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Research Article

Moment Estimation Inequalities Based on g_λ Random Variable on Sugeno Measure Space

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The definitions and properties of moment of g_λ random variable are provided on Sugeno measure space. Then some important moment estimation inequalities based on g_λ random variable are presented and proven.

1. Introduction

In 1974, the Japanese scholar Sugeno [1] presented a kind of typical nonadditive measure, Sugeno measure, which is an important generalization of probability measure [2–6]. As we all know, the definitions and properties of moment of random variable play an important role in probability theory [7–9]. Likewise, they are also very important for Sugeno measure. In this paper we present the analogous definitions and properties based on g_λ random variable on Sugeno measure space. Then some important moment estimation inequalities based on g_λ random variable are presented and proven.

2. Preliminaries

Let us recall some definitions and facts from [5].

Definition 2.1. Let X be a nonempty set, let ζ be a nonempty class of subsets of X, and let μ be a nonnegative real valued set function defined on ζ. Therefore μ satisfies the σ-λ rule (on ζ)

if and only if there exists

$$\lambda \in \left(-\frac{1}{\sup \mu}, \infty\right) \cup \{0\} \tag{2.1}$$

such that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \begin{cases} \dfrac{1}{\lambda}\left\{\displaystyle\prod_{i=1}^{\infty}[1+\lambda \cdot \mu(E_i)] - 1\right\}, & \text{as } \lambda \neq 0, \\ \displaystyle\sum_{i=1}^{\infty}\mu(E_i), & \text{as } \lambda = 0, \end{cases} \tag{2.2}$$

for any disjoint sequence $\{E_i\}$ of sets in ζ whose union is also in ζ.

Definition 2.2. Let $\mathcal{F}$ be a σ-algebra of subsets of X. And μ is called Sugeno measure on $\mathcal{F}$ if and only if it satisfies the σ-λ rule and $\mu(X) = 1$. Usually, Sugeno measure on $\mathcal{F}$ is denoted by g_λ.

We call the triple $(X, \mathcal{F}, g_\lambda)$ a Sugeno measure space, denoted by g_λ space, where $\lambda \in (-1, \infty)$. In the following, our discussion will be restricted to this space.

Theorem 2.3. *For all $E, F \in \mathcal{F}, E \subset F$ imply that $g_\lambda(E) \leq g_\lambda(F)$ (monotonicity).*

Theorem 2.4. *Let g_λ be a Sugeno measure on $\mathcal{F}$. Then, for any $E \in \mathcal{F}$ and $F \in \mathcal{F}$,*

$$\begin{aligned} g_\lambda(E \cup F) &= \frac{g_\lambda(E) + g_\lambda(F) - g_\lambda(E \cap F) + \lambda g_\lambda(E) g_\lambda(F)}{1 + \lambda g_\lambda(E \cap F)}, \\ g_\lambda(E - F) &= \frac{g_\lambda(E) - g_\lambda(E \cap F)}{1 + \lambda g_\lambda(E \cap F)}, \\ g_\lambda(E^c) &= \frac{1 - g_\lambda(E)}{1 + \lambda g_\lambda(E)}. \end{aligned} \tag{2.3}$$

In order to present the analogous definitions and properties based on g_λ random variable on Sugeno measure space, we recall some definitions and facts from [10].

Definition 2.5. Let ξ be a function mapping from $(X, \mathcal{F}, g_\lambda)$ to real line $\mathbb{R}$. Then ξ is called a g_λ random variable.

Definition 2.6. Let ξ be a g_λ random variable. Then the distribution function of ξ is defined by

$$F_{g_\lambda}(x) = g_\lambda\{\xi \leq x\}, \quad \forall x \in \mathbb{R}. \tag{2.4}$$

Definition 2.7. Let $F_{g_\lambda}(x)$ be the distribution function of g_λ random variable ξ. Then ξ is called continuous g_λ random variable if there exists a nonnegative real valued function $f_{g_\lambda}(x)$ such that

$$F_{g_\lambda}(x) = \int_{-\infty}^{x} f_{g_\lambda}(t)dt, \quad \forall x \in \mathbb{R} \tag{2.5}$$

is valid. The function $f_{g_\lambda}(x)$ is called a density function of ξ.

In the following, our discussion will be restricted to the continuous g_λ random variable.

Definition 2.8. Let $F_{g_\lambda}(x)$ be the distribution function of g_λ random variable ξ. If $\int_{-\infty}^{\infty}|x|dF_{g_\lambda}(x) < \infty$, then we call $\int_{-\infty}^{\infty}xdF_{g_\lambda}(x)$ an expected value of g_λ random variable ξ, denoted by $E_{g_\lambda}(\xi)$.

