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MOMENT INEQUALITIES FOR THE MAXIMUM CUMULATIVE SUM[1]

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0. Summary. Assume $E(X_i) \equiv 0$. For $\nu \geqq 2$, bounds on the νth moment of $\max_{1 \leqq k \leqq n} |\sum_{a+1}^{a+k} X_i|$ are deduced from assumed bounds on the νth moment of $|\sum_{a+1}^{a+n} X_i|$. The inequality due to Rademacher–Mensov for $\nu = 2$ and orthogonal X_i's is generalized to $\nu \geqq 2$ and other types of dependent rv's. In the case $\nu > 2$, a second result is obtained which is considerably stronger than the first for asymptotic applications.

1. Introduction. Let $\{X_i\}_{-\infty}^{\infty}$ be a sequence of rv's having finite variances $\{\sigma_i^2\}$. Assume throughout (without loss of generality) that $E(X_i) \equiv 0$. For each vector $\mathbf{X}_{a,n} = (X_{a+1}, \cdots, X_{a+n})$ of n consecutive X_i's, let $F_{a,n}$ denote the joint df and let

$$S_{a,n} = \sum_{a+1}^{a+n} X_i \quad \text{and} \tag{1.1}$$

$$M_{a,n} = \max\{|S_{a,1}|, \cdots, |S_{a,n}|\}. \tag{1.2}$$

Thus $M_{a,n}$ is the largest magnitude for the n consecutive partial sums formed from the n consecutive X_i's commencing with X_{a+1}. The concern of this paper is to provide bounds on $E(M_{a,n}^\nu)$ in terms of given bounds on $E|S_{a,n}|^\nu$, where $\nu \geqq 2$.

It is not assumed that the X_i's are independent. The only restrictions on the dependence will be those, if any, imposed by the assumed bounds on $E|S_{a,n}|^\nu$. Such bounds may or may not reflect the presence of a dependence restriction. For example, in the case that $E|X_i|^\nu \leqq K$, all i, Minkowski's inequality implies that $E|S_{a,n}|^\nu = O(n^\nu)$ uniformly in a, as $n \to \infty$, regardless of the dependence of the X_i's. However, under a suitable dependence restriction (e.g., mutual independence, martingale differences, strong mixing, or the like), the quantity $O(n^\nu)$ may be replaced by $O(n^{\frac{1}{2}\nu})$, a stronger conclusion that need not hold without such a further assumption.

Bounds on $E(M_{a,n}^\nu)$ are of use in deriving convergence properties of $S_{a,n}$ as $n \to \infty$ (see [7] [8]), probability inequalities for $M_{a,n}$, and tightness criteria for certain sequences of random functions (see [1]). For development of such results under various dependence restrictions, the theorems of this paper reduce the problem of placing appropriate bounds on $E(M_{a,n}^\nu)$ to the typically easier problem of placing appropriate bounds on $E|S_{a,n}|^\nu$.

Two theorems are presented, whose comparison is as follows. In Theorem A the bounds may involve parameters of the joint df of $X_{a+1}, \cdots, X_{a+n}$, a flexibility particularly useful with non-identically distributed variables. In Theorem B the

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bounds are required to be functions of n alone, but are of lower order than those of Theorem A, an advantage for asymptotic applications. In Theorem A bounds on $E|S_{a,n}|^{\nu}$ are assumed for some $\nu \geqq 2$, while in Theorem B it is required that $\nu > 2$. In either case, however, bounds on $E(M_{a,n}^{r})$ may be concluded, by Hölder's inequality, for all values of $r \leqq$ the ν of the hypothesis.

