Теорія Ймовір. та Матем. Статист.
Вип. 90, 2014

Theor. Probability and Math. Statist.
No. 90, 2015, Pages 175–181
http://dx.doi.org/10.1090/tpms/958
Article electronically published on August 10, 2015

MOMENT MEASURES OF MIXED EMPIRICAL RANDOM POINT PROCESSES AND MARKED POINT PROCESSES IN COMPACT METRIC SPACES. 2

UDC 519.21

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Abstract. This is a continuation of the paper by M. G. Semeĭko, *Moment measures of mixed empirical random point processes and marked point processes in compact metric spaces. I*, Theor. Probability and Math. Statist. **88** (2014), 161–174. Moment measures of mixed empirical marked random point processes are investigated by using the probability generating functions of random counting measures.

This is a continuation of the paper [1]. The numbering of equations in the current paper continues the numbering in [1].

7. Main definitions of the theory of random mixed empirical ordered marked point processes

Every trajectory E^* of a finite strictly simple ordered marked point process

$$\mathcal{D} = (\mathcal{E}^*, \mathfrak{X}^*, P^*)$$

in a bounded space $(Y = X \times K, \mathfrak{A}_Y = \mathfrak{A}_X \otimes \mathfrak{A}_K, \mathcal{B}_Y = \mathcal{B}_X \odot \mathcal{B}_K)$ is a thinned set in the Cartesian product $Y = X \times K$ [2]. If X is a compact metric space of states endowed with a measure ϑ, metric $\rho_X(x_i, x_j)$, and natural structures of measurable sets $\mathfrak{A}_X$ and bounded sets $\mathfrak{B}_X$ and if the space of marks K is an interval $[a, b] \subset R^1$, then every trajectory E^* of an ordered marked point process consists of a finite sequence of points: $E^* = (y_1, \dots, y_i, \dots, y_n) = ([x_1; k_1], \dots, [x_i; k_i], \dots, [x_n; k_n])$, where $y_i = [x_i; k_i]$, x_i is a state and k_i is its mark. The phase space $Y = X \times K$ can be endowed with the structure of a metric space by defining the distance between the points $y_i = [x_i; k_i]$ and $y_j = [x_j; k_j]$, $i \neq j$, by

$$\rho_Y([x_i; k_i], [x_j; k_j]) = \rho_X(x_i, x_j) + |k_i - k_j|.$$

Consider the following random procedure of constructing an ordered marked point process. Introduce the following three random variables: $x = x(\omega)$, $k = k(\omega)$, and $\nu = \nu(\omega)$ and assume that these random variables satisfy the following conditions.

7.1. The random variables $x(\omega)$, $k(\omega)$, and integer valued nonnegative random variable $\nu(\omega)$ are defined on the main probability space $(\Omega, \mathfrak{F}, \mathsf{P})$.

2010 *Mathematics Subject Classification.* Primary 60G55.
Key words and phrases. Mixed empirical point process, marked point process, probability generating function, moment measures.

7.2. The random variables $x(\omega)$, $k(\omega)$, and $\nu(\omega)$ assume values in the sample spaces $(X, \mathfrak{A}_X, P_x)$, $(K, \mathfrak{A}_K, P_k)$, and $(Z_+, \mathfrak{A}_{Z_+}, P_\nu)$, respectively, where

$$P_x(B_X) = \mathsf{P}\{\omega\colon x(\omega) \in B_X\} = \mu_1(B_X), \qquad B_X \in \mathfrak{A}_X,$$
$$P_k(B_K) = \mathsf{P}\{\omega\colon k(\omega) \in B_K\} = \mu_2(B_K), \qquad B_K \in \mathfrak{A}_K,$$

and

$$P_\nu(B_{Z_+}) = \mathsf{P}\{\omega\colon \nu(\omega) \in B_{Z_+}\}, \qquad B_{Z_+} \in \mathfrak{A}_{Z_+}.$$

7.3. The distribution $P_x(B_X)$ of the random variable $x = x(\omega)$ is absolutely continuous with respect to the measure ϑ in the measurable space $(X, \mathfrak{A}_X)$.