Theorem 2.9. *Let ξ,η be g_λ random variables; let C and D be constants. Then*

$$E_{g_\lambda}(C\xi + D\eta) = CE_{g_\lambda}(\xi) + DE_{g_\lambda}(\eta). \tag{2.6}$$

Definition 2.10. Let ξ be a g_λ random variable. If $E_{g_\lambda}\{[\xi - E_{g_\lambda}(\xi)]^2\}$ exists, then $E_{g_\lambda}\{[\xi - E_{g_\lambda}(\xi)]^2\}$ is called the variance of ξ, denoted by $D_{g_\lambda}(\xi)$.

3. Moment Estimation Inequalities Based on g_λ Random Variable

We begin this section with a short lemma (see [11]), which will be useful in the sequel.

Lemma 3.1. *Let ξ be a g_λ random variable whose Sugeno density function f_{g_λ} exists. If the Lebesgue integral*

$$\int_{0}^{+\infty} g_\lambda\{\xi \geq r\}dr - \int_{-\infty}^{0} g_\lambda\{\xi \leq r\}dr + \lambda\int_{0}^{+\infty} g_\lambda\{\xi \geq r\} \cdot g_\lambda\{\xi \leq r\}dr \tag{3.1}$$

is finite, then

$$E_{g_\lambda}(\xi) = \int_{0}^{+\infty} g_\lambda\{\xi \geq r\}dr - \int_{-\infty}^{0} g_\lambda\{\xi \leq r\}dr + \lambda\int_{0}^{+\infty} g_\lambda\{\xi \geq r\} \cdot g_\lambda\{\xi \leq r\}dr. \tag{3.2}$$

Theorem 3.2. *Let ξ be a nonnegative g_λ random variable. When $\lambda \geq 0$, the inequality*

$$\sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\} \leq E_{g_\lambda}(\xi) \leq (1+\lambda)\left(1+\sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\}\right) \tag{3.3}$$

is valid; when $\lambda < 0$, the inequality

$$(1+\lambda)\sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\} \leq E_{g_\lambda}(\xi) \leq 1+\sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\} \tag{3.4}$$

holds true.

Proof. (I) When $\lambda \geq 0$, since $g_\lambda\{\xi \geq r\}$ is a monotone decreasing function of r, we have

$$\begin{aligned}
E_{g_\lambda}(\xi) &= \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{\xi \geq r\}\cdot g_\lambda\{\xi \leq r\}dr \\
&\geq \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr \\
&= \sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq r\}dr \\
&\geq \sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq i\}dr \\
&= \sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\}, \\
E_{g_\lambda}(\xi) &= \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{\xi \geq r\}\cdot g_\lambda\{\xi \leq r\}dr \\
&\leq \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{\xi \geq r\}dr \\
&= (1+\lambda)\int_0^{+\infty} g_\lambda\{\xi \geq r\}dr \\
&= (1+\lambda)\sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq r\}dr \\
&\leq (1+\lambda)\sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq i-1\}dr \\
&= (1+\lambda)\left(1+\sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\}\right).
\end{aligned} \tag{3.5}$$

(II) When $\lambda < 0$, owing to the monotonicity of $g_\lambda\{\xi \geq r\}$ we also have

$$\begin{aligned}
E_{g_\lambda}(\xi) &= \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{\xi \geq r\} \cdot g_\lambda\{\xi \leq r\}dr \\
&\geq \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{\xi \geq r\}dr \\
&= (1+\lambda)\int_0^{+\infty} g_\lambda\{\xi \geq r\}dr \\
&= (1+\lambda)\sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq r\}dr \\
&\geq (1+\lambda)\sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq i\}dr \\
&= (1+\lambda)\sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
E_{g_\lambda}(\xi) &= \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{\xi \geq r\} \cdot g_\lambda\{\xi \leq r\}dr \\
&\leq \int_0^{+\infty} g_\lambda\{\xi \geq r\}dr \\
&= \sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq r\}dr \\
&\leq \sum_{i=1}^{\infty}\int_{i-1}^{i} g_\lambda\{\xi \geq i-1\}dr \\
&= 1 + \sum_{i=1}^{\infty} g_\lambda\{\xi \geq i\}.
\end{aligned}$$

□

Definition 3.3. Let ξ be a g_λ random variable and k a positive number. Then (1) the expected value $E_{g_\lambda}(\xi^k)$ is called the kth moment, (2) the expected value $E_{g_\lambda}(|\xi|^k)$ is called the kth absolute moment, (3) the expected value $E_{g_\lambda}\{[\xi - E_{g_\lambda}(\xi)]^k\}$ is called the kth central moment, and (4) the expected value $E_{g_\lambda}\{[|\xi - E_{g_\lambda}(\xi)|]^k\}$ is called the kth absolute central moment.