2. Theorem A: generalization of the Rademacher-Mensov inequality. In the theory of sequences of orthogonal rv's (i.e., $E(X_i X_j) = 0$ if $i \neq j$), a basic lemma is

THEOREM. (Rademacher–Mensov). *If $X_{a+1}, \cdots, X_{a+n}$ are mutually orthogonal* rv'*s, then*

$$E(M_{a,n}^2) \leqq (\log_2 4n)^2 \textstyle\sum_{a+1}^{a+n} \sigma_i^2. \tag{2.1}$$

The result is given and used in Doob ([2] 156) and, more recently, in Révész ([5] 83). Concerning more general results, Billingsley ([1] 102) indicates how to prove

THEOREM. (Billingsley). *Let $\nu \geqq 1$, $\alpha \geqq 1$. Suppose that there exist nonnegative numbers $\{u_i\}$ such that*

$$E|S_{a,n}|^{\nu} \leqq (\textstyle\sum_{a+1}^{a+n} u_i)^{\alpha} \qquad (\textit{all } a, \textit{all } n \geqq 1). \tag{2.2}$$

Then

$$E(M_{a,n}^{\nu}) \leqq (\log_2 4n)^{\nu} (\textstyle\sum_{a+1}^{a+n} u_i)^{\alpha} \qquad (\textit{all } a, \textit{all } n \geqq 1). \tag{2.3}$$

In the case of mutually orthogonal X_i's, (2.2) holds with $u_i = \sigma_i^2$, $\nu = 2$, $\alpha = 1$ and (2.3) reduces to (2.1). More generally, if $E(S_{a,n}^2) = O(n^{\alpha})$ uniformly in a, then (2.2) holds with $\nu = 2$, $\alpha = 1$ and $u_i = A$, where A is a constant suitably large. Indeed, if $E|S_{a,n}|^{\nu} = O(n^{\alpha})$, where $\nu \geqq 1$, $\alpha \geqq 1$, then (2.2) holds for suitable choice of $u_i \equiv$ constant. However, if $1 \leqq \nu < 2$, but $\alpha \geqq 1$, then the condition (2.2) is weaker than in the case $\nu = 2$, $\alpha = 1$, and accordingly the conclusion (2.3) is weaker than in the case $\nu = 2$, $\alpha = 1$. Hence the case $1 \leqq \nu < 2$, $\alpha = 1$ is interesting only when ν is the highest value for which an assumption of form (2.2) is available. On the other hand, the case $\nu \geqq 2$, $\alpha < \frac{1}{2}\nu$ is somewhat restricted in applicability since, if condition (2.2) were met in this case, we would have $E(S_{a,n}^2) \leqq (\sum_{a+1}^{a+n} u_i)^{\delta}$ for a $\delta < 1$, a condition unrealistic in many applications.

Restricting attention, therefore, to situations in which (2.2) is assumed to hold for some $\nu \geqq 2$ and $\alpha \geqq \frac{1}{2}\nu$, the above theorem is a special case of Theorem A below, which permits the quantity $(\sum_{a+1}^{a+n} u_i)$ to be replaced by quantities of other types. The result is obtained by an approach somewhat different from those underlying the above theorems.

In the following, the function $g(F_{a,n})$ denotes a functional depending on the joint df of $X_{a+1}, \cdots, X_{a+n}$. Examples are $g(F_{a,n}) = n$ or $g(F_{a,n}) = \sum_{a+1}^{a+n} \sigma_i^2$. The value a_0 is arbitrary but fixed.

THEOREM A. *Let $\nu \geqq 2$. Suppose that there exists a function $g(F_{a,n})$ satisfying*

$$g(F_{a,k}) + g(F_{a+k,l}) \leqq g(F_{a,k+l}) \qquad (\textit{all } a \geqq a_0,\ 1 \leqq k < k+l), \tag{2.4}$$

such that

(2.5) $$E|S_{a,n}|^{\nu} \leqq g^{\frac{1}{2}\nu}(F_{a,n}) \qquad (\textit{all } a \geqq a_0, \textit{ all } n \geqq 1).$$

Then

(2.6) $$E(M_{a,n}^{\nu}) \leqq (\log_2 2n)^{\nu} g^{\frac{1}{2}\nu}(F_{a,n}) \qquad (\textit{all } a \geqq a_0, \textit{ all } n \geqq 1).$$

PROOF. Let $N > 1$ be given and put $m = [\frac{1}{2}(N+1)]$, where $[\cdot]$ denotes greatest integer part. Then $N = 2m$ or $2m-1$. Let $a \geqq a_0$.

