding cells. We introduce a $(\mathbb{Z}\oplus\mathbb{Z}^m)$-grading on the cochains by setting

$$\operatorname{mdeg} \kappa(\omega, \sigma)^* = \bigl(-|\omega|, 2V(\sigma) + 2\omega\bigr),$$

where we think of both $V(\sigma)$ and ω as vectors in $\{0,1\}^m \subset \mathbb{Z}^m$. The cellular differential does not change the $\mathbb{Z}^m$-part of the multigrading, so we obtain a decomposition

$$C^*(\mathcal{Z}_{\mathcal{S}}) = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^m} C^{*,2\boldsymbol{a}}(\mathcal{Z}_{\mathcal{K}})$$

into a sum of subcomplexes. In fact the only nontrivial subcomplexes are those for which $\boldsymbol{a}$ is in $\{0,1\}^m$. The cellular cohomology of $\mathcal{Z}_{\mathcal{S}}$ thereby acquires an additional grading, and we may define the *multigraded Betti numbers* $b^{-i,2\boldsymbol{a}}(\mathcal{Z}_{\mathcal{S}})$ by

$$b^{-i,2\boldsymbol{a}}(\mathcal{Z}_{\mathcal{S}}) = \operatorname{rank} H^{-i,2\boldsymbol{a}}(\mathcal{Z}_{\mathcal{S}}), \qquad i = 1,\dots,m, \quad \boldsymbol{a} \in \mathbb{Z}^m.$$

For the ordinary Betti numbers we have $b^k(\mathcal{Z}_{\mathcal{S}}) = \sum_{2|\boldsymbol{a}|-i=k} b^{-i,2\boldsymbol{a}}(\mathcal{Z}_{\mathcal{S}})$.

The face ring $\mathbb{Z}[\mathcal{S}]$ acquires a $\mathbb{Z}[v_1,\dots,v_m]$-algebra structure via the map $\mathbb{Z}[v_1,\dots,v_m]\to\mathbb{Z}[\mathcal{S}]$ sending each v_i identically. (Unlike the case of simplicial complexes, this map is generally not surjective.) The $(\mathbb{Z}\oplus\mathbb{Z}^m)$-graded Tor-algebra of $\mathbb{Z}[\mathcal{S}]$ is defined in the standard way [2, § 3.4]:

$$\operatorname{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{S}],\mathbb{Z})=\bigoplus_{i\geqslant 0,\boldsymbol{a}\in\mathbb{Z}^m}\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{S}],\mathbb{Z}).$$

Note that the first degree is always nonpositive, which is because we number the terms in a free resolution by nonpositive integers.

Theorem 3.5.

There is a graded ring isomorphism

$$H^*(\mathcal{Z}_{\mathcal{S}};\mathbb{Z})\cong\operatorname{Tor}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{S}],\mathbb{Z})$$

whose graded components are given by the group isomorphisms

$$H^p(\mathcal{Z}_{\mathcal{S}};\mathbb{Z})\cong\bigoplus_{-i+2|\boldsymbol{a}|=p}\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{S}],\mathbb{Z})\tag{7}$$

in each degree p. Here $|\boldsymbol{a}|=j_1+\dots+j_m$ for $\boldsymbol{a}=(j_1,\dots,j_m)$.

Using the Koszul resolution for the trivial $\mathbb{Z}[v_1,\dots,v_m]$-module $\mathbb{Z}$ (see [2, Lemma 3.29]) we restate the above theorem as follows:

Theorem 3.6.

There is a graded ring isomorphism

$$H^*(\mathcal{Z}_{\mathcal{S}};\mathbb{Z})\cong H\big[\Lambda[u_1,\dots,u_m]\otimes\mathbb{Z}[\mathcal{S}],d\big].$$

Here on the right hand side stands the cohomology of a differential $(\mathbb{Z}\oplus\mathbb{Z}^m)$-graded ring with

$$\operatorname{mdeg}u_i=(-1,2\boldsymbol{e}_i),\qquad \operatorname{mdeg}v_\sigma=(0,2V(\sigma)),\qquad du_i=v_i,\qquad dv_\sigma=0,$$

where $\boldsymbol{e}_i\in\mathbb{Z}^m$ is the i^{th} basis vector, for $i=1,\dots,m$.

Proof. The proof given here structurally resembles the proof of [14, Theorem 4.7] (for the case of $\mathcal{Z}_{\mathcal{K}}$). However, algebraic arguments used in the proof for $\mathcal{Z}_{\mathcal{K}}$ do not work in the case of simplicial posets. Instead, we use topological and categorical arguments at the appropriate places of this proof.