7.4. The distribution $P_k(B_K)$ of the random variable $k = k(\omega)$ is absolutely continuous with respect to the Lebesgue measure in the measurable space $(\mathbf{R}^1, \mathfrak{A}_{\mathbf{R}^1})$.

7.5. $x(\omega)$, $k(\omega)$, and $\nu(\omega)$ are jointly independent random variables.

Consider the product of probability measures $P_y = P_x \otimes P_k$ on the σ-algebra of Borel sets $\mathfrak{A}_Y = \mathfrak{A}_X \otimes \mathfrak{A}_K$ of the phase space $Y = X \times K$. Then $(Y, \mathfrak{A}_Y, P_y)$ can be viewed as a sample probability space for the two dimensional random variable $y(\omega) = [x(\omega); k(\omega)]$ that generates the probability measure

$$\begin{aligned} P_y(B_Y) &= P_y(B_X \times B_K) = \mathsf{P}\{\omega\colon [x(\omega); k(\omega)] \in B_X \times B_K\} \\ &= \mathsf{P}\{\omega\colon x(\omega) \in B_X, k(\omega) \in B_K\} = \mathsf{P}\{\omega; x(\omega) \in B_X\}\, \mathsf{P}\{\omega\colon k(\omega) \in B_K\} \\ &= P_x(B_X) P_k(B_K) = \mu_1(B_X)\mu_2(B_K) = \mu(B_X \times B_K) = \mu(B_Y), \end{aligned}$$

where $B_Y = B_X \times B_K \in \mathfrak{C}_Y = \mathfrak{A}_Y \cap \mathcal{B}_Y$.

Let G_1 and G_2 be two independent random experiments corresponding to the sample probability spaces $(X, \mathfrak{A}_X, P_x)$ and $(K, \mathfrak{A}_K, P_k)$. Then $G = (G_1, G_2)$ is a “compound” random experiment with the sample probability space $(Y, \mathfrak{A}_Y, P_y)$. A number $n \in Z_+$ is chosen at random and then every trajectory of the ordered marked point process

$$E^* = ([x_1; k_1], \dots, [x_i; k_i], \dots, [x_n; k_n])$$

of size n is formed as a result of n independent repetitions of the same “compound” random experiment $G = (G_1, G_2)$ that can be described as a simple random choice without repetition of a pair $y_i = [x_i; k_i]$, $i = 1, \dots, n$, from the phase space $Y = X \times K$: the state point x_i is chosen from the space X (experiment G_1), while its mark k_i is chosen from the space K (experiment G_2).

Thus a trajectory E^* can be viewed as a realization in the sample measurable space $(Y, \mathfrak{A}_Y)$ of the following sequence:

$$\begin{aligned} E^* = E^*(\omega) &= (y_1(\omega), \dots, y_i(\omega), \dots, y_{\nu(\omega)}(\omega)) \\ &= ([x_1(\omega); k_1(\omega)], \dots, [x_i(\omega); k_i(\omega)], \dots, [x_{\nu(\omega)}(\omega); k_{\nu(\omega)}(\omega)]) \end{aligned}$$

of a random size $\nu(\omega)$ of independent and identically distributed random elements (two dimensional random variables) defined in the main probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and distributed according to the probability measure P_y, where $y_i(\omega) = [x_i(\omega); k_i(\omega)]$.

It is clear that the random variable $\nu(\omega)$ admits the following representation:

$$\nu(\omega) = N^* = N^*(E^*, Y) = \operatorname{card}[E^* \cap Y] = \sum_{y \in E^*} I_Y(y),$$

where $N^*(E^*, Y)$ is the random variable determining the number of points in the set E^* of the space Y and where $I_Y(y)$ is the characteristic function of the space Y.

Definition 7.1. A random process $\mathcal{D} = (\mathcal{E}^*, \mathfrak{X}^*, P^*)$ is called a strictly simple mixed empirical ordered marked point process with independent marks in a bounded space $(Y, \mathfrak{A}_Y, \mathcal{B}_Y)$ [2, 3].