Theorem 3.4. *Let ξ be a nonnegative g_λ random variable and k a positive number. Then*

$$E_{g_\lambda}\left(\xi^k\right) = k\int_0^{+\infty} r^{k-1} g_\lambda\{\xi \geq r\}dr + k\lambda\int_0^{+\infty} r^{k-1} g_\lambda\{\xi \geq r\} \cdot g_\lambda\{\xi \leq r\}dr. \tag{3.7}$$

Proof. From Lemma 3.1, we infer

$$
\begin{aligned}
E_{g_\lambda}\left(\xi^k\right) &= \int_0^{+\infty} g_\lambda\left\{\xi^k \ge x\right\}dx + \lambda\int_0^{+\infty} g_\lambda\left\{\xi^k \ge x\right\}\cdot g_\lambda\left\{\xi^k \le x\right\}dx \\
&= \int_0^{+\infty} g_\lambda\{\xi \ge r\}dr^k + \lambda\int_0^{+\infty} g_\lambda\{\xi \ge r\}\cdot g_\lambda\{\xi \le r\}dr^k \\
&= k\int_0^{+\infty} r^{k-1}g_\lambda\{\xi \ge r\}dr + k\lambda\int_0^{+\infty} r^{k-1}g_\lambda\{\xi \ge r\}\cdot g_\lambda\{\xi \le r\}dr.
\end{aligned} \tag{3.8}
$$

□

Similar to the case of credibility theory [12], we have the following: Theorems 3.5, 3.6, and 3.7.

Theorem 3.5. *Let ξ be a g_λ random variable that takes values in $[m, n]$ and has expected value $E_{g_\lambda}(\xi)$, and let $f(x)$ be a convex function on $[m, n]$. Then*

$$
E_{g_\lambda}[f(\xi)] \le \frac{n - E_{g_\lambda}(\xi)}{n-m}f(m) + \frac{E_{g_\lambda}(\xi) - m}{n-m}f(n). \tag{3.9}
$$

Theorem 3.6. *Let ξ be a g_λ random variable that takes values in $[m, n]$ and has expected value $E_{g_\lambda}(\xi)$. Then*

$$
D_{g_\lambda}(\xi) \le \left[E_{g_\lambda}(\xi) - m\right]\left[n - E_{g_\lambda}(\xi)\right]. \tag{3.10}
$$

Theorem 3.7. *Let ξ be a g_λ random variable that takes values in $[m, n]$ and has expected value μ. Then, for any positive integer k,*

$$
\begin{aligned}
E_{g_\lambda}\left(|\xi|^k\right) &\le \frac{n-\mu}{n-m}|m|^k + \frac{\mu - m}{n-m}|n|^k, \\
E_{g_\lambda}\left(|\xi - \mu|^k\right) &\le \frac{n-\mu}{n-m}|\mu - m|^k + \frac{\mu - m}{n-m}|n-\mu|^k.
\end{aligned} \tag{3.11}
$$

Theorem 3.8. *Let ξ be a g_λ random variable and $t > 0$. Then $E_{g_\lambda}(|\xi|^t) < \infty$ if and only if $\sum_{i=1}^{\infty} g_\lambda\{|\xi| > i^{1/t}\} < \infty$.*

Proof. From $g_\lambda\{|\xi|^t \ge i\} = g_\lambda\{|\xi| \ge i^{1/t}\}$ and Theorem 3.2, the conclusion is valid. □

Theorem 3.9. *Let ξ be a g_λ random variable and $t > 0$. If $E_{g_\lambda}(|\xi|^t) < \infty$, then $\lim_{x\to\infty} x^t g_\lambda\{|\xi| \ge x\} = 0$. Conversely, if there exists one positive number t such that $\lim_{x\to\infty} x^t g_\lambda\{|\xi| \ge x\} = 0$, then $E_{g_\lambda}(|\xi|^s) < \infty$ for any s, where $0 \le s < t$.*

Proof. (1) When $\lambda \geq 0$, we have

$$E_{g_\lambda}\left(|\xi|^t\right) = \int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr + \lambda\int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\} \cdot g_\lambda\left\{|\xi|^t \leq r\right\}dr$$
$$\geq \int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr. \tag{3.12}$$

Since $E_{g_\lambda}(|\xi|^t) < \infty$, we obtain $\int_0^{+\infty} g_\lambda\{|\xi|^t \geq r\}dr < \infty$. Consequently,

$$\lim_{x\to\infty}\int_{x^t/2}^{\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr = 0. \tag{3.13}$$

