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Moment Generating Function of the Bivariate Generalized Exponential Distribution

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Abstract

Recently a new distribution, named a bivariate generalized exponential (BVGE) distribution has been introduced by Kundu and Gupta (2008). In this paper we obtain joint moments and the moment generating function for (BVGE) which is in closed form, and convenient to use in practice.

Keywords: bivariate generalized exponential distribution; joint and marginal moments; moment generating function; marginal moment generating function.

1. Introduction

The two parameter generalized exponential distribution was introduced by Gupta and Kundu (1999). Gupta and Kundu (2001a) observed that it can be used quite effectively to analyze positive life time data , particularly , in place of the two-parameter gamma or two-parameter Weibull distributions . Since the distribution function of the generalized exponential is in closed form, it can be used quit easily for analyzing censored data also. The frequentest and Bayesian inferences have been developed for the unknown parameters of the generalized exponential distribution. The readers are referred to the review article of Gupta and Kundu (2007) for a current account on generalized exponential distribution.

Although quite bit of work has been done in the recent years on generalized exponential distribution, but not much attempt has made to extend this to the multivariate case. Recently Kundu and Gupta (2008) define a bivariate generalized exponential distribution (BVGE) distribution as : The bivariate vector (X_1, X_2) has a bivariate generalized exponential distribution with the shape parameters α_1, α_2 and α_3, denoted by $BGE(\alpha_1, \alpha_2, \alpha_3)$. if (X_1, X_2) has the joint probability density function

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$$f(x_1,x_2)=\begin{cases} f_1(x_1,x_2) & \text{if} \quad 0<x_1<x_2<\infty \\ f_2(x_1,x_2) & \text{if} \quad 0<x_2<x_1<\infty \\ f_0(x) & \text{if} \quad 0<x_1=x_2=x. \end{cases} \qquad (1.1)$$

where

$$f_1(x_1,x_2)=\alpha_2(\alpha_1+\alpha_3)(1-e^{-x_1})^{\alpha_1+\alpha_3-1}(1-e^{-x_2})^{\alpha_2-1}e^{-x_1-x_2}$$
$$f_2(x_1,x_2)=\alpha_1(\alpha_2+\alpha_3)(1-e^{-x_2})^{\alpha_2+\alpha_3-1}(1-e^{-x_1})^{\alpha_1-1}e^{-x_1-x_2}$$
$$f_0(x)=\alpha_3(1-e^{-x})^{\alpha_1+\alpha_2+\alpha_3-1}e^{-x}$$

for $\alpha_1,\alpha_2,\alpha_3>0$ .Kundu and Gupta (2008) observed that the joint cumulative distribution function and the joint survival distribution function can be expressed in compact forms, they discussed several properties of this distribution, and used the EM algorithm to compute the maximum likelihood estimators of the unknown parameters and obtained the observed and expected Fisher information matrices. One data set has been re-analyzed and it is observed that the bivariate generalized exponential distribution provides a better fit than the bivariate exponential distribution.

The main aim of this paper is to provide joint and marginal moments of the bivariate generalized exponential distribution , and the joint moment generating function which is in closed form, and convenient to use in practice. The paper is organized as follows. The joint and marginal moments are provided in section 2. In section 3 we introduce the joint moment generating function.

2. Joint and Marginal Moments

In this section we drive the rth and sth joint moments of X_1 and X_2 , as well as the marginal moments .

Theorem2.1. The rth and sth joint moments of the bivariate generalized exponential distribution, denoted by $\mu'_{r,s}$ is given by

$$\mu'_{r,s}=E(X_1^r,X_2^s)=\alpha_2(\alpha_1+\alpha_3)\,r!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{k_{ij}}{(j+1)^{r+1}}\left[\frac{\Gamma(s+1)}{(i+1)^{s+1}}-\sum_{k=0}^{r}\frac{\Gamma(s+k+1)}{k!(i+2)^{s+k+1}}\right]$$
$$+\alpha_1(\alpha_2+\alpha_3)\,s!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{k'_{ij}}{(j+1)^{s+1}}\left[\frac{\Gamma(r+1)}{(i+1)^{r+1}}-\sum_{k=0}^{s}\frac{\Gamma(r+k+1)}{k!(i+2)^{r+k+1}}\right]$$
$$+\alpha_3\sum_{i=0}^{\infty}k_i\frac{\Gamma(r+s+1)}{(i+1)^{r+s+1}}. \qquad r,s=1,2,3,...$$