Given n, $N^* = n$, and an arbitrary bounded measurable set $B_Y = B_X \times B_K$, $B_X \in \mathfrak{E}_X$, $B_K \in \mathfrak{A}_K$, consider a random empirical counting measure of the ordered marked point process

$$N^*(B_Y) = N^*(E^*, B_Y) = \operatorname{card}[E^* \cap B_Y] = \sum_{i=1}^{n} I_{B_Y}([x_i; k_i]).$$

The counting measure $N^*(B_Y)$ has the binomial distribution $B(n, \mu(B_Y))$ with parametric measure $\mu(B_Y)$ [3]:

$$\begin{aligned} P^*\{N^*(B_Y) = k \mid N^* = n\} &= C_n^k \mu^k(B_Y)[1 - \mu(B_Y)]^{n-k} \\ &= C_n^k \mu_1^k(B_X)\mu_2^k(B_K)[1 - \mu_1(B_X)\mu_2(B_K)]^{n-k}, \\ k &= 0, 1, \dots, n. \end{aligned} \tag{34}$$

If $\left\{B_Y^j = B_X^j \times B_K^j : j = 1, \dots, s, s \geq 2, \bigcup_{j=1}^s B_Y^j = Y\right\}$ is an arbitrary finite sequence of disjoint bounded measurable sets of the phase space Y, $k_j \in Z_+$, $j = 1, \dots, s$, $k_1 + k_2 + \cdots + k_s = n$, then, given $N^* = n$, the joint conditional distribution of counting measures $\{N^*(B_Y^j), j = 1, \dots, s\}$ has the polynomial distribution [4]:

$$P^*\left\{N^*(B_Y^j) = k_j, j = 1, \dots, s \;\middle|\; N^* = n\right\} = \frac{n!}{k_1! \dots k_s!} \mu^{k_1}\left(B_Y^1\right) \dots \mu^{k_s}\left(B_Y^s\right), \tag{35}$$

where $\sum_{j=1}^s \mu(B_Y^j) = 1$.

8. Moment measures of a mixed empirical random ordered marked point process with independent marks

We construct the joint probability generating function of the counting measures

$$\left\{N^*(B_Y^j), j = 1, \dots, s\right\},$$

where $\bigcup_{j=1}^s B_Y^j = Y (B_Y^i \cap B_Y^r = \varnothing, 1 \leq i, r \leq s, i \neq r)$, by using the joint conditional distribution (35) and a similar evaluation presented in Section 4 of [1]:

$$\begin{aligned} \Pi_{N^*(B_Y^1), \dots, N^*(B_Y^s)}(z_1, \dots, z_s) &= \Pi_{N^*}(z_1\mu(B_Y^1) + \cdots + z_s\mu(B_Y^s)) \\ &= \Pi_{N^*}(z_1\mu_1(B_X^1)\mu_2(B_K^1) + \cdots + z_s\mu_1(B_X^s)\mu_2(B_K^s)). \end{aligned} \tag{36}$$

Introduce the following notation for the measures of an ordered marked point process $\mathcal{D}$:

1. $\nu_{\mathcal{D}}^{(h)}(B_Y^j) = M[\{N^*(B_Y^j)\}^h]$ is the moment measure of order h, $h = 1, 2, \dots$;
2. $\nu_{\mathcal{D}}^{(h)}(B_Y^{j_1} \times \cdots \times B_Y^{j_h}) = M[N^*(B_Y^{j_1}) \dots N^*(B_Y^{j_h})]$ is the mixed moment measure of order h, $1 \leq j_1 < \cdots < j_h \leq s$, $h = 1, \dots, s$);
3. $\alpha_{\mathcal{D}}^{(h)}(B_Y^j) = M[N^*(B_Y^j)(N^*(B_Y^j) - 1) \dots (N^*(B_Y^j) - h + 1)]$ is the factorial moment measure of order h;
4. $\nu_{\mathcal{D}}^{(2)}(B_Y^{j_1} \times B_Y^{j_2}) = M[N^*(B_Y^{j_1})N^*(B_Y^{j_2})]$ is the mixed moment measure of the second order, $1 \leq j_1, j_2 \leq s$, $j_1 \neq j_2$;
5. $D(N^*(B_Y^j)) = \nu_{\mathcal{D}}^{(2)}(B_Y^j) - \{\nu_{\mathcal{D}}^{(1)}(B_Y^j)\}^2$ is the variance of the counting measure $N^*(B_Y^j)$;
6. $\operatorname{cov}[N^*(B_Y^{j_1}), N^*(B_Y^{j_2})] = \nu_{\mathcal{D}}^{(2)}(B_Y^{j_1} \times B_Y^{j_2}) - \nu_{\mathcal{D}}^{(1)}(B_Y^{j_1})\nu_{\mathcal{D}}^{(1)}(B_Y^{j_2})$ is the covariance measure of dependence between the measures $N^*(B_Y^{j_1})$ and $N^*(B_Y^{j_2})$.

Reasoning as in Section 4 of [1], we obtain

$$\nu_{\mathcal{D}}^{(h)}\big(B_Y^{j_1}\times\cdots\times B_Y^{j_h}\big)=\mu\big(B_Y^{j_1}\big)\dots\mu\big(B_Y^{j_h}\big)\Pi_{N^*}^{(h)}(1), \tag{37}$$

$$\alpha_{\mathcal{D}}^{(h)}\big(B_Y^{j}\big)=\mu^h\big(B_Y^{j}\big)\Pi_{N^*}^{(h)}(1), \qquad \nu_{\mathcal{D}}^{(1)}\big(B_Y^{j}\big)=\mu\big(B_Y^{j}\big)\Pi'_{N^*}(1), \tag{38}$$

$$\nu_{\mathcal{D}}^{(2)}\big(B_Y^{j}\big)=\mu^2\big(B_Y^{j}\big)\Pi''_{N^*}(1)+\mu\big(B_Y^{j}\big)\Pi'_{N^*}(1), \tag{39}$$

$$\nu_{\mathcal{D}}^{(2)}\big(B_Y^{j_1}\times B_Y^{j_2}\big)=\mu\big(B_Y^{j_1}\big)\mu\big(B_Y^{j_2}\big)\Pi''_{N^*}(1), \tag{40}$$

$$D\big(N^*(B_Y^{j})\big)=\mu^2\big(B_Y^{j}\big)\big[\Pi''_{N^*}(1)-\{\Pi'_{N^*}(1)\}^2\big]+\mu\big(B_Y^{j}\big)\Pi'_{N^*}(1), \tag{41}$$

$$\operatorname{cov}\big[N^*\big(B_Y^{j_1}\big),N^*\big(B_Y^{j_2}\big)\big]=\mu\big(B_Y^{j_1}\big)\mu\big(B_Y^{j_2}\big)\left[\Pi''_{N^*}(1)-\{\Pi'_{N^*}(1)\}^2\right]. \tag{42}$$

9. Mixed empirical Poisson random ordered marked point process with independent marks

Theorem 9.1. *If*

1. *the size of a sample* $\nu=\nu(\omega)=N^*$ *is a random variable with the Poisson distribution with parameter* λ*;*
2. *the random variable* ν *is independent of the random variables*

$$\{[x_i(\omega);k_i(\omega)]\colon i=1,\dots,\nu(\omega)\};$$

3. $N^*(B_Y)=\sum_{i=1}^{\nu}I_{B_Y}([x_i;k_i])$ *is the random empirical counting measure of the ordered marked point process;*
4. $\big\{B_Y^j=B_X^j\times B_K^j\colon j=1,\dots,s,\, s\ge 2\big\}$ *is an arbitrary finite sequence of disjoint bounded measurable sets in the space* Y*:* $B_Y^iB_Y^j=\varnothing$ $i,j=1,\dots,s$, $i\ne j$,