Since

$$\int_{x^t/2}^{\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr \geq \int_{x^t/2}^{x^t} g_\lambda\left\{|\xi|^t \geq r\right\}dr \geq \frac{1}{2}x^t g_\lambda\{|\xi| \geq x\}, \tag{3.14}$$

we have

$$\lim_{x\to\infty} x^t g_\lambda\{|\xi| \geq x\} = 0. \tag{3.15}$$

(2) When $\lambda < 0$, we have

$$E_{g_\lambda}\left(|\xi|^t\right) = \int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr + \lambda\int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\} \cdot g_\lambda\left\{|\xi|^t \leq r\right\}dr$$
$$\geq \int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr + \lambda\int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr \tag{3.16}$$
$$= (1+\lambda)\int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr.$$

Since

$$E_{g_\lambda}\left(|\xi|^t\right) < \infty, \tag{3.17}$$

we obtain

$$(1+\lambda)\int_0^{+\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr < \infty. \tag{3.18}$$

Consequently,

$$\lim_{x\to\infty}(1+\lambda)\int_{x^t/2}^{\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr = 0. \tag{3.19}$$

Since

$$(1+\lambda)\int_{x^t/2}^{\infty} g_\lambda\left\{|\xi|^t \geq r\right\}dr \geq (1+\lambda)\int_{x^t/2}^{x^t} g_\lambda\left\{|\xi|^t \geq r\right\}dr \geq \frac{1}{2}(1+\lambda)x^t g_\lambda\{|\xi| \geq x\}, \tag{3.20}$$

we have

$$\lim_{x\to\infty} x^t g_\lambda\{|\xi| \geq x\} = 0. \tag{3.21}$$

Conversely, if $\lim_{x\to\infty} x^t g_\lambda\{|\xi| \geq x\} = 0$, then there exists one number l such that $x^t g_\lambda\{|\xi| \geq x\} \leq 1$, for all $x \geq l$.

(3) When $\lambda \geq 0$, for any s, where $0 \leq s < t$, we have

$$\begin{aligned}
E_{g_\lambda}(|\xi|^s) &= \int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\}\cdot g_\lambda\{|\xi|^s \leq r\}dr \\
&\leq \int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\}dr + \lambda\int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\}dr \\
&= (1+\lambda)\int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\}dr \\
&= (1+\lambda)\left(\int_0^{l} g_\lambda\{|\xi|^s \geq r\}dr + \int_l^{+\infty} g_\lambda\{|\xi|^s \geq r\}dr\right) \\
&= (1+\lambda)\left(\int_0^{l} g_\lambda\{|\xi|^s \geq r\}dr + \int_l^{+\infty} sr^{s-1} g_\lambda\{|\xi| \geq r\}dr\right) \\
&\leq (1+\lambda)\left(\int_0^{l} g_\lambda\{|\xi|^s \geq r\}dr + s\int_l^{+\infty} r^{s-t-1}dr\right) \\
&\leq (1+\lambda)\left(\int_0^{l} g_\lambda\{|\xi|^s \geq r\}dr + s\int_0^{+\infty} r^{s-t-1}dr\right).
\end{aligned} \tag{3.22}$$

Since $\int_0^{+\infty} r^p dr < \infty$ for any $p < -1$, we have

$$E_{g_\lambda}(|\xi|^s) \leq (1+\lambda)\left(\int_0^{l} g_\lambda\{|\xi|^s \geq r\}dr + s\int_0^{+\infty} r^{s-t-1}dr\right) < \infty. \tag{3.23}$$

(4) When $\lambda < 0$, for any s, where $0 \leq s < t$, we have

$$\begin{aligned} E_{g_\lambda}\left(|\xi|^s\right) &= \int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\} dr + \lambda \int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\} \cdot g_\lambda\{|\xi|^s \leq r\} dr \\ &\leq \int_0^{+\infty} g_\lambda\{|\xi|^s \geq r\} dr \\ &= \int_0^l g_\lambda\{|\xi|^s \geq r\} dr + \int_l^{+\infty} g_\lambda\{|\xi|^s \geq r\} dr \\ &= \int_0^l g_\lambda\{|\xi|^s \geq r\} dr + \int_l^{+\infty} s r^{s-1} g_\lambda\{|\xi| \geq r\} dr \\ &\leq \int_0^l g_\lambda\{|\xi|^s \geq r\} dr + s\int_l^{+\infty} r^{s-t-1} dr \\ &\leq \int_0^l g_\lambda\{|\xi|^s \geq r\} dr + s\int_0^{+\infty} r^{s-t-1} dr. \end{aligned} \tag{3.24}$$

Since $\int_0^{+\infty} r^p dr < \infty$ for any $p < -1$, we have

$$E_{g_\lambda}\left(|\xi|^s\right) \leq \int_0^l g_\lambda\{|\xi|^s \geq r\} dr + s\int_0^{+\infty} r^{s-t-1} dr < \infty. \tag{3.25}$$

□

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