We consider the quotient differential graded ring

$$R^*(\mathcal{S})=\Lambda[u_1,\dots,u_m]\otimes\mathbb{Z}[\mathcal{S}]/\mathcal{I}_R,$$

where $\mathcal{I}_R$ is the ideal generated by the elements

$$u_iv_\sigma\quad\text{with}\quad i\in V(\sigma),\qquad\quad v_\sigma v_\tau\quad\text{with}\quad\sigma\wedge\tau\neq\hat{0}.$$

Note that the latter condition is equivalent to $V(\sigma)\cap V(\tau)\neq\varnothing$.

We claim that the quotient projection

$$\varrho\colon\ \Lambda[u_1,\dots,u_m]\otimes\mathbb{Z}[\mathcal{S}]\ \to\ R^*(\mathcal{S})$$

is a quasi-isomorphism, that is, it induces an isomorphism in cohomology.

Lemma 2.4 implies that $R^*(\mathcal{S})$ is generated, as an abelian group, by the monomials $u_\omega v_\sigma$, where $\omega \subseteq V(\mathcal{S})$, $\sigma \in \mathcal{S}$, $\omega \cap V(\sigma) = \varnothing$, and $u_\omega = u_{i_1} \dots u_{i_k}$ for $\omega = \{i_1, \dots, i_k\}$. In particular, $R^*(\mathcal{S})$ is a free abelian group of finite rank. It is now easy to observe that the map

$$g\colon R^*(\mathcal{S}) \to C^*(\mathcal{Z}_\mathcal{S}), \qquad u_\omega v_\sigma \mapsto \kappa(\omega, \sigma)^* \tag{8}$$

is an isomorphism of cochain complexes. Indeed, the additive bases of the two groups are in one-to-one correspondence, and the differential in $R^*(\mathcal{S})$ acts (in the case $|\omega| = 1$ and $i \notin V(\sigma)$) as

$$d(u_i v_\sigma) = v_i v_\sigma = \sum_{\eta \in i \vee \sigma} v_\eta.$$

This is exactly how the cellular differential in $C^*(\mathcal{Z}_\mathcal{S})$ acts on $\kappa(i, \sigma)^*$. The case of an arbitrary ω is treated similarly. It follows that we have an isomorphism of cohomology groups $H^j[R^*(\mathcal{S})] \cong H^j(\mathcal{Z}_\mathcal{S})$ for all j.

The differential ring $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{S}]$ also may be identified with the cellular cochains of a certain space. Namely, consider the space $\mathcal{Z}_\mathcal{S}\big(S^\infty, S^1\big)$ defined in the same way as (6), but with D^2 replaced by an infinite-dimensional sphere S^∞. The latter is a contractible space which has a cell decomposition with one cell in every dimension. The boundary of every $2k$-dimensional cell is the closure of the $(2k-1)$-cell, while the boundary of an odd-dimensional cell is zero. The cellular cochains of S^∞ can be identified with the Koszul differential ring

$$\Lambda[u] \otimes \mathbb{Z}[v], \qquad \deg u = 1, \quad \deg v = 2, \quad du = v, \quad dv = 0.$$

As in the case of (8), Lemma 2.4 implies that there is an isomorphism of cochain complexes

$$g'\colon \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{S}] \to C^*\big(\mathcal{Z}_\mathcal{S}\big(S^\infty, S^1\big)\big).$$

We also have a deformation retraction $D^2 \hookrightarrow S^\infty \to D^2$. It follows from the standard functoriality arguments that we also have a deformation retraction

$$\mathcal{Z}_\mathcal{S} = \operatorname{colim}\big(D^2, S^1\big)^\mathcal{S} \hookrightarrow \operatorname{colim}\big(S^\infty, S^1\big)^\mathcal{S} \to \operatorname{colim}\big(D^2, S^1\big)^\mathcal{S}$$

onto a cellular subcomplex. Therefore the cochain map $C^*\big(\mathcal{Z}_\mathcal{S}\big(S^\infty, S^1\big)\big) \to C^*(\mathcal{Z}_\mathcal{S})$ induced by the inclusion is a cohomology isomorphism.

Summarising the above observations we obtain the commutative square

$$\begin{array}{ccc} \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{S}] & \xrightarrow{g'} & C^*(\mathcal{Z}_\mathcal{S}(S^\infty, S^1)) \\ {\scriptstyle \varrho}\big\downarrow & & \big\downarrow \\ R^*(\mathcal{S}) & \xrightarrow{g} & C^*(\mathcal{Z}_\mathcal{S}) \end{array} \tag{9}$$

in which the horizontal arrows are isomorphisms of cochain complexes, and the right vertical arrow induces a cohomology isomorphism. It follows that the left arrow is a quasi-isomorphism, as claimed.

Remark 3.7.

There is an obvious inclusion of cochain complexes $\iota\colon R^*(\mathcal{S}) \to \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{S}]$, which is not a ring homomorphism though. It is possible to prove that ϱ is a cohomology isomorphism by constructing a cochain homotopy s between the maps id and $\iota \cdot \varrho$ from $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{S}]$ to itself. However, in the construction of s we cannot use an inductive argument as in [14, Lemma 4.4], and the general formula for s is rather cumbersome.