then

1*. *the counting measure* $N^*(B_Y^j)$, $j=1,\dots,s$, *of the empirical ordered marked point process* $\mathcal{D}=(\mathcal{E}^*,\mathfrak{X}^*,P^*)$ *is distributed by the Poisson law with parametric measure* $\lambda\mu(B_Y^j)$*:*

$$P^*\big\{N^*\big(B_Y^j\big)=k_j\big\}=\frac{\big[\lambda\mu\big(B_Y^j\big)\big]^{k_j}}{k_j!}e^{-\lambda\mu(B_Y^j)},$$

where $\mu(B_Y^j)=P_y(B_Y^j)$, $k_j=0,1,2,\dots$;

2*. *the counting measures* $\{N^*(B_Y^j), j=1,\dots,s\}$ *are jointly independent random variables*:

$$P^*\big\{N^*\big(B_Y^j\big)=k_j, j=1,\dots,s\big\}=\prod_{j=1}^{s}P^*\big\{N^*\big(B_Y^j\big)=k_j\big\}.$$

Proof. The first statement is proved with the help of the full probability formula in view of the binomial distribution (34):

$$\begin{aligned}P^*\big\{N^*\big(B_Y^j\big)=k_j\big\}&=\sum_{n\ge k_j}P^*\big\{N^*\big(B_Y^j\big)=k_j\mid N^*=n\big\}P^*\{N^*=n\}\\&=\sum_{n-k_j\ge 0}\frac{n!}{k_j!\,(n-k_j)!}\mu^{k_j}\big(B_Y^j\big)\big[1-\mu\big(B_Y^j\big)\big]^{n-k_j}\frac{\lambda^{n-k_j}\lambda^{k_j}}{n!}e^{-\lambda}\\&=\frac{\big[\lambda\mu\big(B_Y^j\big)\big]^{k_j}}{k_j!}e^{-\lambda}\sum_{m\ge 0}\frac{\big[\lambda\big(1-\mu\big(B_Y^j\big)\big)\big]^{m}}{m!},\end{aligned}$$

where $m = n - k_j$. Since

$$\sum_{m\geq 0} \frac{[\lambda(1-\mu(B_Y^j))]^m}{m!} = e^{\lambda} e^{-\lambda\mu(B_Y^j)},$$

we obtain

$$P^*\{N^*(B_Y^j) = k_j\} = \frac{[\lambda\mu(B_Y^j)]^{k_j}}{k_j!} e^{-\lambda\mu(B_Y^j)}.$$

To prove the second statement about the joint independence of the random variables $\{N^*(B_Y^j), j = 1,\dots,s\}$ we use the full probability formula, polynomial law (35), and assume that $B_Y^{s+1} = \left(\bigcup_{j=1}^s B_Y^j\right)^c$, $k = \sum_{j=1}^s k_j$, $k_{s+1} = n - k$ for $n \geq k$. Then

(43)

$$\begin{aligned} &P^*\{N^*(B_Y^j) = k_j, j = 1,\dots,s\} \\ &\quad = \sum_{n\geq k} P^*\{N^*(B_Y^j) = k_j, j = 1,\dots,s+1 \mid N^* = n\} P^*\{N^* = n\} \\ &\quad = \sum_{n\geq k} \frac{n!}{k_1!\dots k_s!(n-k)!} \mu^{k_1}(B_Y^1)\dots\mu^{k_s}(B_Y^s)\mu^{n-k}(B_Y^{s+1}) \frac{\lambda^k \lambda^{n-k}}{n!} e^{-\lambda} \\ &\quad = \sum_{n-k\geq 0} \frac{n!}{k_1!\dots k_s!(n-k)!} \mu^{k_1}(B_Y^1)\dots\mu^{k_s}(B_Y^s)\mu^{n-k}(B_Y^{s+1}) \frac{\lambda^{k_1+\dots+k_s}\lambda^{n-k}}{n!} e^{-\lambda} \\ &\quad = \prod_{j=1}^s \frac{[\lambda\mu(B_Y^j)]^{k_j}}{k_j!} e^{-\lambda} \left[\sum_{m\geq 0} \frac{[\lambda\mu(B_Y^{s+1})]^m}{m!}\right] \qquad (m = n-k). \end{aligned}$$