Where

$$k_{ij} = (-1)^{i+j}\binom{\alpha_2 - 1}{i}\binom{\alpha_1 + \alpha_3 - 1}{j}, \quad k'_{ij} = (-1)^{i+j}\binom{\alpha_1 - 1}{i}\binom{\alpha_2 + \alpha_3 - 1}{j},$$
$$k_i = (-1)^{i}\binom{\alpha_1 + \alpha_2 + \alpha_3 - 1}{i},$$

and

$$\Gamma(x) = \int_0^\infty u^{x-1}\, e^{-u} du, \qquad x > 0 \quad \text{is the complete gamma function.}$$

Proof. Starting with

$$E(X_1^r, X_2^s) = \int_0^\infty\int_0^\infty x_1^r x_2^s f(x_1, x_2) dx_1 dx_2 \qquad r, s = 1,2,3,...$$

and substituting for $f(x_1, x_2)$ from (1.1) ,we get

$$E(X_1^r, X_2^s) = \alpha_2(\alpha_1 + \alpha_3)\int_0^\infty\int_0^{x_2} x_1^r x_2^s (1 - e^{-x_1})^{\alpha_1+\alpha_3-1}(1 - e^{-x_2})^{\alpha_2-1} e^{-x_1-x_2} dx_1 dx_2$$
$$+ \alpha_1(\alpha_2 + \alpha_3)\int_0^\infty\int_0^{x_1} x_1^r x_2^s (1 - e^{-x_2})^{\alpha_2+\alpha_3-1}(1 - e^{-x_1})^{\alpha_1-1} e^{-x_1-x_2} dx_2 dx_1$$
$$+ \alpha_3\int_0^\infty x^{r+s}(1 - e^{-x})^{\alpha_1+\alpha_2+\alpha_3-1}\, e^{-x} dx$$

upon using the binomial series expansion

$$(1 - e^{-x})^{\alpha-1} = \sum_{i=1}^\infty (-1)^i \binom{\alpha - 1}{i} e^{-ix}$$

and the fact that

$$\gamma(c, z) = \int_0^z u^{c-1} e^{-u} du = \Gamma(c)[1 - e^{-z}\sum_{k=0}^{c-1}\frac{z^k}{k!}]; \qquad c \in N^* \qquad (2.1)$$

where $\gamma(c, z)$ is an incomplete gamma function,

The expression for $E(X_1^r, X_2^s)$ is derived.

Comment(2.1). From theorem 2.1, it easily to obtain different marginal moments of X_1 and X_2 in terms of infinite series by putting $r = 0$ and $s = 0$ respectively ,that is the rth marginal moment of X_1 will be

$$\mu'_{r0} = E(X_1^r) = \alpha_2(\alpha_1+\alpha_3)\,r!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{k_{ij}}{(j+1)^{r+1}}\,[\frac{1}{(i+1)}-\sum_{k=0}^{r}\frac{1}{(i+2)^{k+1}}]$$

$$+\alpha_1(\alpha_2+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{k'_{ij}}{(j+1)}\,[\frac{\Gamma(r+1)}{(i+1)^{r+1}}-\frac{\Gamma(r+1)}{(i+2)^{r+1}}]$$

$$+\alpha_3\sum_{i=0}^{\infty}k_i\,\frac{\Gamma(r+1)}{(i+1)^{r+1}}. \qquad r=1,2,3,...$$

Similarly, the sth marginal moment of X_2 is given by

$$\mu'_{0s} = E(X_2^s) = \alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{k_{ij}}{(j+1)}\,[\frac{\Gamma(s+1)}{(i+1)^{s+1}}-\frac{\Gamma(s+1)}{(i+2)^{s+1}}]$$

$$+\alpha_1(\alpha_2+\alpha_3)\,s!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{k'_{ij}}{(j+1)^{s+1}}\,[\frac{1}{(i+1)}-\sum_{k=0}^{s}\frac{1}{(i+2)^{k+1}}]$$

$$+\alpha_3\sum_{i=0}^{\infty}k_i\,\frac{\Gamma(s+1)}{(i+1)^{s+1}}. \qquad s=1,2,3,...$$

We observe that the infinite series is summable and it has only a finite number of terms if the shape parameters are integers.