The additive isomorphism of (7) now follows from (9). To establish the ring isomorphism we need to analyse the multiplication of cellular cochains in $C^*(\mathcal{Z}_\mathcal{S})$.

We consider the diagonal approximation map $\widetilde{\Delta}\colon D^2 \to D^2 \times D^2$, defined in polar coordinates $z = \rho e^{i\varphi} \in D^2$, $0 \leqslant \rho \leqslant 1$, $0 \leqslant \varphi < 2\pi$, as follows:

$$\rho e^{i\varphi} \mapsto \begin{cases} \left(1+\rho(e^{2i\varphi}-1), 1\right) & \text{for } 0 \leqslant \varphi \leqslant \pi, \\ \left(1, 1+\rho(e^{2i\varphi}-1)\right) & \text{for } \pi \leqslant \varphi < 2\pi. \end{cases}$$

This is a cellular map homotopic to the diagonal $\Delta\colon D^2 \to D^2 \times D^2$. Taking an m-fold product, we obtain a cellular diagonal approximation

$$\widetilde{\Delta}\colon \left(D^2\right)^m \to \left(D^2\right)^m \times \left(D^2\right)^m.$$

It restricts to a map $B_\sigma \to B_\sigma \times B_\sigma$ for every $\sigma \in \mathcal{S}$ and gives rise to a map of diagrams

$$\left(D^2, S^1\right)^\mathcal{S} \to \left(D^2, S^1\right)^\mathcal{S} \times \left(D^2, S^1\right)^\mathcal{S}.$$

By definition, the colimit of the latter is $\mathcal{Z}_{\mathcal{S}*\mathcal{S}}$, which is identified with $\mathcal{Z}_\mathcal{S} \times \mathcal{Z}_\mathcal{S}$. We therefore obtain a cellular approximation $\widetilde{\Delta}\colon \mathcal{Z}_\mathcal{S} \to \mathcal{Z}_\mathcal{S} \times \mathcal{Z}_\mathcal{S}$ for the diagonal map of $\mathcal{Z}_\mathcal{S}$. It induces a ring structure on the cellular cochains via the composition

$$C^*(\mathcal{Z}_\mathcal{S}) \otimes C^*(\mathcal{Z}_\mathcal{S}) \xrightarrow{\ \times\ } C^*(\mathcal{Z}_\mathcal{S} \times \mathcal{Z}_\mathcal{S}) \xrightarrow{\ \widetilde{\Delta}^*\ } C^*(\mathcal{Z}_\mathcal{S}).$$

We claim that, with this multiplication in $C^*(\mathcal{Z}_\mathcal{S})$, the map (8) becomes a differential graded ring isomorphism. To see this we first observe that since (8) is a linear map, it is enough to consider the product of two generators $u_\omega v_\sigma$ and $u_\psi v_\tau$. If any two of the subsets ω, $V(\sigma)$, ψ and $V(\tau)$ have nonempty intersection, then $u_\omega v_\sigma \cdot u_\psi v_\tau = 0$. Otherwise (if all of the four subsets are complementary) we have

$$g(u_\omega v_\sigma \cdot u_\psi v_\tau) = g\left(u_{\omega \sqcup \psi} \cdot \sum_{\eta \in \sigma \vee \tau} v_\eta\right) = \sum_{\eta \in \sigma \vee \tau} \kappa(\omega \sqcup \psi, \eta)^*. \tag{10}$$

We also observe that

$$\widetilde{\Delta}\kappa(\chi, \eta) = \sum_{\substack{\omega \sqcup \psi = \chi \\ \sigma \vee \tau \ni \eta}} \kappa(\omega, \sigma) \times \kappa(\psi, \tau)$$

whenever $\chi \cap V(\eta) = \varnothing$. Therefore,

$$g(u_\omega v_\sigma) \cdot g(u_\psi v_\tau) = \kappa(\omega, \sigma)^* \cdot \kappa(\psi, \tau)^* = \widetilde{\Delta}^*\left(\kappa(\omega, \sigma) \times \kappa(\psi, \tau)\right)^* = \sum_{\eta \in \sigma \vee \tau} \kappa(\omega \sqcup \psi, \eta)^*.$$

Comparing this with (10) we deduce that (8) is a ring map, concluding the proof of Theorem 3.6. □

Remark 3.8.

Using the monoid structure on D^2 as in [14, Lemma 4.2] one easily sees that the construction of $\mathcal{Z}_\mathcal{S}$ is functorial with respect to maps of simplicial posets. This together with Proposition 2.6 makes the isomorphism of Theorem 3.5 functorial.

We have the following important corollary.

Corollary 3.9.