Since

$$\sum_{m\geq 0} \frac{[\lambda\mu(B_Y^{s+1})]^m}{m!} = e^{\lambda\mu(B_Y^{s+1})}, \qquad \sum_{j=1}^{s+1} \mu(B_Y^j) = 1,$$

we conclude that

(44)

$$\begin{aligned} \exp\{-\lambda[(\cdot)]\} &= \exp\left\{-\lambda\left(\sum_{j=1}^{s+1}\mu(B_Y^j)\right)\right\} \exp\{\lambda\mu(B_Y^{s+1})\} \\ &= \exp\left\{-\lambda\sum_{j=1}^{s}\mu(B_Y^j)\right\}. \end{aligned}$$

Substituting (44) into (43) we get

$$\begin{aligned} P^*\{N^*(B_Y^j) = k_j, j = 1,\dots,s\} &= \prod_{j=1}^s \frac{[\lambda\mu(B_Y^j)]^{k_j}}{k_j!} e^{-\lambda\mu(B_Y^j)} \\ &= \prod_{j=1}^s P^*\{N^*(B_Y^j) = k_j\}. \end{aligned}$$

□

Definition 9.1. A random process $\mathcal{D} = (\mathcal{E}^*, \mathfrak{X}^*, P^*)$ satisfying the assumptions of Theorem 9.1 is called a strictly simple mixed empirical Poisson ordered marked point process with independent marks in a bounded space $(Y, \mathfrak{A}_Y, \mathcal{B}_Y)$ [2, 3].

Corollary 9.1. *If the random variable N^* is distributed by the homogeneous Poisson law with parameter λ, then, using the general results* (37)–(42), *we obtain the moment* *characteristics of the mixed empirical Poisson ordered marked point process $\mathcal{D}$ with independent marks*:

$$\nu^{(1)}_{\mathcal{D}}\big(B^j_Y\big) = \nu^{(1)}_{\mathcal{D}}\big(B^j_X \times B^j_K\big) = \lambda\mu\big(B^j_X \times B^j_K\big) = \lambda\mu_1\big(B^j_X\big)\mu_2\big(B^j_K\big) = \nu^{(1)}_{\widetilde{\mathcal{D}}}\big(B^j_X\big)\mu_2\big(B^j_K\big),$$

where $\nu^{(1)}_{\widetilde{\mathcal{D}}}(B^j_X) = \lambda\mu_1(B^j_X)$ *is the moment measure of the first order of the Poisson ordered point process* $\widetilde{\mathcal{D}}$ *of state points considered in Section* 5 *of* [1],

$$\nu^{(h)}_{\mathcal{D}}\big(B^{j_1}_Y \times \cdots \times B^{j_h}_Y\big) = \lambda^h\mu\big(B^{j_1}_Y\big)\dots\mu\big(B^{j_h}_Y\big) = \nu^{(1)}_{\mathcal{D}}\big(B^{j_1}_Y\big)\dots\nu^{(1)}_{\mathcal{D}}\big(B^{j_h}_Y\big),$$

$$\lambda^{(h)}_{\mathcal{D}}\big(B^j_Y\big) = \lambda^h\mu^h\big(B^j_Y\big) = \big[\nu^{(1)}_{\mathcal{D}}\big(B^j_Y\big)\big]^h, \qquad \nu^{(2)}_{\mathcal{D}}\big(B^j_Y\big) = \lambda^2\mu^2\big(B^j_Y\big) + \lambda\mu\big(B^j_Y\big),$$

$$\nu^{(2)}_{\mathcal{D}}\big(B^{j_1}_Y \times B^{j_2}_Y\big) = \lambda^2\mu\big(B^{j_1}_Y\big)\mu\big(B^{j_2}_Y\big),$$

$$D\big(N^*(B^j_Y)\big) = \lambda\mu\big(B^j_Y\big), \qquad \operatorname{cov}\big[N^*\big(B^{j_1}_Y\big), N^*\big(B^{j_2}_Y\big)\big] = 0.$$