3. Joint Moment Generating Function

The following theorem gives the joint moment generating function of (X_1, X_2).

Theorem 3.1 : The moment generating function for the BVGED is given by

$$\begin{aligned} M(t_1,t_2) &= \alpha_2 B(1-t_2,\alpha_1+\alpha_2+\alpha_3)\times \\ &\qquad {}_3F_2[1-t_2,\alpha_1+\alpha_3,t_1;\alpha_1+\alpha_3+1,\alpha_1+\alpha_2+\alpha_3+1-t_2;1] \\ &+\alpha_1 B(1-t_1,\alpha_1+\alpha_2+\alpha_3)\times \\ &\qquad {}_3F_2[1-t_1,\alpha_2+\alpha_3,t_2;\alpha_2+\alpha_3+1,\alpha_1+\alpha_2+\alpha_3+1-t_1;1] \\ &+\alpha_3 B(1-(t_1+t_2),\alpha_1+\alpha_2+\alpha_3), \qquad (3.1) \end{aligned}$$

where

$$B(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du \quad \text{, is a beta function,}$$

$${}_pF_q(b_1,...,b_p;c_1,...,c_q;u)=\sum_{i=0}^{\infty}\frac{(b_1)_i...(b_p)_i}{(c_1)_i...(c_q)_i}\frac{u^i}{i!}$$, is a hypergeometric function,

and $(b)_i = b(b+1)...(b+i-1)=\dfrac{\Gamma(b+i)}{\Gamma(b)}(b\neq 0, i=1,2,...),$, and p,q are nonnegative integers.

Proof: The joint moment generating function of (X_1, X_2) is given by

$$M(t_1,t_2)=E[e^{t_1x_1+t_2x_2}]=\int_0^{\infty}\int_0^{\infty}e^{t_1x_1+t_2x_2}f(x_1,x_2)\,dx_1dx_2$$

substituting for $f(x_1,x_2)$ from (1.1) we get

$$M(t_1,t_2)=\alpha_2(\alpha_1+\alpha_3)\int_0^{\infty}(1-e^{-x_2})^{\alpha_2-1}e^{-x_2(1-t_2)}\int_0^{x_2}(1-e^{-x_1})^{\alpha_1+\alpha_3-1}e^{-x_1(1-t_1)}dx_1dx_2$$

$$+\alpha_1(\alpha_2+\alpha_3)\int_0^{\infty}(1-e^{-x_1})^{\alpha_1-1}e^{-x_1(1-t_1)}\int_0^{x_1}(1-e^{-x_2})^{\alpha_2+\alpha_3-1}e^{-x_2(1-t_2)}dx_2dx_1$$

$$+\alpha_3\int_0^{\infty}e^{(t_1+t_2)x}\,(1-e^{-x})^{\alpha_1+\alpha_2+\alpha_3-1}dx. \qquad (3.2)$$

Upon using the fact that

$$B_x(\alpha,\beta)=\int_0^x u^{\alpha-1}(1-u)^{\beta-1}du=\frac{x^{\alpha}}{\alpha}\,{}_2F_1(\alpha,1-\beta;\alpha+1;x), \qquad (0\le x\le 1)$$

where $B_x(\alpha,\beta)$ is an incomplete beta function
and the identity [see Sarhan and Balakrishnan (2007)]

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\;{}_2F_1(c,d;\rho;u)du=B(\alpha,\beta)\;{}_3F_2(\alpha,c,d;\rho,\alpha+\beta;1)$$

for $\alpha,\beta>0$ and $d+\beta-\alpha-c>0,$
we can drive the expression for $M(t_1,t_2)$ given in (3.1) .