The groups $\mathrm{Tor}^{-i, 2\boldsymbol{a}}_{\mathbb{Z}[v_1, \dots, v_m]}(\mathbb{Z}[\mathcal{S}], \mathbb{Z})$ *vanish for* $\boldsymbol{a} \notin \{0,1\}^m$.

Proof. The multigraded component $R^{-i, 2\boldsymbol{a}}(\mathcal{S})$ is zero for $\boldsymbol{a} \notin \{0,1\}^m$. □

Denote by $\mathcal{S}_{\boldsymbol{a}}$ the subposet of $\mathcal{S}$ consisting of those σ for which $V(\sigma) \subset \boldsymbol{a}$. As a further corollary of Theorems 3.5 and 3.6 we obtain the following generalisation of Hochster's theorem to simplicial posets.

Corollary 3.10.

For every $\boldsymbol{a} \in \{0,1\}^m$ there is an isomorphism

$$\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{S}],\mathbb{Z}) \cong \widetilde{H}^{|\boldsymbol{a}|-i-1}\big(|\mathcal{S}_{\boldsymbol{a}}|\big);$$

here we follow the standard convention that $\widetilde{H}^{-1}(\varnothing) = \mathbb{Z}$.

Proof. The argument is identical to that of [14, Theorem 5.1]: there is an isomorphism of cellular cochain complexes

$$\widetilde{C}^*\big(|\mathcal{S}_{\boldsymbol{a}}|\big) \to C^{*+1-|\boldsymbol{a}|,2\boldsymbol{a}}(\mathcal{Z}_{\mathcal{K}}), \qquad \sigma^* \mapsto \kappa\big(\boldsymbol{a}\backslash V(\sigma),\, \sigma\big)^*,$$

inducing the required isomorphisms in cohomology. □

Remark 3.11.

The statement of Corollary 3.10 was obtained by Duval [7] (with field coefficients, and without considering the ring structure in Tor).

It is clear from Corollary 3.10 that the cohomology of $\mathcal{Z}_{\mathcal{S}}$ may contain an arbitrary amount of additive torsion; just take $|\mathcal{S}|$ to be a triangulation of a space with the appropriate torsion in cohomology.

The *multigraded algebraic Betti numbers* of $\mathbb{Z}[\mathcal{S}]$ are defined as

$$\beta^{-i,2\boldsymbol{a}}(\mathbb{Z}[\mathcal{S}]) = \operatorname{rk}\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbb{Z}[v_1,\dots,v_m]}(\mathbb{Z}[\mathcal{S}],\mathbb{Z}) = \operatorname{rk} H^{-i,2\boldsymbol{a}}(\mathcal{Z}_{\mathcal{S}})$$

for $i = 1,\dots,m$, $\boldsymbol{a} \in \mathbb{Z}^m$. We also set $\beta^{-i}(\mathbb{Z}[\mathcal{S}]) = \sum_{\boldsymbol{a}\in\mathbb{Z}^m} \beta^{-i,2\boldsymbol{a}}(\mathbb{Z}[\mathcal{S}])$.

Example 3.12.

Let us see how the isomorphism of Theorem 3.6 looks in the case of the simplicial poset $\mathcal{S}$ of Example 2.3.1. The elements 1, v_1, v_2, v_σ and v_τ of $R^{0,*}$ are all cocycles. Moreover, v_1, v_2 and $v_\sigma + v_\tau$ are coboundaries, the latter because $d(u_1v_2) = v_1v_2 = v_\sigma + v_\tau$. It therefore follows that $\beta^{0,(0,0)}(\mathbb{Z}[\mathcal{S}]) = \beta^{0,(2,2)}(\mathbb{Z}[\mathcal{S}]) = 1$, while $\beta^{0,(2,0)}(\mathbb{Z}[\mathcal{S}]) = \beta^{0,(0,2)}(\mathbb{Z}[\mathcal{S}]) = 0$. Also, a direct computation shows that $\beta^{-i,2\boldsymbol{a}}(\mathbb{Z}[\mathcal{S}]) = 0$ for $i > 0$. This implies that $\mathbb{Z}[\mathcal{S}]$ is a free $\mathbb{Z}[v_1,v_2]$-module with two generators, 1 and v_σ. The multigraded decomposition (7) in cohomology of $\mathcal{Z}_{\mathcal{S}} \cong S^4$ is as follows: $H^0(\mathcal{Z}_{\mathcal{S}}) = H^{0,(0,0)}(\mathcal{Z}_{\mathcal{S}}) \cong \mathbb{Z}$ and $H^4(\mathcal{Z}_{\mathcal{S}}) = H^{0,(2,2)}(\mathcal{Z}_{\mathcal{S}}) \cong \mathbb{Z}$.