Now, for $m < n \leqq N$, we have

$$S_{a,n}^2 = (S_{a,m} + S_{a+m,n-m})^2 = S_{a,m}^2 + S_{a+m,n-m}^2 + 2S_{a,m}S_{a+m,n-m},$$

so that, for $m < n \leqq N$,

(2.7) $$S_{a,n}^2 \leqq M_{a,m}^2 + M_{a+m,N-m}^2 + 2|S_{a,m}|M_{a+m,N-m}.$$

Also, for $1 \leqq n \leqq m$, we have $S_{a,n}^2 \leqq M_{a,m}^2$ and hence (2.7) for $1 \leqq n \leqq m$. Therefore,

(2.8) $$M_{a,N}^2 \leqq M_{a,m}^2 + M_{a+m,N-m}^2 + 2|S_{a,m}|M_{a+m,N-m}$$

and, by Minkowski's inequality,

(2.9) $$[E(M_{a,N}^{\nu})]^{2/\nu} \leqq [E(M_{a,m}^{\nu})]^{2/\nu} + [E(M_{a+m,N-m}^{\nu})]^{2/\nu} + 2[E(|S_{a,m}M_{a+m,N-m}|^{\nu/2})]^{2/\nu}.$$

Suppose now that the conclusion (2.6) of the theorem is true for $n < N$. Then, defining

(2.10) $$f(k) = (\log_2 2k)^2 \qquad (k \geqq 1),$$

we have, using (2.6) and Schwarz' inequality,

(2.11) $$\begin{aligned}[E(M_{a,N}^{\nu})]^{2/\nu} &\leqq f(m)g(F_{a,m}) + f(N-m)g(F_{a+m,N-m}) \\ &\quad + 2\{[E|S_{a,m}|^{\nu}]^{\frac{1}{2}}[E(M_{a+m,N-m}^{\nu})]^{\frac{1}{2}}\}^{2/\nu}. \\ &\leqq f(m)g(F_{a,m}) + f(N-m)g(F_{a+m,N-m}) \\ &\quad + 2(E|S_{a,m}|^{\nu})^{1/\nu} f^{\frac{1}{2}}(N-m)g^{\frac{1}{2}}(F_{a+m,N-m}).\end{aligned}$$

Then, by (2.4), (2.5), the inequality $2AB \leqq A^2 + B^2$, and the fact that $f(N-m) \leqq f(m)$, it follows that

(2.12) $$[E(M_{a,N}^{\nu})]^{2/\nu} \leqq [f(m) + f^{\frac{1}{2}}(m)]g(F_{a,N}).$$

Now note that

(2.13) $$f(2k) = [(\log_2 2k) + 1]^2 \geqq f(k) + 2f^{\frac{1}{2}}(k), \qquad k \geqq 1,$$

and, since $2^{\frac{1}{2}}(2k-1) \geqq 2k$ if $k \geqq 2$,

(2.14) $$f(2k-1) = [\log_2 2^{\frac{1}{2}}(2k-1) + \tfrac{1}{2}]^2 \geqq f(k) + f^{\frac{1}{2}}(k), \qquad k \geqq 2.$$

Hence, whether $N = 2m$ or $N = 2m-1$,

$$f(N) \geqq f(m)+f^{\frac{1}{2}}(m), \qquad N > 1, \tag{2.15}$$

so that (2.12) yields

$$E(M_{a,N}^{\nu}) \leqq (\log_2 2N)^{\nu} g^{\frac{1}{2}\nu}(F_{a,N}). \tag{2.16}$$

Therefore, since the conclusion of the theorem is true for $N = 1$ by the hypothesis (2.5), it follows by induction for all $N = 1, 2, \cdots$.