10. Mixed empirical negative binomial ordered marked point process with independent marks

Definition 10.1. A random process $\mathcal{D} = (\mathcal{E}^*, \mathfrak{X}^*, P^*)$ is called a strictly simple mixed empirical negative binomial ordered marked point process with independent marks in a bounded space $(Y, \mathfrak{A}_Y, \mathcal{B}_Y)$ if the size of a sample N^* is a random variable with the negative binomial distribution.

Similarly to Section 6 of [1], one can evaluate the following probability generating functions:

$$\Pi_{N^*(B^1_Y),\dots,N^*(B^s_Y)}(z_1,\dots,z_s) = \left[1 + \beta\sum_{j=1}^{s}\mu\big(B^j_Y\big)(1 - z_j)\right]^{-\alpha}, \tag{45}$$

$$\Pi_{N^*(B^j_Y)}(z_j) = \Big[1 + \beta\mu\big(B^j_Y\big)(1 - z_j)\Big]^{-\alpha}, \qquad j = 1,\dots,s. \tag{46}$$

Then, using (37)–(42) and (45), (46), we obtain the following moment characteristics for a family of bounded measurable disjoint sets $\{B^j_Y, j = 1,\dots,s\}$ that form a partition of the phase space $Y = X \times K$:

$$\nu^{(h)}_{\mathcal{D}}\big(B^{j_1}_Y \times \cdots \times B^{j_h}_Y\big) = \beta^h\prod_{i=1}^{h}(\alpha + i - 1)\mu\big(B^{j_1}_Y\big)\dots\mu\big(B^{j_h}_Y\big)$$

$$1 \le j_1 < \cdots < j_h \le s, \qquad h = 1,\dots,s,$$

$$\alpha^{(h)}_{\mathcal{D}}\big(B^j_Y\big) = \beta^h\prod_{i=1}^{h}(\alpha + i - 1)\mu^h\big(B^j_Y\big), \qquad \nu^{(1)}_{\mathcal{D}}\big(B^j_Y\big) = \lambda\beta\mu\big(B^j_Y\big),$$

$$\nu^{(2)}_{\mathcal{D}}(B^j_Y) = \lambda\beta\mu(B^j_Y)[(\alpha + 1)\beta\mu(B^j_Y) + 1],$$

$$\nu^{(2)}_{\mathcal{D}}\big(B^{j_1}_Y \times B^{j_2}_Y\big) = \alpha(\alpha + 1)\beta^2\mu\big(B^{j_1}_Y\big)\mu\big(B^{j_2}_Y\big),$$

$$D\big(N^*\big(B^j_y\big)\big) = \alpha\beta\mu\big(B^j_Y\big)\big[\beta\mu\big(B^j_Y\big) + 1\big],$$

$$\operatorname{cov}\big[N^*\big(B^{j_1}_Y\big), N^*\big(B^{j_2}_Y\big)\big] = \alpha\beta^2\mu\big(B^{j_1}_Y\big)\mu\big(B^{j_2}_Y\big), \qquad 1 \le j_1,\ j_2 \le s,\ j_1 \ne j_2. \tag{47}$$

Considering (45)–(47) we make the following conclusions:

a) Counting measures $N^*(B^1_Y),\dots,N^*(B^s_Y)$ form a family of mutually correlated identically distributed random variables with negative binomial distribution with parameters $\beta\mu(B^j_Y) > 0$, $\alpha > 0$, $j = 1,\dots,s$.

b) There is a positive correlation between the counting measures $N^*(B_Y^i)$ and $N^*(B_Y^r)$ $(1 \leq i, r \leq s, i \neq r)$.

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Received 15/SEP/2011
Translated by S. KVASKO