The reader may compare this with similar computations of [14, Examples 4.8, 5.7] in the case of moment-angle complexes $\mathcal{Z}_{\mathcal{K}}$. Note that unlike the case of simplicial complexes, $\beta^0(\mathbb{Z}[\mathcal{S}])$ may be bigger than 1. In fact, Corollary 3.10 implies the following.

Proposition 3.13.

The number of generators of $\mathbb{Z}[\mathcal{S}]$ as a $\mathbb{Z}[v_1,\dots,v_m]$-module equals

$$\beta^0(\mathbb{Z}[\mathcal{S}]) = \sum_{\boldsymbol{a}\subset[m]} \operatorname{rk}\widetilde{H}^{|\boldsymbol{a}|-1}\big(|\mathcal{S}_{\boldsymbol{a}}|\big).$$

We finish this section by considering a poset $\mathcal{S}$ slightly more complicated than the toy examples we saw before, and calculating the cohomology of $\mathcal{Z}_{\mathcal{S}}$ accordingly.

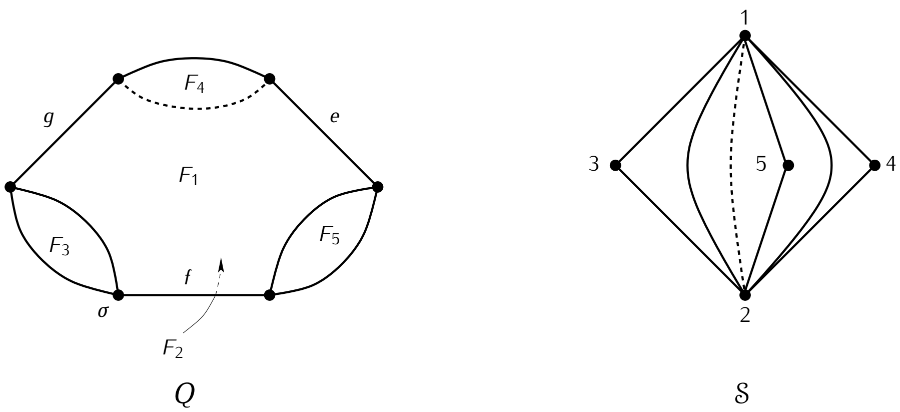

Figure 2. Manifold with corners Q and the dual poset $\mathcal{S}$.

Example 3.14.

Let Q be the 3-dimensional manifold with corners shown in Figure 2 (left). It is a 3-ball with $m = 5$ facets $F_1, \dots, F_5$ numbered as shown. We denote the edges e, f, g and the vertex σ of Q as shown. The corresponding moment-angle complex $\mathcal{Z}_Q$ is an 8-dimensional manifold.

The inverse face poset of Q is the simplicial poset $\mathcal{S}$ shown in Figure 2 (right). Note that the facets of Q correspond to the 5 vertices of $\mathcal{S}$, while σ corresponds to a certain 2-simplex of $\mathcal{S}$. The face ring $\mathbb{Z}[\mathcal{S}]$ is the quotient of the polynomial ring

$$\mathbb{Z}[\mathcal{S}] = \mathbb{Z}[v_1, \dots, v_5, v_e, v_f, v_g], \qquad \deg v_i = 2, \qquad \deg v_e = \deg v_f = \deg v_e = 4$$

by the relations

$$v_1 v_2 = v_e + v_f + v_g,$$
$$v_3 v_4 = v_3 v_5 = v_4 v_5 = v_3 v_e = v_4 v_f = v_5 v_g = v_e v_f = v_e v_g = v_e v_f = 0.$$

The other generators and relations in the original presentation can be derived from these; e.g., $v_\sigma = v_3 v_f$.

Given a vector $\boldsymbol{a} \in \{0,1\}^m$ regarded as a subset of $[m]$, set

$$Q_{\boldsymbol{a}} = \bigcup_{i \in \boldsymbol{a}} F_i \subset Q.$$

It is a subspace in the boundary of Q. Using the barycentric subdivision it is easy to see that $|\mathcal{S}_{\boldsymbol{a}}|$ is a deformation retract of $Q_{\boldsymbol{a}}$. Then Theorem 3.5 and Corollary 3.10 give the following formula for the multigraded cohomology of $\mathcal{Z}_Q$:

$$H^{-i,2\boldsymbol{a}}(\mathcal{Z}_Q) \cong \widetilde{H}^{|\boldsymbol{a}|-i-1}(Q_{\boldsymbol{a}}). \tag{11}$$