REMARKS. (i) The result (as well as its corollaries stated below) may be applied to obtain strong laws of large numbers, and convergence rates thereof, under restrictions merely on the moments of sums $S_{a,n}$. This treatment is given in [7]. (ii) Better *asymptotic* results are given by Theorem B, when ν may be taken > 2. Thus the corollaries below shall, for simplicity, confine attention to the case $\nu = 2$. (iii) As concerns the factor $(\log_2 2n)^{\nu}$ in (2.6), note that it is a slight improvement over the corresponding factor in (2.3). A slight further improvement could be obtained since $(\log_2 2n)^{\nu}$ may be replaced by $f^{\frac{1}{2}\nu}(n)$, for any $f(n)$ satisfying $f(1) \geqq 1$ and the inequalities in (2.13) and (2.14). No such $f(n)$, however, may be smaller than $O(\log_2{}^2 n)$, so any possible improvement by the above method of proof is trivial.

It is easily seen that the functional $g(F_{a,n}) = \sum_{a+1}^{a+n} \sigma_i^2$ satisfies condition (2.4). Moreover, in the case of orthogonal X_i's, this quantity is exactly $E(S_{a,n}^2)$. Hence we have

COROLLARY A1. *If* $\{X_i\}$ *is a sequence of mutually orthogonal* rv's, *then*

$$E(M_{a,n}^2) \leqq (\log_2 2n)^2 \textstyle\sum_{a+1}^{a+n} \sigma_i^2. \tag{2.17}$$

The flexibility of conditions (2.4) and (2.5) is illustrated by the following result.

COROLLARY A2. *Suppose that there exist nonnegative constants* $r_j (j = 0, 1, \cdots)$ *such that*

$$E(X_i X_{i+j}) \leqq r_j \qquad (\textit{all } i, \textit{all } j \geqq 0). \tag{2.18}$$

Then

$$E(M_{a,n}^2) \leqq (\log_2 2n)^2 n(r_0 + 2\textstyle\sum_1^{n-1} r_j). \tag{2.19}$$

PROOF. The functional $g(F_{a,n}) = n(r_0+2\sum_1^{n-1} r_j)$ trivially satisfies (2.4) and it is not difficult to see that

$$E(S_{a,n}^2) \leqq n(r_0+2\textstyle\sum_1^{n-1} r_j) \tag{2.20}$$

if (2.18) is satisfied.

It should be noted that the condition (2.20) is much broader than (2.2). Moreover, no dependence restriction whatsoever on the sequence $\{X_i\}$ is imposed by condition (2.20) or condition (2.18). Rather, (2.18) is in the nature of a stationarity restriction. In particular, we have

COROLLARY A2.1. *If* $\{X_i\}$ *is a* weakly stationary *sequence with* $|E(X_i X_{i+j})| = r_j (j = 0, 1, \cdots)$, *then* (2.19) *holds.*

In Corollary A1, the functional $g(F_{a,n})$ depends upon $F_{a,n}$ through specific parameters as well as through a and n. In Corollary A2.1, the dependence upon $F_{a,n}$ is less specific but involves more than n alone. In the following, we consider the implication (for $v = 2$) of Theorem A when the relevant functional is a function of n alone.