To show that $M(0,0)=1$,set $t_1=t_2=0$ in (3.1)
Since

$${}_3F_2[1,\alpha_i+\alpha_3,0;\alpha_i+\alpha_3+1,\alpha_1+\alpha_2+\alpha_3+1;1]=1 \quad , \quad i=1,2$$

then

$$M(0,0)=(\alpha_1+\alpha_2+\alpha_3)\cdot B[1,\alpha_1+\alpha_2+\alpha_3]=1.$$

Another form for the moment generating function can also be obtained in the form of a series which is finite or infinite depending on whether the shape parameters is integers or not . By using the binomial series expansion ,(3.2) can be written as

$$
\begin{aligned}
M(t_1,t_2) = {} & \alpha_2(\alpha_1+\alpha_3)\int_0^\infty \sum_{i=0}^\infty (-1)^i \binom{\alpha_2-1}{i} e^{-x_2(1-t_2+i)} \\
& \times \int_0^{x_2} \sum_{j=0}^\infty (-1)^j \binom{\alpha_1+\alpha_3-1}{j} e^{-x_1(1-t_1+j)} dx_1 dx_2 \\
& + \alpha_1(\alpha_2+\alpha_3)\int_0^\infty \sum_{i=0}^\infty (-1)^i \binom{\alpha_1-1}{i} e^{-x_1(1-t_1+i)} \\
& \times \int_0^{x_1} \sum_{j=0}^\infty (-1)^j \binom{\alpha_2+\alpha_3-1}{j} e^{-x_2(1-t_2+j)} dx_2 dx_1 \\
& + \alpha_3 \sum_{i=0}^\infty (-1)^i \binom{\alpha_1+\alpha_2+\alpha_3-1}{i} e^{-x[1-(t_1+t_2)]}\, dx
\end{aligned}
$$

Since the quantity inside the summation is absolutely integrable, interchanging the summation and integration, and by using (2.1) we have

$$
\begin{aligned}
M(t_1,t_2) = {} & \alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^\infty \sum_{j=0}^\infty k_{ij} \; \frac{1}{1-t_1+j} \; \left[\frac{1}{1-t_2+i} - \frac{1}{2-t_2+i}\right] \\
& + \alpha_1(\alpha_2+\alpha_3)\sum_{i=0}^\infty \sum_{j=0}^\infty k'_{ij} \; \frac{1}{1-t_2+j} \; \left[\frac{1}{1-t_1+i} - \frac{1}{2-t_1+i}\right] \\
& + \alpha_3 \sum_{i=0}^\infty k_i \; \frac{1}{1-t_1-t_2+i} \qquad (3.3) \\
& \qquad (t_1<1,\, t_2<1)
\end{aligned}
$$

where

$$
k_{ij} = (-1)^{i+j}\binom{\alpha_2-1}{i}\binom{\alpha_1+\alpha_3-1}{j}, \quad k'_{ij} = (-1)^{i+j}\binom{\alpha_1-1}{i}\binom{\alpha_2+\alpha_3-1}{j},
$$

and $k_i = (-1)^i\binom{\alpha_1+\alpha_2+\alpha_3-1}{i}$,

We observe that the infinite series is summable , differentiable and it has only a finite number of terms if the shape parameters are integers. Hence we can obtain different joint moments by differentiating and evaluating at $t_1 = t_2 = 0$.

Comment 3.1. From equation (3.3), it easy to obtain the marginal moment generating functions of X_1 and X_2 in terms of infinite series by putting $t_2 = 0$ and $t_1 = 0$ respectively, that is the marginal moment generating function of X_1 will be

$$M(t_1,0)=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k_{ij}\;\frac{1}{1-t_1+j}\;[\frac{1}{1+i}-\frac{1}{2+i}]$$
$$+\alpha_1(\alpha_2+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k'_{ij}\;\frac{1}{1+j}\;[\frac{1}{1-t_1+i}-\frac{1}{2-t_1+i}]$$
$$+\alpha_3\sum_{i=0}^{\infty}k_i\;\frac{1}{1-t_1+i}$$

Similarly, the marginal moment generating function of X_2 is given by

$$M(0,t_2)=\alpha_2(\alpha_1+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k_{ij}\;\frac{1}{1+j}\;[\frac{1}{1-t_2+i}-\frac{1}{2-t_2+i}]$$
$$+\alpha_1(\alpha_2+\alpha_3)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k'_{ij}\;\frac{1}{1-t_2+j}\;[\frac{1}{1+i}-\frac{1}{2+i}]$$
$$+\alpha_3\sum_{i=0}^{\infty}k_i\;\frac{1}{1-t_2+i}\;.$$

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