Using this formula we calculate the nontrivial cohomology groups of $\mathcal{Z}_Q$ as follows:

$$\begin{aligned}
H^{0,(0,0,0,0,0)}(\mathcal{Z}_Q) &= \widetilde{H}^{-1}(\varnothing) = \mathbb{Z} && 1\\
H^{-1,(0,0,2,2,0)}(\mathcal{Z}_Q) &= \widetilde{H}^{0}(F_3 \cup F_4) = \mathbb{Z} && u_3 v_4\\
H^{-1,(0,0,2,0,2)}(\mathcal{Z}_Q) &= \widetilde{H}^{0}(F_3 \cup F_5) = \mathbb{Z} && u_5 v_3\\
H^{-1,(0,0,0,2,2)}(\mathcal{Z}_Q) &= \widetilde{H}^{0}(F_4 \cup F_5) = \mathbb{Z} && u_4 v_5\\
H^{-2,(0,0,2,2,2)}(\mathcal{Z}_Q) &= \widetilde{H}^{0}(F_3 \cup F_4 \cup F_5) = \mathbb{Z} \oplus \mathbb{Z} && u_5 u_3 v_4,\ u_5 u_4 v_3\\
H^{0,(2,2,0,0,0)}(\mathcal{Z}_Q) &= \widetilde{H}^{1}(F_1 \cup F_2) = \mathbb{Z} \oplus \mathbb{Z} && v_e,\ v_f\\
H^{-1,(2,2,2,0,0)}(\mathcal{Z}_Q) &= \widetilde{H}^{1}(F_1 \cup F_2 \cup F_3) = \mathbb{Z} && u_3 v_e
\end{aligned}$$

$$H^{-1,(2,2,0,2,0)}(\mathcal{Z}_Q) = \widetilde{H}^1(F_1 \cup F_2 \cup F_4) = \mathbb{Z} \qquad u_4 v_f$$
$$H^{-1,(2,2,0,0,2)}(\mathcal{Z}_Q) = \widetilde{H}^1(F_1 \cup F_2 \cup F_5) = \mathbb{Z} \qquad u_5 v_g$$
$$H^{-2,(2,2,2,2,2)}(\mathcal{Z}_Q) = \widetilde{H}^2(F_1 \cup \ldots \cup F_5) = \mathbb{Z} \qquad u_5 u_4 v_3 v_f = u_5 u_4 v_\sigma.$$

It follows that the ordinary (1-graded) Betti numbers of $\mathcal{Z}_Q$ are given by the sequence $(1,0,0,3,4,3,0,0,1)$. In the right column of the table above we include the cocycles in the differential graded ring $\Lambda[u_1,\ldots,u_5]\otimes\mathbb{Z}[\mathcal{S}]$ representing generators of the corresponding cohomology group. This allows us to determine the ring structure in $H^*(\mathcal{Z}_\mathcal{S})$. For example,

$$[u_5u_3v_4]\cdot[v_f] = [u_5u_3v_4v_f] = 0 = [u_5u_4v_3]\cdot[v_e].$$

On the other hand,

$$[u_5u_3v_4]\cdot[v_e] = -[u_3u_5v_4v_e] = -[u_3u_4v_5v_e] = [u_3u_4v_5v_f] = [u_5u_4v_3v_f] = [u_5u_4v_3]\cdot[v_f].$$

Here we have used the relations $d(u_3u_4u_5v_e) = u_3u_4v_5v_e - u_3u_5v_4v_e$ and $d(u_1u_3u_4v_2v_5) = u_3u_4v_5v_e + u_3u_4v_5v_f$. In fact, all nontrivial products come from the Poincaré duality. These calculations may be summarised by the cohomology ring isomorphism

$$H^*(\mathcal{Z}_Q) \;\cong\; H^*\Big(\big(S^3\times S^5\big)^{\#3} \,\#\, \big(S^4\times S^4\big)^{\#2}\Big),$$

where the manifold on the right hand side is the connected sum of three copies of $S^3\times S^5$ and two copies of $S^4\times S^4$. We expect that this cohomology isomorphism is induced by a homeomorphism; one might be able to prove this by using the surgery techniques of [10].

4. Almost free torus actions

Halperin's *toral rank conjecture* states that if a torus T^k acts almost freely on a finite-dimensional space X, then the "total amount of homology" of X is at least that of the torus, that is,

$$\sum_i \operatorname{rk} H^i(X) \geqslant 2^k.$$

(An action is *almost free* if all isotropy subgroups are finite.) We refer to $\sum_i \operatorname{rk} H^i(X)$ as the *homology rank of* X and denote it $\operatorname{hrk} X$.

It has been shown in the recent works of Cao–Lü [5] and Ustinovsky [18] that the toral rank conjecture holds for the restricted torus action on the moment-angle complex $\mathcal{Z}_\mathcal{K}$. Here we show that the same holds for $\mathcal{Z}_\mathcal{S}$.