COROLLARY A3. *Suppose that*

$$E(S_{a,n}^2) \leqq g(n) \qquad (\textit{all } a \geqq a_0, \textit{ all } n \geqq 1), \tag{2.21}$$

where $g(k)+g(l) \leqq g(k+l)$ *for* $1 \leqq k < k+l$. *Then*

$$E(M_{a,n}^2) \leqq (\log_2 2n)^2 g(n) \qquad (\textit{all } a \geqq a_0, \textit{ all } n \geqq 1). \tag{2.22}$$

An important special case is

COROLLARY A3.1. *Suppose that*

$$E(S_{a,n}^2) \leqq An^\delta \qquad (\textit{all } a \geqq a_0, \textit{ all } n \geqq 1), \tag{2.23}$$

where $0 < A < \infty$ *and (without loss of generality)* $1 \leqq \delta \leqq 2$. *Then*

$$E(M_{a,n}^2) \leqq A(\log_2 2n)^2 n^\delta \qquad (\textit{all } a \geqq a_0, \textit{ all } n \geqq 1). \tag{2.24}$$

In particular, if (2.18) holds with $\sum_1^\infty r_j$ convergent, then (2.23) holds with $\delta = 1$ and $A = r_0 + 2\sum_1^\infty r_j$.

3. Theorem B: asymptotically optimal inequality. This result provides a bound for $E(M_{a,n}^v)$ which is asymptotically optimal (as $n \to \infty$) in the sense that it is of the same order of magnitude as the bound assumed for $E|S_{a,n}|^v$. Roughly speaking, the factor $(\log_2 2n)^v$ occurring in the bound given by Theorem A becomes eliminated. As a consequence, the scope of useful asymptotic applications becomes greatly enlarged. This gain over Theorem A is achieved at the expense of requiring that the bound assumed on $E|S_{a,n}|^v$ be for a value of $v > 2$ and be a function depending upon $F_{a,n}$ only through n. The crucial difference between the proofs of the two theorems is that Minkowski's inequality is exploited to obtain Theorem A but must be avoided in the following.

THEOREM B. *Let* $v > 2$. *Suppose that*

$$E|S_{a,n}|^v \leqq g^{\frac{1}{2}v}(n) \qquad (\text{all } a \geqq a_0, \text{ all } n \geqq 1), \tag{3.1}$$

where $g(n)$ *is nondecreasing*, $2g(n) \leqq g(2n)$, *and* $g(n)/g(n+1) \to 1$ *as* $n \to \infty$. *Then there exists a finite constant* K *(which may depend on* v, g *and the joint distributions of the* X_i*'s) such that*

$$E(M_{a,n}^v) \leqq K g^{\frac{1}{2}v}(n) \qquad (\textit{all } a \geqq a_0, \textit{ all } n \geqq 1). \tag{3.2}$$

PROOF. Define k to be $v-1$ if v is an integer and otherwise to be $[v]$. Let $\varepsilon = v-k$. Then $0 < \varepsilon \leqq 1$. It follows that the function

$$w(x) = \textstyle\sum_{j=0}^{k-1} \binom{k}{j} x^{-(j+\varepsilon)/v} + \sum_{j=1}^{k} \binom{k}{j} x^{-j/v} \tag{3.3}$$

tends to 0 as $x \to \infty$.

Since $\nu > 2$, δ may be chosen such that $2/\nu < \delta < 1$. Since $w(x)\downarrow 0$ as $x \to \infty$, $\exists\, x_0$ such that

$$x \geqq x_0 \Rightarrow w(x) \leqq 2^{\frac{1}{2}\nu\delta} - 2. \tag{3.4}$$

Also, since $g(n) \sim g(n+1)$, $\exists\, n_0$ such that

$$n \geqq n_0 \Rightarrow g(n) \leqq 2^{1-\delta} g(n-1). \tag{3.5}$$

Now, by the hypothesis of the theorem, the quantity

$$q_n = \sup_{a \geqq a_0} E(M_{a,n}^\nu)/g^{\frac{1}{2}\nu}(n) \tag{3.6}$$

is finite. Define

$$K = \max\{q_1, q_2, \cdots, q_{n_0}, x_0\}. \tag{3.7}$$

Thus, for K given by (3.7), the conclusion of the theorem holds for all $n \leqq n_0$. We shall now show that it holds for any $N > n_0$ if it is assumed true for all $n < N$. By induction, (3.2) will then hold for all $N = 1, 2, \cdots$.

Let $N \geqq n_0$ be given and put $m = [\frac{1}{2}(N+1)]$. Let $a \geqq a_0$. For $m < n \leqq N$, we have

$$\begin{aligned} |S_{a,n}|^\nu &\leqq (|S_{a,m}| + M_{a+m,N-m})^\nu \\ &\leqq |S_{a,m}|^\nu + M_{a+m,N-m}^\nu + \textstyle\sum_{j=0}^{k-1} \binom{k}{j} |S_{a,m}|^{j+\varepsilon} M_{a+m,N-m}^{k-j} \\ &\quad + \textstyle\sum_{j=1}^{k} \binom{k}{j} |S_{a,m}|^{j} M_{a+m,N-m}^{k-j+\varepsilon}. \end{aligned} \tag{3.8}$$

For $1 \leqq n \leqq m$, we have $|S_{a,n}|^\nu \leqq M_{a,m}^\nu$. It follows that

$$\begin{aligned} M_{a,N}^\nu \leqq M_{a,m}^\nu + M_{a+m,N-m}^\nu &+ \textstyle\sum_{j=0}^{k-1} \binom{k}{j} |S_{a,m}|^{j+\varepsilon} M_{a+m,N-m}^{k-j} \\ &+ \textstyle\sum_{j=1}^{k} \binom{k}{j} |S_{a,m}|^{j} M_{a+m,N-m}^{k-j+\varepsilon}. \end{aligned} \tag{3.9}$$

Now, by Hölder's inequality, for $r \geqq 0$, $s \geqq 0$ and $r+s > 0$,

$$E(|S_{a,m}|^r M_{a+m,N-m}^s) \leqq (E|S_{a,m}|^{r+s})^{r/(r+s)} (E M_{a+m,N-m}^{r+s})^{s/(r+s)}. \tag{3.10}$$

Suppose now that (3.2) holds for all $n < N$. Then, by (3.1) and (3.2), and by (3.10) with $r+s = \nu$, we have

$$\begin{aligned} E(|S_{a,m}|^r M_{a+m,N-m}^s) &\leqq K^{s/\nu} g^{\frac{1}{2}r}(m) g^{\frac{1}{2}s}(N-m) \\ &\leqq K^{s/\nu} g^{\frac{1}{2}\nu}(m) \end{aligned} \tag{3.11}$$

since $N-m \leqq m$ and g is nondecreasing. Application of (3.11) in each term on the right-hand side of (3.9) yields

$$E(M_{a,N}^\nu) \leqq K g^{\frac{1}{2}\nu}(m)[2 + w(K)]. \tag{3.12}$$

Since $K \geqq x_0$ and $2m \geqq n_0$, (3.4) and (3.5) and the assumptions on $g(\cdot)$ imply

$$\begin{aligned} E(M_{a,N}^\nu) &\leqq K 2^{\frac{1}{2}\nu\delta} g^{\frac{1}{2}\nu}(m) = K 2^{\frac{1}{2}\nu(\delta-1)} [2g(m)]^{\frac{1}{2}\nu} \\ &\leqq K 2^{\frac{1}{2}\nu(\delta-1)} g^{\frac{1}{2}\nu}(2m) \\ &\leqq K g^{\frac{1}{2}\nu}(2m-1) \\ &\leqq K g^{\frac{1}{2}\nu}(N), \end{aligned} \tag{3.13}$$

i.e., (3.2) holds for $n = N$. This completes the proof.

The application of Theorem B to obtain a law of the iterated logarithm, and convergence rates thereof, is dealt with in [7].