We define the *toral rank* $\operatorname{trk}\mathcal{Z}_\mathcal{S}$ as the maximal dimension of a subtorus $T^k \subset T^m$ acting almost freely on $\mathcal{Z}_\mathcal{S}$. Assume that $\dim\mathcal{S} = n-1$; then $\dim\mathcal{Z}_\mathcal{S} = m+n$. The isotropy subgroups of the T^m-action on $\mathcal{Z}_\mathcal{S}$ are coordinate subtori in T^m of the form

$$T^\sigma = \big\{(z_1,\ldots,z_m)\in T^m : z_i = 1 \text{ for } i\notin V(\sigma)\big\}, \tag{12}$$

where $\sigma\in\mathcal{S}$. The maximal dimension of these subgroups is n, hence $\operatorname{trk}\mathcal{Z}_\mathcal{S} \leqslant m-n$.

Let $\boldsymbol{t} = (t_1,\ldots,t_n)$ be a sequence of linear (degree-two) elements in $\mathbb{Z}[\mathcal{S}]$. We may write

$$t_i = \lambda_{i1}v_1 + \cdots + \lambda_{im}v_m, \qquad i = 1,\ldots,n. \tag{13}$$

Given $\sigma\in\mathcal{S}$, define the *restriction homomorphism*

$$s_\sigma\colon\ \mathbb{Z}[\mathcal{S}] \to \mathbb{Z}[\mathcal{S}]/(v_\tau : \tau \nleqslant \sigma).$$

Its image may be identified with the polynomial ring $\mathbb{Z}[\sigma]$ on $|\sigma|$ generators. Note that s_σ is induced by the inclusion of posets $\mathcal{S}_{\leqslant\sigma}\to\mathcal{S}$. Remember that $\boldsymbol{t}$ is called an *lsop* (*linear system of parameters*) in $\mathbb{Z}[\mathcal{S}]$ if it consists of algebraically independent elements and $\mathbb{Z}[\mathcal{S}]$ is a finitely generated $\mathbb{Z}[t_1,\ldots,t_n]$-module (equivalently, $\mathbb{Z}[\mathcal{S}]/(\boldsymbol{t})$ has finite rank as an abelian group).

Lemma 4.1.

A degree-two sequence $\boldsymbol{t} = (t_1, \dots, t_n)$ is an lsop in $\mathbb{Z}[\mathcal{S}]$ if and only if for every $\sigma \in \mathcal{S}$ the elements $s_\sigma(t_1), \dots, s_\sigma(t_n)$ generate the positive degree ideal $\mathbb{Z}[\sigma]_+$.

Proof. Assume (13) is an lsop. Every s_σ induces an epimorphism of the quotient rings:

$$\mathbb{Z}[\mathcal{S}]/(\boldsymbol{t}) \to \mathbb{Z}[\sigma]/s_\sigma(\boldsymbol{t}).$$

Since $\boldsymbol{t}$ is an lsop, $\mathbb{Z}[\mathcal{S}]/(\boldsymbol{t})$ has finite rank as a group. Therefore, $\mathbb{Z}[\sigma]/s_\sigma(\boldsymbol{t})$ is also of finite rank, which happens only if $s_\sigma(\boldsymbol{t})$ generates $\mathbb{Z}[\sigma]_+$.

The other direction is proved by considering the sum of the restrictions:

$$\mathbb{Z}[\mathcal{S}] \to \bigoplus_{\sigma \in \mathcal{S}} \mathbb{Z}[\sigma].$$

This is an injective $\mathbb{Z}[t_1, \dots, t_n]$-module map by [13, Lemma 5.6]. Since $\mathbb{Z}[t_1, \dots, t_n]$ is a Noetherian ring and $\bigoplus_{\sigma \in \mathcal{S}} \mathbb{Z}[\sigma]$ is finitely generated as a $\mathbb{Z}[t_1, \dots, t_n]$-module by assumption, its submodule $\mathbb{Z}[\mathcal{S}]$ is also finitely generated. This implies that $\boldsymbol{t}$ is an lsop. □

We organise the coefficients in (13) into an $(n \times m)$-matrix $\Lambda = (\lambda_{ij})$. For any $\sigma \in \mathcal{S}$ denote by Λ_σ the $(n \times |\sigma|)$-submatrix formed by the elements λ_{ij} with $j \in V(\sigma)$. The matrix Λ defines homomorphisms $\mathbb{Z}^m \to \mathbb{Z}^n$ and $\lambda\colon T^m \to T^n$. Let $T_\Lambda = \ker \lambda \subset T^m$.

Theorem 4.2.

The following conditions are equivalent:

(a) *the sequence (13) is an lsop in the rational face ring $\mathbb{Q}[\mathcal{S}]$;*

(b) *for every $\sigma \in \mathcal{S}$ the matrix Λ_σ has rank $|\sigma|$;*

(c) *T_Λ is the product of an $(m-n)$-torus and a finite group, and T_Λ acts almost freely on $\mathcal{Z}_\mathcal{S}$.*

Proof. The equivalence of (a) and (b) is the $\mathbb{Q}$-version of Lemma 4.1. Now, (b) holds if and only if $T_\Lambda \cap T^\sigma$ is a finite group for every $\sigma \in \mathcal{S}$, which means that T_Λ acts almost freely on $\mathcal{Z}_\mathcal{S}$ (see (12)). The fact that T_Λ contains an $(m-n)$-torus also follows from (b), because there is $\sigma \in \mathcal{S}$ with $|\sigma| = n$. □

Corollary 4.3.

If $\mathcal{S}$ is of rank n with m vertices, then $\operatorname{trk} \mathcal{Z}_\mathcal{S} = m - n$.

Proof. Consider the ring $\mathbb{Q}[\mathcal{K}_\mathcal{S}]$. Since it is generated by the degree-two elements, it has an lsop $\boldsymbol{t}$ (this is where we need the $\mathbb{Q}$-coefficients). Since $\mathbb{Q}[\mathcal{S}]$ is integral over $\mathbb{Q}[\mathcal{K}_\mathcal{S}]$ by [17, Lemma 3.9], $\boldsymbol{t}$ is also an lsop for $\mathbb{Q}[\mathcal{S}]$. By multiplying by a common denominator, we may assume that $\boldsymbol{t}$ is in $\mathbb{Z}[\mathcal{S}]$ (although it may fail to be an *integral* lsop). Then there is an $(m-n)$-subtorus acting almost freely on $\mathcal{Z}_\mathcal{S}$ by Theorem 4.2. □

There is also an integral version of Theorem 4.2, which is proved similarly:

Theorem 4.4.

The following conditions are equivalent:

(a) *the sequence (13) is an lsop in $\mathbb{Z}[\mathcal{S}]$;*

(b) *for every $\sigma \in \mathcal{S}$ the columns of Λ_σ form a part of a basis of $\mathbb{Z}^n$;*

(c) *T_Λ is an $(m-n)$-torus acting freely on $\mathcal{Z}_\mathcal{S}$.*

Remark 4.5.
Unlike the case of $\mathbb{Q}[\mathcal{S}]$, an lsop in $\mathbb{Z}[\mathcal{S}]$ may fail to exist, which means the there is no $(m-n)$-subtorus acting *freely* on the corresponding $\mathcal{Z}_\mathcal{S}$. The maximal dimension $s(\mathcal{S})$ of a subtorus $T^s \subset T^m$ acting freely on $\mathcal{Z}_\mathcal{S}$ is also known as the *Buchstaber invariant* of $\mathcal{S}$. It is a much more subtle characteristic than $\operatorname{trk} \mathcal{Z}_\mathcal{S}$ and is usually difficult to determine. For more information about the Buchstaber invariant for polytopes and simplicial complexes see [8] and [9].

Proposition 4.6.
We have that $\operatorname{hrk} \mathcal{Z}_\mathcal{S} \geqslant \operatorname{hrk} \mathcal{Z}_{\mathcal{K}_\mathcal{S}}$.

Proof. The folding map $|\mathcal{S}| \to |\mathcal{K}_\mathcal{S}|$ has an obvious section, which means that it is a retraction. It follows that $\operatorname{rk} \widetilde{H}^i(|\mathcal{S}|) \geqslant \operatorname{rk} \widetilde{H}^i(|\mathcal{K}_\mathcal{S}|)$. The same holds for every subposet $\mathcal{S}_\sigma$. Now the result follows from Theorem 3.5 and Corollary 3.10. □

Corollary 4.7.
The toral rank conjecture holds for the restricted torus action on $\mathcal{Z}_\mathcal{S}$, that is, $\operatorname{hrk} \mathcal{Z}_\mathcal{S} \geqslant 2^{\operatorname{trk} \mathcal{Z}_\mathcal{S}}$.

Proof. We have that $\operatorname{trk} \mathcal{Z}_\mathcal{S} = \operatorname{trk} \mathcal{Z}_{\mathcal{K}_\mathcal{S}} = m-n$ by Corollary 4.3, and $\operatorname{hrk} \mathcal{Z}_{\mathcal{K}_\mathcal{S}} \geqslant 2^{m-n}$ by [5, Corollary 1.4] or [18, § 3]. Therefore,

$$\operatorname{hrk} \mathcal{Z}_\mathcal{S} \geqslant \operatorname{hrk} \mathcal{Z}_{\mathcal{K}_\mathcal{S}} \geqslant 2^{m-n},$$

as claimed. □

Remark 4.8.
In fact, according to [18, Theorem 3.2], the sharper bound $\operatorname{hrk} \mathcal{Z}_\mathcal{S} \geqslant 2^{m-\operatorname{mrk} \mathcal{S}}$ holds, where $\operatorname{mrk} \mathcal{S}$ is the minimal rank of maximal elements in $\mathcal{S}$. It equals n (the rank of $\mathcal{S}$) if and only if $\mathcal{S}$ is pure, that is, all maximal elements of $\mathcal{S}$ have the same rank.

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