COROLLARY B1. *Let* $\nu > 2$. *Suppose that*

$$E|S_{a,n}|^{\nu} \leqq Mn^{\frac{1}{2}\nu\delta} \qquad (\text{all } a \geqq a_0, \text{ all } n \geqq 1), \tag{3.14}$$

where $0 < M < \infty$ *and* (*without loss of generality*) $1 \leqq \delta \leqq 2$. *Then there exists a finite constant K such that*

$$E(M_{a,n}^{\nu}) \leqq Kn^{\frac{1}{2}\nu\delta} \qquad (\text{all } a \geqq a_0, \text{ all } n \geqq 1). \tag{3.15}$$

PROOF. Since $\delta \geqq 1$, the function $g(n) = n^{\delta}$ satisfies the hypothesis of Theorem B.

Condition (3.14) with $\delta < 2 < \nu$ is implied by various dependence restrictions quite different in nature. For example, let us confine attention to sequences $\{X_i\}$ satisfying

$$E|X_i|^{\nu} \leqq M_0 \qquad (\text{all } i), \tag{3.16}$$

for some $\nu > 2$ and $M_0 < \infty$, and

$$E(S_{a,n}^2) \sim An \quad \text{uniformly in} \quad a, n \to \infty, \tag{3.17}$$

for some positive finite constant A. Then, under certain additional restrictions, (3.14) may hold with $\delta < 2$. In particular, Ibragimov [4] shows that (3.14) holds with $\delta = 1$ if $\{X_i\}$ is strictly stationary and obeys a certain dependence restriction (I). (We shall not define (I) here but merely comment that it includes the cases of independent rv's, m-dependent rv's and Markov processes satisfying Doeblin's condition.) In [8], it is shown that if the ν in (3.16) is an even integer, then (3.14) holds with $\delta = 1$ in the case of X_i's which are *multiplicative of order* ν:

$$E(X_{i_1} \cdots X_{i_j}) = 0 \quad \text{if} \quad 1 \leqq j \leqq \nu \quad \text{and} \quad i_1 < \cdots < i_j. \tag{3.18}$$

This condition is stronger than mutual orthogonality (if $\nu > 2$) but includes the cases of independent rv's and sequences of martingale differences. In [6], it is shown that (3.14) holds with $\delta = 2(\nu-1)/\nu$ if the X_i's are uniformly bounded. Therefore, we may conclude the following corollaries.

COROLLARY B2. *Let* $\{X_i\}$ *be a strictly stationary sequence satisfying* (3.16) *for some* $\nu > 2$ *and* (3.17). *If dependence restriction* (I) *is satisfied, then* $\exists K < \infty$ *such that*

$$E(M_{a,n}^{\nu}) \leqq Kn^{\frac{1}{2}\nu} \qquad (\text{all } a, \text{ all } n \geqq 1). \tag{3.19}$$

COROLLARY B3. *Let* $\{X_i\}$ *satisfy* (3.16) *for an even integer* $\nu > 2$ *and suppose that* $\{X_i\}$ *is multiplicative of order* ν. *Then* $\exists K < \infty$ *such that* (3.19) *holds.*

COROLLARY B4. *Let* $\{X_i\}$ *be a bounded sequence*: $|X_i| < C$ (*all* i) *for some* $C < \infty$. *Suppose that* (3.17) *is satisfied. Then, for each* $\nu > 2$, $\exists K_{\nu} < \infty$ *such that*

$$E(M_{a,n}^{\nu}) \leqq K_{\nu} n^{\nu-1} \qquad (\text{all } a, \text{ all } n \geqq 1). \tag{3.20}$$

REMARK. It must be emphasized that only the assumptions in Theorem B need involve a moment of $|S_{a,n}|$ of order higher than 2. Under the appropriate assumptions, the *conclusion* of the theorem yields, by Hölder's inequality, bounds on the moments of $M_{a,n}$ of all orders $\leqq$ the ν assumed in (3.1). That is, conclusion (3.2) implies

$$E(M_{a,n}^{\alpha}) \leqq K^{\alpha/\nu} g^{\frac{1}{2}\alpha}(n) \qquad (\text{all } a \geqq a_0, \text{ all } n \geqq 1), \tag{3.21}$$

for any $\alpha \leqq \nu$.

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